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Adaptive λ-tracking for Systems with Higher Relative Degree

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Stuttgart, April 2001 Eric Bullinger

Contents

Summary

For many control applications, good models are not available or their parameters are not precisely known. A possibility to control these systems is to use an adaptive λ-tracking controller. For designing this controller, only the knowledge of the model structure, not of precise parameter values is needed. Therefore, the controller is robust for a large class of uncertainties.

Exact tracking is not required for many applications. Based on the necessary performance and on the measurement quality, the user can specify a tolerance for the tracking error which should be achieved. The objective of λ-tracking is that the tracking error asymptotically tends to $[-\lambda, \lambda]$ where λ is a tolerance specified by the user.

Most controllers achieving λ-tracking can only be used for systems having a relative degree of one. The adaptive λ-tracking controller proposed in this thesis extends the system class to systems with higher relative degree. This is achieved by having in the controller an observer which estimates the output of the system and its first $r-1$ derivatives. Another component of the controller is an observer-based state-feedback. Both the controller and the state-feedback include a high-gain parameter, the controller gain k. For a sufficiently large value of this parameter, the controller is guaranteed to achieve λ-tracking. Instead of fixing this parameter a priori, the following adaptation scheme is used. The parameter k is increased if the output is outside of the λ-strip and kept constant within. This allows to start with a relatively small value for k and, nevertheless, being robust for a large class of uncertainties.

A further advantage of the proposed adaptive λ-tracking controller is its relatively simple structure which is helpful for implementing it and for understanding how the controller works. The main drawbacks of the adaptive λ-tracking controller are that the performance is not directly addressed and that the parameter k might become large. On the one hand, this increases the sensitivity to measurement noise. Also peaking is then more likely. On the other hand, a small k usually means that the tracking error is for quite a long time relatively large.

Adaptive λ-tracking controllers have been applied to several control problems. In this thesis, the adaptive λ-tracking controller is applied to a control problem in anesthesia. The control objective is to keep the endtidal concentration of anesthetic gas close to a target value chosen by the anesthetist. This target is changed several times during a surgery. It is shown that, after suitable modifications, the adaptive λ-tracking controller achieves the task of λ-tracking in a satisfactory manner.

Zusammenfassung

Für viele Regelungsanwendungen sind keine guten Modelle vorhanden, oder die Modellparameter sind nur ungenau bekannt. Bei der Regelung solcher Systeme kann ein adaptiver λ-Trajektorienfolgeregler angewendet werden. Um diesen Regler zu entwerfen, ist nur die Kenntnis der Modellstruktur nötig, nicht die der genauen Parameterwerte. Dadurch ist der Regler sehr robust gegenüber Modellunsicherheiten.

Exaktes Trajektorienfolgen ist bei vielen Anwendungen nicht notwendig. Auf Grund der benötigten Regelgüte und der Qualität der Messungen kann der Benutzer angeben, wie klein der Regelfehler sein soll. Das Regelziel bei λ-Trajektorienfolgen ist, daß der Ausgangsfehler asymptotisch gegen $[-\lambda, \lambda]$ geht, wobei λ die vom Benutzer spezifizierte Toleranz ist.

Die meisten Regler, die λ-Trajektorienfolgen erreichen, können nur für Systeme mit Relativgrad eins angewendet werden. Der in dieser Arbeit vorgeschlagene adaptive λ-Trajektorienfolgeregler erweitert die Systemklasse auf Systeme mit höherem Relativgrad. Dies wird dadurch erreicht, daß der Regler einen Beobachter enthält, der den Systemausgang und dessen erste $r-1$ Ableitungen schätzt. Eine weitere Komponente des Reglers ist eine beobachterbasierte Zustandsrückführung. Sowohl der Beobachter als auch die Zustandsrückführung enthalten einen Parameter mit großer Verstärkung, die Reglerverstärkung k. Für genügend große Werte dieses Parameters erreicht der Regler λ-Trajektorienfolgen. Anstatt diesen Parameter a priori festzulegen, wird folgendes Adaptionsschema benutzt: Der Parameter k wird erhöht, falls der Ausgang außerhalb des λ-Streifens ist, und wird im Streifen konstant gehalten. Dies erlaubt es, mit einem relativ kleinen Wert für k zu starten und trotzdem Robustheit für eine große Klasse von Unsicherheiten zu erreichen.

Ein weiterer Vorteil des in dieser Arbeit vorgestellten adaptiven λ-Trajektorienfolgereglers ist seine relativ einfache Struktur. Sie erleichtert es, das Funktionieren des Reglers zu verstehen und ihn zu implementieren. Hauptnachteil des adaptiven λ-Trajektorienfolgereglers ist, daß die Regelgüte nicht direkt in den Entwurf eingeht und die Reglerverstärkung k groß werden kann. Einerseits erhöht eine große Reglerverstärkung sowohl die Sensitivität gegenüber Meßrauschen als auch die Möglichkeit, daß während der transienten Phase Zustände kurzfristig sehr weit weg getrieben werden. Andererseits kann bei kleiner Verstärkung der Trajektorienfolgefehler für lange Zeit relativ groß sein.

Adaptive λ-Trajektorienfolgeregler sind auf verschiedene Regelungsprobleme angewendet worden. In dieser Arbeit wird der vorgestellte adaptive λ-Trajektorienfolgeregler auf ein Regelungsproblem der Anästhesie angewendet. Das Regelziel ist, die endtidale Konzentration von Anästhesiegas nahe bei dem vom Anästhesisten gewählten Referenzwert zu halten. Dieser Wert ändert sich mehrere Male während einer Operation. Es wird gezeigt, daß nach geeigneten Modifikationen der adaptive λ-Trajektorienfolgeregler das Regelziel

zufriedenstellend erreicht.

Introduction

Motivation

Many controller designs are based on the assumption that good models are available. It is then possible to use these models to design controllers guaranteeing stability and good performance. Finding good models is, however, often a lengthy and therefore costly task, if at all possible. Furthermore, at each change of the plant, the model needs to be adapted. This can also happen gradually, for example due to wearout. Another difficulty in modeling comes from the fact that similar systems can have significantly different model parameters. A good example for a system that is uncertain and difficult to model is the human body. Especially, some model parameters differ much from one person to the other. Some of them, like the body weight, can easily be measured. Much more difficult to quantify are for example the differences caused by smoking or by regular physical exercise.

For many applications, a controller has to be designed without knowing the model parameters precisely. A possibility is to use a controller design method requiring only the knowledge of the structure of the model. A single controller designed in such a way should then stabilize any system having a certain model structure which can include uncertainty in the parameters and in the system order. Having an uncertain model usually also means that the controller should be an output feedback one, as the observation of the states is rather difficult without a precise model. Other common objectives are that the controller is able to stabilize the system at several set-points and achieves good control performance.

In many cases it is important that the controller is not too complicated. On the one hand, this simplifies the implementation. On the other hand, a controller with a clear internal structure makes it a lot easier to understand how it functions. This allows to tune the controller parameter in a straightforward manner and makes it possible for system operators to better understand the controller's behavior. A controller achieving robust stabilization while being relatively simple is the adaptive λ-tracking controller. Its two main features, approximate stabilization and adaptation, are presented next.

Even though approximate stabilization is often seen as a negative property of a controller, it is a very natural objective to tolerate a certain output error. Several reasons exists for using such an objective. In many applications, the measurements are subject to noise or have a finite precision. In such a case, the user should be able to specify which precision of the output he can tolerate and would like to obtain. Another case in which approximate stabilization is advantageous is if precise stabilization requires too much input energy. A typical example is position control of a system subject to stiction. Here, too, the user should be able to define a tolerable output error. This is exactly what λ-stabilization is about.

In λ-stabilization, the user defines by the positive parameter λ the output error around the set-point he is willing to tolerate. A straightforward extension of this is to allow the set-point to be a time-varying reference signal. Asymptotic tracking with a tolerance of λ is called λ-tracking.

To increase the robustness, controllers achieving λ-tracking can be adapted. The adaptation is usually of the following form: The controller has one adaptation parameter that is increased whenever the control objective is not attained and kept constant as long as the output is in a λ-neighborhood of the reference signal. This adaptation increases the set of systems that are stabilizable by a specific non-adaptive λ-tracking controller while achieving the same objectives.

In this thesis, special emphasis is placed on those adaptive controllers that can be used to stabilize or λ-track nonlinear systems of high relative degree. A new adaptive controller is proposed λ-stabilizing a large class of nonlinear systems in a robust manner. This controller is simple and can track practically all relevant reference signals.

As an application example, the proposed λ-tracking controller is applied to control the endtidal concentration in anesthesia. In simulations and in an experiment it is shown that this controller is well suited for applications where precise models are difficult to obtain or where the time varying nature of the process and system parameters makes it difficult to apply standard control strategies.

Introduction to adaptive λ-tracking

In this thesis, adaptive output-feedback controllers achieving λ-tracking for nonlinear systems are presented. These controllers only require the knowledge of structural information of the system, not of specific system parameters. With a fixed adaptation parameter, λ-tracking is achieved for a relatively large set of systems. The adaptation increases the set of systems that can be λ-tracked. Therefore, these controllers are very robust. The adaptation of the controller is guaranteed to converge and the control objective, namely that the output error is not larger than the user-defined parameter λ, is attained asymptotically.

λ-stabilization and -tracking

A classical control objective is that the output y of a system is asymptotically regulated to a constant reference y_{ref}, i.e. for $y(t) \in \mathbb{R}$

$$y(t) \to y_{ref} \text{ for } t \to \infty.$$

In many practical applications, such an objective is either not achievable or too restrictive. Instead, a certain output error is often a better choice. For example, if an upper and lower bound for a temperature during normal operation is specified, why should the controller keep the temperature constant up to the precision of the thermometer? Asymptotic λ-stabilization, usually just called λ-stabilization, is a suitable control objective for such applications. The output is not required to converge exactly to the steady-state y_{ref}, but to a ball of radius $\lambda > 0$ around it, i.e. for $y(t) \in \mathbb{R}$,

$$y(t) - y_{ref} \to [-\lambda, +\lambda] \text{ for } t \to \infty,$$

see also Figure 1.

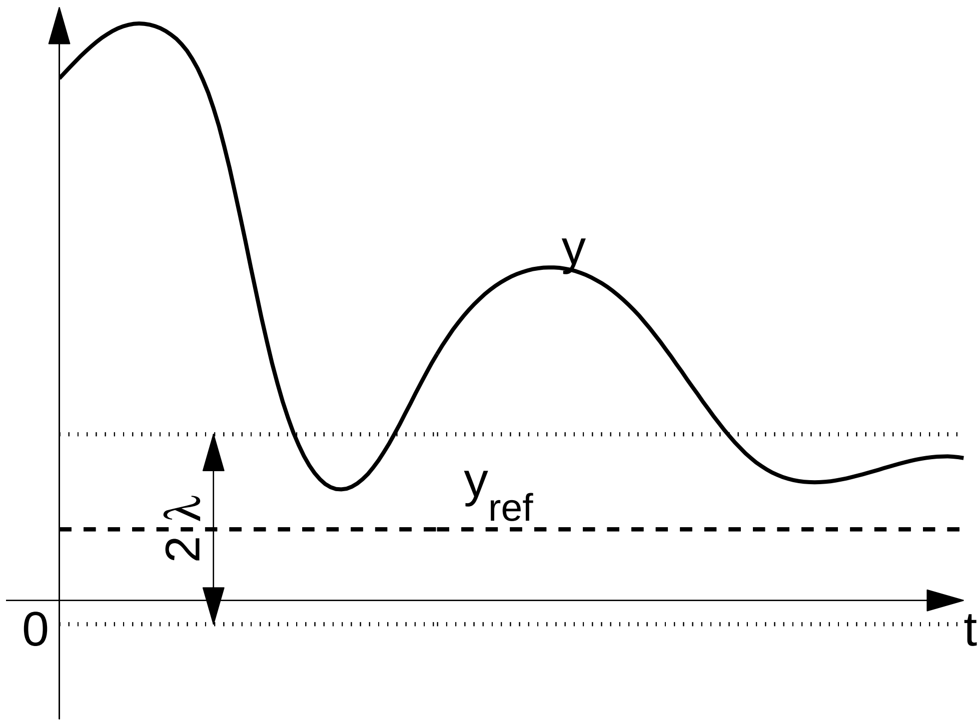


Figure 1: Sketch of λ-stabilization. Output y in solid, reference y_{ref} in dashed, λ-strip as dotted lines.

The concept of approximate tracking is not new. A classical example of approximate tracking is relay controllers that stabilize a system up to a limit cycle (see for example

Gibson, 1963; Föllinger, 1969). If the limit cycle lies inside the λ-strip, this is a sort of λ-stabilization. Another well established result is the steady-state offset: Linear systems without an open-loop pole at the origin do not converge to a constant, non-zero reference under proportional control (see for example Unbehauen, 1989). Closely related is also the concept of strong practical stability (La Salle and Lefschetz, 1961). In contrary to λ-tracking, where the λ-strip is only attractive, strong practical stability also requires the λ-strip to be invariant, i.e. that the output does not leave the λ-strip and that the output error during the transient can be made arbitrarily small. Thus, it guarantees that there is no peaking (see Sussmann and Kokotović, 1991)

The concept of approximate tracking was introduced in the field of adaptive controllers by Mareels (1984). The term λ-tracking was coined by Ilchmann and Ryan (1994), (see also Ilchmann, 1993; Ryan, 1994). If the reference is a steady-state, then this control objective is called λ-stabilization (Ryan, 1994).

Asymptotic output tracking can be achieved by including a model of the reference signal in the controller (internal model principle, see Francis and Wonham (1975)). For minimum phase, relative degree one systems, asymptotic tracking can also be achieved without an internal model by using a discontinuous controller of the following form (Ryan, 1992)

$$u = -k\big((y - y_{ref}) + \text{sign}(y - y_{ref})\big)$$

for a sufficiently large k. This controller is similar to sliding mode controllers (see for example Khalil, 1996), where y is then the height above the sliding surface $\{\boldsymbol{x}|y(\boldsymbol{x}) = 0\}$. Such a controller requires arbitrary fast switching and will usually lead to chattering. Therefore, for many applications, a continuous controller is preferable.

Objectives of λ-tracking controllers

λ-tracking is not only a very natural specification. It also allows to stabilize a system close to any point, even if this requires a non-zero input. Another advantage of tolerating an output error λ is that this enables to treat a rather large class of nonlinear systems.

Besides achieving λ-tracking, a λ-tracking controller should guarantee boundedness of the states of the closed-loop. In the case of an adaptive controller, the adaptation parameter has to remain bounded. A simple structure and a clear meaning of the controller and its parameters is certainly advantageous. Furthermore, the controller should have a certain robustness with respect to model uncertainty. Lastly, the main control objectives should be attained even if the controller parameters are chosen in a less fortunate manner.

Principle of adaptive λ-tracking control

To simplify the presentation, it is assumed in the following argument that the system is linear, first order with scalar input u and scalar output y:

$$\dot{y} = ay + bu, \tag{1}$$

where a is a non-zero and b a positive constant. This enables to present the main features of adaptive λ-tracking controllers in a simple way. The general case will be treated in Chapter 1.

One approach to λ-track a constant reference y_{ref} with (1) is a proportional output feedback controller

$$u = -k(y - y_{ref}), \tag{2}$$

with sufficiently large controller gain k. Then the closed loop is given by

$$\dot{y} = (a - bk)y + bky_{ref}.$$

The closed-loop is stable if $a - bk < 0$ which is achievable for sufficiently large k. The steady-state y_{ss} is then given by

$$y_{ss} = \left(1 + \frac{a}{bk - a}\right) y_{ref},$$

which is close to y_{ref} for $k \gg |\frac{a}{b}|$. Particularly, for $|\frac{a}{bk-a}|\ |y_{ref}| < \lambda$, the output asymptotically enters the λ-strip.

The main disadvantage of the above controller is that in order to choose k, bounds on the parameters a and b as well as on the reference y_{ref} are needed. Instead of fixing k, potentially at a much too large value, it is possible to adapt this parameter. For example, the following adaptation can be used (Mareels, 1984):

$$\dot{k} = \begin{cases} (|y - y_{ref}| - \lambda)^2, & |y - y_{ref}| \geq \lambda, \\ 0, & |y - y_{ref}| < \lambda. \end{cases} \tag{3}$$

This adaptation can be described in the following way. Whenever the output is outside of the λ-strip around the reference, the controller gain is increased, see Figure 2.

The controller (2) together with the adaptation (3), i.e.

$$u = -k(y - y_{ref}), \tag{4a}$$

$$\dot{k} = \begin{cases} (|y - y_{ref}| - \lambda)^2, & |y - y_{ref}| \geq \lambda, \\ 0, & |y - y_{ref}| < \lambda. \end{cases} \tag{4b}$$

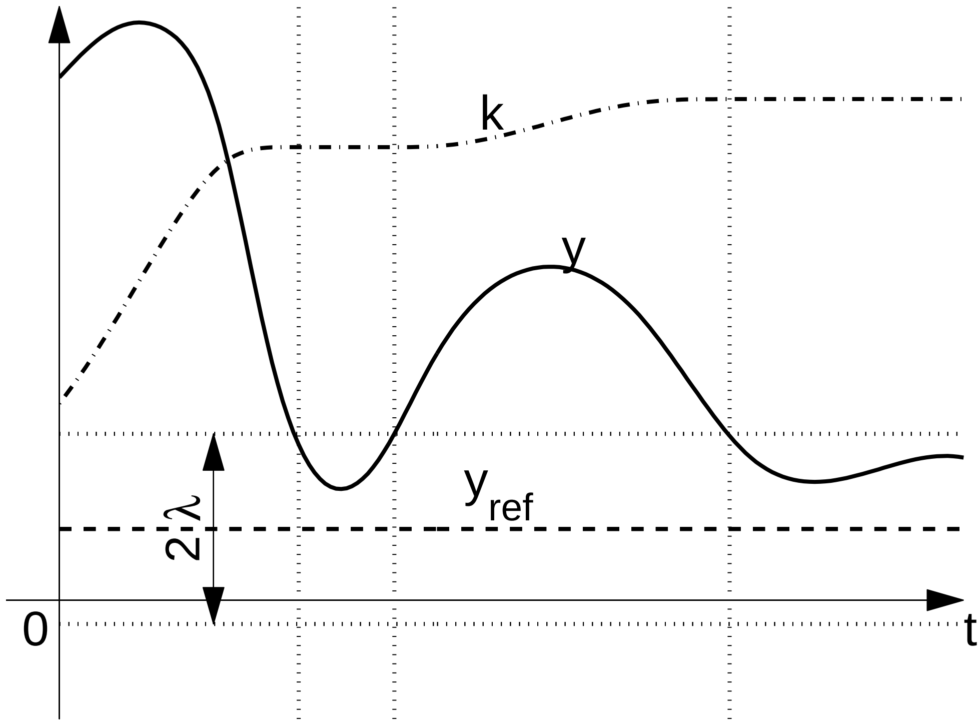


Figure 2: Sketch of the gain adaptation in adaptive λ-stabilization. Output y in solid, reference y_{ref} in dashed, adaptation parameter k in dash-dotted, λ-strip and times when the output enters or leaves the λ-strip as dotted lines.

achieves λ-tracking and boundedness of the state of the system as well as of the adaptation parameter k. These objectives are also achieved for time-varying references $y_{ref}(\cdot)$ and for a larger class of systems, see Chapter 1.

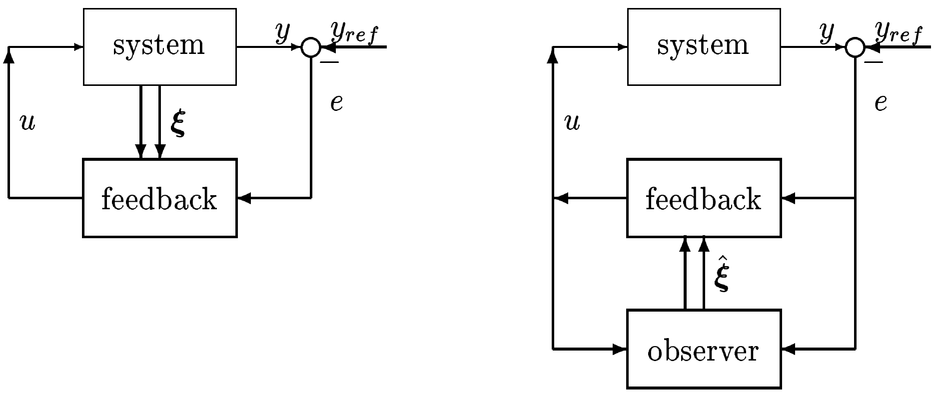
Even though the system class discussed in this section is very specific, the main features of adaptive λ-tracking controllers are already present in the controller (4). The control objective could be achieved by choosing any fixed, sufficiently large controller gain k. With any larger gain, the objective is also attained. Such a controller is called high-gain controller.

Instead of fixing the gain, the adaptive λ-tracking controller uses an adaptation to increase the gain whenever the control objective is not achieved, i.e. when $|y(t) - y_{ref}(t)| > \lambda$. The control objective is attained, independently of the unknown system parameters. However, they influence the transient behavior and the terminal gain, $k_\infty = \lim_{t\to\infty} k(t)$. The controller is very simple and has only one tuning parameter, namely λ, which has to be specified by the user.

Despite its simplicity, the adaptive λ-tracking controller (4) achieves λ-tracking for a larger class of systems as the one specified in (1). But this is still a restricted class of systems, particularly as the relative degree must not exceed two. The main contribution of this thesis is to propose a dynamical adaptive high-gain controller scheme that can achieve λ-tracking for a large class of nonlinear systems with arbitrary relative degree.

Dynamic adaptive λ-tracking controllers

Many controller design methods consist of the following two steps. First, a static state-feedback controller is designed which uses all or some of the system states $\boldsymbol{x}$, i.e. $u = q(\boldsymbol{\xi})$ where $x \mapsto \boldsymbol{\xi}$ is usually either the identity or a projection to a lower dimensional space. The closed-loop consisting of the system and the state-feedback controller, see Figure 3(a), should achieve the control objectives. In the tracking case, the controller also needs the reference signal y_{ref} or the tracking error e, see Figure 3(a). In a second step, an observer is designed estimating the states $\boldsymbol{\xi}$ based on the system input u and output y. Using the observer states $\hat{\boldsymbol{\xi}}$ instead of the system states $\boldsymbol{\xi}$ in the control law then yields an observer-based state-feedback controller as depicted in Figure 3(b).



(a) State-feedback controller.

(b) Observer-based state-feedback controller.

Figure 3: Comparison of the structure of the closed loop using a state-feedback and an observer-based state feedback controller.

The controllers discussed in this thesis usually consist of a high-gain partial state-feedback and a high-gain observer. Either the gains of the state-feedback and of the observer can be fixed a priori if sufficient information on the system to be controlled is available. Or, an adaptation is used increasing these gains as long as the control objective, i.e. λ-tracking, is not attained. In other words, a single gain k can be used for both the state-feedback and the observer, see Figure 4.

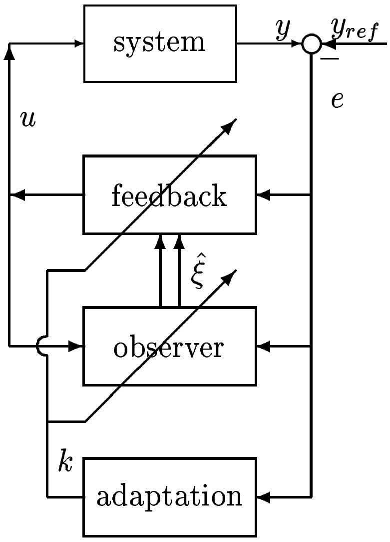


Figure 4: Structure of the closed loop using an adaptive observer-based state feedback controller.

Summary

This section gives a short overview of adaptive λ-tracking controllers. Their different components are described in more details in Chapter 1 before these parts are combined to different adaptive λ-tracking controllers in Section 2.2.

Adaptive λ-tracking controllers are relatively simple controllers that are remarkably robust against a large set of uncertainties, making them attractive for many practical applications.

Applications of adaptive λ-tracking controllers

Adaptive λ-tracking controllers are not only interesting from a theoretical point of view. Several successful applications of adaptive λ-tracking controllers have been reported with applications ranging from process control (Allgöwer and Ilchmann, 1995; Allgöwer et al., 1997) to bio-processes (Ilchmann et al., 1998; Ilchmann and Pahl, 1998; Ilchmann, 1997) as well as to mobile robots (Mazur and Hossa, 1997). The application of adaptive λ-tracking control in the field of anesthesia depth control is described in Chapter 4 in this thesis, see also (Bullinger et al., 2000b).

Main contribution

This thesis gives an overview of adaptive high-gain controllers with a special emphasis on controllers achieving λ-tracking for systems with high relative degree. Most high-gain controllers for high relative degree systems proposed in the literature use a reduced-order observer. As shown in this thesis, it is also possible to have a λ-tracking controller that uses a full-order observer. This work proves that for a large class of nonlinear systems and reference trajectories, this controller is guaranteed to achieve λ-tracking while the adaptation converges and all states remain bounded.

The proposed adaptive λ-tracking controller is applied to the control of the endtidal concentration of volatile anesthetics in anesthesia. Simulations and experiments demonstrate that the concept of λ-tracking can be successfully applied to this kind of uncertain and difficult to model systems.

Thesis overview

CHAPTER 1: The concept of λ-stabilization and tracking and the different components of an adaptive λ-tracking controller are presented: the state-feedback, the high-gain-observer and the adaptation. Furthermore, an overview of non-adapted high-gain controllers is given.

CHAPTER 2: Different adaptive high-gain controllers found in the literature are presented in this chapter. After a historical overview of universal stabilization (covering work of the early 1980's), different adaptive high-gain controllers, stabilizing systems of relative degree larger than one are compared.

CHAPTER 3: A new adaptive λ-tracking controller is proposed in this chapter. This controller incorporates a full-order high-gain observer. It is shown that for a large class of systems, this controller achieves λ-tracking of most practically relevant reference trajectories. The chapter starts with a description of the controller's components (state-feedback, observer, adaptation) which are then combined to form an adaptive λ-tracking controller. Theorem 3.1 contains the main theoretical result of the thesis.

CHAPTER 4: The controller proposed in Chapter 3 is applied to anesthesia depth control. The control objective is to keep the endtidal concentration of the anesthetic close to a reference value which is adjusted several times during a surgery. The adaptive λ-tracking controller achieves this task in a satisfactory manner.

Chapter 1

Objective and Components of Adaptive λ-tracking Controllers

This chapter presents the main elements of adaptive λ-tracking controllers. Section 1.1 introduces the concept of λ-stabilization. The principal structural properties needed in this thesis are discussed in Section 1.2. Section 1.3 describes the non-adaptive high-gain controller: the basis of an adaptive λ-tracking controller. For systems of higher relative degree it consists of a high-gain state-feedback (see Section 1.3.1) and a high-gain observer (Section 1.3.2). Compared to exact stabilization, λ-stabilization is achieved for a larger class of systems. However, the design of the high-gain controller is the same for λ- or exact stabilization. The control objective only influences the magnitude of the controller gain. Instead of choosing the controller gain a priori, it is possible to adapt the gain as shown in Section 1.4. Chapter 2 combines the adaptation with the high-gain controllers to different adaptive controllers achieving exact- or λ-stabilization or -tracking.

1.1 λ-stabilization and λ-tracking

The following presents the concepts of λ-stabilization and -tracking (see Ilchmann and Ryan, 1994; Ilchmann, 1993; Ryan, 1994).

1.1.1 λ-tracking

The objective of λ-tracking is to have the output $y(\cdot)$ of the system to be controlled, approach asymptotically a given reference trajectory $y_{ref}(\cdot)$ with a tolerance of λ:

$$|y(t) - y_{ref}(t)| \to [0, \lambda] \text{ for } t \to \infty,$$

where the parameter $\lambda > 0$ is a user-defined constant, see also Figure 1.1.

