# Certificate of infeasibility and cutting planes from lattice-point-free polyhedra

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Joint work with K. Andersen (Copenhagen), R. Weismantel (Magdeburg)

# Outline

- Split Cuts
- Lattice-Point-Free Polyhedra
- Integral Farkas Lemma for Systems with Inequalities
- Extension to the mixed case
- Cutting Planes from Lattice-Point-Free Polyhedra
- Conclusion

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# Split cuts

## The algebra

Based on a disjunction

$$\pi^T x \leq \pi_0 \quad \text{or} \quad \pi^T x \geq \pi_0 + 1$$

is valid for $x \in \mathbb{Z}^n$ when $\pi, \pi_0$ are integer.

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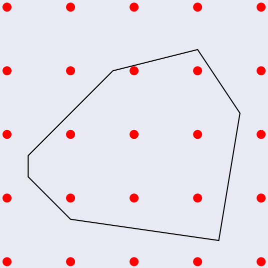

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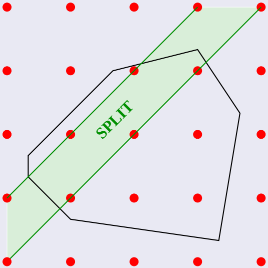

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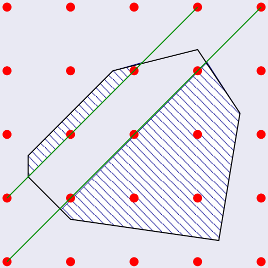

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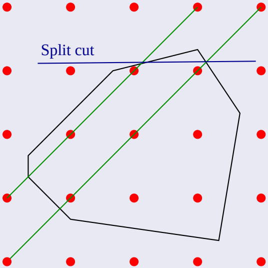

# More about split cuts

## The split closure

Consider a polyhedron $P \subseteq \mathbb{R}^n$, the intersection of all split cuts of $P$ is called the (first) split closure of $P$, denoted by $\mathrm{SC}(P)$.

## Some previous results

- Cook, Kannan, Schrijver [1990] The split closure is a polyhedron
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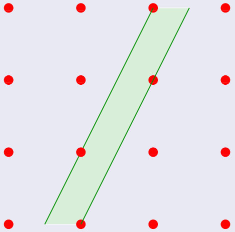

A basic split set in $\mathbb{R}^2$ is a lattice-point-free polyhedron

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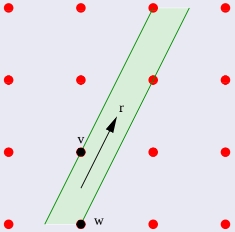


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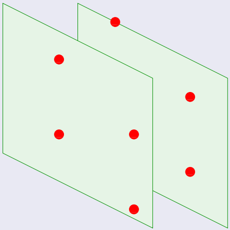

A basic split set in $\mathbb{R}^3$ is a lattice-point-free polyhedron

# Towards high-dimensional splits

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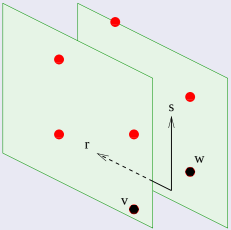


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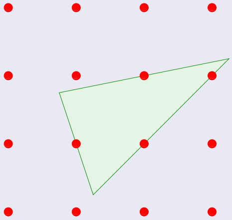

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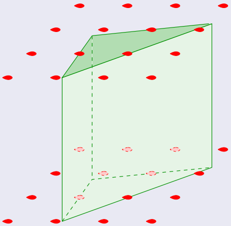

- A triangle in $\mathbb{R}^2$ can be lattice-point-free
- It can be lifted to a lattice-point-free polyhedron in $\mathbb{R}^3$

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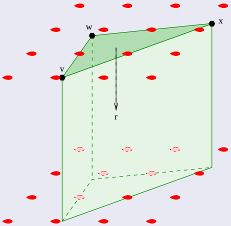


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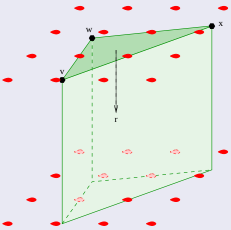


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## Definition of the split dimension

A lattice-point-free polyhedron $P \subseteq \mathbb{R}^n$ can be written as

$$P = \text{conv}\{v^1, \ldots, v^p\} + \text{cone}\{w^1, \ldots, w^q\} + \text{span}\{r^1, \ldots, r^{n-d}\}.$$

The split-dimension of $P$ is $d$.

