Absorbing layers for shallow water models

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Absorbing layer

as open boundary treatment

Ingoing waves
Outgoing waves
Absorbing layer

as open boundary treatment

- Extension of the computational domain
  
  The fields are subject to a particular treatment

- Prescribe progressively the external forcing

- Minimize the reflection of outgoing waves
The best layer for linear gravity wave
Different kinds of layers
Comparison of layers
The best layer for linear gravity wave
Different kinds of layers
Comparison of layers
Absorbing layer
for linear gravity wave

Basic equations

\[ \frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} = 0 \]
\[ \frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0 \]
Absorbing layer
for linear gravity wave

Equations with absorption terms

\[
\frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} = -\sigma \eta \\
\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = -\sigma u
\]
Absorbing layer
for linear gravity wave

Equations with absorption terms

\[ \frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} = -\sigma \eta \]
\[ \frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = -\sigma u \]

Which absorption coefficient?
The best absorption coefficient

A discrete problem

Reflection coefficient

σ uniform value
The best absorption coefficient

A discrete problem

In the continuous context, the reflection coefficient is 0 if

\[ \int_{\text{Layer}} \sigma(x) \, dx = +\infty \]
The best absorption coefficient

A discrete problem

In the continuous context, the reflection coefficient is 0 if

$$\int_{\text{Layer}} \sigma(x) \, dx = +\infty$$

In the discrete context, the fields variations must also be represented by the discrete scheme.
The best absorption coefficient

Optimization procedure
The best absorption coefficient

Optimization procedure

Minimize Energy norm of the reflected signal

\[ \int_{\text{Domain}} \left[ \frac{1}{2} g (\eta - \eta_{\text{reference}})^2 + \frac{1}{2} h (u - u_{\text{reference}})^2 \right] \, dx \]

Function of the absorption coefficient
The best absorption coefficient

Discrete optimum


Shifted hyperbola: \[ \frac{\sqrt{gh}}{\delta} \frac{x}{x - \delta} \]
The best layer for linear gravity wave

Different kinds of layers

Comparison of layers
Flow relaxation scheme (FRS)

An easy-used layer


\[
\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = -\sigma (H - H^{ext})
\]

\[
\begin{align*}
\frac{\partial (Hu)}{\partial t} &+ \frac{\partial (gH^2/2)}{\partial x} + \frac{\partial (Hu^2)}{\partial x} + \frac{\partial (Huv)}{\partial y} - fHv = -\sigma (Hu - Hu^{ext}) \\
\frac{\partial (Hv)}{\partial t} &+ \frac{\partial (gH^2/2)}{\partial y} + \frac{\partial (Huv)}{\partial x} + \frac{\partial (Hv^2)}{\partial y} + fHu = -\sigma (Hv - Hv^{ext})
\end{align*}
\]

Relaxation terms
Adapted FRS
An other easy-used layer


\[
\begin{align*}
\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial ( Hv)}{\partial y} &= -(\sigma_x + \sigma_y)(H - H_{ext}) \\
\frac{\partial (Hu)}{\partial t} + \frac{\partial (gH^2/2)}{\partial x} + \frac{\partial (Hu^2)}{\partial x} + \frac{\partial (Huv)}{\partial y} - fHv &= -\sigma_x(Hu - Hu_{ext}) \\
\frac{\partial (Hv)}{\partial t} + \frac{\partial (gH^2/2)}{\partial y} + \frac{\partial (Huv)}{\partial x} + \frac{\partial (Hv^2)}{\partial y} + fHu &= -\sigma_y(Hv - Hv_{ext}) \\
\end{align*}
\]

Other relaxation terms
Perfectly matched layer (PML)