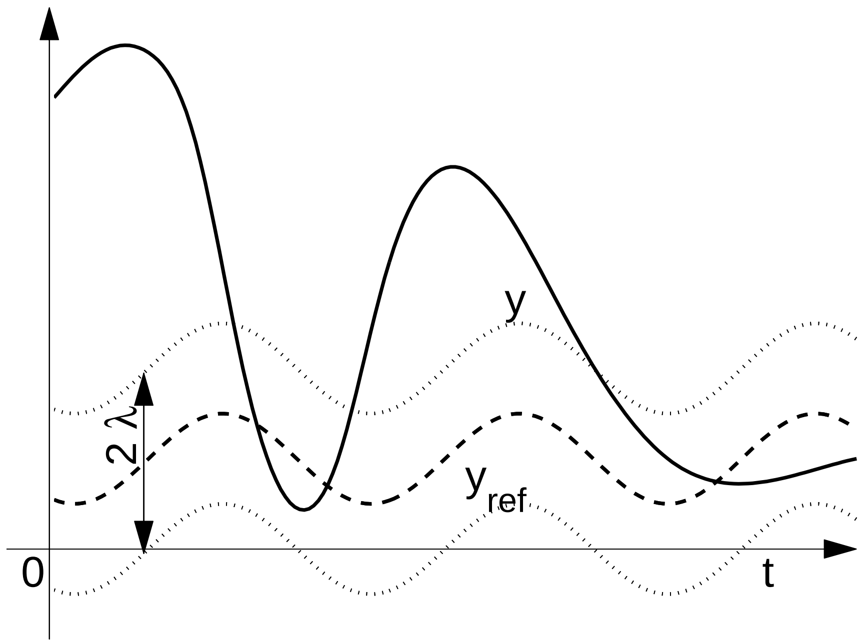


Figure 1.1: Sketch of λ-tracking. Output y in solid, reference y_{ref} in dashed, λ-strip in dotted lines.

1.1.2 λ-stabilization

A special case of λ-tracking, called λ-stabilization corresponds to tracking a steady-state reference.

1.1.3 Strong λ-tracking

Miller and Davison (1991a) propose a controller achieving strong practical stability for the tracking error $e = y - y_{ref}$ (see La Salle and Lefschetz, 1961). More precisely, the tracking error should remain bounded by $e(0) + \epsilon$ during the transient phase up to some time T and then stay smaller than λ:

$$\begin{aligned} |e(t)| &\leq |e(0)| + \epsilon, && t < T \\ |e(t)| &\leq \lambda, && t \geq T, \end{aligned}$$

where ϵ, T and λ are arbitrary positive constants chosen by the user. In contrary to λ-tracking, these bounds are hard, not only asymptotic. Figure 1.2 shows this control objective.

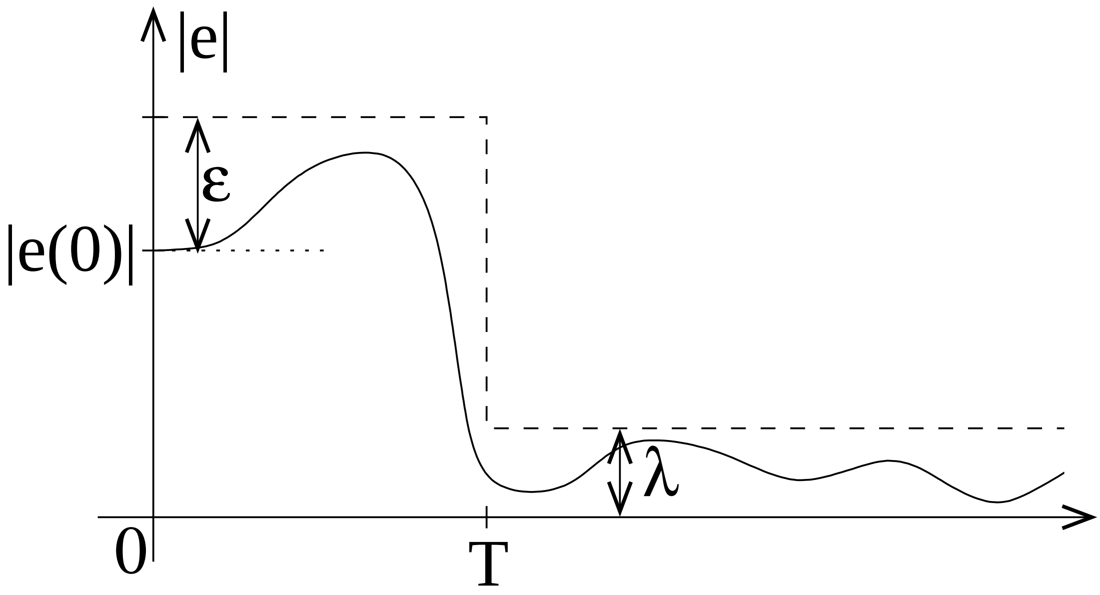


Figure 1.2: Sketch of strong λ-tracking (Miller and Davison, 1991a).

1.1.4 Comparison

In λ-tracking, the tracking error $y(\cdot) - y_{ref}(\cdot)$ approaches the λ-strip $[-\lambda, \lambda]$ asymptotically. λ-stabilization is a special case of λ-tracking where y_{ref} is constant and equal to a steady-state. Strong λ-tracking is a much stricter requirement as, after a user-specified transient time T, the tracking error is strictly smaller in magnitude than λ and, furthermore, the tracking error is arbitrarily bounded during the transient phase. In all cases, λ is a user-defined parameter.

There are several reasons for using λ-tracking. On the one hand, it is a very natural control objective as many specifications include a tolerance, e.g. $\pm 5\%$. For example, if the output is corrupted by noise, exact tracking requires control energy that can be saved if the λ-strip is large enough to tolerate these measurement errors. On the other hand, λ-tracking allows to control a larger class of systems with adaptive high-gain controllers, see Appendix B.3.

1.2 System class

This thesis only treats time-invariant single-input single-output (SISO) systems affine in the input. They can be described by:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \tag{1.1a}$$

$$y = \boldsymbol{h}(\boldsymbol{x}), \tag{1.1b}$$

where the input is denoted by u, the output by y. The state $\boldsymbol{x}$ lives on $\mathbb{R}^n$, $\boldsymbol{f}(\cdot)$ and $\boldsymbol{g}(\cdot)$ are continuous and locally Lipschitz functions mapping $\mathbb{R}^n \to \mathbb{R}^n$. Furthermore, the system should satisfy the following assumptions 1.1, 1.2 and 1.3.

Assumption 1.1 *The system is transformable into Byrnes-Isidori normal form (Byrnes and Isidori, 1984; Byrnes and Isidori, 1985; Isidori, 1995):*

$$y = \xi_1 \tag{1.2a}$$

$$\dot{\xi}_i = \xi_{i+1} \quad \text{for } i = 1, \ldots, r-1 \tag{1.2b}$$

$$\dot{\xi}_r = \alpha(\boldsymbol{\xi}, \boldsymbol{\eta}) + g(\boldsymbol{\xi}, \boldsymbol{\eta})u \tag{1.2c}$$

$$\dot{\boldsymbol{\eta}} = \tilde{\boldsymbol{\theta}}(\boldsymbol{\xi}, \boldsymbol{\eta}), \;\; \boldsymbol{\eta}(t) \in \mathbb{R}^{n-r} \tag{1.2d}$$

where the function $(\boldsymbol{\xi}, \boldsymbol{\eta}) \mapsto \alpha(\boldsymbol{\xi}, \boldsymbol{\eta})$ *is linearly bounded.*

Assumption 1.2 (Internal dynamics) *The zero-dynamics of the system* (1.2) *can be decomposed as*

$$\dot{\boldsymbol{\eta}} = \tilde{\boldsymbol{\theta}}(\boldsymbol{0}, \boldsymbol{\eta}) = \boldsymbol{\theta}(\boldsymbol{\eta}) + \tilde{\boldsymbol{w}}(\boldsymbol{\eta})$$

where $\tilde{\boldsymbol{w}}(\cdot)$ *is bounded and the dynamics* $\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta})$ *is globally exponentially stable.*

Assumption 1.3 (Positive high-frequency gain) *The high-frequency-gain* $g(\boldsymbol{\xi}, \boldsymbol{\eta})$ *of system* (1.2) *is strictly positive and bounded in magnitude. A definition of the high-frequency gain can be found in Definition 8 in Appendix A.3, page 89.*

Remark 1.1 *A system satisfying Assumption 1.1 with non-zero high-frequency gain has a strong relative degree r (Byrnes and Isidori, 1984).* ◊

Remark 1.2 *Assumption 1.2 includes systems with stable zero-dynamics, e.g.* $\tilde{\boldsymbol{w}} \equiv 0$. *But it is not restricted to them.* ◊

1.3 High-gain controllers

The main component of an adaptive λ-tracking controller is a high-gain controller. In this section, non-adapted high-gain controllers are presented, first, for relative degree one systems. Then, for the higher relative degree case, an observer-based realization is presented. The high-gain controller then consists of a high-gain state-feedback (see Section 1.3.1) and a high-gain observer (Section 1.3.2).

1.3.1 High-gain state-feedback

A very simple feedback is the following proportional output feedback:

$$u = -ky. \tag{1.3}$$

This controller has only one parameter, namely k. It can for example be used to stabilize linear systems of dimension one, i.e.

$$\dot{y} = \alpha y + \beta u \tag{1.4}$$

with positive high-frequency gain β. It is easy to see that if a specific system (1.4) can be stabilized by the controller (1.3) with $k = k^*$, then also any larger k stabilizes (1.4). A controller with this property is called high-gain controller, see Definition 1 below.

Definition 1 (High-gain controller, parameter, stabilization) *A controller parameterized by a single parameter k stabilizing a given system for any $k \geq k^*$ is called high-gain controller, its parameter k high-gain parameter. A system stabilizable by a high-gain controller is called high-gain stabilizable.*

Definition 1 is a generalization of the definition in (Corless, 1991; Ilchmann, 1993) where the controller is a static output feedback.

For linear systems,

$$\frac{y(s)}{u(s)} = g\frac{s^m + \beta_{m-1}s^{m-1} + \ldots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0},$$

Corless (1991) lists the necessary and sufficient conditions a system has to satisfy to be high-gain stabilizable by the controller (1.3) as:

1. minimum-phase, i.e. $s^m + \beta_{m-1}s^{m-1} + \ldots + \beta_0$ has zeros only in $\mathbb{C}^-$,
2. positive high-frequency gain g,
3. relative degree one or two ($m = n - 1$ or $m = n - 2$).

If the relative degree is two, then it is also necessary that $\alpha_{n-1} > \beta_{n-3}$. For a system with dimension $n = 2$, i.e.

$$\ddot{y} + \alpha_1\dot{y} + \alpha_2 y = gu.$$

the conditions for high-gain stabilizability are $g > 0$ and $\alpha_1 > 0$.

Using the Lemmata B.4 and B.5, this result can be extended to nonlinear systems and to systems having internal dynamics.

In the following part, the static high-gain output feedback described above is extended to dynamic high-gain output feedback. To simplify the presentation, it is assumed that the system is linear and in controller normal form with positive high-frequency gain g, see also Figure 1.3:

$$\dot{\xi}_i = \xi_{i+1}, \; i = 1, \ldots, r-1 \tag{1.5a}$$
$$\dot{\xi}_r = \boldsymbol{\alpha}^T \boldsymbol{\xi} + gu. \tag{1.5b}$$

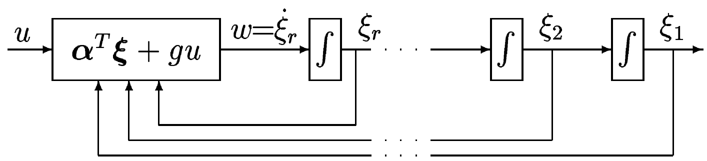


Figure 1.3: Linear system in controller normal form.

As the system (1.5) is controllable, there exist positive constants $\tilde{q}_i$, $i = 0, \ldots, r-1$ such that

$$u(t) = -\sum_{i=0}^{r-1} \tilde{q}_i \xi_{i+1}(t)$$

stabilizes the closed loop. By choosing the parameters $\tilde{q}_i$ as $\tilde{q}_i = q_i \kappa^{r-i}$, the controller is parameterized by a parameter κ which is the high-gain parameter for this state-feedback controller. It is shown in Appendix B.2 that there exist parameters q_i such that the controller

$$u(t) = -\sum_{i=0}^{r-1} q_i \kappa^{r-i} \xi_{i+1}(t) \tag{1.6}$$

stabilizes for sufficiently large κ any system (1.5) with a high-frequency gain $g \geq \underline{g}$ where $\underline{g}$ is a known positive constant, see Appendix B.2. The condition on the parameters q_i is that the polynomial

$$s^r + g \sum_{i=0}^{r-1} q_i s^i$$

is Hurwitz for all $g \geq \underline{g}$. The parameter κ scales the poles of the closed loop: the larger κ, the faster the closed-loop. This can easily be seen if the system is simply a chain of integrators, i.e. if $\alpha = 0$ and $g = 1$. Then the closed-loop is

$$\dot{\boldsymbol{\xi}} = (J - \boldsymbol{b}\boldsymbol{q}_\kappa)\,\boldsymbol{\xi}$$

where $\{J, \boldsymbol{b}\}$ is a prime tuple (see Definition 10, page 90) and

$$\boldsymbol{q}_\kappa = [q_0\kappa^r, \ldots, q_{r-1}\kappa]\,.$$

In the coordinates $\zeta_i = \kappa^{1-i}\boldsymbol{\xi}_i$, the closed-loop is

$$\dot{\boldsymbol{\zeta}} = \kappa\,(J - \boldsymbol{b}\boldsymbol{q})\,\boldsymbol{\zeta}$$

where

$$\boldsymbol{q} = [q_0, \ldots, q_{r-1}]\,.$$

Thus, κ is a time-scale for the closed-loop in this special case.

The controller (1.6) requires the measurement of ξ_1 and of its derivatives ξ_2 to ξ_r. If ξ_1 is the output, i.e. $y = \xi_1$, then the same result is achievable by the $PD \ldots D^{r-1}$-controller

$$u(s) = -\sum_{i=0}^{r-1} q_i\kappa^{r-i}s^i y(s) = q_\kappa(s)y(s). \tag{1.7}$$

This is a dynamic high-gain controller with high-gain parameter κ. Note, that with this output, the system has no internal dynamics.

The system class of high-gain stabilizable systems can be extended to nonlinear systems as well as to systems having internal dynamics by the Lemmata B.4 and B.5. A similar result holds for λ-stabilization, see Lemma B.6 and B.7.

The controller (1.7) has two main drawbacks. The first is that the first $r-1$ derivative of the output are needed. For practical applications, they need to be estimated via numerical differentiation, or by the use of an observer. Different observer schemes usable for such a control problem are presented in Section 1.3.2. The second drawback is that the controller gain needs to be sufficiently large. But its minimal value depends on the system parameters. If these are not known with sufficient precision, the controller gain has to be adapted. Possible adaptation schemes are presented in Section 1.4; the adaptive high-gain controller in Chapter 2.

1.3.2 High-gain observers

For the high-gain controller presented in Section 1.3.1, an observer is needed to estimate the derivatives of the output y. In this thesis, the term *observer* is used for a device estimating some states of a system. It is neither required that all states are estimated, nor that the observer error decreases asymptotically to zero. For example, a biased estimation is tolerated. This section presents two observer schemes that can be used together with the high-gain controller of Section 1.3.1.

Full-order observer

For a system in nonlinear controller normal form (Krener, 1987; Zeitz, 1989) (the nonlinear generalization of Figure 1.3):

$$\dot{\xi}_i = \xi_{i+1},\ i = 1, \ldots, r-1 \tag{1.8a}$$

$$\dot{\xi}_r = \alpha(\boldsymbol{\xi}) + g(\boldsymbol{\xi})u, \tag{1.8b}$$

a high-gain observer as proposed by (Nicosia et al., 1989; Tornambè, 1992) is well-suited. Figure 1.4 shows the structure of this observer (called full-order observer in the following). It consists of a copy of the integrator chain at the output (drawn in thick lines) corrected by the observer output error, the difference between the outputs of the two integrator chains y and $\hat{y}$, denoted by $\hat{e}$:

$$\dot{\hat{\boldsymbol{\xi}}} = \hat{A}_{\hat{\kappa}}\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{b}}y = J\hat{\boldsymbol{\xi}} + \boldsymbol{p}_{\hat{\kappa}}(y - \hat{y}) \tag{1.9a}$$

$$\hat{y} = \hat{\boldsymbol{c}}^T\hat{\boldsymbol{\xi}} \tag{1.9b}$$

where $(J, \hat{\boldsymbol{c}}^T)$ is a prime tuple (see Definition 10, page 90) and

$$\boldsymbol{p}_{\hat{\kappa}} = \begin{bmatrix} p_{r-1}\hat{\kappa}, & \ldots & p_0\hat{\kappa}^r \end{bmatrix}^T,$$

where the p_i's are the coefficients of the Hurwitz polynomial $s^r + p_{r-1}s^{r-1} + \ldots + p_0$. The input to the integrator chain, namely $w = \alpha(\boldsymbol{\xi}) + g(\boldsymbol{\xi})u$, is neglected. Designing

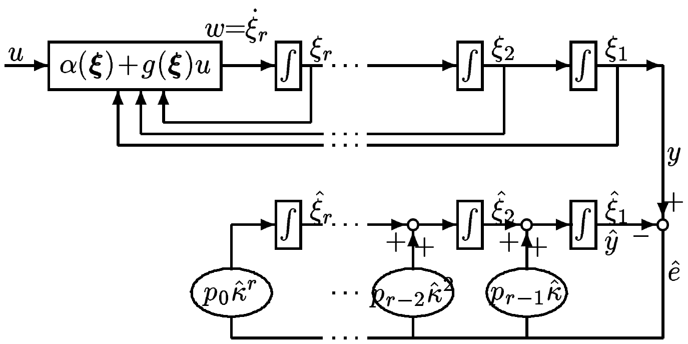


Figure 1.4: Full-order high-gain observer.

this observer only requires the knowledge of the relative degree r. If a bound for $w =$

$\alpha(\boldsymbol{\xi}) + g(\boldsymbol{\xi})u$ is known, the observer output error $\hat{e} = y - \hat{y}$ can be "λ-stabilized" by choosing the parameter $\hat{\kappa}$ sufficiently large (Nicosia et al., 1989). Therefore, $\hat{\kappa}$ is a high-gain parameter for this observer. No further information on the functions $\alpha(\cdot)$ and $g(\cdot)$ is needed. If a bound on w is known sufficiently well, it is possible to fix the parameter $\hat{\kappa}$. An alternative is to adapt $\hat{\kappa}$ as will be described in Section 1.4.2. This ensures that the observer output error $y - \hat{y}$ tends to the λ-strip, see Bullinger et al. (1998). Another approach (Bullinger and Allgöwer, 1997) is to increase the parameter $\hat{\kappa}$ in a discrete manner as in Section 1.4.1 on page 29, but with $|y|$ in (1.23a) replaced by $|y| - \lambda$. The high-gain observers as presented here estimate the whole state of a system only if the system has no internal dynamics. In other cases, only the states on the direct path between the input and the output are estimated, see Figure 1.5. Not estimated are the states of the internal dynamic. The effect of the input u and of all the states $\boldsymbol{\xi}$ on $w = \dot{\xi}_r$ is neglected. Therefore, the internal dynamics needs to be bounded.

Since the dimension of the full-order observer is always equal to the relative degree, it can be much smaller than the dimension of the system.

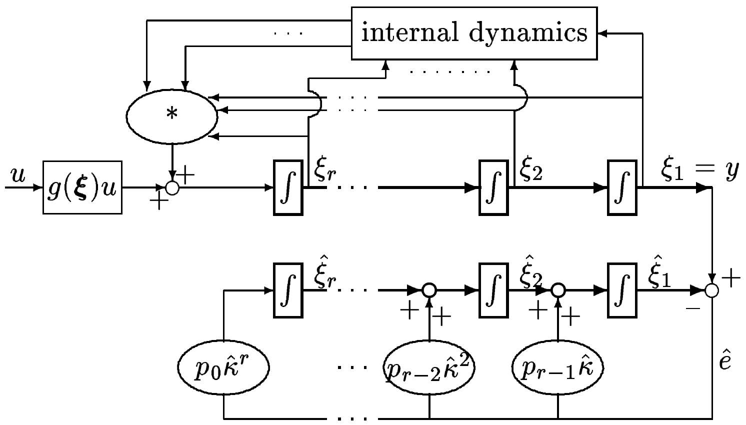


Figure 1.5: Full-order high-gain observer for a system with internal dynamics.

For a system in controller normal form and relative degree r equal to the system dimension, the state $\hat{\xi}_i$ of the full-order high-gain observer approximates the state ξ_i of the system as, for fixed $\hat{\kappa}$,

$$\hat{\xi}_i(s) = s^{i-1}\left(1 - \frac{\left(\frac{s}{\hat{\kappa}}\right)^r + \ldots + \left(\frac{s}{\hat{\kappa}}\right)^{r-i+1} p_{r-i+1}}{\left(\frac{s}{\hat{\kappa}}\right)^r + \left(\frac{s}{\hat{\kappa}}\right)^{r-1} p_{r-1} + \cdots + p_0}\right) y(s),$$

which is at "low" frequencies, i.e. for small $\left(\frac{s}{\hat{\kappa}}\right)$, approximately equal to the (i-1)-th derivative of the output y:

$$\hat{\xi}_i(s) \approx s^{i-1} y(s).$$

With coordinates $\zeta_i = \hat{\kappa}^{1-i}\hat{\xi}_i$, the observer equations become

$$\dot{\boldsymbol{\zeta}} = \hat{\kappa}\left(J - \bar{\boldsymbol{p}}\boldsymbol{c}^T\right)\boldsymbol{\zeta}, \tag{1.10}$$

where $\hat{\kappa}$ is assumed to be constant and

$$\bar{\boldsymbol{p}} = \begin{bmatrix} p_{r-1}, & \dots & p_0 \end{bmatrix}^T.$$

From (1.10) follows that the parameter $\hat{\kappa}$ scales the observer time constants.

Reduced-order observer

The full-order observer estimates the output $y = \xi_1$ by $\hat{y}$ even though it is measured. This is not the case for the reduced-order observer. Thus, its dimension is reduced by one. It is best understood if the system is given in the following normal form ((1.11) shows the linear case with $n = r$ to simplify the presentation):

$$\begin{aligned} \dot{y} &= \alpha_{r-1}y + g\xi_1 \\ \dot{\tilde{\boldsymbol{\xi}}} &= \tilde{\boldsymbol{\alpha}}y + \tilde{A}\tilde{\boldsymbol{\xi}} + \tilde{\boldsymbol{b}}u \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} \tilde{\boldsymbol{\xi}} &= \begin{bmatrix} \xi_2, & \dots, & \xi_r \end{bmatrix}^T \\ \tilde{\boldsymbol{\alpha}}^T &= \begin{bmatrix} \alpha_{r-2}, & \dots, & \alpha_0 \end{bmatrix} \\ \tilde{\boldsymbol{b}}^T &= \begin{bmatrix} 0, & \dots, & 1 \end{bmatrix} \end{aligned}$$

and $\tilde{A}$ is a known Hurwitz matrix, see also Figure 1.6 where, for simplicity, $\tilde{A} = J - \bar{\boldsymbol{p}}\boldsymbol{c}^T$. This normal form goes back to Luenberger (1964) who showed that, given an arbitrary matrix $\hat{A}$, any observable linear system can be transformed into normal form (1.11) and that

$$\dot{\hat{\boldsymbol{\xi}}} = \tilde{\boldsymbol{\alpha}}y + \tilde{A}\hat{\boldsymbol{\xi}} + \tilde{\boldsymbol{b}}u \tag{1.12}$$

is an observer for the state $\tilde{\boldsymbol{\xi}}$ with an observer error $\boldsymbol{e} = \hat{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}$ converging to zero for any Hurwitz matrix $\tilde{A}$ as

$$\dot{\boldsymbol{e}} = \tilde{A}\boldsymbol{e}.$$

The estimation of y by $\hat{y}$ is not performed, see Figure 1.6, where the observer $(\tilde{A}, \tilde{\boldsymbol{b}})$ is in observer normal form (Krener, 1987; Zeitz, 1989). The reduced-order observer does not include a correction term proportional to $y - \hat{y}$ as the full-order observer. It can be regarded as a parallel model driven by u and y.

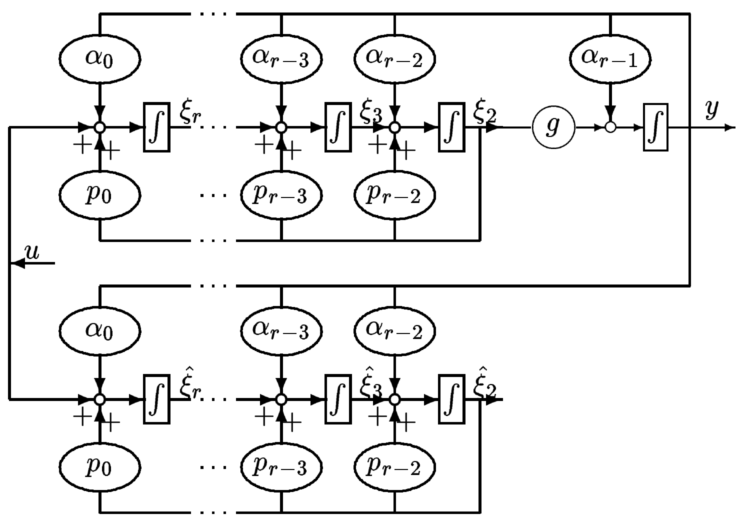


Figure 1.6: Reduced-order observer.

The reduced-order high-gain observer is very similar to the observer (1.12), but neglects the influence of $\tilde{\boldsymbol{\alpha}}y$:

$$\dot{\hat{\boldsymbol{\xi}}} = \tilde{A}\hat{\boldsymbol{\xi}} + \tilde{\boldsymbol{b}}u.$$

As $\tilde{A} \in \mathbb{R}^{r-1\times r-1}$ is a design parameter, the observer design requires only the knowledge of the relative degree r. The observer error $\boldsymbol{e} = \hat{\boldsymbol{\xi}} - \tilde{\boldsymbol{\xi}}$ is therefore not autonomous, but depends on the output y:

$$\dot{\boldsymbol{e}} = \tilde{A}\boldsymbol{e} + \tilde{\boldsymbol{\alpha}}y.$$

Thus, the observer error converges to zero only if the output y goes to zero. Also, the reduced-order high-gain observer does not include a correction term proportional to $y - \hat{y}$. It is a parallel model driven only by u. In open-loop, the reduced-order observer does not include a correction term. As for the full-order observer, the states of the internal dynamics are neglected: they are neither observed nor is their influence on the other states taken into account.

In contrast to the full-order case, the reduced-order observer (1.16) is not an explicit high-gain observer. In closed loop as in Section 1.3.3, the error dynamics are time-scaled via the input u which contains a high-gain linear combination of the observer states. As for the full-order observer, the reduced-order observer can also be used for a relative large class of nonlinear systems. Khalil and Saberi (1987) and Saberi and Lin (1990) use a reduced-order observer which is explicitly a high-gain observer.

If $n > r$, the system has internal dynamics. The reduced-order observer is designed as if $n = r$, thus it does not estimate the states of the internal dynamics and therefore neglects their influence.

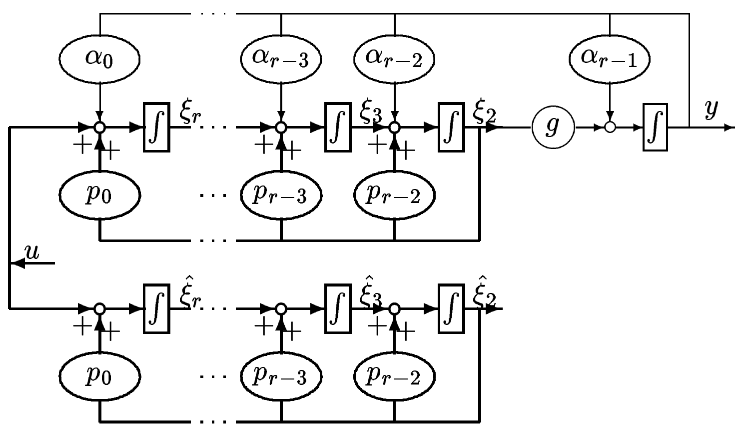


Figure 1.7: Reduced-order high-gain observer.

Summary

For uncertain systems of known relative degree r, it is possible to estimate the states of the input-output path with the help of high-gain observers. These estimate the states even if the parameters of the system are not known. These observers can have r (full-order observer) or $r-1$ (reduced-order observer) observer states. The full-order observer can be used in open-loop. This is not the case for the reduced-order observer as its states only converge to those of the system if the system output is zero.

In a closed loop, both observers can be used together with high-gain state feedback controllers. Chapters 2 and 3 discuss this in greater detail.

1.3.3 Output-feedback high-gain controllers

In the previous sections, high-gain state-feedback and high-gain observers have been presented. In this section, these components are combined to form high-gain output-feedback controllers for linear systems having a known relative degree r and positive high-frequency gain.