# Classical Farkas Lemma

**The continuous Farkas Lemma [Farkas, 1902]**

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$,

$$\begin{array}{l} Ax \leq b \\ x \geq 0 \\ x \in \mathbb{R}^n \end{array} \quad \text{is empty if and only if} \quad \begin{array}{l} y^T A \geq 0 \\ y^T b < 0 \\ \text{for some } y \in \mathbb{R}^m. \end{array}$$

**Example**

$$\begin{array}{rrcl} (1) & 10x_1 + 14x_2 & \leq & 35 \\ (2) & -x_1 + x_2 & \leq & 0 \\ (3) & -x_2 & \leq & -2 \end{array}$$

A certificate of infeasibility

$$y = (1 \quad 8 \quad 21)^T$$

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$$\begin{array}{lrcl} (1) & 3x_1 + x_2 - 5x_3 + x_4 - 7x_5 & = & 1 \\ (2) & 7x_1 - 3x_2 - 3x_3 - 2x_4 + 5x_5 & = & 5 \\ (3) & 2x_1 + x_2 + x_3 + 6x_4 & = & 1 \end{array}$$

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$$\sum \quad 7x_1 - x_2 - 3x_3 + x_4 + x_5 = \frac{13}{3}$$

# Geometric interpretation of the Integral Farkas Lemma

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| $Ax = b$ | $\{v^*\} + \text{span}\{w^1, \dots, w^d\}$ |
| $y^T A$ | subset of $\text{span}\{w^1, \dots, w^d\}^\perp$ |
| $y^T b \notin \mathbb{Z}$ | there exists $\pi \in \text{span}\{w^1, \dots, w^d\}^\perp \cap \mathbb{Z}^n$ with $\pi^T v^* \notin \mathbb{Z}$. |

Equivalent to say that $L = \{\lfloor \pi^T v^* \rfloor \leq \pi^T x \leq \lceil \pi^T v^* \rceil\}$ contains $Ax = b$ in its interior.

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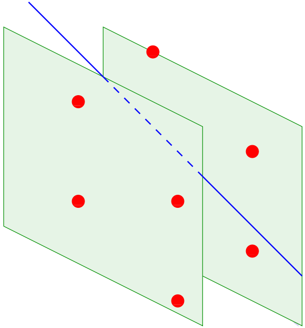

# Natural extension for one range inequality

## Integral Farkas Lemma with one range inequality [Andersen, L. , Weismantel 2007]

$$\begin{array}{l} Ax = b \\ l \le cx \le u \\ x \in \mathbb{Z}^n \end{array} = \emptyset \quad \text{iff} \quad \exists y \in \mathbb{Q}^m, z \in \mathbb{Q}_+ \text{ with } \begin{array}{l} (y^T \; z) \begin{pmatrix} A \\ c \end{pmatrix} \in \mathbb{Z}^n \\ [y^T b + zl, y^T b + zu] \cap \mathbb{Z} = \emptyset \end{array}$$

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The certificate

$$y = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \end{pmatrix}, z = \frac{1}{5}$$

$$\begin{array}{lll} \frac{2}{5}(1) & \frac{6}{5} = & \frac{4}{5}x_1 + \frac{2}{5}x_2 + \frac{6}{5}x_3 - \frac{2}{5}x_4 = \frac{6}{5} \\ \frac{1}{5}(2) & 1 = & \frac{6}{5}x_1 - \frac{1}{5}x_2 - \frac{2}{5}x_3 + \frac{1}{5}x_4 = 1 \\ \frac{1}{5}(3) & 1 \le & \frac{4}{5}x_2 + \frac{1}{5}x_3 - \frac{4}{5}x_4 \le \frac{8}{5} \end{array}$$

