MODEL REDUCTION TECHNIQUES IN NONLINEAR DYNAMICS

USING PROPER ORTHOGONAL DECOMPOSITION

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Finite element simulations are increasingly large

Accurate and detailed modelling

High number of DOFs

High computational time

[D’Otreppe, 2009]

[Cenaero]
The size of the model and the computational time have to be reduced

From the detailed mathematical model, the goal is to create a reduced model which

- represents well the dynamics
- is robust w.r.t. parameter changes
- reduces the computational time
Outline

1. Galerkin projection
2. Non-linear beam formulation
3. Proper orthogonal decomposition
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The accuracy is closely related to the choice of the projection basis.

**Full Space** $\mathbb{R}^N$
- $R(\dot{x}, x, t) = 0$
- $x(t = 0) = x^0$

**Reduced Space** $\mathbb{V}^k$
- $\tilde{x} = \Phi x_r$

**Reduced Transform**
- $x \approx \tilde{x} = \Phi x_r$

**Reduced Problem**
- $\Phi^T R(\Phi \dot{x}_r, \Phi x_r, t) = 0$
- $x_r(t = 0) = \Phi^T x^0$

Galerkin projection  Non-linear beam formulation  Proper orthogonal decomposition
The computational time is not reduced as much as expected in non-linear dynamics.

\[ \Phi^T M \Phi \mathbf{x}_r + \Phi^T F_{NL} (\Phi \dot{x}_r, \Phi \mathbf{x}_r, t) = \Phi^T F_{ext} \]

\[ k \times N \quad N \times 1 \]

Computational complexity of non-linear term still depends on \( N \)
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3  Proper orthogonal decomposition
The beam formulation takes into account the large displacements

Kinematic assumptions
- Initially straight
- Beam cross-sections remain plane
- Deformation of the neutral axis allowed
- Rotational kinetic energy of cross-sections

[Cardona & Lens, 2008]
Non-linear beam discretization

\[
\mathbf{\textbf{q}} = \begin{bmatrix}
\mathbf{x}_1 \\
\mathbf{\psi}_1 \\
\mathbf{x}_2 \\
\mathbf{\psi}_2
\end{bmatrix}
\]

\[
\mathbf{x}(s) = N_1(s) \mathbf{x}_1 + N_2(s) \mathbf{x}_2
\]

\[
\mathbf{\psi}(s) = N_1(s) \mathbf{\psi}_1 + N_2(s) \mathbf{\psi}_2
\]

\[
\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}^{int}(\mathbf{q}) = \mathbf{g}^e
\]
The home-made Matlab results agree with Oofelie results

[Simo et al 1988, Cardona & Lens 2007]
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The Proper Orthogonal Decomposition provides optimal projection basis

Basis definition

\[
\min_{\Phi} \sum_{i} \| \mathbf{x}(t = t_i) - \tilde{\mathbf{x}}(t = t_i) \|^2_K
\]

subject to

\[
(\Phi, \Phi)_K = \mathbb{I}_{k \times k}
\]

with the \(K\)-inner product

\[
(u, v)_K = \overline{u}^T K \overline{v}
\]

Properties

- data-driven method
- minimization of the error between \(\mathbf{x}\) and \(\tilde{\mathbf{x}}\)
The quality of the basis depends on the snapshots matrix

Simulation

Pre-process

\[ \mathbf{x}(t_i) = \mathbf{x}(t_i) - \bar{\mathbf{x}} \quad i = 1, \ldots, T \]

Eigenvalue problem

\[ \frac{1}{T} \mathbf{X}^T \mathcal{K} \mathbf{X} \Psi = \lambda \Psi \]

Orthonormalization

\[ \Psi \rightarrow \Phi : \Phi^T \Phi = \mathbb{I}, \text{span}(\Phi) = \text{span}(\Psi) \]

\[ \tilde{\mathbf{x}} = \bar{\mathbf{x}} + \Phi \mathbf{x}_r \]

Galerkin projection Non-linear beam formulation Proper orthogonal decomposition
The Proper Orthogonal values give information about the size of the basis

Truncation error connected with the eigenvalues and the amount of « energy » included in the basis span

Galerkin projection  Non-linear beam formulation  Proper orthogonal decomposition
The weighted POD improves the reduced model

Cartesian: $\mathcal{K} = \mathbb{I}$

Stiffness: $\mathcal{K} = \mathbf{K}_{tg}(t = t_0)$

« Energy » related to the linear potential energy $\mathcal{V} = \frac{1}{2} \mathbf{x}^T \mathbf{K}_{tg} \mathbf{x}$
The POD-based reduced model is not robust w.r.t parameter changes

Parameter: geometry, excitation, initial conditions,…

Range of validity of POD-based reduced model is small

Galerkin projection Non-linear beam formulation Proper orthogonal decomposition
The robustness can be increased by interpolation or extrapolation/expansion

Interpolation: differential geometry principle, interpolation in the tangent space of the Grassmann manifold

Extrapolation/expansion: use of the POD basis sensitivities

Normalized Mean Square Error

Classic POD

Interpolated POD

Thickness variation (%)
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Conclusions and further work

- Reduction of non-linear problems by Galerkin projection
- Non-linear beams benchmark
- Projection basis computed by proper orthogonal decomposition
- Influence of the metric: cartesian/stiffness
- Increase of the robustness by interpolation/expansion

- Efficient discretization of the parameter space
- Reduction of the complexity of non-linear terms
Further research is still needed

Reduction in the complexity of the non-linear term:

- Empirical Interpolation method

Discretization of the parameter space:

- Greedy method

Error estimator without the full response:

- Dual-Weighted-Residual