The relative degree influences the structure of the high-gain controllers presented here. In a first part, a controller for systems having relative degree one or two is presented. This is simply a proportional controller. In the case of higher relative degree, the observers presented in the previous section are combined with a high-gain state-feedback as presented in Section 1.3.1.

Relative degree one or two

Minimum phase systems with relative degree one can be stabilized by a proportional controller with a sufficiently large gain k, as can be seen from a root locus analysis (Evans, 1950):

$$u = -ky. \tag{1.13}$$

This scheme has been applied in a variety of controllers, see Ilchmann (1993) for a survey.

There are systems of relative degree two that can be stabilized by high-gain output feedback, see (Corless, 1991) or Section 1.3.1. To stabilize a system of arbitrary relative degree a dynamic controller is required. In the following section, two approaches for dynamic controllers are presented.

Higher relative degree

If the system has a relative degree larger than two, it cannot be stabilized by a static high-gain controller. Then a dynamical high-gain controller is required, see Appendix C or (Corless, 1991). An observer-based realization will be used and two possible design schemes are presented.

Full-order observer based controller The states of the full-order high-gain observer $\hat{\xi}_i$ approximate the state of the system on the input-output path. This motivates to use them instead of the not measured states ξ_i in (1.7). The feedback is then:

$$\begin{aligned} u &= -\sum_{i=0}^{r-1} q_i \kappa^{r-i} \hat{x}_{i+1}, \\ \dot{\hat{\boldsymbol{x}}} &= \hat{A}_{\hat{\kappa}} \hat{\boldsymbol{x}} + \hat{\boldsymbol{b}}_{\hat{\kappa}} y, \end{aligned} \tag{1.14}$$

where the controller (1.14) consists of the observer part and the feedback part with time scaling parameters, $\hat{\kappa}$ and κ respectively. As shown in Appendix C, the observer time-scale has to be at least as fast as the feedback one. The controller structure (1.14) is used by (Teel and Praly, 1995) for fixed κ, $\hat{\kappa}$ and in Chapter 3 for adapted κ, $\hat{\kappa}$.

For $r = 1$, the controller does not reduce to (1.13). It then consists of a first-order low pass:

$$u(s) = -\frac{\kappa}{\frac{s}{\hat{\kappa}} + 1} y(s), \tag{1.15}$$

which, for low frequencies, i.e. $s \ll \hat{\kappa}$, is almost equal to the controller (1.13).

Controllers of the form (1.14) achieve semi-global stabilization, (see Khalil, 1999, for a survey). In semi-global stabilization stability is guaranteed for initial conditions within given bounds, where increasing the parameters of the controller enlarges the domain of attraction. The semi-global stabilizers differ from the previously discussed controllers by the fact that bounds on the initial conditions and on the system need to be known a priori to fix the controller parameters. In Section 3, the controller structure (1.14) with $\hat{A}$, $\hat{b}$ in observer normal form is used in an adaptive controller. It is shown that this controller achieves λ-tracking for a large class of nonlinear systems.

Reduced-order observer based controller When stabilizing a system with a reduced-order observer, it is necessary that the output y enters the feedback explicitly, as the reduced-order observer does not estimate y:

$$u = -\kappa\hat{\kappa}^{r-1}q_0 y - \sum_{i=1}^{r-1} q_i \hat{\kappa}^i \hat{\boldsymbol{\xi}}; \tag{1.16a}$$

$$\dot{\hat{\boldsymbol{\xi}}} = \hat{A}_{\hat{\kappa}}\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{b}}u. \tag{1.16b}$$

where the observer $\{\hat{A}_{\hat{\kappa}}, \hat{\boldsymbol{b}}\}$ is of dimension $r-1$.

Figure 1.8 shows the signal flow diagram of this controller, where for simplicity, $\{\hat{A}, \hat{\boldsymbol{b}}\}$ is chosen in controllability normal form and $\hat{A}_{\hat{\kappa}} = \tilde{A}$, the Hurwitz matrix resulting from the chosen transformation in (1.11), is independent of $\hat{\kappa}$. This controller is a generalization of the controller (1.13) to systems of relative degree $r \geq 1$ as for $r = 1$ the controller (1.16) is equal to (1.13).

The state-feedback depends on an adaptation parameters κ and $\hat{\kappa}$. The observer matrix $\hat{A}_{\hat{\kappa}}$, and therefore the parameters p_i, $i = 0, \ldots, r-1$, depends on $\hat{\kappa}$ only in a few design schemes which either use fixed high-gain parameters or adapt them in a discrete manner, (see e.g. Khalil and Saberi, 1987).

Most reduced-order high-gain observers do not depend explicitly on the high-gain parameter, but due to the high-gain observer state-feedback, they can be regarded as high-gain observers (see for example Saberi and Sannuti, 1990). To show this, the controller is rewritten as

$$u = -q_0\kappa\hat{\kappa}^{r-1}y - \sum_{i=1}^{r-1} q_i\hat{\kappa}^{r-i}\tilde{\xi}_i$$

$$\dot{\hat{\xi}} = \left(\hat{A} - \hat{\boldsymbol{b}}\left[q_1\hat{\kappa}^{r-1}, \quad q_{r-1}\hat{\kappa}\right]\right)\hat{\xi} - q_0\kappa\hat{\kappa}^{r-1}\hat{\boldsymbol{b}}y.$$

To simplify the presentation, it is assumed that $\{\hat{A}, \hat{\boldsymbol{b}}\}$ is in controller normal form, i.e.

$$\hat{A} = J - \hat{\boldsymbol{b}}\boldsymbol{p}^T, \; \hat{\boldsymbol{b}} = \begin{bmatrix}0, & \ldots, & 0, & 1\end{bmatrix}^T.$$

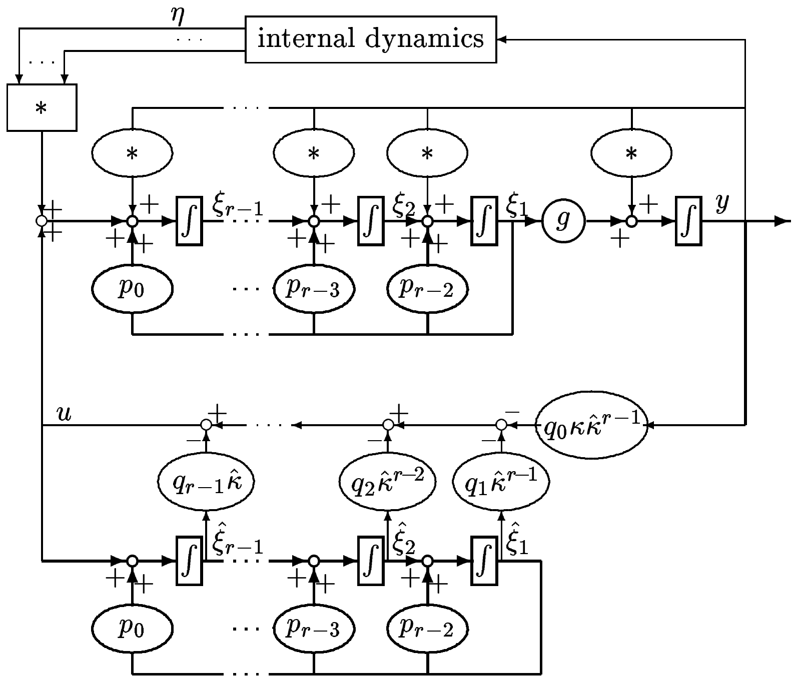


Figure 1.8: Reduced-order observer based high-gain controller.

For fixed κ, $\hat{\kappa}$, the observer in the coordinates $\hat{\zeta}_i = \kappa^{-1}\hat{\kappa}^{1-i}\hat{\boldsymbol{\zeta}}_i$ becomes

$$\dot{\hat{\zeta}} = \hat{\kappa}\left(\hat{A}_{\hat{\kappa}}\hat{\zeta} - q_0\frac{\kappa}{\hat{\kappa}}\hat{\boldsymbol{b}}y\right),$$

where

$$\hat{A}_{\hat{\kappa}} = J - \hat{\boldsymbol{b}}\left[q_1 - p_{r-2}\hat{\kappa}^{-(r-1)}, \quad q_{r-1} - p_0\hat{\kappa}^{-1}\right]$$

and the state-feedback is

$$u = -\hat{\kappa}^{r-1}\left(\kappa q_0 y + \sum_{i=1}^{r-1} q_i\hat{\boldsymbol{\zeta}}_i\right).$$

In these coordinates, it is apparent that $\hat{\kappa}$ is the time-scale for the observer while κ is the output gain. It is not possible to separate observer and state-feedback scaling parameters as it is in the case of the full-order observer-based controller.

Several controllers very similar to (1.16) have been proposed in the literature (see Mareels, 1984; Khalil and Saberi, 1987; Mudgett and Morse, 1989; Saberi and Lin, 1990; Ye, 1999, and Section 2.1.3 where these controllers are discussed in more details).

Unknown relative degree

The controllers (1.16), (1.13) and (1.14) require the exact knowledge of the relative degree r, see Appendix C. A controller stabilizing minimum phase plants of relative degree $r \leq \rho+1$ with positive high-frequency gain has been proposed by Mårtensson (1986):

$$u(s) = k^{2^{\rho+1}} \frac{p(s)}{(s+k^2)(s+k^4)\dots(s+k^{2^\rho})} \tag{1.17}$$

where $p(\cdot)$ is an arbitrary monic Hurwitz polynomial of degree ρ. As noted in (Mårtensson, 1986), this controller ensures that for $k \to \infty$, $n-r$ closed-loop poles tend to the zeros of the plant, ρ to the zeros of $p(\cdot)$, and the remaining r ones go to infinity in $\mathbb{C}^-$ at different speed. This last property is the main difference to the controllers (1.14) and (1.16) which can be viewed as having only three time scales: the zero-dynamics (plant zeros), the controlled plant (controller zeros) and the observer error (observer poles).

A similar controller, though only for systems of relative degree not larger than three has been proposed in (Morse, 1987). The controller consists of a reduced-order observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -\lambda_1 \hat{x}_1 + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -\lambda_2 \hat{x}_1 + u \end{aligned}$$

and the following observer-state feedback

$$u = -k^4 y - k^3 \hat{x}_1 - k^2 \hat{x}_2$$

for some positive constants λ_1, λ_2. This results in

$$u(s) = -\frac{k^4(s^2 + s\lambda_1 + \lambda_2)}{s^2 + s(k^2+\lambda_1) + (k^3 + k^2\lambda_1 + \lambda_2)} y(s). \tag{1.19}$$

For large k, $k^2 \gg k \gg \lambda_1$ and $k^3 \gg k^2\lambda_1 + \lambda_2$. Therefore, the poles of (1.19) move approximately as the roots of

$$s^2 + s(k^2 + k) + k^3 = (s+k)(s+k^2).$$

Thus, the poles of the controller tend to infinity at different rates, as does the controller (1.17). This feature seems to be necessary for the stabilization of systems where only an upper bound of the relative degree in known, see Appendix C.

Another approach in the case of unknown relative degree is to search for a stabilizing controller in a set of controllers containing controllers for each possible relative degree. This will be discussed in Section 2.1.3 on page 42.

Summary

If sufficient information on the system is given, the controllers presented in this section can be used for stabilization. It is then necessary to take large enough values for the observer and state-feedback time scales, $\hat{\kappa}$ and κ respectively. If this information is not available or not precise, a good alternative is to adapt these parameters. That way, there is an adapted parameter in the feedback and one in the observer. But as shown in the following section, they can be combined such that the controllers are parameterized by a single parameter, denoted k.

1.4 Adaptation of the gain

To a priori choose the right gain for a high-gain controller, it is necessary to know the magnitude of the system parameters and, for nonlinear systems, also a bound of the initial condition. An alternative is to adapt the gains online. The different adaptation schemes presented in the following are all based on

$$\dot{k} = |y|^2. \tag{1.20}$$

Such an adaptation only increases the gain k. Therefore, this adaptation can only be applied to high-gain controllers, i.e. controllers achieving the control objective for any gain k larger than some minimal gain.

In a first part, adaptation laws for achieving exact stabilization are presented. Then adaptation in the context of λ-tracking is treated.

1.4.1 Adaptation for exact stabilization

The adaptation (1.20) is well-known for first-order linear systems:

$$\dot{y} = ay + bu, \ \ b > 0$$

which can be stabilized by the controller

$$u = -ky$$

together with the adaptation (1.20) (see for example Willems and Byrnes, 1984). Such an adaptive controller is called *non-identifier-based* (Mareels and Polderman, 1996) as the controller parameter k is chosen in a predefined manner, namely by (1.20), only the adaptation speed depends on the measured error. The controller parameters are not based on some estimate of the system to be controlled.

The controllers in Section 1.3 often use two parameters which have to be adapted, κ and $\hat{\kappa}$. These can be coupled to one adaptation by taking

$$\hat{\kappa} = k^{\alpha},$$
$$\kappa = k^{\beta}$$

for some positive constants α and β.

When parameters are adapted, it is necessary to augment the control objective, e.g. asymptotic stability or λ-tracking, with another objective: the convergence of the adaptation. Furthermore, stability of the closed-loop for a fixed adaptation parameter k does not guarantee stability in the case of adaptation. As it has been shown by Desoer (1969), the time variation has to be sufficiently slow. For some high-gain controllers, especially those for higher relative degree systems, the adaptation (1.20) is too fast for large k. A sufficiently slow adaptation seems to be a necessary condition for proving boundedness of the adaptation parameter, see e.g. Chapter 3. A similar requirement has been observed by Morse (1996) who describes that slowing down the tuning was the essential step in proving boundedness and asymptotic tracking of the model reference adaptive controller in the higher relative degree case.

Different adaptation schemes slowing down the adaptation for large values of k are presented in the following. For clarity of presentation, not all details are treated. The presentation is focused on the characteristic features.

Continuous adaptation

The adaptation (1.20) is used in many adaptive high-gain controllers (see Ilchmann, 1993, for a survey).

A generalization of the adaptation presented in (1.20) is to slow the adaptation for large k. This can be done by utilizing an adaptation law of the form

$$\dot{k} = k^{-\gamma} |y|^2, \tag{1.21a}$$

or, equivalently,

$$\frac{d}{dt}\left(k^{\gamma+1}\right) = (\gamma + 1)|y|^2, \tag{1.21b}$$

see Figure 1.9 for a sketch. For systems of relative degree $r \leq 2$, γ is usually chosen to be equal to zero.

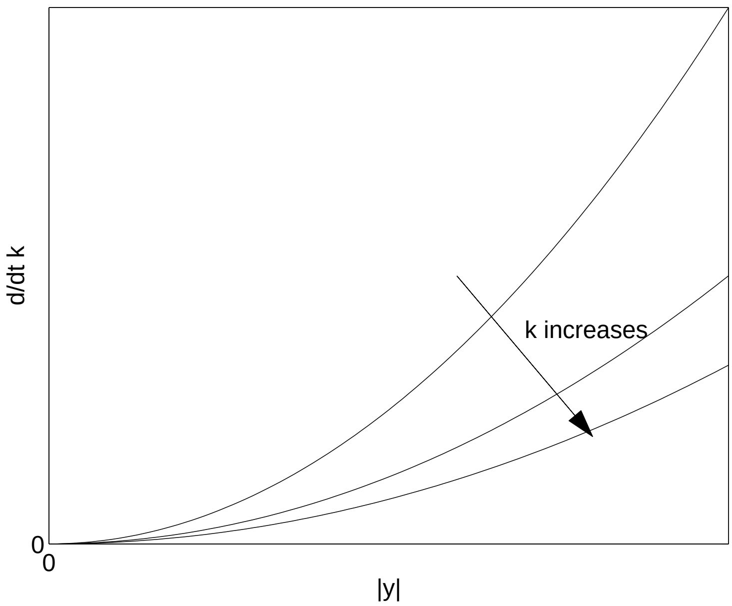


Figure 1.9: Sketch of $\dot{k}$ as in (1.21a).

Another possibility is to bound the adaptation, e.g.

$$\dot{k} = \begin{cases} |y|^2 & \text{for } |y| < \hat{\lambda} \\ \hat{\lambda}^2 & \text{for } |y| \geq \hat{\lambda}. \end{cases} \tag{1.22}$$

Mareels (1984) uses this adaptation in a high-gain controller to asymptotically stabilize linear systems of relative degree larger than one. A necessary assumptions for using this adaptation is that the limit controller, i.e. the one with $k = \lim_{t\to\infty} k(t)$ exponentially stabilizes the system.

Discrete adaptation

When using discrete adaptation, the adaptation parameter k takes only values out of a set of possible, increasing values $\{k_i\}$. A parameter, s, is adapted as in (1.20). Whenever s is equal to a value of the set $\{k_i\}$, this k_i is taken as new value for the adaptation parameter k:

$$\dot{s}(t) = |y(t)|^2, \tag{1.23a}$$

$$k(t) = k_i \tag{1.23b}$$

where i is such that

$$k_i \leq s(t) < k_{i+1}. \tag{1.23c}$$

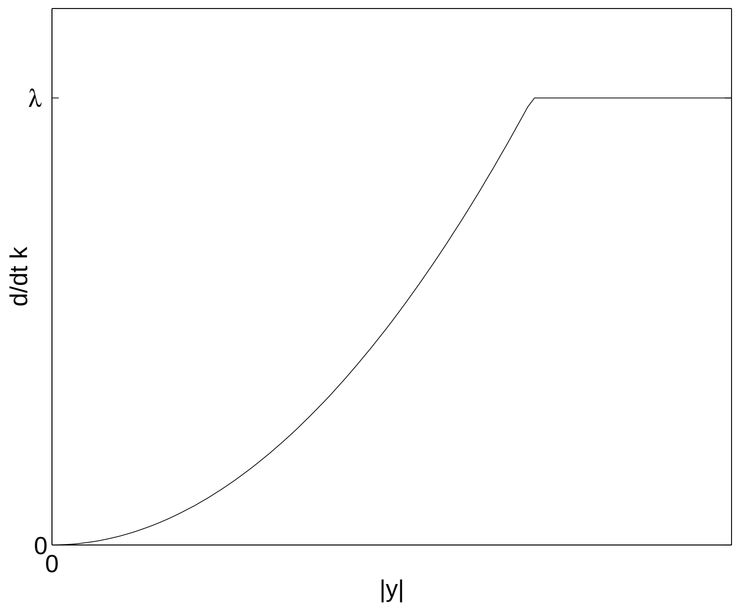


Figure 1.10: Sketch of $\dot{k}$ as in (1.22).

The main advantage of this scheme is that it simplifies the proof of boundedness of the adaptation parameter k as compared to the continuous case. For example, for a linear system and discrete adaptation the closed loop is linear on each time interval where the adaptation parameter is constant. For a continuous adaptation, the closed loop is automatically nonlinear. The disadvantages of such a scheme are the switching which is state-dependent and that the height of the steps usually needs to increase at least exponentially. Such an adaptation has been used by (Khalil and Saberi, 1987; Saberi and Lin, 1990).

1.4.2 Adaptation for λ-tracking

As described in Section 1.1, the control objective in λ-tracking is that the tracking error $y_{ref} - y$ tends asymptotically to the λ-strip. Similarly as in the case of exact stabilization, as long as the control objective is not attained, the adaptation parameter is increased. This section describes different adaptation mechanisms for this purpose.

The adaptation scheme used in (Ilchmann and Ryan, 1994; Ryan, 1994; Allgöwer and Ilchmann, 1995; Allgöwer et al., 1997; Ye, 1999) is

$$\dot{k} = \begin{cases} |y|\,(|y| - \lambda) & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases} . \tag{1.24}$$

As shown in Chapter 3, it is also possible to use the following adaptation

$$\dot{k} = \begin{cases} (|y| - \lambda)^2 & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases}. \tag{1.25}$$

The adaptation laws (1.24) and (1.25) hold for λ-stabilization. For λ-tracking, y has to be replaced by the tracking error $y - y_{ref}$. It is not clear which of the above adaptations results in a better control performance. As Figure 1.11 shows, the adaptation rate for (1.24) is larger than that for (1.25), but the first one has the drawback of not being smooth at $|y| = \lambda$.

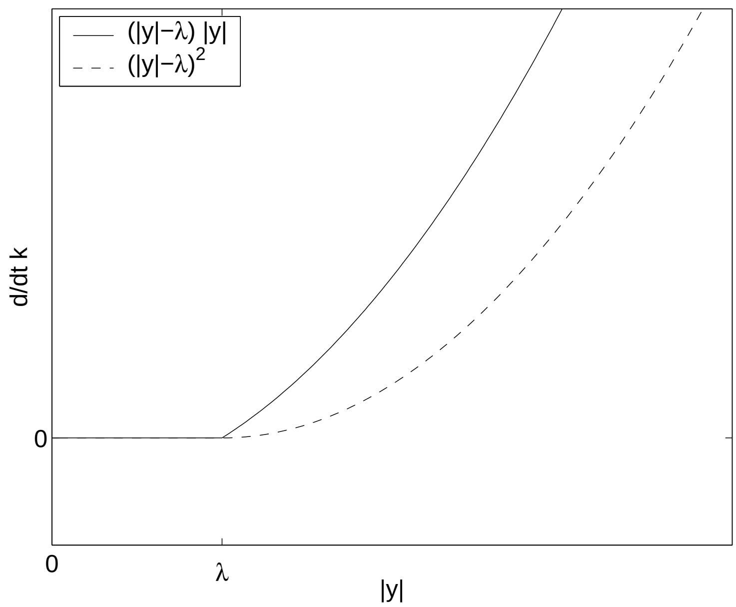


Figure 1.11: Qualitative behavior of the adaptation schemes (1.24) and (1.25).

Some adaptive λ-tracking controllers require that the adaptation is slowed down for large k. Then the following adaptation can be used for some $\gamma > 0$:

$$\frac{d}{dt}k^{\gamma} = \begin{cases} (|y| - \lambda)^2 & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases}. \tag{1.26}$$

This adaptation scheme is used in the controller presented in Chapters 3 as well as in (Bullinger et al., 2001).

Another possibility is to bound $\dot{k}$. If it is not possible to use the adaptation (1.22), as in λ-tracking of nonlinear systems, there is no guarantee that the limit system is stable. An alternative to (1.22) is to use two adaptation parameters. The first one, s, is adapted by (1.20). The second one k is rate-bounded and the one that is actually used in the controller. If the two parameters differ, a sort of proportional controller (1.27c) makes k increase more.

$$\dot{s} = |y|^{2}, \tag{1.27a}$$

$$\dot{k} = \begin{cases} |\tilde{y}|^{2} & \text{for } |\tilde{y}| < \hat{\lambda}, \\ \hat{\lambda}^{2} & \text{for } |\tilde{y}| \geq \hat{\lambda}, \end{cases} \tag{1.27b}$$

$$\tilde{y} = y + \tilde{\gamma}(s - k), \tag{1.27c}$$

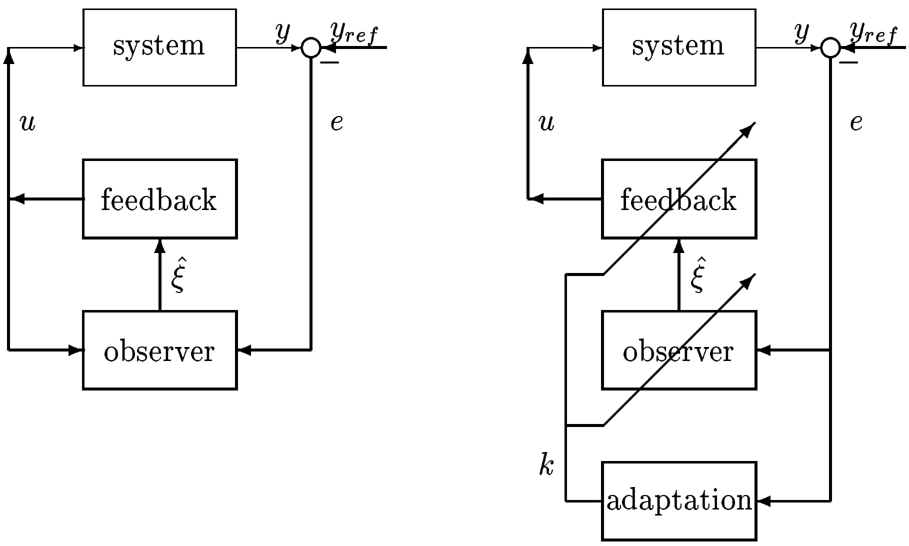
for some $\tilde{\gamma} > 0$. The advantage of this scheme is that the adaptation can be slowed down to an arbitrary rate $\hat{\lambda}$.

The schemes (1.26) and (1.27) both reduce the speed of the adaptation: The first one inverse proportionally to k, the second by a fixed parameter.

1.5 Summary

This chapter has introduced the control problems to be solved, namely λ-stabilization and λ-tracking, together with the main elements of the adaptive λ-tracking controller: λ-stabilization, high-gain feedback, high-gain observer and adaptation. The controllers have one of the following forms: In Figure 1.12, the structure of controllers using full-order high-gain observers is depicted while Figure 1.13 shows the structure of controllers using a reduced-order high-gain observer.

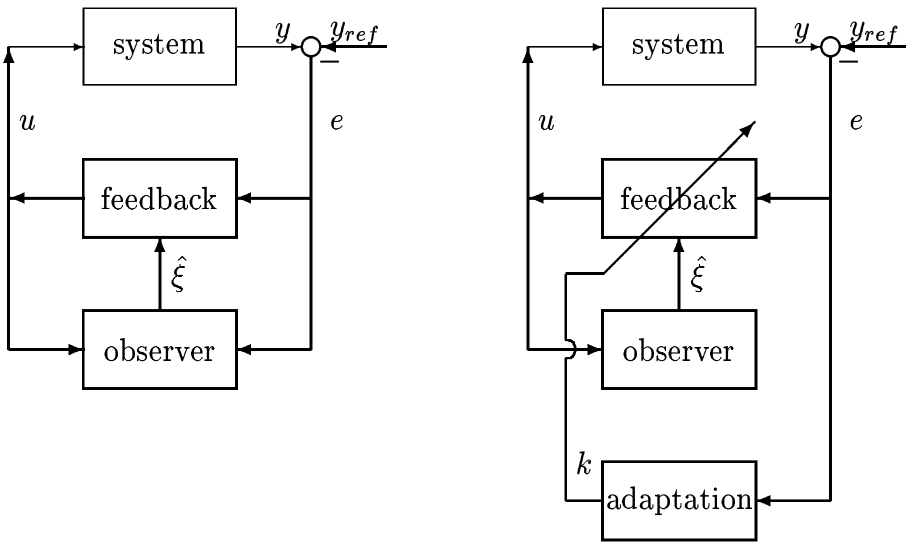
In the following Chapter 2, feedback, observer and adaptation are combined to form different adaptive λ-tracking controllers.



(a) Structure of a general full-order observer-based state feedback controller.

(b) Structure of the proposed full-order adaptive λ-tracking controller.

Figure 1.12: Comparison of the structures of a full-order observer-based state feedback controller and of an adaptive λ-tracking controller. The controllers are drawn in thick, the system in thin lines. Besides the adaptation, the main difference between the two controller schemes is that the observer in the adaptive λ-tracking controller does not need the system input u.



(a) Structure of a general reduced observer-based state feedback controller.

(b) Structure of an adaptive λ-tracking controller using a reduced-order observer.

Figure 1.13: Comparison of the structures of a reduced-order observer-based state feedback controller and of an adaptive λ-tracking controller. The controllers are drawn in thick, the system in thin lines. Besides the adaptation, the main difference between the two controller schemes is that the observer in the adaptive λ-tracking controller does not need the system output y.

Chapter 2

Adaptive High-gain Controllers

The previous chapter presented the key components of adaptive λ-tracking and adaptive high-gain controllers. This chapter gives an overview on how these components can be combined to achieve stabilization or λ-tracking.

The chapter starts with a historical review of the research on universal stabilizers for relative degree one systems. Section 2.1.3 and Section 2.2 then give an overview of adaptive high-gain controllers for systems of higher relative degree achieving stabilization and λ-stabilization, respectively.

2.1 Adaptive high-gain control

Adaptive high-gain control can be split into two main direction. The first requires as little information as possible on the system to be controlled and is often called universal stabilization. The second assumes that some structural information is available, for example the sign of the high-frequency gain.

2.1.1 Historical overview of universal stabilization

The research on universal stabilization greatly influenced research on high-gain stabilizers. This section gives an overview of the rather fast evolution in the 1980's in the field of universal stabilizers.

The term universal stabilization is used here to define controllers which stabilize any system of a class of systems Σ, independent of the initial conditions, see (Ilchmann, 1993; Mareels and Polderman, 1996). An important driving force in the research on universal stabilizers was the minimal information on a certain system class Σ required to designing a single controller for any system in it.