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$$\begin{array}{llrl} \frac{2}{5}(1) & \frac{6}{5} = & \frac{4}{5}x_1 + \frac{2}{5}x_2 + \frac{6}{5}x_3 - \frac{2}{5}x_4 & = \frac{6}{5} \\ \frac{1}{5}(2) & 1 = & \frac{6}{5}x_1 - \frac{1}{5}x_2 - \frac{2}{5}x_3 + \frac{1}{5}x_4 & = 1 \\ \frac{1}{5}(3) & 1 \leq & \frac{4}{5}x_2 + \frac{1}{5}x_3 - \frac{4}{5}x_4 & \leq \frac{8}{5} \end{array}$$

$$\sum \quad \frac{16}{5} \leq 2x_1 + x_2 + x_3 - x_4 \leq \frac{19}{5}$$

# Geometry of the Farkas Lemma with one range inequality

$$Ax = b$$
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$E^* + \text{span}\{w^1, \ldots, w^d\}$,
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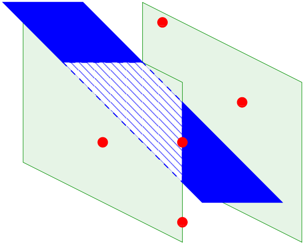

## Idea

$$\begin{aligned} Ax &= b \\ Cx &\leq d \\ x &\in \mathbb{Z}^n \end{aligned} \qquad (1)$$

The bigger rank$(C)$, the more complicate the certificate of infeasibility.
(1) is infeasible if and only if $\{Ax = b, Cx \leq d\}$ is contained in the interior of a lattice-point-free polyhedron of split-dimension equal to rank$(C)$.

## Integral Farkas Lemma for Systems with Equalities and Inequalities

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A certificate of infeasibility of (1) is an integral infeasible linear system (derived from the rows of (1)) with as many variables as rank$(C)$.

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# Example with $\text{rank}(C) = 2$

$$\begin{array}{lrcl}
(1) & x_1 + 2x_2 + 3x_3 & = & 0 \\
(2) & -3x_1 + 4x_2 & \leq & 0 \\
(3) & -x_1 - 2x_2 & \leq & -3 \\
(4) & 2x_1 - x_2 & \leq & 5
\end{array}$$

A certificate

$$\begin{array}{ll}
\frac{1}{3}(1) + \frac{1}{12}(2): & x_2 + x_3 \leq -\frac{1}{12}x_1 \\
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It is a system with 2 variables and 3 inequalities

$$y \leq -\frac{1}{12}x_1 \qquad y \geq -\frac{1}{2}x_1 + \frac{1}{2} \qquad y \geq \frac{1}{3}x_1 - \frac{5}{3}$$

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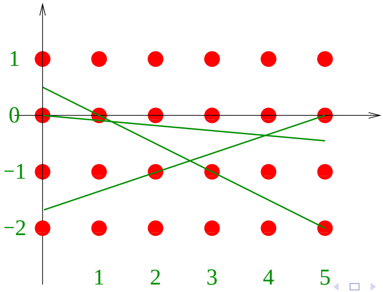

# Sketch of the proof on the rank 2 example

$X = \{x \in \mathbb{R}^n | Ax = b, Cx \leq d\}$ with $\text{rank}(C) = 2$.
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We can rewrite the system using 2 variables corresponding to $v^1$ and $v^2$ respectively.