A layer with theoretical justification

\[ \frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = - (\sigma_x + \sigma_y)(H - H_{\text{ext}}) - q_H \]

\[ \frac{\partial (Hu)}{\partial t} + \frac{\partial (gH^2/2 + Hu^2)}{\partial x} + \frac{\partial (Huv)}{\partial y} - f H_u = - (\sigma_x + \sigma_y)(Hu - Hu_{\text{ext}}) - q_{Hu} \]

\[ \frac{\partial (Hv)}{\partial t} + \frac{\partial (gH^2/2 + Huv)}{\partial y} + \frac{\partial (Hv^2)}{\partial x} + f H_v = - (\sigma_x + \sigma_y)(Hv - Hv_{\text{ext}}) - q_{Hv} \]

With the additional equations:

\[ \frac{\partial q_H}{\partial t} = \sigma_x \sigma_y (H - H_{\text{ext}}) + \sigma_y \frac{\partial (Hu - Hu_{\text{ext}})}{\partial x} + \sigma_x \frac{\partial (Hv - Hv_{\text{ext}})}{\partial y} \]

\[ \frac{\partial q_{Hu}}{\partial t} = (\sigma_x + \sigma_y)(-f Hv + (f Hv)_{\text{ext}}) + \sigma_x \sigma_y (Hu - (Hu)_{\text{ext}} + \hat{q}_{Hu}) \]

\[ + \sigma_y \frac{\partial [(gH^2/2 + Hu^2) - (gH^2/2 + Hu^2)_{\text{ext}}]}{\partial x} + \sigma_x \frac{\partial [(Huv) - (Huv)_{\text{ext}}]}{\partial y} \]

\[ \frac{\partial q_{Hv}}{\partial t} = (\sigma_x + \sigma_y)(f Hu - (f Hu)_{\text{ext}}) + \sigma_x \sigma_y (Hv - (Hv)_{\text{ext}} + \hat{q}_{Hv}) \]

\[ + \sigma_y \frac{\partial [(Huv) - (Huv)_{\text{ext}}]}{\partial x} + \sigma_x \frac{\partial [(gH^2/2 + Hv^2) - (gH^2/2 + Hv^2)_{\text{ext}}]}{\partial y} \]

\[ \frac{\partial \hat{q}_{Hu}}{\partial t} = -f Hv + (f Hv)_{\text{ext}} \quad \frac{\partial \hat{q}_{Hv}}{\partial t} = f Hu - (f Hu)_{\text{ext}} \]
The best layer for linear gravity wave
Different kinds of layers
Comparison of layers
Test case

Collapse of the Gaussian-shaped mound of water

![Diagram showing a rectangular domain with a layer inside it. The axes are labeled x and y.]
Test case

Collapse of the Gaussian-shaped mound of water
Elevation and error using the shifted hyperbola

Elevation after 9h

Error with the FRS

Error with the adapted FRS

Error with the PML
Reflection ratio

A measure of the layer efficiency

\[
\text{Reflection ratio} = \frac{\text{Energy norm of the reflected signal}}{\text{Energy norm of the initial fields}}
\]
Reflection ratio

A measure of the layer efficiency

\[
\text{Reflection ratio} = \frac{\text{Energy norm of the reflected signal}}{\text{Energy norm of the initial fields}}
\]

- **Shifted hyperbola**
  \[\sqrt{g\delta} \frac{x}{(\delta-x)}\]

- **Optimum parabola**
  \[\sigma_m \left(\frac{x}{\delta}\right)^2\]

Graph showing results for:
- FRS
- Adapted FRS
- PML
Reflection ratio

A measure of the layer efficiency

\[
\text{Reflection ratio} = \frac{\text{Energy norm of the reflected signal}}{\text{Energy norm of the initial fields}}
\]

Shifted hyperbola

\[ \sqrt{\frac{gh}{\delta}} \frac{x}{(\delta-x)} \]

Parabola

\[ \sigma_m \left( \frac{x}{\delta} \right)^2 \]
Summary and conclusion

✔ Absorbing layer
  ▶ The PML gives the best results
    A layer with a theoretical justification
    Additional fields and equations
  ▶ The adapted FRS is better than the FRS
    Easy to use

✔ Absorption coefficient
  ▶ The choice of this coefficient is a discrete problem
  ▶ The shifted hyperbola is the best for gravity wave
    No additional parameters to adjust
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