A simple example of a class Σ are first order systems of the form (2.1). As already outlined before, it is well-known that these systems

$$\dot{y} = ay + u \tag{2.1a}$$

$$\dot{y} = ay - u \tag{2.1b}$$

with input $u \in \mathbb{R}$, output $y \in \mathbb{R}$ and unknown constant $a \in \mathbb{R}$ can be stabilized by feedback with sufficiently large gain k if the sign of the gain of u, which is the sign of the high-frequency gain, is known, i.e.

$$\begin{aligned} u &= -ky \text{ for (2.1a), and} \\ u &= +ky \text{ for (2.1b).} \end{aligned}$$

In 1982, Morse conjectured that it was impossible to stabilize the one-dimensional system (2.1) via smooth feedback, without the knowledge of the sign of the high-frequency gain (Morse, 1993).

Shortly afterwards, this was proven wrong by Nussbaum (1983) who presented the following controller:

$$\begin{aligned} \dot{k} &= y(k^2+1) \\ u &= N(k)y \end{aligned} \tag{2.2}$$

with $N(k)$ a *Nussbaum* gain[1] satisfying for some $k_0 \in \mathbb{R}$

$$\begin{aligned} \sup_{k \geq k_0} \int_{k_0}^{k} \frac{1}{k-k_0} N(\nu) d\nu &= \infty \text{ and} \\ \inf_{k \geq k_0} \int_{k_0}^{k} \frac{1}{k-k_0} N(\nu) d\nu &= -\infty, \end{aligned} \tag{2.3}$$

see Ilchmann (1993). Conditions (2.3) imply that $\int N(\nu)d\nu$ grows faster than k both towards $+\infty$ and towards $-\infty$. The example given by Nussbaum (1983) for a function satisfying (2.3) was

$$N(k) = \big(k^2+1\big)\cos(\frac{\pi}{2}k)\exp(k^2).$$

The main feature of this function is that it changes sign. The controller $u = N(k)y$ then "tries" positive and negative feedback alternatively until the adaptation converges.

[1]This term was already used by Morse (1984).

In Willems and Byrnes (1984), a simpler controller than (2.2) was proposed which stabilizes any linear systems of relative degree one. The controller is given by

$$\begin{aligned} \dot{k} &= y^2 \\ u &= N(k)y \end{aligned} \tag{2.4}$$

with

$$N(k) = k\cos(\sqrt{|k|}\,) \tag{2.5}$$

or, yielding a non-smooth feedback,

$$N(k) = k \cdot \begin{cases} +1 & n^2 \le |k| \; < (n+1)^2, \quad n \text{ even}, \\ -1 & n^2 \le |k| \; < (n+1)^2, \quad n \text{ odd}. \end{cases} \tag{2.6}$$

Another possible Nussbaum functions has been proposed by Morse (1984) (see also Ilchmann, 1993, for further discussions):

$$N(k) = k^2 \cos(k). \tag{2.7}$$

The Nussbaum function (2.6) is particularly interesting. As already pointed out by Willems and Byrnes (1984), this Nussbaum gain can be seen as a switching between two regions, namely positive or negative feedback. Willems and Byrnes encouraged to try this scheme on other union of regions. Several controllers using such a search scheme have been proposed in the literature, see (Byrnes and Willems, 1984; Mårtensson, 1985; Mårtensson, 1986; Mårtensson and Polderman, 1993; Ilchmann, 1997; Miller, 1994; Miller, 1998). The controller by Mårtensson (1985) requires only the knowledge of the order of a stabilizing controller, an assumption which was shown to be almost necessary in (Byrnes et al., 1986). This means that the controller searches not only the sign but also the necessary controller dimension. Miller proved that any stabilizable and detectable linear MIMO system can be controlled by a single continuous, though either time-varying controller (Miller, 1994), or non-smooth controller (Miller, 1998).

Morse (1984) uses the Nussbaum gain (2.7) in a model reference adaptive controller with augmented error as in (Morse, 1980). This controller enables to track arbitrary reference signals $y_{ref}(.)$ satisfying for a known Hurwitz polynomial $\alpha(\cdot)$

$$\alpha(s)y_{ref}(s) = r(s),$$

where $r(\cdot)$ is any piecewise-continuous signal. The disadvantage of this scheme, and that of its predecessors (Åström and Wittenmark, 1973; Monopoli, 1974; Feuer and Morse, 1978; Egardt, 1979; Morse, 1980; Narendra et al., 1980) is the high complexity of the controller.

In the 1980's and early 90's, the research on the field of universal or non-identifier based adaptive control boomed, see (Ilchmann, 1991; Ilchmann, 1993) for surveys and (Morse, 1996) for a historical overview similar to this section.

Controllers using a Nussbaum gain are mostly of theoretical interest as they regularly destabilize the loop until a stabilizing controller is found. As stated in (Mårtensson, 1985) "The regulator [...] is absolutely useless for every practical purpose". In the following, some knowledge on the system will therefore be required. Particularly, the sign of the high-frequency gain and the relative degree of the system will be assumed to be known.

2.1.2 Stabilization of nonlinear, relative degree one systems

There are numerous articles on adaptive high-gain stabilization of linear, relative degree one systems. In this thesis, the main emphasis is on the higher relative degree case. For good overview articles treating system with relative degree one, refer to (Ilchmann, 1991; Ilchmann, 1993) and, for infinite-dimensional systems, to (Logemann and Townley, 1997).

But for nonlinear systems of relative degree one, only few adaptive high-gain controllers have been proposed. The n-dimensional nonlinear system is assumed to be of the following form:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + \boldsymbol{e} + \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(u + \boldsymbol{g}(\boldsymbol{x})), \\ y &= \boldsymbol{c}^T\boldsymbol{x} \end{aligned} \tag{2.8}$$

where the three kinds of nonlinearities present in (2.8) are:

1. linear bounded: there exists an M such that for all $\boldsymbol{x}$: $\|\boldsymbol{f}(\boldsymbol{x})\| \leq M\|\boldsymbol{x}\|$,
2. "matched" linear bounded perturbation: $\|\boldsymbol{g}(\boldsymbol{x})\| \leq M\|\boldsymbol{x}\|$ for some $M > 0$,
3. integrable perturbation: $|\int_0^t e(\tau)d\tau|$ is bounded.

The controller by (Ilchmann et al., 1987) stabilizes an n-dimensional, relative degree one system which is linear up to a linearly bounded nonlinearity. This result was extended by (Owens et al., 1987; Prätzel-Wolters et al., 1989) to also cope with matched linearly bounded perturbations. Another approach is taken by (Helmke, 1988) where integrable perturbation are allowed in systems of order one.

2.1.3 Stabilization of systems with a relative degree larger than one

In Section 1.3.1, high-gain state-feedback controllers achieving stabilization of higher-relative degree have been discussed. As shown in Corless (1991) or in Section 1.3.1, a static output high-gain feedback, i.e. $u = -ky$, can stabilize relative degree one and some relative degree two systems. For higher relative degree systems, the high-gain controller needs to be dynamical. Such controllers with fixed high-gain parameter k have been presented in Section 1.3.3. In the following, the main results in the field of non-identifier based control of systems with a higher relative degree are presented. The controllers are classified according to their adaptation.

Continuous adaptation with bounded adaptation rate

Besides (Willems and Byrnes, 1984) and (Morse, 1984), a third key paper was published in 1984: (Mareels, 1984). There, it is shown that the controller depicted in Figure 2.1 stabilizes any minimum-phase linear plant if its relative degree, the sign and an upper bound of the high-frequency gain are known. For a fixed adaptation parameter k, the controller can be described by the transfer function

$$u(s) = -k^r \left(1 + \sum_{i=1}^{r-1} k^{r-i} \frac{g_i}{\rho_i(s)}\right)^{-1} y(s) \tag{2.9}$$

where $\rho_i(\cdot)$, $i = 1, \ldots, r-1$ are monic Hurwitz polynomials of degree $r - i$. The coefficients g_i, $i = 1, \ldots, r-1$ have to be chosen in such a way that

$$s^r + \sum_{i=1}^{r-1} s^i g_i + g_0$$

is a Hurwitz polynomial. This requires the knowledge of an upper bound on the high-frequency gain g_0, see Appendix C.4. The resulting controller has dimension $\frac{r(r-1)}{2}$. The dimension can be brought to $r - 1$ if the polynomials $\rho_i(\cdot)$ are chosen to be

$$\begin{aligned} \rho_{r-1}(s) &= s + \lambda_{r-1} \\ \rho_i(s) &= \rho_{i+1}(s + \lambda_i),\ i = 1 \ldots r-2 \end{aligned} \tag{2.10}$$

for some $\lambda_i > 0$, $i = 1 \ldots r-1$. This means that the transfer functions $\frac{1}{\rho_i(\cdot)}$ have the maximum possible number of poles in common. For large k's, the controller (2.9) with condition (2.10) is approximately

$$\frac{u(s)}{y(s)} \approx -k^r \frac{(s+\lambda_1)\ldots(s+\lambda_{r-1})}{s^{r-1} + \sum_{i=0}^{r-2} g_{i+1} k^{r-i-1} s^i}, \tag{2.11}$$

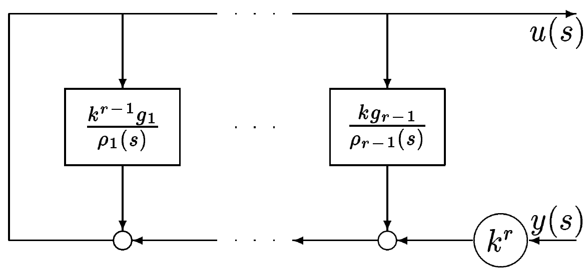


Figure 2.1: Signal flow of the controller of Mareels (1984).

which is equivalent to the controller of (Bullinger et al., 2001) achieving λ-tracking for nonlinear systems.

For a relative degree one system, the controller (2.9) is simply:

$$u = -ky.$$

Together with the adaptation $\dot{k} = y^2$, this is the same as the controller in (Willems and Byrnes, 1984) with the Nussbaum gain replaced by $-\operatorname{sign}(g_0)$.

The adaptation used in Mareels (1984) in the higher relative degree case is

$$\dot{k} = \frac{1}{k^{r-1}} f(y) \tag{2.12}$$

where $f(y)$ is any function $\mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

- $0 < f(y) < \hat{\lambda}$ for all $y \neq 0$ and some $\hat{\lambda} > 0$,
- $f(y_1) \leq f(y_2)$ for all $|y_1| \leq |y_2|$,
- $G(Y) = \int_0^Y \frac{f(y)}{y} dy$ is continuous.

The main feature of this adaptation is that it is bounded by $\hat{\lambda}$ and slowed down for large k. For example,

$$f(y) = \begin{cases} |y|^2 & \text{for } y \leq \hat{\lambda}, \\ \hat{\lambda} & \text{for } y > \hat{\lambda}. \end{cases}$$

satisfies these assumptions. Such an adaptation requires that the limit system is stable, which does not hold for general nonlinear systems under λ-tracking.

Continuous adaptation with unbounded adaptation rate

The controller of Mudgett and Morse (1989) is a reduced-order observer based controller as in (1.16) on page 24 and uses the adaptation law

$$\dot{k} = y^2.$$

This adaptation is neither slowed down for large k's nor is $\dot{k}$ bounded. This is possible as there is a flaw in the proof. The part of the proof of (Mudgett and Morse, 1989) between Equations (22) and (23) can be restated as follows. In the time derivative of the Lyapunov function candidate

$$\boldsymbol{x}^T P \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^n,\ P > 0 \in \mathbb{R}^n,$$

the following matrix

$$\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix} P + P \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix}$$

is shown to be positive definite. But, in general, this is not true. A counter-example is $P = \left[\begin{smallmatrix} 1 & 4 \\ 4 & 17 \end{smallmatrix}\right]$. A possible solution is to slow down the adaptation as

$$\dot{k} = k^{-\gamma} |y|^2$$

for some γ depending on r.

Discrete adaptation

Khalil and Saberi (1987) and Saberi and Lin (1990) present the following high-gain controller

$$u(s) = \kappa_i \hat{\kappa}_i^{r-1} \frac{s^{r-1} + \sum_{i=0}^{r-1} p_i s^i}{s^{r-1} + \sum_{i=0}^{r-1} q_i \hat{\kappa}_i^{r-1-i}} y(s)$$

which stabilizes a class of nonlinear systems. The adaptation of k in (1.23) is used to determine $\hat{\kappa} = k^\alpha$ and $\kappa = k^\beta$ for some $\alpha > \beta > 0$. The lower bound for the discrete values of the adaptation parameters κ_i and $\hat{\kappa}$ is relatively complicated: they grow more than exponentially, see Khalil and Saberi (1987). The adaptation ensures that κ and $\frac{\hat{\kappa}}{\kappa}$ get sufficiently large such that the controller stabilizes the system before the next increase of the adaptation parameters is due.

Adaptation corresponds to searching in a set

In (Mårtensson, 1985; Mårtensson, 1986; Mårtensson and Polderman, 1993) it is shown that for universal controller design it is sufficient to know the dimension of a stabilizing controller. The stabilizing controller itself is found by searching through a set of controllers. From a theoretical point of view, this is a very interesting result as it shows that for controller design it is only necessary to know the order of a stabilizing controller, not necessarily some information on the system itself. This assumption was shown to be almost necessary in (Byrnes et al., 1986). But Miller proved that any stabilizable and detectable linear system can be controlled by a single continuous controller, though either time-varying (Miller, 1994) or non-smooth (Miller, 1998). For nonlinear systems of relative degree one, a universal controller has been proposed by (Ilchmann, 1997).

Similar controllers achieving other objectives than stabilization are the "hard constraint λ-tracking controller" by Miller and Davison (1991a) and controllers for linear systems lying in a known compact set where exponential convergence can be guaranteed, see (Fu and Barmish, 1986) for the relative degree one case and (Miller and Davison, 1991b) for the unknown relative degree case. Overviews over these controllers can be found in (Ilchmann, 1993; Morse, 1995; Miller et al., 1997).

2.2 Adaptive λ-tracking controllers

In the following, adaptive high-gain controllers achieving λ-tracking are presented.

2.2.1 Adaptive λ-tracking controllers for relative degree one systems

Adaptive λ-tracking controllers have been introduced by (Ilchmann and Ryan, 1994; Ryan, 1994). These controllers achieve λ-tracking for relative degree one systems of unknown sign of the high-frequency gain, using the following adaptation

$$\dot{k} = \begin{cases} |y|\,(|y| - \lambda) & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases},$$

see also (Ilchmann, 1993).

The following two controllers extend this to nonlinear systems. The controller by (Allgöwer and Ilchmann, 1995) to MIMO systems of known sign of the high-frequency gain, the one in (Allgöwer et al., 1997) to SISO systems with unknown sign of the high-frequency gain.

2.2.2 Adaptive λ-tracking controllers for higher relative degree systems

In the following, controllers achieving λ-tracking for systems with a relative degree larger than one are presented. As in the case of exact stabilization, the different controllers achieving λ-tracking are split in groups of similar adaptation schemes.

Bounded continuous adaptation

For linear systems, Mareels (1984) proposed a controller stabilizing a system of arbitrary, but known relative degree r having a high-frequency gain that is positive and has a known upper bound. This controller has been presented in Section 2.1.3. As shown in (Mareels, 1984), with the adaptation law

$$\dot{k} = \begin{cases} \frac{1}{k^{r-1}} f(|y| - \lambda), & |y| \geq \lambda, \\ 0 & |y| < \lambda, \end{cases}$$

the controller achieves λ-stabilization. If augmented by an internal model of the reference signal, the controller achieves λ-tracking.

Continuous adaptation — reduced-order observer based controller

In (Ye, 1999) the first λ-tracking controller for nonlinear systems of higher relative degree was proposed. This controller uses a reduced-order observer as discussed in Section 1.3.2 with

$$\hat{A} = \begin{bmatrix} -\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & -\lambda_{r-1} \end{bmatrix}, \ \hat{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The observer-state feedback law is computed via backstepping, see (Krstić et al., 1995). The advantage of this approach is that the adaptation does not need to be slowed down for large k. Therefore the same adaptation law can be used as in the relative degree one case:

$$\dot{k} = \begin{cases} |y|\,(|y| - \lambda) & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases}.$$

The disadvantage of this controller is that the feedback law is much more complicated than all other schemes presented here as it is constructed by recursion and includes higher-order terms in y. These are necessary to cope with fast adaptation, i.e. compensate the terms in

$\frac{\dot{k}}{k}$. By using a Nussbaum gain, this controller can cope with an unknown high-frequency gain. In the case of a system of relative degree two and known sign of the high-frequency gain, the simplest controller of the family of controllers proposed by Ye (1999) is

$$\begin{aligned} u &= \lambda_1 \xi_1 - (\xi_1 + 2ke)\left(1 + 4d_\lambda^2(e)e^4 + k^2\left(\xi_1^2 + (1 + d_\lambda(e))^2\right)\right), \\ \dot{k} &= 2d_\lambda(e)|e|, \end{aligned}$$

where λ_1 is the time-constant of the observer, ξ_1 the observer state, $e = y - y_{ref}$ the tracking error and

$$d_\lambda(e) = \begin{cases} |e| - \lambda, & |e| \geq \lambda \\ 0, & |e| < \lambda, \end{cases}$$

the distance between e and the λ-strip. Note that even for a relative degree two system, the feedback law is rather complicated and highly nonlinear in e and ξ_1.

A similar controller as the controller by Mudgett and Morse (1989) can be used for λ-tracking a rather large class of nonlinear systems (see Bullinger et al., 2001). This controller consists of a reduced-order observer and the high-gain feedback

$$u = -q_0 k^r (y - y_{ref}) - \sum_{i=1}^{r-1} q_i k^{r-i} \hat{\xi}_i, \tag{2.13}$$

see (1.16) or Figure 1.8, page 25. This feedback is affine in the tracking error $e = y - y_{ref}$ and in the observer states ξ_i, $i = 1, \ldots, r-1$. The observer is more general than the one used in (Ye, 1999) as it is possible to chose for the matrix $\hat{A}$ any Hurwitz lower triangular matrix with ones on the superdiagonal[2]:

$$\hat{A} = \begin{bmatrix} * & 1 & & \\ \vdots & \ddots & \ddots & \\ & & \ddots & 1 \\ & \cdots & \cdots & * \end{bmatrix}$$

where the stars stand for arbitrary values.

The adaptation law in (Bullinger et al., 2001) includes a "damping" for large k

$$\dot{k} = \begin{cases} k^{-\gamma}(|y| - \lambda)^2 & \text{for } |y| \geq \lambda \\ 0 & \text{for } |y| < \lambda \end{cases}, \quad \gamma > 0.$$

Continuous adaptation — full-order observer based controller

A different adaptive λ-tracking controller is proposed in Chapter 3. There, the observer is a full-order one. The feedback part is similar to (2.14), but instead of the tracking error

[2]The superdiagonal consists of the elements directly above the diagonal (Weisstein, 2000)

$y - y_{ref}$, another observer state, which is a filtered version of the tracking error, is used:

$$u = -\sum_{i=1}^{r} q_i k^{r-i} \hat{\xi}_i. \tag{2.14}$$

The adaptation is similar to the reduced-order case as it also includes some "damping" in the adaptation as in (2.13).

Discrete adaptation

For linear systems, an adaptive λ-tracking controller with a discrete adaptation has been proposed by (Bullinger et al., 1999). It uses a full-order high-gain observer and a step-wise increase of the adaptation parameter as in (1.23). In principle, this controller is similar to the one of Khalil and Saberi (1987) and Saberi and Lin (1990) but it is directly designed in the state space. The minimum gain increase is given by

$$k_{i+1} - k_i > \gamma^i k_i^{2(r-1)} \quad \text{for all } i \geq i^*. \tag{2.15}$$

for some $\gamma > 0$ and $i^* > 0$.

2.3 Conclusions

In this chapter, results on adaptive high-gain control and adaptive λ-tracking have been reviewed. In the next chapter, an adaptive λ-tracking controller for *nonlinear, higher relative degree* systems is proposed. As already outlined, this controller has a simple structure and is easier to design as the one in (Ye, 1999).

Chapter 3

An Adaptive λ-tracking Controller for Nonlinear Systems with High Relative Degree

This chapter proposes an adaptive λ-tracking controller for nonlinear systems of higher relative degree. This controller consists of a full-order observer and an observer-state feedback. The controller structure is relatively simple, as opposed to the controller by Ye (1999) which utilizes a complex observer-state feedback calculated via backstepping. Theorem 3.1 in Section 3.4 shows that under mild assumptions on the system and the reference signal, λ-tracking can be achieved. Furthermore, the adaptation converges and all states remain bounded.

The advantage of using a full-order observer is that the output does not enter directly the feedback part of the controller. Instead the output enters in a filtered way. This is advantageous when output noise is present.

3.1 Motivation

The controller consists of a full-order high-gain observer, see (1.9a), page 18 and a high-gain state-feedback as in (1.14), page 23, see also Figure 1.8.

The motivation for such an observer-based controller comes from the fact that for fixed adaptation parameters κ and $\hat{\kappa}$ the transfer function from y to $\hat{\boldsymbol{x}}_i$, the i-th state of the observer,

$$\hat{\boldsymbol{x}}_i(s) = s^{i-1}\left(1 - \frac{\left(\frac{s}{\hat{\kappa}}\right)^r + \ldots + \left(\frac{s}{\hat{\kappa}}\right)^{r-i+1} r_{r-i+1}}{\left(\frac{s}{\hat{\kappa}}\right)^r + \left(\frac{s}{\hat{\kappa}}\right)^{r-1} p_{r+1} + \cdots + p_n}\right) y(s),$$

is for "low" frequencies, i.e. for small $\left(\frac{s}{\hat{\kappa}}\right)$

$$\hat{\boldsymbol{x}}_i(s) \approx s^{i-1} y(s).$$

The observer therefore approximates a series of differentiators with a band-width proportional to the observer gain $\hat{\kappa}$. Thus, the observer states approximate the derivatives of the output y. Therefore, the controller can be seen as an approximation of a $PD \ldots D^{r-1}$ controller:

$$u(s) \approx -\sum_{i=0}^{r-1} q_i s^i \kappa^{r-i} y(s) = -\kappa^r \sum_{i=0}^{r-1} q_i \left(\frac{s}{\kappa}\right)^i y(s).$$

3.2 Setup

This section first presents the system class the proposed adaptive observer-based state-feedback controller can be applied to. Then the controller components, the observer, the state-feedback and the adaptation are presented separately.

3.2.1 System class

The proposed controller is applicable to single-input single-output systems that are affine in the input:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot u, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{3.1a}$$

$$y = h(\boldsymbol{x}). \tag{3.1b}$$

For the results to hold, the following assumptions on the system must be satisfied.

Assumption 3.1 (Known relative degree) *The relative degree r is known and strong, i.e. for all $\boldsymbol{x} \in \mathbb{R}^n$*

$$L_{\boldsymbol{g}} L_{\boldsymbol{f}}^i h(\boldsymbol{x}) \equiv 0, \; i = 0, \ldots, r-2,$$

$$g(\boldsymbol{x}) = L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{r-1} h(\boldsymbol{x}) \neq 0 \tag{3.2}$$

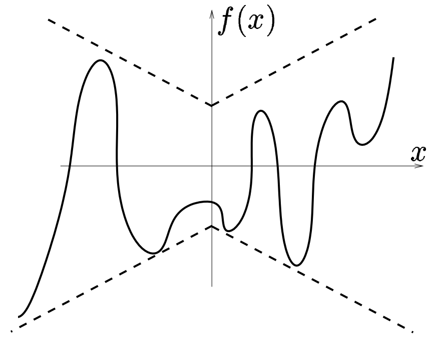


Figure 3.1: Example of a one-dimensional function $f \in \mathcal{A}$.

Assumption 3.2 (Positive high-frequency gain) *The high-frequency gain $g(\boldsymbol{x})$ is strictly positive and globally bounded away from zero by some known constant $\underline{g} > 0$,*

$$g(\boldsymbol{x}) \geq \underline{g} \quad \text{for all } \boldsymbol{x}.$$

The following definition is needed for describing the system class.

Definition 2 (Affine sector bound) *A function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is in the set $\mathcal{A}$ if for some $m > 0$ it can be decomposed as*

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}_0(\boldsymbol{x}) + F(\boldsymbol{x})\boldsymbol{x}$$

where

$$\begin{aligned} \boldsymbol{f}_0(\cdot) &: \mathbb{R}^n \to \mathbb{R}^n, \ \|\boldsymbol{f}_0(\boldsymbol{x})\| \leq m, \\ F(\cdot) &: \mathbb{R}^n \to \mathbb{R}^{n \times n}, \ \|F(\cdot)\| \leq m. \end{aligned}$$

Remark 3.1 *Special cases of functions $\boldsymbol{f}(.) \in \mathcal{A}$ are functions satisfying for some constants m_1, m_2*

$$\|\boldsymbol{f}(\boldsymbol{x})\| \leq m_1 + m_2 \|\boldsymbol{x}\|,$$

or

$$\lim_{\|\boldsymbol{x}\| \to \infty} \frac{\|\boldsymbol{f}(\boldsymbol{x})\|}{\|\boldsymbol{x}\|} = m_2.$$

Therefore, the class $\mathcal{A}$ is a generalization of sector bounded nonlinearities. Figure 3.1 shows a sketch of a possible one-dimensional function. ♢

This enables to state the main assumption on the system class.

Assumption 3.3 (Bounded nonlinearities) *There exists a coordinate transformation* $T(\boldsymbol{x}) = [\boldsymbol{\xi}^T, \boldsymbol{\eta}^T]^T$ *transforming* (3.1) *into input-normalized Byrnes-Isidori normal form* (3.3) *(see (Byrnes and Isidori, 1984; Byrnes and Isidori, 1985; Isidori, 1995)*

$$\begin{aligned} y &= \xi_1 \\ \dot{\xi}_i &= \xi_{i+1} \quad \text{for } i = 1, \dots, r-1 \\ \dot{\xi}_r &= \alpha(\boldsymbol{\xi}, \boldsymbol{\eta}) + g(\boldsymbol{\xi}, \boldsymbol{\eta})u \\ \dot{\boldsymbol{\eta}} &= \tilde{\boldsymbol{\theta}}(\boldsymbol{\xi}, \boldsymbol{\eta}) \end{aligned} \tag{3.3}$$

with

$$\begin{aligned} \boldsymbol{\xi}(t) &= [\xi_1(t), \dots, \xi_r(t)]^T \in \mathbb{R}^r \\ \boldsymbol{\eta}(t) &\in \mathbb{R}^{n-r}, \end{aligned}$$

where the following conditions hold:

1. $\alpha(\cdot) \in \mathcal{A}$,
2. $g(\cdot)$ *is bounded,*
3. $\xi_1 \mapsto \tilde{\boldsymbol{\theta}}(\xi_1, \xi_2, \dots, \xi_r, \boldsymbol{\eta}) \in \mathcal{A}$ *for all* $(\xi_2, \dots, \xi_r, \boldsymbol{\eta}) \in \mathbb{R}^{n-1}$,
4. $(\xi_2, \dots, \xi_r, \boldsymbol{\eta}) \mapsto \tilde{\boldsymbol{\theta}}(\xi_1, \xi_2, \dots, \xi_r, \boldsymbol{\eta})$ *is bounded for all* $\xi_1 \in \mathbb{R}$.