Final System

# Integral Farkas Lemma for Systems with Equalities and Inequalities

**Theorem** [Andersen, L. , Weismantel 2007]

Let $A \in \mathbb{Z}^{m\times n}$, $C \in \mathbb{Z}^{p\times n}$ with $\text{rank}(C) = L$.

$$Ax = b$$
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is empty if and only if

- $\exists y^1, \ldots, y^t \in \mathbb{Q}^m \times \mathbb{Q}^p_+$
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$$(y^k)^T \begin{bmatrix} A \\ C \end{bmatrix} = \sum_{i=1}^{L} \lambda_i^k v^i \quad \in \mathbb{Z}^n \text{ with } \lambda_i^k \in \mathbb{Z}$$

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# Extension to the mixed case

## The Mixed-Integer Farkas Lemma for equality systems

Let $A \in \mathbb{Z}^{m \times n}$, $G \in \mathbb{R}^{m \times p}$, $b \in \mathbb{Z}^m$,

$$\begin{array}{l} Ax + Gs = b \\ x \in \mathbb{Z}^n, s \in \mathbb{R}^p \end{array} \quad \text{is empty if and only if} \quad \exists y \in \mathbb{Q}^m \text{ with } \begin{array}{l} y^T A \in \mathbb{Z}^n \\ y^T G = 0 \\ y^T b \notin \mathbb{Z} \end{array}$$

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**Example**

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The certificate

$$y = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\begin{array}{ll} \frac{1}{3}(1) & \frac{1}{3}x_1 + \frac{2}{3}x_2 + x_3 + \frac{2}{3}s = \frac{4}{3} \\ \frac{2}{3}(2) & \frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}s = 0 \end{array}$$

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# The philosophy for the mixed case

A set $X \subseteq \mathbb{R}^{n+p}$ has no mixed-integer solutions, namely $X \cap (\mathbb{Z}^n \times \mathbb{R}^p) = \emptyset$, if and only if the projection to the "integer space" has no integral solution.

**Theorem** [Andersen, L. , Weismantel 2007]

Let $A \in \mathbb{Z}^{m \times n}, G \in \mathbb{R}^{m \times p}, C \in \mathbb{Z}^{q \times n}, H \in \mathbb{R}^{q \times p}$ with $\text{rank}([C, H]) = L$.

$$\begin{aligned} Ax + Gs &= b \\ Cx + Hs &\leq d \\ x \in \mathbb{Z}^n&, s \in \mathbb{R}^p \end{aligned}$$

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the system in variables $z$ (representing $(v^i)^T x$)

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has no integral solution.

# Mixed-Integer Farkas Lemma for Systems with Equalities and Inequalities

Theorem [Andersen, L. , Weismantel 2007]

Let $A \in \mathbb{Z}^{m\times n}, G \in \mathbb{R}^{m\times p}, C \in \mathbb{Z}^{q\times n}, H \in \mathbb{R}^{q\times p}$ with $\text{rank}([C, H]) = L$.

$$\begin{aligned} Ax + Gs &= b \\ Cx + Hs &\le d \\ x \in \mathbb{Z}^n&, s \in \mathbb{R}^p \end{aligned}$$

is empty if and only if $\exists y^1, \ldots, y^t \in \mathbb{Q}^m \times \mathbb{Q}^q_+$ such that

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The projection operation does not change rank($[C, H]$) (or at least does not increase it) !

**Example :**

$$\begin{aligned} x_1 + x_2 \quad\quad - s &= 1 \\ 0 \le x_1 + 2x_2 + 3x_3 + s &\le 1 \end{aligned}$$

Rank of the inequality system is 1.

**Projecting out the $s$ variable**
Using $s = x_1 + x_2 - 1$,

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# Using the lattice-point-free polyhedra to generate cuts

## The algebra

Let $P \subseteq \mathbb{R}^{n+m}$ be a polyhedron and $L \subseteq \mathbb{R}^n$ be a lattice-point-free polyhedron. We define a set of cuts, valid for $\{(x,y) \in \mathbb{R}^{n+m} | x \in P \cap \mathbb{Z}^n\}$ as

$$\text{cuts}_P(L) = \text{conv}\{(x,y) \in \mathbb{R}^{n+m} | (x,y) \in P \text{ and } x \notin \text{int}(L)\}.$$

## The geometry

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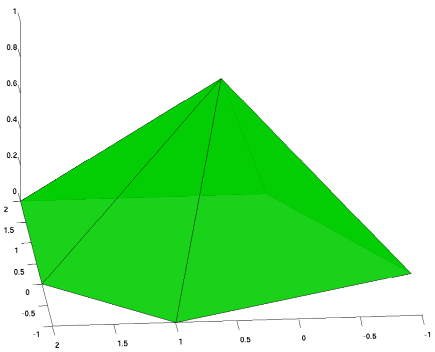