Assumption 3.4 (Zero dynamics) *The zero-dynamics of* (3.3) *can be decomposed as*

$$\dot{\boldsymbol{\eta}} = \tilde{\boldsymbol{\theta}}(\mathbf{0}, \boldsymbol{\eta}) = \boldsymbol{\theta}(\boldsymbol{\eta}) + \tilde{\boldsymbol{w}}(\boldsymbol{\eta})$$

where $\tilde{\boldsymbol{w}}(\cdot)$ *is bounded and the dynamics* $\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta})$ *are globally exponentially stable.*

Under Assumption 3.3 and 3.4, (3.3) can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= J\boldsymbol{\xi} + \boldsymbol{b}\big(\boldsymbol{\psi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{\phi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\eta} \\ &\qquad + g(\boldsymbol{\xi}, \boldsymbol{\eta})u + v(\boldsymbol{\xi}, \boldsymbol{\eta})\big)\big) \end{aligned} \tag{3.4a}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})y + \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{w}(\boldsymbol{\xi}, \boldsymbol{\eta}) \tag{3.4b}$$

$$y = \boldsymbol{c}^T\boldsymbol{\xi}, \tag{3.4c}$$

with $\boldsymbol{\xi}(t) \in \mathbb{R}^r$, $\boldsymbol{\eta}(t) \in \mathbb{R}^{n-r}$ and $\{J, \boldsymbol{b}, \boldsymbol{c}^T\}$ a prime triple (see Definition 10, page 90). All functions are bounded:

$$\begin{aligned} g(\boldsymbol{\xi}, \boldsymbol{\eta}) &\geq g\_, \\ v, \boldsymbol{\theta}, \boldsymbol{w}, \boldsymbol{\chi}, \boldsymbol{\phi}, \boldsymbol{\psi} &\in \mathcal{L}_\infty(\mathbb{R}^n). \end{aligned}$$

Remark 3.2 *The zero-dynamics of* (3.4) *are*

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{w}(\mathbf{0}, \boldsymbol{\eta}). \tag{3.5}$$

As $\boldsymbol{w}(\cdot)$ *is bounded,* $\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta})$ *can be considered as the unperturbed zero-dynamics. Assumption 3.4 guarantees the existence of a* C^2 *Lyapunov function for the unperturbed zero-dynamics* (3.5)*. Especially, for some constants* μ, δ, $\lambda > 0$, $\tau \geq 0$, *the following holds:*

$$\begin{aligned}
&\boldsymbol{\theta}(\mathbf{0}) = 0, \\
&\|\eta(t)\| \leq \mu\|\boldsymbol{\eta}(\tau)\| \exp(-\delta(t-\tau))\forall t \geq \tau, \forall \boldsymbol{\eta}(\tau) \in \mathbb{R}^{n-r}, \\
&\frac{\partial}{\partial \eta}\boldsymbol{\theta}(\boldsymbol{\eta}) \leq \lambda\|\boldsymbol{\eta}\|,
\end{aligned}$$

see (Vidyasagar, 1993, Section 5.7). ◇

Remark 3.3 *Assumptions 3.1–3.4 are not very restrictive. They are essentially the same as for the relative degree one case.* ◇

Remark 3.4 *Sannuti (1983) show that any linear system with relative degree* r *is transformable into*

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= J\boldsymbol{\xi} + \boldsymbol{b}(\boldsymbol{\psi}^T\boldsymbol{\xi} + \boldsymbol{\phi}^T\boldsymbol{\eta} + gu) \\
\dot{\boldsymbol{\eta}} &= \boldsymbol{\chi}y + H\boldsymbol{\eta} \\
y &= \boldsymbol{c}^T\boldsymbol{\xi}.
\end{aligned}$$

The system (3.4) *can be seen as a nonlinear generalization of this.* ◇

3.2.2 Objective

The control objective is to asymptotically track a reference signal $y_{ref}(\cdot)$ while tolerating a tracking error smaller than a user-defined λ (λ-tracking):

$$|y - y_{ref}| \to [0, \lambda].$$

All states should remain bounded, i.e. $\boldsymbol{x} \in \mathcal{L}_\infty([0,\infty))$. The reference signal $y_{ref}(\cdot)$ is considered to be in $W^{r,\infty}$, the set of all bounded functions that are absolutely continuous on compact subintervals and whose first r derivatives are essentially bounded.

For the given system and objective, an adaptive output-feedback “state-space” controller is designed. It consists of an adaptive high-gain observer and an adaptive high-gain state-feedback controller, described in the following.

3.3 Controller structure

The adaptive λ-tracking controller can be decomposed into a high-gain observer (Section 3.3.1) and a high-gain state-feedback (Section 3.3.2). The adaptation of the gains is described in Section 3.3.3.

3.3.1 Full-order observer

The observer is an adaptive version of the high-gain observer introduced by Nicosia and Tornambè (1989) (see also Tornambè, 1992) as proposed in (Bullinger et al., 1998). The observer is presented in observability normal form (Zeitz, 1989) and given by

$$\dot{\hat{\boldsymbol{x}}} = \hat{A}_{\hat{\kappa}} \hat{\boldsymbol{x}} + \boldsymbol{p}_{\hat{\kappa}} e \tag{3.6a}$$

$$e = y - y_{ref} \tag{3.6b}$$

with $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^r$ and

$$\hat{A}_{\hat{\kappa}} = \begin{bmatrix} -p_{r-1} \cdot \hat{\kappa} & 1 & 0 & & \\ -p_{r-2} \cdot \hat{\kappa}^2 & 0 & 1 & & \\ \vdots & & & \ddots & \\ -p_1 \cdot \hat{\kappa}^{r-1} & 0 & 0 & & 1 \\ -p_0 \cdot \hat{\kappa}^r & 0 & 0 & & 0 \end{bmatrix} = J - \boldsymbol{p}_{\hat{\kappa}} \boldsymbol{c}^T, \; \boldsymbol{p}_{\hat{\kappa}} = \begin{bmatrix} p_{r-1} \cdot \hat{\kappa} \\ p_{r-2} \cdot \hat{\kappa}^2 \\ \vdots \\ p_1 \cdot \hat{\kappa}^{r-1} \\ p_0 \cdot \hat{\kappa}^r \end{bmatrix}.$$

The parameters p_i are chosen such that $p(s) = s^r + \sum_{i=0}^{r-1} p_i s^i$ is Hurwitz. For any positive value of the observer gain $\hat{\kappa}$, the spectrum of $\hat{A}_{\hat{\kappa}}$ lies in the open left half plane and the observer dynamics are stable. No further knowledge of the system besides that of the relative degree is needed for the design of the observer. The observer gain $\hat{\kappa}$ is adapted according to the adaptation law described below.

3.3.2 Observer-state feedback controller

The controller is an observer-state feedback

$$u = -\boldsymbol{q}_{\kappa} \hat{\boldsymbol{x}}, \tag{3.7}$$

where

$$\boldsymbol{q}_{\kappa} = \left[q_0 \cdot \kappa^r, \quad \cdots, \quad q_{r-1} \cdot \kappa \right]^T.$$

The parameters q_i are chosen such that

$$q_g(s) = s^r + g \sum_{i=0}^{r-1} q_i s^i \tag{3.8}$$

is Hurwitz for all $g \geq \underline{g}$ where $\underline{g}$ is a lower bound of the high-frequency gain of the system.

For a relative degree $r \leq 2$, this is trivially achieved by choosing q_1 and q_0 positive. For a higher relative degree, it is necessary to choose $\sum_{i=0}^{r-1} q_i s^i$ to be Hurwitz, see Appendix C.5. Then there exists a g^* such that for any $g \geq g^*$, $q_g(\cdot)$ is Hurwitz. By re-scaling the q_i's, g^* can be made sufficiently small to ensure $g^* \leq \underline{g}$.

For any positive values of the controller gain κ, the spectrum of $J - g\boldsymbol{b}\boldsymbol{q}_\kappa^T$ lies in the open left half plane. Only the relative degree r and a lower bound of the high-frequency gain $\underline{g}$ are needed for the controller design. The adaptation law for the controller gain κ is described below.

3.3.3 Gain adaptation

The adaptation for the observer gain $\hat{\kappa}$ and the controller gain κ is chosen such that the gains are increased as long as the amplitude of the tracking error e is larger than the user-defined bound λ (the control objective).

The observer and controller gains are given by

$$\hat{\kappa} = k^\alpha \tag{3.9a}$$

$$\kappa = k^\beta \tag{3.9b}$$

where the parameters α and β have to satisfy

$$\alpha > \beta > 0. \tag{3.10}$$

For given polynomials $p(\cdot)$ and $q_{\tilde{g}}(\cdot)$ there exist positive constants ϵ and μ such that for all $\tilde{g} \geq \underline{g}$, $p(\cdot)$ and $q_{\tilde{g}}(\cdot)$ are in $H(\epsilon, \mu)$, see Definition 3, page 87. The parameters ϵ and μ can be interpreted as a measure of robustness with respect to time scaling for ϵ and the decay rate of the differential equation corresponding to the polynomial for μ.

With $\lambda > 0$, $\gamma > 0$ and $k(0) = k_0 > 0$, the adaptation parameter k itself is given by

$$\dot{k} = d_\lambda^2(e, k), \tag{3.11a}$$

$$d_\lambda(e, k) = \frac{\gamma}{k^{\tilde{\gamma}}} \begin{cases} |e| - \lambda & \text{for } |e| > \lambda, \\ 0 & \text{for } |e| \leq \lambda, \end{cases} \tag{3.11b}$$

where $\tilde{\gamma}$ has to satisfy

$$\tilde{\gamma} > 2\alpha\epsilon + (\alpha - \beta)(2r - 3) - \frac{1}{2} \qquad \text{for } r > 1, \tag{3.11c}$$

$$\tilde{\gamma} \geq 2\beta\epsilon - \frac{1}{2} \qquad \text{for } r = 1. \tag{3.11d}$$

Remark 3.5 *The parameters α and β can be used to tune individually the "gains" of the observer and the controller, respectively.* $\diamondsuit$

Remark 3.6 *This adaptation law ensures a monotonic increase of the observer and controller gains. Also the observer gain $\hat{\kappa}$ grows faster than the controller gain κ for $k \geq 1$.* $\diamondsuit$

Remark 3.7 *The parameter $\tilde{\gamma}$ slows down the adaptation, particularly when k is large. Its lower bound depends on the relative degree, the choice of the exponents α and β and on the polynomials $p(\cdot)$ and $q(\cdot)$. For systems with relative degree one, ϵ can be chosen to be zero, and $\tilde{\gamma} = 0$ is a valid choice for the relative degree one case.* $\diamondsuit$

Remark 3.8 *Systems with negative high frequency gain, i.e. $g \leq \underline{g} < 0$ instead of* (3.2)*, can easily be treated by changing the control law* (3.7) *to*

$$u = +\boldsymbol{q}_{\kappa}^{T}\hat{\boldsymbol{x}}. \qquad \diamondsuit$$

Remark 3.9 *The adaptive λ-tracking controller consisting of the observer* (3.6)*, the adaptive controller* (3.7) *and the adaptation law* (3.11) *with* (3.9) *and* (3.10) *is the same as the adaptive λ-tracking controller for linear systems with higher relative degree (Bullinger et al., 2000a). In the following section, it is shown that the same objectives, namely λ-stabilization, convergence of the adaptation and boundedness of the states, can be achieved for the considered class of nonlinear systems.* $\diamondsuit$

3.4 Result on λ-tracking and convergence of the adaptation

The main result in this chapter is Theorem 3.1. It states that combining the adaptive observer (3.6) with the adaptive controller (3.7) and using the adaptation law (3.11) with (3.9) and (3.10) to close the loop for an arbitrary system of class (3.1), satisfying Assumptions 3.1 to 3.4 yields that the tracking error asymptotically converges to the λ-strip. Furthermore, the adaptation converges, no finite escape time can occur and all states remain bounded.

Theorem 3.1 (Full-order adaptive λ-tracking controller)
Define the constants $\epsilon > 0$ and $\mu > 0$ so that the polynomials $p(\cdot)$ and $q_{\tilde{g}}(\cdot)$ are in $H(\epsilon, \mu)$ for all $\tilde{g} \geq \underline{g}$. Then the application of the λ-tracking controller (3.6), (3.7), (3.11) *with*

$\hat{\kappa} = k^{\alpha}$, $\kappa = k^{\beta}$ and $\alpha > \beta > 0$ to any system satisfying Assumptions 3.1 to 3.4 with any reference signal $y_{ref}(\cdot) \in W^{r,\infty}$ results in a closed-loop system which, independently of the initial values $\boldsymbol{x}(0) \in \mathbb{R}^n$, $\hat{\boldsymbol{x}}(0) \in \mathbb{R}^r$ and $k(0) > 0$ has a unique solution which exists on the whole half axis $t \in [0, \infty)$ and, moreover,

a) $(\boldsymbol{x}(\cdot), \hat{\boldsymbol{x}}(\cdot), k(\cdot)) \in \mathcal{L}_{\infty}\big([0, \omega)\big)$,
b) $\lim_{t\to\infty} |y(t) - y_{ref}(t)| \leq \lambda$.

The proof of Theorem 3.1 consists of five steps. The first part shows that k cannot go to infinity on the maximal domain of existence. Then, boundedness of the observer states $\hat{\boldsymbol{x}}$ and thus of the plant input u is proven. Part three shows boundedness of the plant states $\boldsymbol{x}$. Step four yields that the solution of the differential equations exists for all times. A consequence of the first and fourth step is the convergence of the adaptation parameter k and by that of κ and $\hat{\kappa}$. The proof concludes by showing that the tracking error converges to the λ-strip.

Proof (of Theorem 3.1)

3.1.a) Boundedness of the adaption parameters. Since this part of the proof is rather tedious, a short sketch of the proof is given. First, the closed loop is transformed into a coordinate system with states for the tracking and the observer error and their time-derivatives ($\bar{\boldsymbol{x}}$-coordinates). Then a k-dependent time scaling is applied ($\bar{\bar{\boldsymbol{x}}}$-coordinates). In the resulting coordinates it is possible to define a Lyapunov-like function V such that it can be used to bound $\dot{k}$: $\dot{k} \leq \bar{M}V(\bar{\bar{\boldsymbol{x}}}, k)$ for some $\bar{M} > 0$. The boundedness of k is concluded by contradiction: It is assumed that k grows to infinity. Upper bounding the derivative of V along closed-loop trajectories yields that $\dot{V} \leq -2\tilde{\mu}V = \dot{V}_{\max}$ for some $\tilde{\mu} > 0$. An upper bound for V can be derived by the integration of $\dot{V}_{\max}$. This bound is then used together with $\dot{k} \leq \bar{M}V(\bar{\bar{\boldsymbol{x}}}, k)$ to show that k cannot grow to infinity. Therefore, the adaptation parameter k has to remain bounded.

The nonlinear closed-loop system is given by (3.4), (3.6),(3.7), (3.11), and takes the form

$$\dot{\boldsymbol{\xi}} = \big(J + \boldsymbol{b}\boldsymbol{\psi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\big)\,\boldsymbol{\xi} + \boldsymbol{b}\,\big(\boldsymbol{\phi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\eta} + g(\boldsymbol{\xi}, \boldsymbol{\eta})(-\boldsymbol{q}_{\kappa}^T\hat{\boldsymbol{x}}) + v(\boldsymbol{\xi}, \boldsymbol{\eta})\big) \tag{3.12a}$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})y + \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{w}(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{3.12b}$$
$$\dot{\hat{\boldsymbol{x}}} = \hat{A}_{\hat{\kappa}}\hat{\boldsymbol{x}} + \boldsymbol{p}_{\hat{\kappa}}e, \tag{3.12c}$$
$$\dot{k} = d_{\lambda}^2(e, k), \tag{3.12d}$$
$$e = \boldsymbol{c}^T\boldsymbol{\xi} - y_{ref} \tag{3.12e}$$

with

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \in \mathbb{R}^r, \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0 \in \mathbb{R}^{n-r}$$
$$\hat{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_0 \in \mathbb{R}^r, k(0) = k_0 > 0.$$

In the following, the arguments of $g(\cdot)$, $\boldsymbol{\phi}(\cdot)$, $\boldsymbol{\chi}(\cdot)$, $\boldsymbol{\theta}(\cdot)$, $v(\cdot)$ and $\boldsymbol{w}(\cdot)$ are dropped to increase readability.

For the reference signal and its derivatives, the following notation is used:

$$\hat{\boldsymbol{y}}_{ref} = \begin{bmatrix} y_{ref}, & \dot{y}_{ref}, & \dots, & y_{ref}^{(r-1)} \end{bmatrix}^T \in \mathbb{R}^r,$$
$$\boldsymbol{y}_{ref} = \begin{bmatrix} y_{ref}, & \dot{y}_{ref}, & \dots, & y_{ref}^{(r)} \end{bmatrix}^T \in \mathbb{R}^{r+1}.$$

It clearly holds that

$$\dot{\hat{\boldsymbol{y}}}_{ref} = J\hat{\boldsymbol{y}}_{ref} + \boldsymbol{b}y_{ref}^{(r)}.$$

Introduce the coordinates

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \bar{\boldsymbol{x}}_1 \\ \bar{\boldsymbol{x}}_2 \\ \bar{\boldsymbol{x}}_3 \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \boldsymbol{\eta} \\ \hat{\boldsymbol{e}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi} - \hat{\boldsymbol{y}}_{ref} \\ \boldsymbol{\eta} \\ \hat{\boldsymbol{y}}_{ref} - \hat{\boldsymbol{x}} \end{bmatrix},$$

where $\hat{\boldsymbol{\xi}} = \begin{bmatrix} e & \dot{e} & \dots & e^{(r-1)} \end{bmatrix}^T$ denotes the tracking error and its derivatives, and $\hat{\boldsymbol{e}}$ is the observer error, the closed-loop system is given by

$$\dot{\bar{\boldsymbol{x}}} = \left[\begin{array}{c} \left(J + \boldsymbol{b}\bar{\boldsymbol{\psi}}^T(\cdot)\right)(\bar{\boldsymbol{x}}_1 + \hat{\boldsymbol{y}}_{ref}) + \boldsymbol{b}\bar{\boldsymbol{\phi}}^T(\cdot)\bar{\boldsymbol{x}}_2 \\ -\boldsymbol{b}\bar{g}(\cdot)\boldsymbol{q}_\kappa^T(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_3) + \boldsymbol{b}\bar{v}(\cdot) - J\hat{\boldsymbol{y}}_{ref} - \boldsymbol{b}y_{ref}^{(r)} \\ \hline \bar{\boldsymbol{\chi}}(\cdot)(\boldsymbol{c}^T\bar{\boldsymbol{x}}_1 + y_{ref}) + \bar{\boldsymbol{\theta}}(\bar{\boldsymbol{x}}_2) + \bar{\boldsymbol{w}}(\cdot) \\ \hline (J + \boldsymbol{b}\bar{\boldsymbol{\psi}}^T(\cdot))(\bar{\boldsymbol{x}}_1 + \hat{\boldsymbol{y}}_{ref}) + \boldsymbol{b}\bar{\boldsymbol{\phi}}^T(\cdot)\bar{\boldsymbol{x}}_2 - \boldsymbol{b}\bar{g}(\cdot)\boldsymbol{q}_\kappa^T(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_3) \\ +\boldsymbol{b}\bar{v}(\cdot) - J\hat{\boldsymbol{y}}_{ref} - \boldsymbol{b}y_{ref}^{(r)} - (J - \boldsymbol{p}_{\hat{\kappa}}\boldsymbol{c}^T)(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_3) - \boldsymbol{p}_{\hat{\kappa}}\boldsymbol{c}^T\bar{\boldsymbol{x}}_1 \end{array}\right]$$

$$= \left[\begin{array}{c} \left(J + \boldsymbol{b}\left(\bar{\boldsymbol{\psi}}^T(\cdot) - \bar{g}(\cdot)\boldsymbol{q}_\kappa^T\right)\right)\bar{\boldsymbol{x}}_1 + \boldsymbol{b}\bar{\boldsymbol{\phi}}^T\bar{\boldsymbol{x}}_2 \\ +\boldsymbol{b}\bar{g}(\cdot)\boldsymbol{q}_\kappa^T\bar{\boldsymbol{x}}_3 + \boldsymbol{b}\bar{v}(\cdot) + \bar{B}\boldsymbol{y}_{ref} \\ \hline \bar{\boldsymbol{\chi}}(\cdot)\boldsymbol{c}^T\bar{\boldsymbol{x}}_1 + \boldsymbol{\theta}(\cdot) + \bar{\boldsymbol{w}}(\cdot) + \bar{\boldsymbol{\chi}}(\cdot)y_{ref} \\ \hline \boldsymbol{b}\left(\bar{\boldsymbol{\psi}}^T(\cdot) - \bar{g}(\cdot)\boldsymbol{q}_\kappa^T\right)\bar{\boldsymbol{x}}_1 + \boldsymbol{b}\bar{\boldsymbol{\phi}}^T(\cdot)\bar{\boldsymbol{x}}_2 \\ + \left(\boldsymbol{b}\bar{g}(\cdot)\boldsymbol{q}_\kappa^T + \hat{A}_{\hat{\kappa}}\right)\bar{\boldsymbol{x}}_3 + \boldsymbol{b}\bar{v}(\cdot) + \bar{B}\boldsymbol{y}_{ref} \end{array}\right] \tag{3.13a}$$

$$e = \boldsymbol{c}^T\bar{\boldsymbol{x}}_1 \tag{3.13b}$$

with

$$\bar{B} = \boldsymbol{b}\left[\boldsymbol{\psi}^T(\cdot)| - 1\right]$$
$$\bar{v}(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2, \hat{\boldsymbol{y}}_{ref}) = v(\boldsymbol{\xi}, \boldsymbol{\eta}),$$
similarly for $\bar{\boldsymbol{w}}(\cdot), \bar{\boldsymbol{\chi}}(\cdot), \bar{\boldsymbol{\theta}}(\cdot), \bar{\boldsymbol{\phi}}(\cdot), \bar{\boldsymbol{\psi}}(\cdot)$ and $\bar{g}(\cdot)$.

Using that $\hat{\kappa} = k^\alpha$ and $\kappa = k^\beta$, the matrices $J - \bar{g}(\cdot)\boldsymbol{b}\boldsymbol{q}_\kappa^T$ and $\hat{A}_{\hat{\kappa}} = J - \hat{\boldsymbol{p}}_{\hat{\kappa}}\boldsymbol{c}^T$ can both be factored with the help of the matrix

$$K_r = \text{diag}\{1, k, \ldots, k^{r-1}\}$$

as

$$\begin{aligned} J - \bar{g}(\cdot)\boldsymbol{b}\boldsymbol{q}_\kappa^T &= k^\beta K_r^\beta \bar{\bar{A}}_{11} K_r^{-\beta} \\ \hat{A}_{\hat{\kappa}} = J - \boldsymbol{p}_{\hat{\kappa}}\boldsymbol{c}^T &= k^\alpha K_r^\alpha \bar{\bar{A}}_{33} K_r^{-\alpha} \end{aligned} \tag{3.14}$$

with

$$\begin{aligned} \bar{\bar{A}}_{11}(\bar{g}(\cdot)) &= J - \bar{g}(\cdot)\boldsymbol{b}\boldsymbol{q}^T \text{ with } \boldsymbol{q} = \boldsymbol{q}_\kappa|_{\kappa=1}, \\ \bar{\bar{A}}_{33} &= J - \boldsymbol{p}\boldsymbol{c}^T \text{ with } \boldsymbol{p} = \hat{\boldsymbol{b}}_{\hat{\kappa}}\Big|_{\hat{\kappa}=1}. \end{aligned}$$

By assumption, the polynomials $p(\cdot)$ and $q_{\tilde{g}}(\cdot)$ are in $H(\epsilon, \mu)$ for all $\tilde{g} \geq \underline{g}$. Therefore, the matrices $\bar{\bar{A}}_{11}(\bar{g}(\cdot))$ and $\bar{\bar{A}}_{11}(\bar{g}(\cdot))$ are in the set $H(\epsilon, \mu)$ for all $g \geq \underline{g}$, see Definition 4.

A k-dependent time-scaling is now applied to (3.13) by defining new coordinates $\bar{\bar{\boldsymbol{x}}}$ via a gain-dependent transformation:

$$\bar{\bar{\boldsymbol{x}}} = \bar{\bar{C}}\bar{\bar{K}}^{-1}\bar{\boldsymbol{x}} \tag{3.15}$$

where

$$\begin{aligned} \bar{\bar{C}} &= \text{diag}\{c_1 I_r, c_2 I_m, c_3 I_r\} \\ c_i(k) &= k^{\tilde{c}_i}, \ \tilde{c}_i \in \mathbb{R}, \ i = 1, 2, 3, \\ \bar{\bar{K}} &= \text{diag}\{K_r^\beta, I_m, K_r^\alpha\}. \end{aligned}$$

The matrix $\bar{\bar{K}}^{-1}$ can be seen as a k-dependent time scaling of (3.13).

The coefficients $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ have to satisfy the following inequalities:

$$-(r - \frac{1}{2})\beta < \tilde{c}_2 - \tilde{c}_1 < \frac{1}{2}\beta, \tag{3.16a}$$

$$-(r - \frac{1}{2})(\alpha - \beta) < \tilde{c}_1 - \tilde{c}_3 < -(r - \frac{3}{2})(\alpha - \beta), \tag{3.16b}$$

$$-(r - \frac{1}{2})\alpha < \tilde{c}_2 - \tilde{c}_3, \tag{3.16c}$$

$$\tilde{c}_2 < \tilde{c}_1, \tag{3.16d}$$

$$\tilde{c}_3 - (r-1)\alpha < \tilde{c}_1 + \frac{1}{2}\alpha, \tag{3.16e}$$

$$0 \geq \max\{\beta\epsilon + \tilde{c}_1, \alpha\epsilon + \tilde{c}_3\} - \frac{1}{2}\tilde{c}_1, \tag{3.16f}$$

$$\tilde{\gamma} + \tilde{c}_1 \geq -\frac{1}{2}. \tag{3.16g}$$

Remark 3.10 *The inequalities* (3.16a) *to* (3.16f) *are needed to bound* $\dot{V}$. *More precisely, the inequalities* (3.16a), (3.16b) *and* (3.16c) *are necessary for the compensation by quadratic expansion of the cross terms* $\|\bar{\bar{\boldsymbol{x}}}_i\| \, \|\bar{\bar{\boldsymbol{x}}}_j\|$ *with* $i \neq j$ *in* (3.36). *The inequalities* (3.16d) *and* (3.16e) *are coming from the linear terms in* $\|\bar{\bar{\boldsymbol{x}}}_i\|$ *in* (3.36). *Inequality* (3.16f) *ensures that the factor of* $\frac{\dot{k}}{k}$ *in* $\frac{d}{dt}V$ *is non-positive, see* (3.32). *Finally, inequality* (3.16g) *makes it possible to bound* $\frac{\dot{k}}{k}$ *by* V, *see* (3.28). $\diamondsuit$

Lemma B.9 shows that the inequalities (3.16) are solvable for $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$.

The time derivative of the coordinate transformation matrices $\bar{\bar{C}}$ and $\bar{\bar{K}}^{-1}$ is

$$\frac{d}{dt}\bar{\bar{C}} = \operatorname{diag}\{\frac{d}{dt}c_1 I_r, \frac{d}{dt}c_2 I_m, \frac{d}{dt}c_3 I_r\}, \qquad \frac{d}{dt}c_i = \tilde{c}\frac{\dot{k}}{k}c_i,$$

$$\frac{d}{dt}K_r^{-\alpha} = -\alpha\frac{\dot{k}}{k}\Delta K_r^{-\alpha}, \qquad \frac{d}{dt}K_r^{-\beta} = -\beta\frac{\dot{k}}{k}\Delta K_r^{-\beta}$$

with

$$\Delta = \operatorname{diag}\{0, 1, \ldots, r-1\}.$$

In the $\bar{\bar{\boldsymbol{x}}}$-coordinates, the closed-loop differential equations are

$$\begin{aligned}
\frac{d}{dt}\bar{\bar{x}} &= \left[\begin{array}{c}
K_r^{-\beta}\left(\bar{\bar{A}}_{11} + \boldsymbol{b}\bar{\bar{\boldsymbol{\psi}}}^T(\cdot)\right)K_r^{\beta}\bar{\bar{\boldsymbol{x}}}_1 + \frac{c_1}{c_2}K_r^{-\beta}\boldsymbol{b}\boldsymbol{\phi}^T\bar{\bar{\boldsymbol{x}}}_2 \\
+\frac{c_1}{c_3}K_r^{-\beta}\boldsymbol{b}\bar{\bar{g}}(\cdot)\boldsymbol{q}_\kappa^T K_r^{\alpha}\bar{\bar{\boldsymbol{x}}}_3 + K_r^{-\beta}c_1\boldsymbol{b}\bar{\bar{v}}(\cdot) \\
\hline
\frac{c_2}{c_1}\bar{\bar{\boldsymbol{\chi}}}(\cdot)\boldsymbol{c}^T\bar{\bar{\boldsymbol{x}}}_1 + c_2\bar{\boldsymbol{\theta}}(c_2^{-1}\bar{\bar{\boldsymbol{x}}}_2) + c_2\bar{\bar{\boldsymbol{w}}}(\cdot) \\
\hline
\frac{c_3}{c_1}K_r^{-\alpha}\boldsymbol{b}\left(\bar{\bar{\boldsymbol{\psi}}}^T(\cdot) - \bar{\bar{g}}(\cdot)\boldsymbol{q}_\kappa^T\right)K_r^{\beta}\bar{\bar{\boldsymbol{x}}}_1 + \frac{c_3}{c_2}K_r^{-\alpha}\boldsymbol{b}\boldsymbol{\phi}^T\bar{\bar{\boldsymbol{x}}}_2 \\
+K_r^{-\alpha}\left(\boldsymbol{b}\bar{\bar{g}}(\cdot)\boldsymbol{q}_\kappa^T + \bar{\bar{A}}_{33}\right)K_r^{\alpha}\bar{\bar{\boldsymbol{x}}}_3 + c_3K_r^{-\alpha}\boldsymbol{b}\bar{\bar{v}}(\cdot)
\end{array}\right] \\
&\quad + \bar{\bar{C}}\underbrace{\left[\begin{array}{c} K_r^{-\beta}\bar{B} \\ \hline \bar{\bar{\boldsymbol{\chi}}}(\cdot)\tilde{c} \\ \hline K_r^{-\alpha}\bar{B} \end{array}\right]}_{=:\bar{\bar{B}}}\boldsymbol{y}_{ref} - \frac{\dot{k}}{k}\underbrace{\begin{bmatrix} -\tilde{c}_1 + \beta\Delta \\ -\tilde{c}_2 \\ -\tilde{c}_3 + \alpha\Delta \end{bmatrix}}_{=:\Psi}\bar{\bar{\boldsymbol{x}}} \\
&= \begin{bmatrix} k^{\beta}\bar{\bar{A}}_{11}\bar{\bar{\boldsymbol{x}}}_1 \\ \bar{\bar{\boldsymbol{\theta}}}(\bar{\bar{\boldsymbol{x}}}_2) \\ k^{\alpha}\bar{\bar{A}}_{33}\bar{\bar{\boldsymbol{x}}}_3 \end{bmatrix} + \begin{bmatrix} \bar{\bar{E}}_{11} & \bar{\bar{E}}_{12} & \bar{\bar{E}}_{13} \\ \bar{\bar{E}}_{21} & \bar{\bar{E}}_{22} & \bar{\bar{E}}_{23} \\ \bar{\bar{E}}_{31} & \bar{\bar{E}}_{32} & \bar{\bar{E}}_{33} \end{bmatrix}\begin{bmatrix} \bar{\bar{\boldsymbol{x}}}_1 \\ \bar{\bar{\boldsymbol{x}}}_2 \\ \bar{\bar{\boldsymbol{x}}}_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} c_1K_r^{-\beta}\boldsymbol{b}\bar{\bar{v}}(\cdot) \\ c_2\bar{\bar{\boldsymbol{w}}}(\cdot) \\ c_3K_r^{-\alpha}\boldsymbol{b}\bar{\bar{v}}(\cdot) \end{bmatrix} + \bar{\bar{C}}\bar{\bar{B}}\boldsymbol{y}_{ref} - \frac{\dot{k}}{k}\Psi\bar{\bar{\boldsymbol{x}}}
\end{aligned} \tag{3.17}$$

$$e = c_1^{-1}\boldsymbol{c}^T\bar{\bar{\boldsymbol{x}}}_1 \tag{3.18}$$

with

$$\bar{\bar{E}}_{11} = K_r^{-\beta} \boldsymbol{b} \bar{\bar{\boldsymbol{\psi}}}^T(\cdot) K_r^{\beta}, \qquad \bar{\bar{E}}_{12} = \frac{c_1}{c_2} K_r^{-\beta} \boldsymbol{b} \boldsymbol{\phi}^T,$$

$$\bar{\bar{E}}_{13} = \frac{c_1}{c_3} K_r^{-\beta} \boldsymbol{b} \bar{\bar{g}}(\cdot) \boldsymbol{q}_\kappa^T K_r^{\alpha}, \qquad \bar{\bar{E}}_{21} = \frac{c_2}{c_1} \bar{\bar{\boldsymbol{\chi}}}(\cdot) \boldsymbol{c}^T,$$

$$\bar{\bar{E}}_{22} = \bar{\bar{E}}_{23} = 0, \qquad \bar{\bar{E}}_{31} = \frac{c_3}{c_1} K_r^{-\alpha} \boldsymbol{b} \left(\bar{\bar{\boldsymbol{\psi}}}^T(\cdot) - \bar{\bar{g}}(\cdot) \boldsymbol{q}_\kappa^T \right) K_r^{\beta},$$

$$\bar{\bar{E}}_{32} = \frac{c_3}{c_2} K_r^{-\alpha} \boldsymbol{b} \boldsymbol{\phi}^T, \qquad \bar{\bar{E}}_{33} = K_r^{-\alpha} \boldsymbol{b} \bar{\bar{g}}(\cdot) \boldsymbol{q}_\kappa^T K_r^{\alpha}.$$

To shorten the notation, the highest exponent of k in any matrix element will be denoted by $\operatorname{ord}_k(\cdot)$ (see Definition 11, page 91). Straightforward calculations give the following bounds:

$$\operatorname{ord}_k \bar{\bar{E}}_{11} = 0, \qquad \operatorname{ord}_k \bar{\bar{E}}_{12} = \left(\frac{\tilde{c}_1}{\tilde{c}_2}\right) - (r-1)\beta < \frac{\beta}{2}$$

$$\operatorname{ord}_k \bar{\bar{E}}_{13} = \left(\frac{\tilde{c}_1}{\tilde{c}_3}\right) + (r-1)(\alpha-\beta) + \beta < \frac{\alpha+\beta}{2}$$

$$\operatorname{ord}_k \bar{\bar{E}}_{21} = \left(\frac{\tilde{c}_2}{\tilde{c}_1}\right) < \frac{\beta}{2}, \quad \operatorname{ord}_k \bar{\bar{E}}_{22} = \operatorname{ord}_k \bar{\bar{E}}_{23} = 0$$

$$\operatorname{ord}_k \bar{\bar{E}}_{31} = \left(\frac{\tilde{c}_3}{\tilde{c}_1}\right) - (r-1)(\alpha-\beta) + \beta < \frac{\alpha+\beta}{2}$$

$$\operatorname{ord}_k \bar{\bar{E}}_{32} = \left(\frac{\tilde{c}_3}{\tilde{c}_2}\right) - (r-1)\alpha < \frac{\alpha}{2}, \quad \operatorname{ord}_k \bar{\bar{E}}_{33} = \beta.$$

$$\bar{\bar{B}}_1 = K_r^{-\beta} \bar{B} \qquad \operatorname{ord}_k \bar{\bar{B}}_1 = -(r-1)\beta$$

$$\bar{\bar{B}}_2 = \bar{\bar{\boldsymbol{\chi}}}(\cdot)\tilde{c} \qquad \operatorname{ord}_k \bar{\bar{B}}_2 = 0$$

$$\bar{\bar{B}}_3 = K_r^{-\alpha} \bar{B} \qquad \operatorname{ord}_k \bar{\bar{B}}_3 = -(r-1)\alpha$$

$$\bar{\bar{\boldsymbol{\theta}}}(\bar{\bar{\boldsymbol{x}}}, k) = c_2 \bar{\boldsymbol{\theta}}(c_2^{-1} \bar{\bar{\boldsymbol{x}}}_2)$$

$$\bar{\bar{g}}(\bar{\bar{\boldsymbol{x}}}, k) = g(\boldsymbol{x}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}_{ref}), \text{ ibid.} \qquad \text{for } \bar{\bar{v}}, \bar{\bar{\boldsymbol{w}}}, \bar{\bar{\boldsymbol{\chi}}}, \bar{\bar{\boldsymbol{\phi}}}, \bar{\bar{\boldsymbol{\psi}}}.$$

As the matrices $\bar{\bar{A}}_{11}(\bar{\bar{g}}(\cdot))$ and $\bar{\bar{A}}_{33}$ are in the set $H(\epsilon, \mu)$ for all $g \geq \underline{g}$, there exist symmetric, positive definite solutions P_1 and P_3 such that for any $g(\cdot) \geq \underline{g}$ the following Lyapunov equations hold:

$$\bar{\bar{A}}_{ii}^T P_i + P_i \bar{\bar{A}}_{ii} \leq -\mu P_i, \qquad i = 1, 2, 3, \tag{3.19a}$$

$$P_i(\Delta + \epsilon I) + (\Delta + \epsilon I) P_i, \geq 0 \qquad i = 1, 3. \tag{3.19b}$$

The functions $\bar{\bar{\boldsymbol{x}}}_1 \mapsto V_1(\bar{\bar{\boldsymbol{x}}}_1) = \bar{\bar{\boldsymbol{x}}}_1^T P_1 \bar{\bar{\boldsymbol{x}}}_1$ and $\bar{\bar{\boldsymbol{x}}}_3 \mapsto V_3(\bar{\bar{\boldsymbol{x}}}_3) = \bar{\bar{\boldsymbol{x}}}_3^T P_3 \bar{\bar{\boldsymbol{x}}}_3$ are used as a sort of Lyapunov function candidates for $\bar{\bar{\boldsymbol{x}}}_1$ and $\bar{\bar{\boldsymbol{x}}}_3$, respectively. By Assumption 3.4, there exist positive constants m_1, m_2, m_3 and m_4 and a function $\eta \mapsto \tilde{V}_2(\eta)$ such that

$$m_1\|\eta\|^2 \le \tilde{V}_2(\eta) \le m_2\|\eta\|^2, \tag{3.20a}$$

$$\frac{\partial}{\partial\eta}\tilde{V}_2(\eta)H_2(\eta) \le -m_3\|\eta\|^2, \tag{3.20b}$$

$$\|\frac{\partial}{\partial\eta}\tilde{V}_2(\eta)\| \le m_4\|\eta\|. \tag{3.20c}$$

This result can for example be found in (Vidyasagar, 1993). In the $\bar{\bar{\boldsymbol{x}}}_2$-coordinates,

$$V_2(\bar{\bar{\boldsymbol{x}}}_2) = c_2^2\tilde{V}_2(c_2^{-1}\bar{\bar{\boldsymbol{x}}}_2)$$

can be chosen as a Lyapunov function candidate. It then follows from (3.20) that

$$m_1\|\bar{\bar{\boldsymbol{x}}}_2\|^2 \le V_2(\bar{\bar{\boldsymbol{x}}}_2) \le m_2\|\bar{\bar{\boldsymbol{x}}}_2\|^2 \tag{3.21a}$$

$$\frac{\partial}{\partial\bar{\bar{\boldsymbol{x}}}_2}V_2(\bar{\bar{\boldsymbol{x}}}_2)\bar{\bar{\boldsymbol{\theta}}}_2(\bar{\bar{\boldsymbol{x}}}_2) \le -m_3\|\bar{\bar{\boldsymbol{x}}}_2\|^2 \tag{3.21b}$$

$$\|\frac{\partial}{\partial\bar{\bar{\boldsymbol{x}}}_2}V_2(\bar{\bar{\boldsymbol{x}}}_2)\| \le m_4\|\bar{\bar{\boldsymbol{x}}}_2\|. \tag{3.21c}$$

Now, the Lyapunov function candidates $V_i(\cdot)$ are combined to a single function

$$V(\bar{\bar{\boldsymbol{x}}}, k) = \frac{1}{2}D^2(\bar{\bar{\boldsymbol{x}}}, k) \tag{3.22a}$$

with

$$D(\bar{\bar{\boldsymbol{x}}}, k) = \begin{cases} \nu(\bar{\bar{\boldsymbol{x}}}) - \rho(k), & \text{if } \nu(\bar{\bar{\boldsymbol{x}}}) \ge \rho(k) \\ 0, & \text{if } \nu(\bar{\bar{\boldsymbol{x}}}) < \rho(k). \end{cases} \tag{3.22b}$$

This is a sort of Lyapunov function candidate for (3.17) where

$$\nu(\bar{\bar{\boldsymbol{x}}}) = \sqrt{V_1(\bar{\bar{\boldsymbol{x}}}_1) + V_2(\bar{\bar{\boldsymbol{x}}}_2) + V_3(\bar{\bar{\boldsymbol{x}}}_3)}, \tag{3.22c}$$

$$\rho(k) = \frac{\lambda}{2}\frac{c_1(k)}{\sqrt{\|P_1^{-1}\|}}. \tag{3.22d}$$

The k-dependent parameter ρ has been chosen in such a way that

$$\nu(\bar{\bar{\boldsymbol{x}}}) \le 2\rho(k) \Rightarrow |e| \le \lambda \Rightarrow \dot{k} = 0. \tag{3.23}$$

To see this, combine (3.15), (3.18) and (3.22) to

$$\begin{aligned} |e| &\le \frac{\|\bar{\bar{\boldsymbol{x}}}_1\|}{c_1} \le \frac{\sqrt{\|P_1^{-1}\|}}{c_1(k)}\nu(\bar{\bar{\boldsymbol{x}}}) \\ &\le \frac{\sqrt{\|P_1^{-1}\|}}{c_1(k)}\sqrt{2V(\bar{\bar{\boldsymbol{x}}}, k)} + \frac{\lambda}{2}. \end{aligned} \tag{3.24}$$

Since

$$\nu(\bar{\boldsymbol{x}}) \leq 2\rho(k) \Leftrightarrow V(\bar{\boldsymbol{x}}, k) \leq \frac{1}{2}\rho^2(k), \tag{3.25}$$

(3.22d) and (3.24) yield

$$\nu(\bar{\boldsymbol{x}}) \leq 2\rho(k) \Rightarrow |e| \leq \lambda,$$

which is in the dead-zone of the gain adaptation, implying that

$$\nu(\bar{\boldsymbol{x}}) \leq 2\rho(k) \Rightarrow \dot{k} = 0.$$

The function $V(\bar{\boldsymbol{x}}, k)$ will be used to upper bound $\dot{k}$. From (3.11), the definition of the adaptation, it holds that

$$\dot{k} \leq \frac{\gamma^2}{k^{2\tilde{\gamma}}}(|e| - \lambda)^2. \tag{3.26}$$

From (3.24) follows that

$$(|e| - \lambda)^2 \leq \frac{\|P_1^{-1}\|}{c_1^2(k)} 2V(\bar{\boldsymbol{x}}, k). \tag{3.27}$$

Combining (3.26) and (3.27) yields

$$\dot{k} \leq \frac{\gamma^2}{k^{2\tilde{\gamma}}} \frac{\|P_1^{-1}\|}{c_1^2(k)} 2V(\bar{\boldsymbol{x}}, k). \tag{3.28}$$

Using (3.16g), it holds for some $\bar{M} > 0$ that

$$\frac{\dot{k}}{k} \leq \bar{M} V(\bar{\boldsymbol{x}}, k). \tag{3.29}$$

(3.29) is the first key inequality of Step 1 of the Proof.

From now on the k-dependency of $V(\cdot)$, $D(\cdot)$ and $\rho(\cdot)$ will be dropped to increase the readability.

From the theory of ordinary differential equations it follows that the initial value problem (3.12) possesses an absolutely continuous solution $(\bar{\boldsymbol{x}}(\cdot), k(\cdot))$: $[0, \omega) \to \mathbb{R}^{n+1}$, maximally extended over $[0, \omega)$ for some $\omega \in (0, \infty]$.

The derivative of V along the trajectory of the system (3.12) (denoted for ease of exposition by $\frac{d}{dt}$) is for all $t \in [0, \omega)$ and for all values of $\bar{\bar{\boldsymbol{x}}}$

$$\frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) = D(\bar{\bar{\boldsymbol{x}}})\frac{d}{dt}D(\bar{\bar{\boldsymbol{x}}}) = D(\bar{\bar{\boldsymbol{x}}})\left(\frac{d}{dt}\nu(\bar{\bar{\boldsymbol{x}}}) - \frac{d}{dt}\rho(k)\right)$$
$$= D(\bar{\bar{\boldsymbol{x}}})\left(\frac{\frac{d}{dt}\left(V_1(\bar{\bar{\boldsymbol{x}}}_1) + V_2(\bar{\bar{\boldsymbol{x}}}_2) + V_3(\bar{\bar{\boldsymbol{x}}}_3)\right)}{2\nu(\bar{\bar{\boldsymbol{x}}})} - \frac{d\rho(k)}{dt}\right)$$

$$\begin{aligned}
= \frac{1}{2}\frac{D(\bar{\bar{\boldsymbol{x}}})}{\nu(\bar{\bar{\boldsymbol{x}}})}\Bigg(& k^{\beta}\bar{\bar{\boldsymbol{x}}}_1^T\left(P_1\bar{\bar{A}}_{11} + \bar{\bar{A}}_{11}^T P_1\right)\bar{\bar{\boldsymbol{x}}}_1 + 2\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}H_2(\bar{\bar{\boldsymbol{x}}}_2) \\
&+ k^{\alpha}\bar{\bar{\boldsymbol{x}}}_3^T\left(P_3\bar{\bar{A}}_{33} + \bar{\bar{A}}_{33}^T P_3\right)\bar{\bar{\boldsymbol{x}}}_3 \\
&+ 2\sum_{j=1}^{3}\bar{\bar{\boldsymbol{x}}}_1{}^T P_1\bar{\bar{E}}_{1j}\bar{\bar{\boldsymbol{x}}}_j + 2\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}\bar{\bar{E}}_{21}\bar{\bar{\boldsymbol{x}}}_1 + 2\sum_{j=1}^{3}\bar{\bar{\boldsymbol{x}}}_3{}^T P_3\bar{\bar{E}}_{3j}\bar{\bar{\boldsymbol{x}}}_j \\
&+ 2\bar{\bar{\boldsymbol{x}}}_1{}^T P_1 c_1 K_r^{-\beta}\boldsymbol{b}v(\cdot) + 2\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}c_2\boldsymbol{w}(\cdot) + 2\bar{\bar{\boldsymbol{x}}}_3{}^T P_3 c_3 K_r^{-\alpha}\boldsymbol{b}v(\cdot) \\
&+ \left(2\bar{\bar{\boldsymbol{x}}}_1{}^T P_1 c_1 K_r^{-\beta}\bar{B} + 2\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}c_2\boldsymbol{\chi}(\cdot)\tilde{c}^T + 2\bar{\bar{\boldsymbol{x}}}_3{}^T P_3 c_3 K_r^{-\alpha}\bar{B}\right)\boldsymbol{y}_{ref} \\
&- \frac{\dot{k}}{k}\bar{\bar{\boldsymbol{x}}}_1{}^T(P_1\Psi_1 + \Psi_1 P_1)\bar{\bar{\boldsymbol{x}}}_1 - 2\frac{\dot{k}}{k}\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}\Psi_2\bar{\bar{\boldsymbol{x}}}_2 \\
&- \frac{\dot{k}}{k}\bar{\bar{\boldsymbol{x}}}_3{}^T(P_3\Psi_3 + \Psi_3 P_3)\bar{\bar{\boldsymbol{x}}}_3\Bigg) - \frac{\dot{k}}{k}\operatorname{ord}_k(\rho)\rho(k)D(\bar{\bar{\boldsymbol{x}}}) \qquad (3.30)
\end{aligned}$$

The last four terms in (3.30) are analyzed first:

$$\begin{aligned}
N(\bar{\bar{\boldsymbol{x}}}) &:= -\frac{D(\bar{\bar{\boldsymbol{x}}})}{2\nu(\bar{\bar{\boldsymbol{x}}})}\Bigg(\frac{\dot{k}}{k}\bar{\bar{\boldsymbol{x}}}_1{}^T(P_1\Delta_1 + \Delta_1 P_1)\bar{\bar{\boldsymbol{x}}}_1 + 2\frac{\dot{k}}{k}\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial\bar{\bar{\boldsymbol{x}}}_2}\Delta_2\bar{\bar{\boldsymbol{x}}}_2 \\
&\qquad + \frac{\dot{k}}{k}\bar{\bar{\boldsymbol{x}}}_3{}^T(P_3\Delta_3 + \Delta_3 P_3)\bar{\bar{\boldsymbol{x}}}_3 + \frac{\dot{k}}{k}\operatorname{ord}_k(\rho)\rho(k)2\nu(\bar{\bar{\boldsymbol{x}}})\Bigg) \\
&= -\frac{D(\bar{\bar{\boldsymbol{x}}})}{\nu(\bar{\bar{\boldsymbol{x}}})}\frac{\dot{k}}{k}\Big((-\beta\epsilon - \tilde{c}_1)\bar{\bar{\boldsymbol{x}}}_1^T P_1\bar{\bar{\boldsymbol{x}}}_1 - \tilde{c}_2\bar{\bar{\boldsymbol{x}}}_2^T P_2\bar{\bar{\boldsymbol{x}}}_2 \\
&\qquad + (-\alpha\epsilon - \tilde{c}_3)\bar{\bar{\boldsymbol{x}}}_3^T P_3\bar{\bar{\boldsymbol{x}}}_3 + \tilde{c}_1\rho(k)\nu(\bar{\bar{\boldsymbol{x}}})\Big), \qquad (3.31)
\end{aligned}$$

where (3.19b) has been used. For $\|\bar{\bar{\boldsymbol{x}}}\| > 2\rho$, (3.31) simplifies to

$$N(\bar{\bar{\boldsymbol{x}}}) \le D(\bar{\bar{\boldsymbol{x}}})\frac{\dot{k}}{k}\left(\max\{\beta\epsilon + \tilde{c}_1, \alpha\epsilon + \tilde{c}_3\} - \frac{\tilde{c}_1}{2}\right)\nu^2(\bar{\bar{\boldsymbol{x}}}) \qquad (3.32)$$

and using (3.16f) to

$$N(\bar{\bar{\boldsymbol{x}}}) \leq 0. \tag{3.33}$$

In the case of $\|\bar{\bar{\boldsymbol{x}}}\| \leq 2\rho$, (3.23) yields

$$N(\bar{\bar{\boldsymbol{x}}}) = 0. \tag{3.34}$$

Combining (3.33) and (3.34), it holds for all $\bar{\bar{\boldsymbol{x}}}$ that

$$N(\bar{\bar{\boldsymbol{x}}}) \leq 0. \tag{3.35}$$

Using (3.35), (3.30) simplifies to

$$\begin{aligned} \frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq -\frac{D(\bar{\bar{\boldsymbol{x}}})}{\nu(\bar{\bar{\boldsymbol{x}}})}\Bigg(& k^{\beta}V_1(\bar{\bar{\boldsymbol{x}}}_1) + \frac{m_3}{m_1}V_2(\bar{\bar{\boldsymbol{x}}}_2) + k^{\alpha}V_3(\bar{\bar{\boldsymbol{x}}}_3) \\ & + \sum_{i=1,3}\sum_{j=1}^{3}\|P_i\|\,\|\bar{\bar{E}}_{ij}\|\,\|\bar{\bar{\boldsymbol{x}}}_i\|\,\|\bar{\bar{\boldsymbol{x}}}_j\| + \|\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial \bar{\bar{\boldsymbol{x}}}_2}\|\,\|\bar{\bar{E}}_{21}\|\,\|\bar{\bar{\boldsymbol{x}}}_1\| \\ & + \sum_{i=1,3} c_i\,\|P_i\|\,\|\bar{\bar{K}}_i^{-1}\boldsymbol{b}\|\,\|\bar{\bar{\boldsymbol{x}}}_i\|\,|v(\cdot)| + c_2\,\|\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial \bar{\bar{\boldsymbol{x}}}_2}\|\,\|\boldsymbol{w}(\cdot)\| \\ & + \Bigg(\sum_{i=1,3} c_i\,\|P_i\|\,\|\bar{\bar{K}}_i^{-1}\bar{B}\|\,\|\bar{\bar{\boldsymbol{x}}}_i\| + c_2\,\|\frac{\partial V_2(\bar{\bar{\boldsymbol{x}}}_2)}{\partial \bar{\bar{\boldsymbol{x}}}_2}\|\,\|\boldsymbol{\chi}(\cdot)\|\,\|\tilde{c}\|\Bigg)\|\boldsymbol{y}_{ref}\|\Bigg). \end{aligned}$$

Now assume that k tends to infinity as $t \to \omega$. This will lead to a contradiction. The assumption that $y_{ref} \in W^{r,\infty}$ implies that $\boldsymbol{y}_{ref}$ is bounded almost everywhere. Also almost everywhere bounded are $\boldsymbol{v}(\cdot)$, $\boldsymbol{w}(\cdot)$ and $\boldsymbol{\chi}(\cdot)$. Therefore, there exists a constant $M > 0$ such that for almost all $t \in [0, \omega)$

$$\begin{aligned} \frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq -\frac{D(\bar{\bar{\boldsymbol{x}}})}{\nu(\bar{\bar{\boldsymbol{x}}})}\Bigg(& \frac{\mu}{2}k^{\beta}V_1(\bar{\bar{\boldsymbol{x}}}_1) + \frac{m_3}{m_1}V_2(\bar{\bar{\boldsymbol{x}}}_2) + \frac{\mu}{2}k^{\alpha}V_3(\bar{\bar{\boldsymbol{x}}}_3) \\ & - M\sum_{i=1}^{3}\sum_{j=1}^{3}\|\bar{\bar{E}}_{ij}\|\,\|\bar{\bar{\boldsymbol{x}}}_i\|\,\|\bar{\bar{\boldsymbol{x}}}_j\| \\ & - M\left(c_1 k^{-(r-1)\beta}\|\bar{\bar{\boldsymbol{x}}}_1\| + c_2\|\bar{\bar{\boldsymbol{x}}}_2\| + c_3 k^{-(r-1)\alpha}\|\bar{\bar{\boldsymbol{x}}}_3\|\right)\Bigg). \end{aligned} \tag{3.36}$$

By Lemma B.1, the bounds on $\|E_{ij}\|$, inequalities (3.16) and monotonicity of k, there exist positive constants $\tilde{\mu}$, M_1 and $\tilde{M}$ and a time $t_1 \in [0, \infty)$ such that the gain $k(t)$ is sufficiently large for almost all $t \in [t_1, \omega)$ to ensure

$$\begin{aligned} \frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) &\leq -\tilde{\mu}\frac{D(\bar{\bar{\boldsymbol{x}}})}{\nu(\bar{\bar{\boldsymbol{x}}})}\left(V_1(\bar{\bar{\boldsymbol{x}}}_1) + V_2(\bar{\bar{\boldsymbol{x}}}_2) + V_3(\bar{\bar{\boldsymbol{x}}}_3) - \tilde{M}c_1^2 k^{-M_1}\right) \\ &\leq -\tilde{\mu}D(\bar{\bar{\boldsymbol{x}}})\left(\nu(\bar{\bar{\boldsymbol{x}}}) - \tilde{M}k^{-M_1}c_1\frac{c_1}{\nu(\bar{\bar{\boldsymbol{x}}})}\right). \end{aligned}$$

By (3.22d) and monotonicity of k, there exists $t_2 \in [t_1, \infty)$ such that for almost all $t \in [t_2, \omega)$

$$\frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq -\tilde{\mu}D(\bar{\bar{\boldsymbol{x}}})\left(\nu(\bar{\bar{\boldsymbol{x}}}) - \rho\frac{\rho}{\nu(\bar{\bar{\boldsymbol{x}}})}\right). \tag{3.37}$$

For $\nu(\bar{\bar{\boldsymbol{x}}}) > \rho$, this reduces to

$$\frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq -\tilde{\mu}D(\bar{\bar{\boldsymbol{x}}})\left(\nu(\bar{\bar{\boldsymbol{x}}}) - \rho\right). \tag{3.38}$$

In the case of $\nu(\bar{\bar{\boldsymbol{x}}}) \leq \rho$, (3.37) is simplified to

$$\frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq 0 \text{ as } D(\bar{\bar{\boldsymbol{x}}}) = 0. \tag{3.39}$$

Thus, by combining (3.38) and (3.39) it holds for all $\bar{\bar{\boldsymbol{x}}}$ and for almost all $t \in [t_2, \omega)$ that

$$\frac{d}{dt}V(\bar{\bar{\boldsymbol{x}}}) \leq -\tilde{\mu}D^2(\bar{\bar{\boldsymbol{x}}}) = -2\tilde{\mu}V^2(\bar{\bar{\boldsymbol{x}}}).$$

Therefore, for all $t \in [t_2, \omega)$,

$$V(\bar{\bar{\boldsymbol{x}}}(t), k(t)) \leq e^{-2\tilde{\mu}(t-t_2)} \cdot V(\bar{\bar{\boldsymbol{x}}}(t_2), k(t_2)). \tag{3.40}$$

Inequality (3.40) is the second key inequality of this part of the Proof. If $\omega < \infty$, then (3.29) and (3.40) yield that $k(\cdot) \in \mathcal{L}_\infty\big([0, \omega)\big)$. If $\omega = \infty$, then by (3.40), V enters in finite time the interval $[0, \rho^2/2]$ which by (3.23) and (3.25) implies that $|e| \leq \lambda$. Whence, the dead-zone in the gain adaptation (3.11) yields that $k(\cdot) \in \mathcal{L}_\infty\big([0, \omega)\big)$. In both cases, this contradicts the assumption on unboundedness of $k(\cdot)$, thus proving boundedness of $k(\cdot)$.