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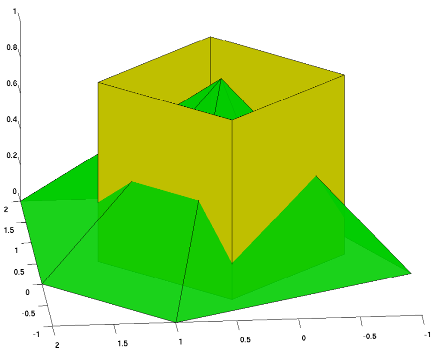

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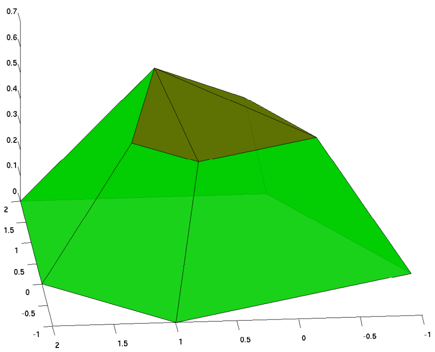

# The high-dimensional split closure

## Definition

The *$d$-dimensional split closure* of $P$ is the set of points in the intersection of all high-dimensional split cuts obtained from $P$ with a split-dimension less or equal to $d$.

## Open question

Is the $d$-dimensional split closure of a polyhedron a new polyhedron ?

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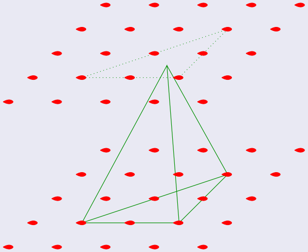

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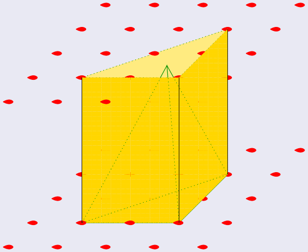

# Family of polyhedra of dimension $n+1$ with an infinite $n$-dimensional split rank

Constructed in the same way :

- a $n$-dimensional lattice-point-free polyhedron with integer points on the interior of each facet
- lifted by an $\epsilon$ in a $(n+1)$th variable

$$P = \text{conv}\{(0,0),(ne_1,0),(ne_2,0),\ldots,(ne_n,0),(\frac{1}{2}\underline{1},\epsilon)\}$$

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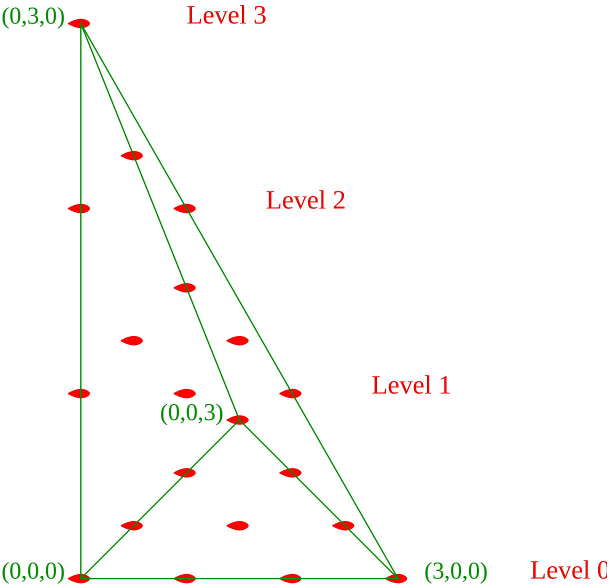
(0,3,0)
Level 3
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- We have characterized the dimension needed in the split bodies in order to be able to cut down to the integer hull
- It gives a lower and an upper bound
- It is determined by the maximum of each face
- Link to the finite cutting plane algorithm [Jörg 2007]

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- How to use them in practice ? Closed form formulae ?
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