3.1.b) Boundedness of the observer states. As $k(\cdot)$ is bounded, $d_\lambda(\cdot) \in \mathcal{L}_2\big([0, \omega)\big)$. From this, (3.11) and the Hölder inequality, it follows that

$$\gamma^{-1}k^{\tilde{\gamma}}(\cdot)d_\lambda(\cdot) \in \mathcal{L}_2\big([0, \omega)\big). \tag{3.41}$$

Using (3.11) again yields

$$\gamma^{-1}k^{\tilde{\gamma}}d_\lambda(e, k) = \begin{cases} |e| - \lambda & \text{for } |e| \geq \lambda, \\ 0 & \text{for } |e| \leq \lambda. \end{cases}$$

Therefore,

$$|e(\cdot)| - \gamma^{-1}k^{\tilde{\gamma}}(\cdot)d_\lambda(e(\cdot), k(\cdot)) \in \mathcal{L}_\infty\big([0, \omega)\big). \tag{3.42}$$

Combining (3.41) and (3.42) yields

$$|e(\cdot)| = \underbrace{|e(\cdot)| - \gamma^{-1}k^{\tilde{\gamma}}(\cdot)d_\lambda(\cdot)}_{\in\mathcal{L}_\infty([0,\omega))} + \underbrace{\gamma^{-1}k^{\tilde{\gamma}}(\cdot)d_\lambda(\cdot)}_{\in\mathcal{L}_2([0,\omega))} \in \mathcal{L}_\infty([0,\omega)). \tag{3.43}$$

Boundedness of $k(\cdot)$ ensures the existence of a k_ω such that

$$k_\omega = \limsup_{t\in[0,\omega)} k(t).$$

Defining $\hat{A} = \hat{A}_{\hat{k}=k_\omega^\alpha}$, $\hat{A}_1 = \hat{A}_{\hat{k}} - \hat{A}$, (3.12c) is equivalent to

$$\dot{\hat{\boldsymbol{x}}} = \hat{A}\hat{\boldsymbol{x}} + \hat{A}_1\hat{\boldsymbol{x}} + \hat{\boldsymbol{b}}e, \tag{3.44}$$

where $\hat{A}$ is Hurwitz, $\|\hat{A}_1\|$ decreases monotonically to zero and $\|\hat{\boldsymbol{b}}\|$ is bounded. Therefore, it follows from Lemma B.8 that $\hat{\boldsymbol{x}}$ is bounded, i.e.

$$\hat{\boldsymbol{x}}(\cdot) \in \mathcal{L}_\infty^r([0,\omega)). \tag{3.45}$$

As $u = -\boldsymbol{q}_k\hat{\boldsymbol{x}}$, this directly ensures that

$$u(\cdot) \in \mathcal{L}_\infty([0,\omega)). \tag{3.46}$$

3.1.c) Boundedness of the system states. The previous part has shown that e and y are bounded almost everywhere.

The internal dynamics of (3.4) are

$$\dot{\boldsymbol{\eta}} = H(\boldsymbol{\eta}) + \boldsymbol{\chi}(\boldsymbol{\xi},\boldsymbol{\eta})y + \boldsymbol{w}(\boldsymbol{\xi},\boldsymbol{\eta}). \tag{3.47}$$

As (3.47) satisfies the assumptions of Lemma B.8, it follows that

$$\boldsymbol{\eta}(\cdot) \in \mathcal{L}_\infty^{n-r}([0,\omega)). \tag{3.48}$$

The remaining states[1], i.e. ξ, satisfy

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= J\boldsymbol{\xi} + \boldsymbol{b}\left(\boldsymbol{\psi}^T(\cdot)\boldsymbol{\xi} + \boldsymbol{\phi}^T(\cdot)\boldsymbol{\eta} + g(\cdot)u(\cdot) + v(\cdot)\right)\\ &= \left(J + \boldsymbol{b}\boldsymbol{\psi}^T(\cdot)\right)\boldsymbol{\xi} + \boldsymbol{b}\tilde{v},\end{aligned} \tag{3.49}$$

where $\tilde{v} \in \mathcal{L}_\infty([0,\omega))$. It is trivial to see that the system (3.49) is observable from ξ_1. Therefore, there exists a $\boldsymbol{l} \in \mathbb{R}^{1\times r}$ s.t.

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= \left(J + \boldsymbol{b}\boldsymbol{\psi}^T(\cdot)\right)\boldsymbol{\zeta} + \boldsymbol{l}\,(y - \boldsymbol{l}\boldsymbol{c})\\ &= \left(J - \boldsymbol{l}\boldsymbol{c} + \boldsymbol{b}\boldsymbol{\psi}^T(\cdot)\right)\boldsymbol{\zeta} + \boldsymbol{l}y\end{aligned}$$

[1]This part of the proof is due to G. Weiss, Imperial College, London.

is an observer for (3.49) in the case of $\tilde{v} \equiv 0$, if $J - \boldsymbol{l}\boldsymbol{c}$ is Hurwitz and sufficiently robust to cope with the perturbation $\boldsymbol{b}\boldsymbol{\psi}^T(\cdot)$ (see e.g. Bellman, 1953, Theorem 2). Boundedness of y ensures boundedness of $\boldsymbol{\zeta}$. The observer error, $\boldsymbol{\xi} - \boldsymbol{\zeta}$ satisfies

$$\frac{d}{dt}(\boldsymbol{\xi} - \boldsymbol{\zeta}) = (J - \boldsymbol{l}\boldsymbol{c} + \boldsymbol{b}\boldsymbol{\psi}^T(\cdot))(\boldsymbol{\xi} - \boldsymbol{\zeta}) + \boldsymbol{b}\tilde{v}.$$

Therefore, it is bounded and

$$\boldsymbol{\xi}(\cdot) \in \mathcal{L}_\infty^r\big([0,\omega)\big). \tag{3.50}$$

3.1.d) Global existence of a unique solution. As k, $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are bounded on $[0,\omega)$, it follows by maximality of ω that $\omega = \infty$.

3.1.e) Convergence of the tracking error. It remains to show that the λ-strip is attractive. This is achieved by showing that $\lim_{t\to\infty} d_\lambda(e(t), k(t)) = 0$. Since $e(\cdot)$ and $k(\cdot)$ are bounded, it follows that $\dot{k}(\cdot) = d_\lambda^2(\cdot) \in \mathcal{L}_1\big([0,\omega)\big)$. Using $\dot{e} = \boldsymbol{c}\,[A\boldsymbol{x} - \boldsymbol{b}\boldsymbol{q}_k\hat{\boldsymbol{x}}] - \dot{y}_{ref}$ and the boundedness of $\boldsymbol{x}(\cdot)$ and $\hat{\boldsymbol{x}}(\cdot)$, it can be concluded that $\dot{e}(\cdot) \in \mathcal{L}_\infty\big([0,\infty)\big)$. As

$$\frac{d}{dt}d_\lambda^2 = 2d_\lambda\left(\frac{\gamma}{k^{\tilde{\gamma}}}\frac{e\dot{e}}{|e|} - \tilde{\gamma}\frac{\dot{k}}{k}d_\lambda\right) \in \mathcal{L}_\infty\big([0,\infty)\big),$$

$d_\lambda^2(\cdot)$ is uniformly continuous. This, together with $d_\lambda^2(\cdot) \in \mathcal{L}_1\big([0,\infty)\big)$ yields, by Barbălat's Lemma (Barbălat, 1959) that $\lim_{t\to\infty} d_\lambda^2(t) = 0$.

This completes the proof of Theorem 3.1. ∎

3.5 Extensions

Theorem 3.1 requires that $\alpha > \beta$. The limiting case $\alpha = \beta$ can also be handled if the following extra conditions are satisfied

1. The matrix $\tilde{A} = J - \boldsymbol{p}\boldsymbol{c}^T - g\boldsymbol{b}\boldsymbol{q}^T$ is Hurwitz for all possible g.
2. The matrix $\begin{bmatrix} A_{11} & \boldsymbol{b}\boldsymbol{q}^T \\ \boldsymbol{b}\boldsymbol{q}^T & \tilde{A} \end{bmatrix}$ is Hurwitz for all possible g.

These conditions are necessary as the state-feedback and the observer are adapted at the same speed. Thus, it is necessary that the observer is stable under the "perturbation" by the state-feedback (first condition) and that the cross-coupling between tracking and observation error is not too large (second condition).

Another possible extension is to allow time-variation of the system. As long as the variations are slow enough, the control objective is attained by the same controller.

3.6 Conclusions

In Theorem 3.1, it has been shown that the adaptive λ-tracking controller with full-order observer is guaranteed to achieve λ-tracking for a large class of nonlinear systems and reference signal. If the system can be transformed Byrnes-Isidori normal form as in Assumption 3.3 and the relative degree and a lower bound of the high-frequency gain $\underline{g}$ are known, then the adaptive λ-tracking controller proposed here is well suited for achieving that the tracking error $y - y_{ref}$ asymptotically converges to the λ-strip. The width of this strip, λ, is a parameter which can be chosen by the user and does usually depend on the specifications, on model uncertainties and on the quality of the measurement. In comparison with other approaches, the controller proposed here uses a full-order high-gain observer. This has the advantage that the output y does not enter the feedback part of the controller directly. As this observer estimates filtered derivatives of the output y and the feedback part is a linear combination of the observer states, the controller has a very simple structure. This makes it much easier to tune the parameters of the controllers as they have a clear meaning.

The practical applicability and performance of the proposed controller are shown in the following chapter where the controller is applied to a control problem in anesthesia.

Chapter 4

Adaptive λ-tracking in Anesthesia

Anesthesia is an important area of modern medicine. But, as of today, there is no commercial anesthesia system with the patient in the feedback loop (Frei, 2000). Due to stricter environmental directives, it is getting necessary to use anesthetics in a more parsimonious manner. As fast and reliable sensors are getting more and more common, automatic control is expected to be part of the coming generation of anesthesia equipments (Baum, 1998). An overview of automatic control for anesthesia can be found in (Schwilden and Stoeckel, 1995; Frei, 2000; Gentilini et al., 2001).

In this chapter, it is shown that one of the possible closed loops in anesthesia control, the control of the endtidal anesthetic concentration of the anesthetic, can be achieved in a satisfactory manner by the adaptive λ-tracking controller proposed in Chapter 3.

4.1 Introduction

The objectives of anesthesia are unconsciousness (hypnosis), insensitivity to pain (analgesia), relaxation and the maintaining of vital functions (Frei, 2000). Unconsciousness of the patient is often achieved through volatile anesthetics (e.g. isoflurane) that are mixed into the inspiratory fresh gas flow, see Figure 4.1.

As unconsciousness cannot be measured, its level has to be inferred from available measurements like mean arterial pressure (MAP), brain activity (EEG) and the endtidal concentration of the anesthetic. The endtidal anesthetic concentration is the concentration of the anesthetic in the expired air measured at the very end of the expiratory phase of a breathing cycle. It closely reflects the actual alveolar concentration and is therefore a good surrogate measure for the arterial partial and brain partial pressure of the anesthetic (Frei, 2000). Furthermore, the endtidal anesthetic concentration has been used in (Zbinden et al., 1986)

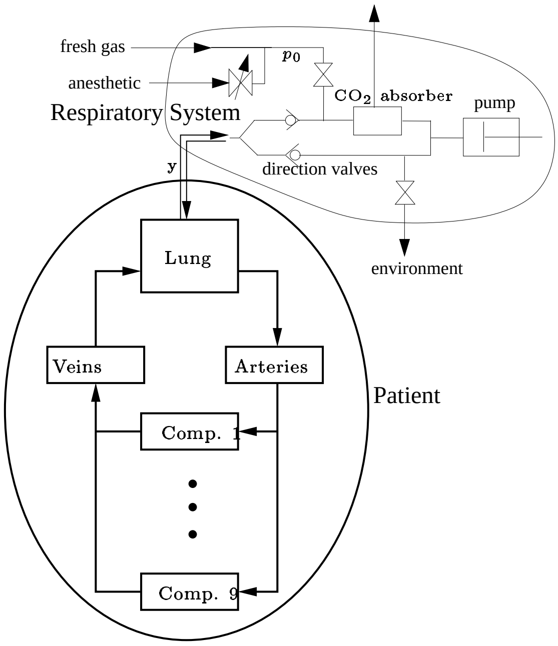


Figure 4.1: Model of the patient with the respiratory system.

in an automated control scheme for maintaining constant levels of hypnosis, for which the endtidal anesthetic concentration also plays the role of a surrogate.

During the various phases of surgery, different levels of unconsciousness are desired. This can be achieved by changing the concentration of the anesthetic added to the fresh gas flow. The control system thus has the endtidal concentration of anesthetic as controlled output and the concentration of anesthetic in the fresh gas flow as manipulated input.

Keeping the endtidal anesthetic concentration constant is a relatively simple task. But to find the right level, the anesthetist usually needs a few trials spaced by several minutes of equilibration. This is due to the long time constants of the patient. This task can be very well be taken over by an automatic controller (Sieber et al., 2000). In that case, the anesthetist only has to fix target values and can concentrate on other tasks, for example monitoring the other available measurements.

Controllers are usually designed based on models of the patient and the respiratory system. Finding good models for automatic control in anesthesia is rather difficult. A physiology-based model is often nonlinear and quite large with parameters that are difficult to estimate. In (Derighetti, 1999) a model derived by (Smith et al., 1972) of dimension 12 is used for controller design. This model has been extended Frei (2000) to a 15 state model that also models the neural activity and two epinephrine concentrations. The parameters are either physiological constants (e.g. solubilities) or patient specific. In the latter case their value is only approximately known. Thus, a model-based controller needs to either identify these parameters before the surgery using input-output data, or the controller has to be robust against changes of these parameters.

An alternative approach is to use a controller design which is not based on a specific system model but only on some of its characteristic features. Further robustness is achieved by tolerating a certain output error. A controller well-suited for this kind of control problem is the adaptive λ-tracking controller as presented in Chapter 3. In the following sections, it is shown how this can be used to control the endtidal isoflurane concentration.

After presenting the control problem and the model used for validating the controller, the controller and its implementation are discussed. The chapter concludes with experimental results are shown in Section 4.6.

4.2 Control objective

The task of the automatic controller is to keep the endtidal isoflurane concentration close to the target level chosen by the anesthetist. This target level is not fixed during a surgery, but has to be adjusted several times at the various stages of the surgery.

In other words, the control objective is that the endtidal anesthetic concentration y tracks the reference signal y_{ref} asymptotically while tolerating a tracking error smaller than a user-defined error bound λ, see Figure 4.2. Graphically speaking, the output y should enter the λ-strip around the reference trajectory y_{ref}. The parameter λ has been chosen as 0.0250%.

Also, the desired endtidal concentration does not need to be attained exactly: The usual precision of the medical instrument is about 0.1%.

Furthermore, all states should remain bounded and the controller should be robust against uncertain model parameters.

Gain adaptation

The adaptation is performed in such a way that the adaptation parameter is increased as long as the amplitude of the tracking error e is larger than the user-defined bound λ from the control objectives, see Figure 4.2.

The fresh gas concentration of anesthetic gas u is limited by 0 and +5%. A wind-up of the adaptation parameter is to be expected when the input saturates. Therefore, the

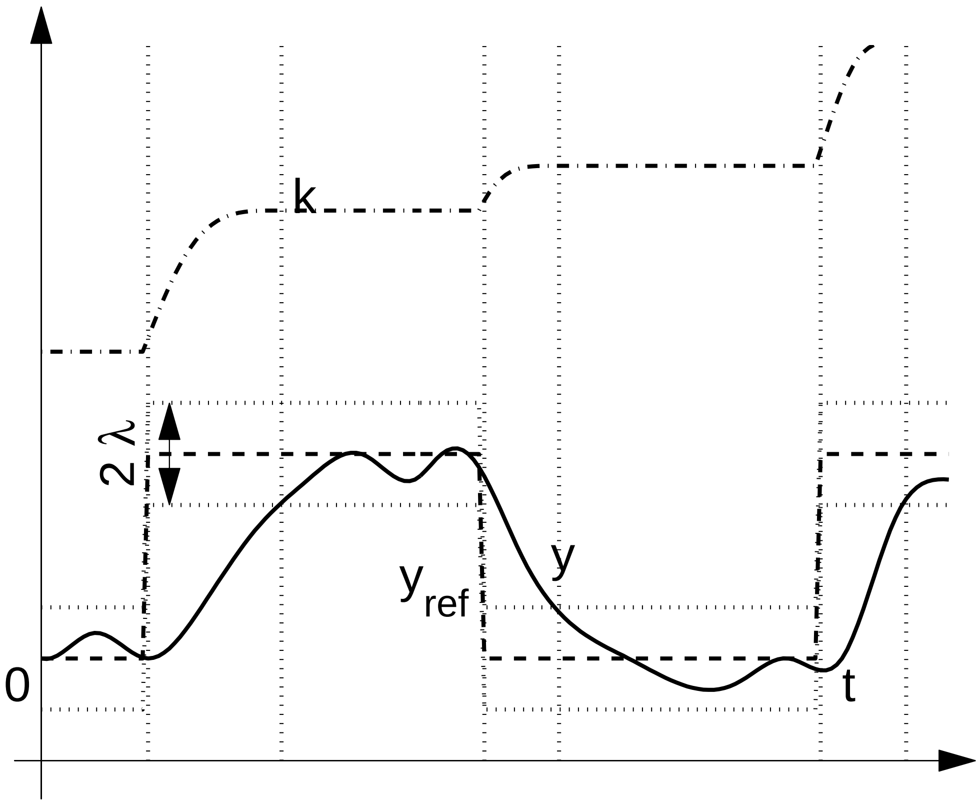


Figure 4.2: Sketch of the adaptation in adaptive λ-stabilization. Output y in solid, reference y_{ref} in dashed, adaptation parameter k in dash-dotted, λ-strip and times when the output enters or leaves the λ-strip in as dotted lines.

adaptation is modified such that the gains are not increased whenever this happens.

4.3 Model used for controller simulation

The patient is well modeled by a model consisting of 12 nonlinear compartments (see Smith et al., 1972; Derighetti, 1999), each represented by a first order dynamical system. These compartments represent lung, arteries, veins, heart and different kinds of tissue. This model has been extended by Frei (2000) to a 15 state model that also models the neural activity and two epinephrine concentrations as this allows to model the mean arterial pressure (MAP). The parameters are either physiological constants (e.g. solubilities) or patient specific. In the latter case, their value is only approximately known.

The state variable for each compartment is concentration or partial pressure of anesthetic in the compartment. For low flow anesthesia the respiratory system adds considerable dynamics which must not be neglected. Although very complex in its details, the respiratory system may well be approximated by a first order dynamic for control purpose. The complete model therefore consists of 13 states, or of 16 if the MAP is also modeled. In Figure 4.1 the model of the patient is shown with thick lines, the respiratory system by thin lines.

In the following, a sub-part of the model, namely the input-output part, is described in detail. The relevant states are p_r and p_l, the anesthetic concentration in the respiratory system and in the lung, respectively. The measured output y is the anesthetic concentration in the outflow

$$\begin{aligned} \dot{p}r &= c_1 u + c_2(p_l - p_r) - c_3 p_l \\ \dot{p}_l &= c_4(\boldsymbol{p})(p_v - p_l) + c_5(p_r - p_l) \\ y &= \frac{1 - V_{AD}}{V_T - V_D} p_l + \frac{V_{AD}}{V_T - V_D} p_r, \end{aligned}$$

where V_T the tidal volume, V_D the dead space (e.g. the trachea) and V_{AD} is the alveolar dead space, i.e. the relative volume of the air which reaches the lung but does not take part in the gas exchange with the blood. The parameters c_1, c_2, c_3 and c_5 are positive constants depending on the fresh gas flow and the different volumes, c_4 depends also on some of the states $\boldsymbol{p}$.

In Derighetti (1999), V_{AD} is assumed to be zero. This results in the system having relative degree two. This has been used in Section 4.6. Subsequent studies have shown that V_{AD} is probably much larger than zero, i.e. in the order of $20 - 25\%$ (Frei, 2000). Then, the system consisting of the respiratory system and the patient has a relative degree of one.

4.4 Controller

The adaptive λ-tracking controller as outlined in Chapter 3 is used:

$$\begin{aligned} \frac{d}{dt}\hat{\boldsymbol{x}} &= \hat{A}_{\kappa}\boldsymbol{x} + \boldsymbol{p}_{\hat{\kappa}} e_1, \\ u &= -\boldsymbol{q}_k^T \hat{\boldsymbol{x}} \end{aligned} \tag{4.1}$$

with $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^r$ and

$$\begin{aligned} \hat{A}_{\hat{\kappa}} &= J - \boldsymbol{p}_{k^\alpha} \boldsymbol{c}^T, \\ \boldsymbol{p}_{k^\alpha} &= \left[p_{r-1} \cdot k^\alpha, \quad \cdots \quad p_0 \cdot k^{\alpha r}\right]^T, \\ \boldsymbol{q}_{k^\beta} &= \left[q_0 \cdot k^\beta r, \quad \cdots, \quad q_{r-1} \cdot k^\beta\right]^T. \end{aligned}$$

To simplify the implementation (see Section 4.5.3), the observer matrix $\hat{A}$ is factored. As shown in Section 3.4 Equation (3.14), the matrix $\hat{A}$ can be written as

$$\hat{A} = k^{\alpha} K^{\alpha} \bar{A} K^{-\alpha},$$

where $K = \text{diag}\{1, k, \dots, k^{r-1}\}$ and $\bar{A} = \hat{A}|_{k^{\alpha}=1}$. The matrix $\bar{A}$ itself can be factored into the matrix of the corresponding eigenvectors, $\bar{V}$, and the diagonal matrix of its eigenvalues, $\bar{D}$, as

$$\bar{A} = \bar{V}\bar{D}\bar{V}^{-1}.$$

Therefore,

$$\hat{A} = K^{\alpha}\bar{V}k^{\alpha}\bar{D}\bar{V}^{-1}K^{-\alpha}. \tag{4.2}$$

Thus, the eigenvectors of $\hat{A}$ are the columns of $\bar{V}^{-1}K^{-\alpha}$, and the corresponding eigenvalues are the diagonal elements of $k^{\alpha}\bar{D}$. By choosing real and distinct eigenvalues, the matrix $\bar{D}$ is real diagonal. As the dimension of the observer is relatively small, the numerical disadvantages of this factorization can be neglected.

The controller (4.1) has been extended to include a feedforward term $u_{ff} = \delta\ y_{ref}$. From the model it follows that at steady-state with concentration $\bar{p}$, the input $\bar{u}$ has to be

$$\bar{u} = \left(1 - \frac{Q_{\Delta}}{V_R}\right)\bar{p},$$

where V_R is the volume of the respiratory circuit and Q_{Δ} the net uptake by the patient. While the first parameter can easily be measured, the second depends on several factors, among them the weight and the cardiac output of the patient (Baum, 1998). The value of $\frac{Q_{\Delta}}{V_R}$ is therefore not precisely known, but is usually much smaller than one. Therefore, $\delta = 1$ has been chosen. The feedforward term reduces the steady-state part of the input signal that the controller has to generate. For the adaptive λ-tracking controller, this is particularly important as it reduces the required adaptation gain.

It is easy to show that the conditions guaranteeing stability of the closed loop consisting of patient, respiratory system and λ-tracking controller are locally satisfied.

4.5 Implementation

For the implementation in a real-time anesthesia system, several additional matters have to be taken into account.

4.5.1 Reference signal filtering

The anesthetist usually wants the endtidal concentration to be kept constant at different levels, depending on other measurements or on the phase of the surgery. Therefore, the reference signal is exemplarily chosen as a series of steps, see also Figure 4.2. The problem of such a trajectory is that it is not feasible due to the response time of the respiratory system and the patient. The output will always change at a slower rate than the reference causing the adaptation parameter to increase at each change of the reference. Furthermore, the patient can be seen as storage compartment for the anesthetic. To reach the new steady-state faster, an overshoot in the reference is helpful. Therefore, the actual reference signal given to the controller consists of filtered steps. The filter is second order: $F_{ref} = \frac{\omega^2}{s^2+2D\omega s+\omega^2}$ with $D = 0.5$ and $\omega = 1.1$, see Figure 4.3. Another motivation for having

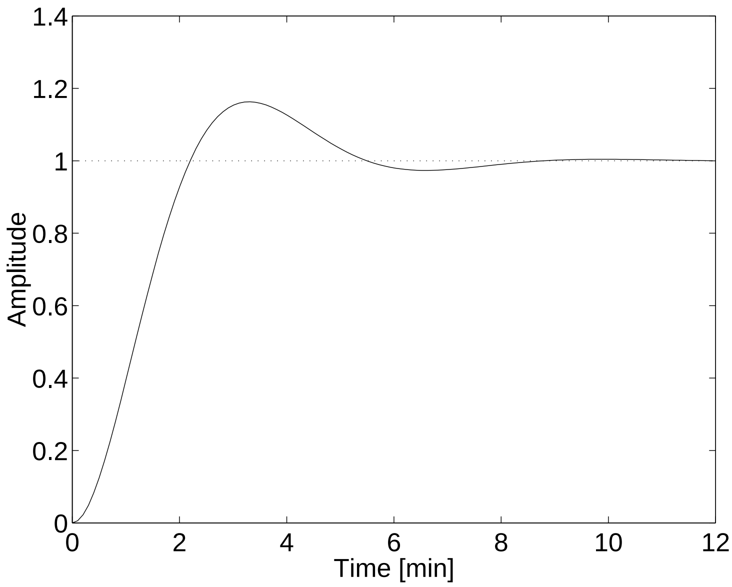


Figure 4.3: Step response of the reference filter.

the overshoot is that, ideally, the anesthetic concentration in the brain is what should be controlled. The overshoot is an approximation of an optimal input signal that would change the brain concentration as quickly as possible without too large deviations in other parts of the body, especially in the lung. The tuning is such that the step is smoothed out and there is a slight overshoot.

4.5.2 Measurement filtering

To remove the measurement noise, a first order filter is used. It is applied to the tracking error $e = y - y_{ref}$, the difference between the measured endtidal concentration y and its reference y_{ref}: $e_f(s) = F_e(s) \cdot e(s) = \frac{1}{sT_e+1} \cdot e(s)$ with $T_e = 0.5\,\text{min}$. This increases the relative degree of the system to be controlled by one, i.e. to three.

4.5.3 Discretization

The input and output measurements are only available in discrete time, namely once per breathing cycle. Furthermore, the controller and the filters can only be implemented in a discretized form. The sample frequency is equal to the respiratory frequency. Because of the controller's special structure (4.2), the controller and the filters can be discretized in a straight-forward manner. Furthermore, the sampling frequency can vary and the amount of online calculation is limited.

Initialization

The initialization of the controller is very simple. If the system is in a quasi steady-state with endtidal concentration p_0, the reference filter output has to be set to p_0 and all the other filter and controller states to zero.

4.5.4 Closed loop

The closed loop system is shown in Figure 4.4. It consists of patient, respiratory system, λ-tracking controller, measurement filter and reference filter. The control input consists of a feedforward term δy_{ref} and the feedback term of the adaptive λ-tracking controller.

Hardware

The controller is implemented on a target (PowerPC with VME bus system) — host (PC) computer system that provides the hardware basis for the experimental platform (Frei et al., 1998). The operating system XOberon (Brega, 1998) of the Robotics Institute of the ETH Zürich provides the required real time and multi task features. The applications are implemented using the object oriented programming language Oberon, a member of the Pascal-Modula family. Making use of the object oriented technology, a framework has been developed which efficiently allows to write new control applications. The compact integration of the computer system equipped with a touch screen on a standard CiceroEM anesthesia workstation from Dräger (see Figure 4.5) contributes significantly to the general acceptance of this prototype system in the operation room (Frei, 2000).

4.6 Experimental results

The discretized version of the controller presented in Section 4.4, Equation (4.1) with $r = 3$ together with the measurement filter as described in Section 4.5.2 was used during a liver surgery for over an hour (Bullinger et al., 2000b). During this time the reference value was changed five times, see the dashed line in Figure 4.6.

In Figure 4.6 the measured endtidal concentration (solid curve) and its reference value (dashed) are shown. For low concentrations the tracking is good, for higher ones there is a relatively large offset. The convergence to the new steady-state is very slow. This can also be seen in Figure 4.7 depicting the tracking error. In simulations, it can be seen that the feedforward term in the controller improves significantly the control performance. The quasi-steady-state offset is relatively small without that the adaptation gain is too high.

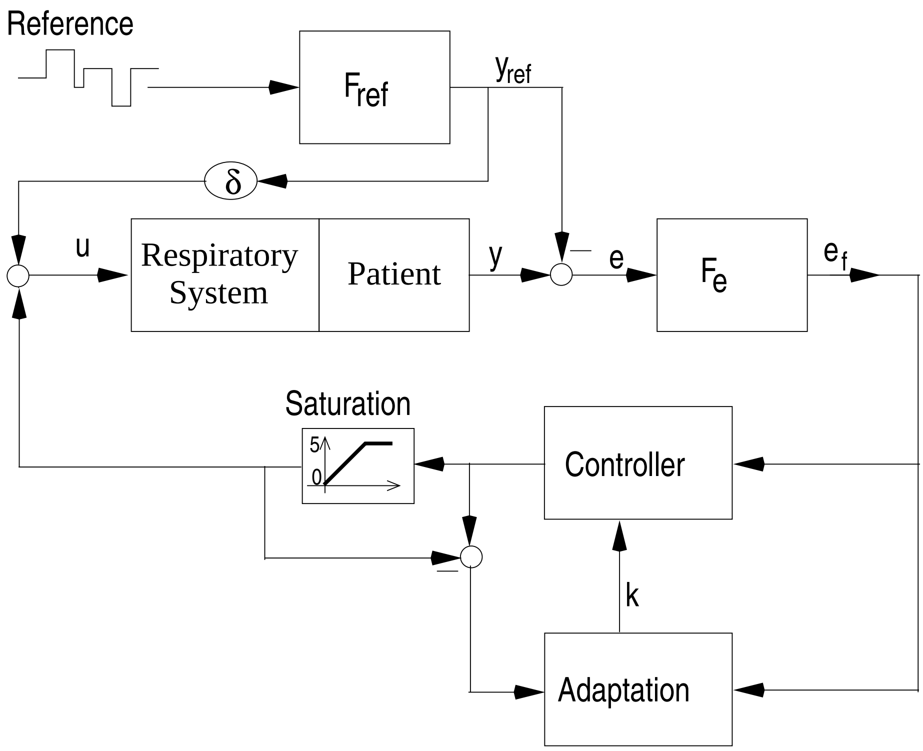


Figure 4.4: Signal flow chart of the closed loop.

Figure 4.5: Anesthesia workplace.

A too large gain would have made the controller particularly sensitive to the measurement noise.

The lower bound on the input severely limits the performance. The transients cannot be much faster, especially when the reference value decreases, see Figure 4.9. This is due to the input saturation: the inflow concentration of isoflurane cannot be negative. A faster decrease could be achieved by increasing the respiratory frequency or the volume of air pumped into the patient.

Around minute 38, the sensor for the endtidal isoflurane concentration re-calibrated itself, thus giving a signal "0". This leads to an undesired excitation of the controller. In the presented control scheme such artifacts are not taken care of. This kind of disturbances can easily be handled, see for example (Frei et al., 1999; Frei et al., 2000; Menold et al., 1999).

Figure 4.8 shows the controller adaptation gain. The step-like behavior of the reference value is clearly visible in the adaptation. Due to the slow convergence, the adaptation does not reach its dead zone, but keeps increasing slowly.

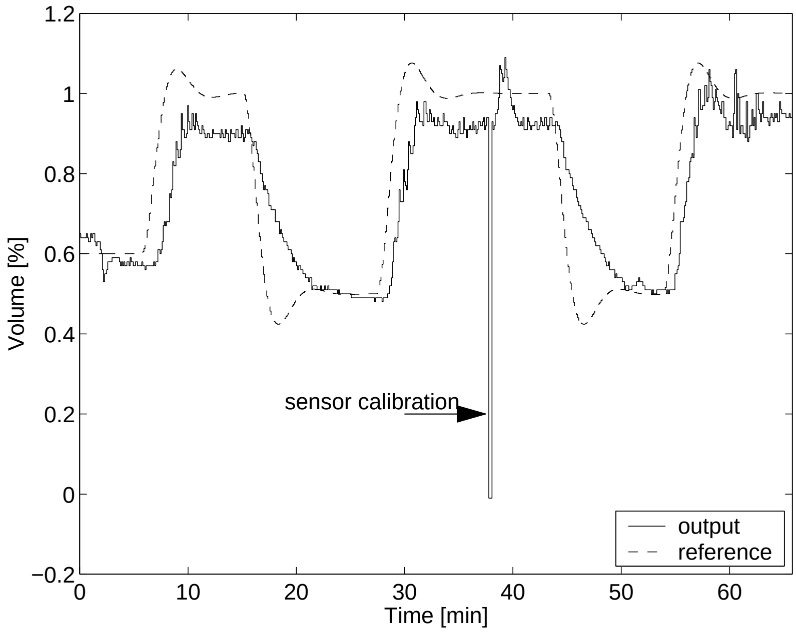


Figure 4.6: Endtidal concentration (controlled variable).

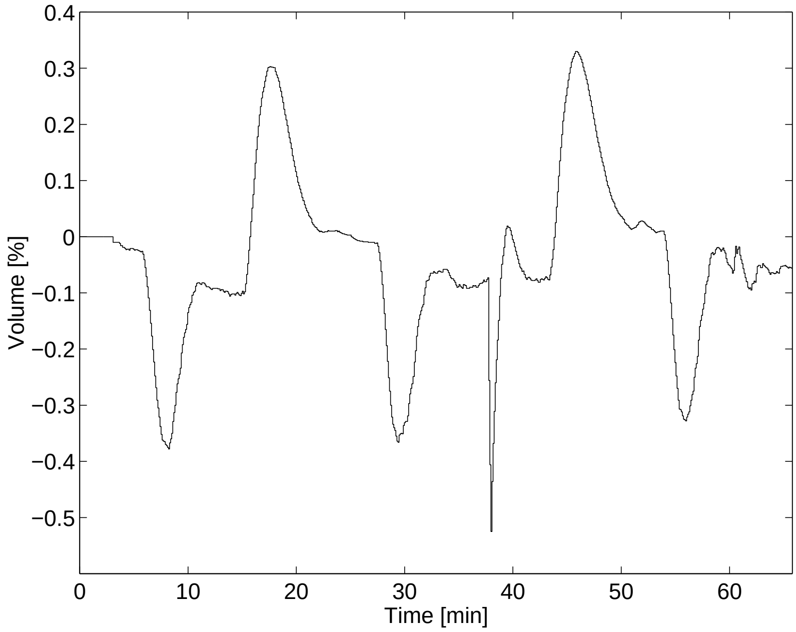


Figure 4.7: Tracking error.

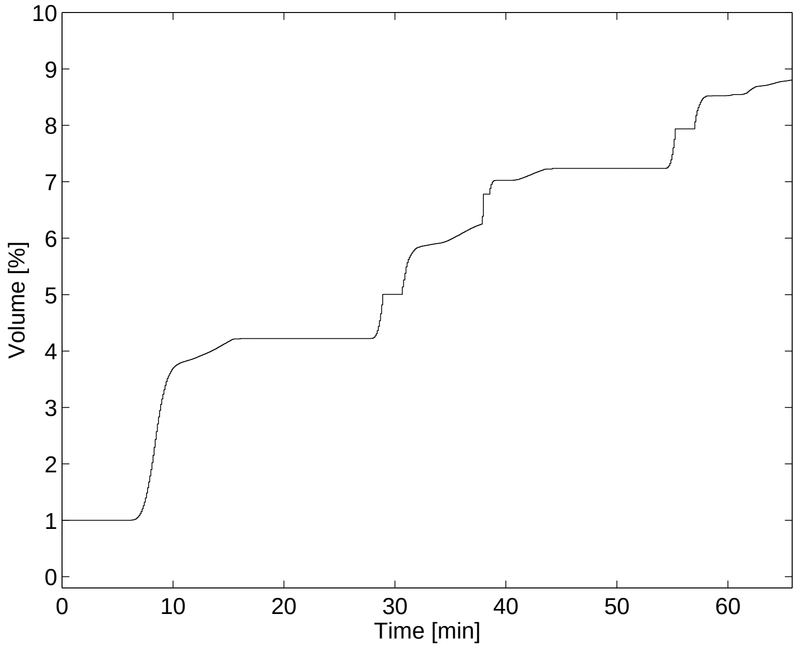


Figure 4.8: Controller gain k^{β} and observer gain k^{α}.

4.7 Conclusions

A full-order adaptive λ-tracking controller has been used both in simulation and in reality for controlling the endtidal anesthetic concentration showing satisfactory results. The result shows that a simple controller can be used for keeping the endtidal concentration at a desired level. As only little knowledge of the model is required, this scheme is very robust.

The controller used in this chapter can be easily improved to cope with measurement artifacts such as the sensor calibration in Figure 4.6.

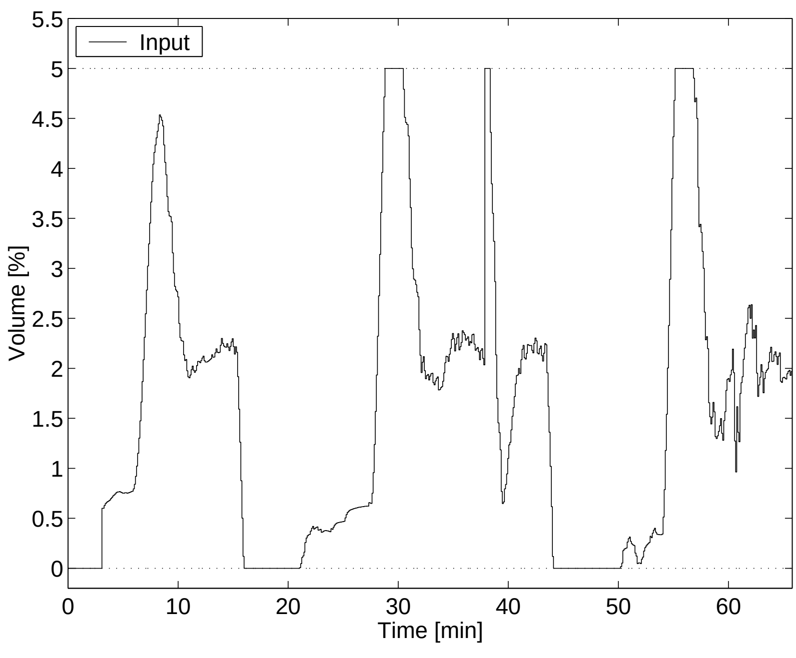


Figure 4.9: Anesthetic concentration in fresh gas (manipulated variable).

Conclusions

A common problem in many practical applications is the necessity to design controllers for uncertain systems. This is the case if no accurate model exists or if the model parameters are not exactly known. A good example are medical applications as it is not possible to model the human body with absolute precision. While certain parameters can easily be measured, others are difficult to estimate, for example the uptake rate of oxygen in the lung. Furthermore, these parameters depend on many factors such as age, weight and health of the patient.

A natural control objective is to tolerate a certain output error. This is for example a good choice if the measurements only have a specific quantization or are subject to noise. Since precise tracking usually requires larger input energy, tolerating a certain output error can significantly reduce the energy needed for control. And, in many cases, a certain tolerance is included in the specifications, for example for a temperature controller or for the purity of chemical products.

A control objective tolerating a certain tracking error is λ-tracking. It includes the tolerable output error λ directly into the control objective. The objective of λ-tracking is the following:

$$|y(t) - y_{ref}(t)| \rightarrow [0, \lambda] \text{ for } t \rightarrow \infty,$$

where $y(\cdot)$ is the output of the system, $y_{ref}(\cdot)$ the reference trajectory and λ the user-defined tolerance for the output error.

One of the advantages of adaptive λ-tracking controllers is that a large class of systems can be robustly stabilized around a reference trajectory. The robustness is due to the fact that a single controller stabilizes a whole class of systems independently of the parameters. Only structural properties like the relative degree (which can be arbitrarily large) are necessary for the controller design.

The adaptive λ-tracking controller presented in Chapter 3, consists of three main components: a state-feedback, an observer and an adaptation. The observer is a full-order observer (see Section 1.3.2) which is parameterized by a single observer gain. Increasing this gain is equivalent to "speeding up" the observer and reducing the observer error. The observer dimension is equal to the relative degree of the systems, which for many practical applications is much smaller than the dimension of the system. The observer estimates the output and its first $r-1$ derivatives and provides them for the state-feedback, i.e. an observer-based state-feedback is used (see Section 1.3.1). The feedback itself is a linear combination of the observer states where the coefficients include a controller gain. Increasing this gain makes the controller more aggressive.

The observer gain and the controller gain are both high-gain parameters, i.e. the control objective is achieved for sufficiently large parameters. They can be fixed a priori if sufficient information on the system, the reference trajectory and on the initial state is available. If this information is not precise, the resulting gains will often be much larger than necessary to achieve λ-tracking. Instead of choosing the gains a priori, it is possible to adapt them. It is possible to design a single adaptation for both the observer and the controller gain. This parameter is increased only if the control objective, namely the output tracking error is smaller than the user-defined parameter λ is violated. Inside the λ-strip around the reference, the adaptation parameter is kept constant.

In Chapter 2, several adaptive high-gain controllers have been presented and compared. Most achieve exact stabilization. A larger class of nonlinear systems can be controlled with a certain high-gain controller if only λ-stabilization must be achieved. The controller proposed in Chapter 3 extends the controller by Mareels (1984) to nonlinear systems and the one of (Allgöwer and Ilchmann, 1995; Allgöwer et al., 1997) to systems with a higher relative degree. In (Ye, 1999), the system class is similar to the one of Chapter 3, but the controllers differ. The controller of Ye (1999) uses a reduced-order observer and a complex state-feedback which is calculated via backstepping whereas the controller of Chapter 3 uses a full-order observer and a simple observer-state feedback. Using a full-order observer has the disadvantage that an extra integrator is required and the advantage that the system output only enters the state-feedback via the observer, i.e. in a filtered way. This is particularly interesting for noise corrupted output signals.

The adaptive λ-tracking controller as proposed in Chapter 3 can be extended in several ways. It is relatively straightforward to extend the system class to time-varying systems. Other extension to systems having multiple inputs and outputs (MIMO) and to systems having an unknown high-frequency gain might be possible. High-gain controllers for systems having a large relative degree can cause peaking of the system states (Sussmann and Kokotović, 1991); i.e. system states can have a large transient deviation. Especially, this has to be expected for large observer and controller gains. Increasing these gains through an adaptation reduces the peaking tendency, compared to choosing the gains at relatively large values a priori. Another possibility to reduce peaking could be to incorporate an input saturation in the controller as presented by Esfandiari and Khalil (1992) (see also Teel and Praly, 1995).

The adaptation as presented in Section 3.3.3 only increases the adaptation gain. This is not optimal, especially for long periods of operation as a large gain often means sensitivity to noise and peaking of the states of the system. Including a σ-modification

$$\dot{k} = -\sigma k + (|y| - \lambda)^2$$

in the adaptation as presented in (Ioannou and Kokotović, 1983) or restarting the adaptation might work well, but no convergence guarantee can then be given. Mareels et al. (1999) have shown that even for systems of small dimension and $\lambda = 0$, the asymptotic dynamics can have limit cycles or can even be chaotic.

The transient behavior is not directly addressed in the design of the adaptive λ-tracking controller. However, to achieve a better performance, it is possible to include more information about the system into the controller. For example, a good tuning of the controller parameters might already result in a sufficiently good transient behavior. Another approach is to add a feedforward control (see Section 4.4). Using an estimate of the input nonlinearity of the system in the observer can also improve the performance of the controller (see also Nicosia et al., 1986; Esfandiari and Khalil, 1987).

The human body is difficult to model and the parameters of such a model are often quite difficult to estimate. Therefore, it is reasonable to use a controller that only requires structural information and is robust against parameter uncertainty and an uncertain model dimension. In Chapter 4, the adaptive λ-tracking controller presented in Chapter 3 is applied to a control problem in anesthesia. One of the objectives an automated anesthesia control system should achieve is to keep the endtidal concentration of the volatile anesthetic close to the target value chosen by the anesthetist. This target value is not always constant and is changed several times according to the different phases of a surgery. The experimental results are satisfactory as the controller is able to adapt relatively quickly without reaching a too high adaptation gain. Dynamically, the controller performs well. A better performance is mostly impeded by the input saturation.

Summary

In this thesis, an overview of adaptive high-gain controllers has been given. A new, simple adaptive controller has been proposed which achieves λ-tracking for a large class of nonlinear systems. Only structural information on the system to be controlled is required for designing the controller. The relative degree is neither restricted to be one nor even to be small. The adaptation is guaranteed to converge for a large class of reference signals and all states of the closed loop remain bounded. The controller has been successfully applied to a control problem in anesthesia.

Appendix A

Definitions

A.1 Polynomials in $H(\epsilon, \mu)$

The controller described in Chapter 3 performs a sort of adaptive time scaling. A Lyapunov function for such a system can be found if the system is stable and the adaptation is not too fast, see Remark A.3. To characterize the possible adaptation speed, it is necessary to introduce measures for the decay rate, μ, and the robustness with respect to time-scaling, ϵ, of a Hurwitz polynomial or matrix.

Definition 3 (Polynomial in $\boldsymbol{H(\epsilon, \mu)}$) *A polynomial $p(\cdot) = s^r + \sum_{i=0}^{r-1} p_i s^i$ belongs to the* class $H(\epsilon, \mu)$ *if there exists a symmetric, positive definite matrix P such that the companion matrix*

$$A_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -p_0 & \ldots & \ldots & -p_{r-1} \end{bmatrix}$$

satisfies for $\Psi_r = \mathrm{diag}\{0,\, 1,\, \ldots,\, r-1\}$ *the inequalities*

$$A_c^T \cdot P + P \cdot A_c \leq -2\mu P, \tag{A.1a}$$

$$\Psi_r \cdot P + P \cdot \Psi_r \geq -2\epsilon P. \tag{A.1b}$$

Remark A.1 *By* (A.1a)*, $H(\epsilon, \mu)$ is a subset of the Hurwitz polynomials. It is shown in Appendix B.2 that for any polynomial $p(\cdot)$ there exist an $\underline{\epsilon}$ and a $\bar{\mu}$ such that*

$$p(\cdot) \in H(\epsilon, \mu) \text{ for all } \epsilon \geq \underline{\epsilon} \text{ and for all } \mu \leq \bar{\mu}. \qquad \diamondsuit$$

A.2 Matrices in $H(\epsilon, \mu)$

Definition 4 (Matrix in $\boldsymbol{H(\epsilon, \mu)}$) *A matrix A_c belongs to the* class $H(\epsilon, \mu)$ *if there exists a symmetric, positive definite matrix P such that for $\Psi_r = \mathrm{diag}\{0,\, 1,\, \ldots,\, r-1\}$ the inequalities* (A.1a) *and* (A.1b) *are satisfied.*

Remark A.2 *If $A_c \in H(\epsilon, \mu)$ then any matrix $A_c - m\Psi_r$ is Hurwitz for $m < \frac{\mu}{\epsilon}$.* ♢

Remark A.3 *Define the system*

$$\dot{\boldsymbol{x}} = kKAK^{-1}\boldsymbol{x}$$

with $A \in \mathbb{R}^r$ a Hurwitz matrix and

$$K = \mathrm{diag}\{1, k, \ldots, k^{r-1}\}$$

where k is a positive function increasing monotonically over time. Define the coordinates

$$\boldsymbol{z} = K^{-1}\boldsymbol{x}.$$

Then

$$\dot{\boldsymbol{z}} = kA\boldsymbol{z} - \frac{\dot{k}}{k}\Psi_r\boldsymbol{z} \tag{A.2}$$

with

$$\Psi_r = \mathrm{diag}\{0,\, 1,\, \ldots,\, r-1\}.$$

In the $\boldsymbol{z}$-coordinates, a Lyapunov-function V can easily be defined:

$$V = k^{-2\gamma}\boldsymbol{z}^T P\boldsymbol{z} \tag{A.3}$$

for some γ and any positive definite matrix P satisfying

$$0 > A^T P + PA. \tag{A.4}$$

Then, along any trajectory of (A.2)*, the time derivative of* (A.3) *is*

$$\dot{V} = k^{-2\gamma}\boldsymbol{z}^T \left(PA + A^T P - \frac{\dot{k}}{k}(P\Psi_r + \Psi_r P + 2\gamma P) \right) \boldsymbol{z}. \tag{A.5}$$

As $k \geq 0$ and $\dot{k} \geq 0$, $\dot{V}$ in (A.5) *is negative definite for any k, $\dot{k}$ only if*

$$0 \geq P\Psi_r + \Psi_r P + 2\gamma P. \tag{A.6}$$

If $A \in H(\epsilon, \mu)$, then there exists a positive definite matrix $\bar{P}$ satisfying (A.4) *such that*

$$-2\bar{\gamma}\bar{P} \geq \bar{P}\Psi_r + \Psi_r\bar{P} \geq -2\epsilon\bar{P}.$$

for any $\bar{\gamma} \geq \gamma$. Thus, for any $\gamma \leq \epsilon$, $V = k^{-2\gamma}\boldsymbol{z}^T P\boldsymbol{z}$ is a Lyapunov function. ♢

Remark A.4 *Analyzing* (A.6) *reveals that* $\tilde{\Psi}_r = \Psi_r + \gamma I$ *satisfies*

$$0 \geq \bar{P}\tilde{\Psi}_r + \tilde{\Psi}_r\bar{P}.$$

The matrix $\tilde{\Psi}_r$ *corresponds to the time-scaling matrix* $\tilde{K} = k^{-\gamma}K$. ♢

A.3 Relative degree, high-frequency gain, zero dynamics

The following definitions can be found in most nonlinear control text books, e.g. (Isidori, 1995; Sastry, 1999).

In the following, it is assumed that the system lives on some manifold $M \subseteq \mathbb{R}^n$ and is affine in the input, i.e. that the system can be described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \qquad \text{(A.7a)}$$

$$y = h(\boldsymbol{x}) \qquad \text{(A.7b)}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are smooth vector fields on M and h is a smooth scalar nonlinear function.

Definition 5 (Lie derivative) *The Lie derivative of a scalar function* $\lambda(\cdot)$ *along a vector field* $\boldsymbol{f}(\cdot)$ *is defined by*

$$L_{\boldsymbol{f}}\lambda(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i}(\boldsymbol{x})\boldsymbol{f}_i(\boldsymbol{x}).$$

The Lie derivative can be seen as the directional derivative of $\lambda(\cdot)$ *along the vector field* $\boldsymbol{f}(\cdot)$.

The following notation will be used:

$$L_{\boldsymbol{f}}^{i+1}\lambda(\boldsymbol{x}) = L_{\boldsymbol{f}}L_{\boldsymbol{f}}^{i}\lambda(\boldsymbol{x}) \text{ for } i \geq 1$$

where, for consistency,

$$L_{\boldsymbol{f}}^{0}\lambda(\boldsymbol{x}) = \lambda(\boldsymbol{x}).$$

Definition 6 (Relative degree) *The system* (A.7) *has a relative degree* r *at* $\tilde{\boldsymbol{x}}$ *if for all* $\boldsymbol{x}$ *in a neighborhood of* $\tilde{\boldsymbol{x}}$

$$L_{\boldsymbol{g}}L_{\boldsymbol{f}}^{i}h(\boldsymbol{x}) \equiv 0 \text{ for } i < r-2, \qquad \text{(A.8a)}$$

$$L_{\boldsymbol{g}}L_{\boldsymbol{f}}^{r-1}h(\tilde{\boldsymbol{x}}) \neq 0. \qquad \text{(A.8b)}$$

Definition 7 (Strong relative degree) *A system* (A.7) *has a strong relative degree* r *if*

$$L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{i} h(\boldsymbol{x}) \equiv 0 \text{ for } i < r-2, \tag{A.9a}$$

and

$$L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{r-1} h(\boldsymbol{x}) \neq 0 \text{ for all } \boldsymbol{x} \in M. \tag{A.9b}$$

Definition 8 (High-frequency gain) *The high-frequency gain of the system* (A.7) *with relative degree* r *is defined by*

$$g(\boldsymbol{x}) = L_{\boldsymbol{g}} L_{\boldsymbol{f}}^{r-1} h(\boldsymbol{x}).$$

Remark A.5 *In case of strong relative degree the high-frequency gain is non-zero everywhere.* ♢

Definition 9 (Zero-dynamics) *The zero-dynamics of the system* (A.7) *with relative degree* r *is the dynamics of*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\bar{u}$$

where $\bar{u}$ *renders the manifold*

$$M = \{\boldsymbol{x} | h(\boldsymbol{x}) = L_f h(\boldsymbol{x}) = \ldots = L_f^{r-1} h(\boldsymbol{x}) = 0\}$$

invariant.

A.4 Prime triple

The following terminology was introduced by Morse (1973) for MIMO systems. In this thesis, only the SISO case is needed.

Definition 10 (Prime triple) *The matrices* $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^{n\times 1}$ *and* $\boldsymbol{c} \in \mathbb{R}^{n\times 1}$ *form a* prime triple $\{\boldsymbol{c}^T, A, \boldsymbol{b},\}$ *if*

$$A = \begin{bmatrix} 0 & 1 & 0 & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$
$$\boldsymbol{c}^T = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$

The pairs $\{A, \boldsymbol{b}\}$ *and* $\{\boldsymbol{c}^T, A\}$ *are called prime tuple.*

A.5 Order of a matrix

The following definition helps writing the bounds on the matrices in a more compact way.

Definition 11 (Order of a polynomial) *The highest exponent of k in any element of a matrix M is denoted* $\mathrm{ord}_k(M)$*, i.e. for the matrix M defined by*

$$M_{i,j} = \sum_{l=l_{\min}^{(i,j)}}^{l_{\max}^{(i,j)}} k^l m_l^{(i,j)},$$

$$\mathrm{ord}_k(M) = \max_{i,j}\{l_{\max}^{(i,j)}\}.$$

For example, $\mathrm{ord}_k(\alpha k^2 + \beta + \gamma k^{-1}) = 2$.

Appendix B

Lemmata

B.1 Quadratic expansion

Lemma B.1 (Quadratic expansion) *For* $\alpha,\ \beta,\ \gamma > 0$

$$\alpha x^2 + \beta y^2 - \gamma xy > 0 \textit{ holds for all } (x, y) \neq (0, 0)$$

if and only if $\alpha\beta > \left(\frac{\gamma}{2}\right)^2$.

Proof (of Lemma B.1)
For any $\delta > 0$,

$$\begin{aligned} \alpha x^2 + \beta y^2 - \gamma xy &\geq \alpha x^2 + \beta y^2 - \gamma xy - \frac{|\gamma|}{2}\left(\delta x - \frac{\gamma}{|\gamma|}\frac{y}{\delta}\right)^2 \\ &\geq \left(\alpha - \frac{|\gamma|}{2}\delta^2\right)x^2 + \left(\beta - \frac{|\gamma|}{2}\delta^{-2}\right)y^2 \end{aligned}$$

Therefore,

$$\alpha x^2 + \beta y^2 - \gamma xy > 0\ \forall (x, y) \neq (0, 0) \text{ if and only if}$$
$$\frac{|\gamma|}{2\beta} < \delta^2 < \frac{2\alpha}{|\gamma|}.$$

This is solvable only if $\gamma^2 < 4\alpha\beta$. ∎

B.2 Matrix inequalities in high-gain control

This section contains two lemmata and is partly based on (Bullinger and Kraus, 1999). Lemma B.2 deals with properties of high-gain polynomials appearing in the closed-loop of full-order observer based high-gain controllers as in Chapters 2 and 3. Lemma B.3 shows that a certain class of matrix inequalities has arbitrarily many solutions. These inequalities appear for example in Definitions 3 and 4 defining $H(\epsilon, \mu)$.

Lemma B.2 (High-gain Hurwitz Polynomial) *Assume that the polynomial*

$$p_g(s) = s^r + g \sum_{i=0}^{r-1} p_i s^i \tag{B.1}$$

is Hurwitz for all $g \geq \underline{g}$ for some $\underline{g} > 0$. Then the following statements are true:

a) $\sum_{i=0}^{r-1} p_i s^i$ is Hurwitz.

b) There exists a symmetric, positive definite matrix P such that for any $g \geq \underline{g}$ the companion matrix

$$A_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -gp_0 & \cdots & \cdots & -gp_{r-1} \end{bmatrix}$$

satisfies for some $\mu > 0$

$$A_c^T \cdot P + P \cdot A_c \leq -2\mu P, \tag{B.2a}$$

$$A_c = J - g\boldsymbol{b}\boldsymbol{b}^T P. \tag{B.2b}$$

Proof (of Lemma B.2 a)) Assume that the open loop is defined by

$$g \frac{\sum_{i=0}^{r-1} p_i s^i}{s^r}.$$

Then the closed loop poles are the zeros of the polynomial

$$s^r + g \sum_{i=0}^{r-1} p_i s^i = 0.$$

For $g \to \infty$, one of the closed loop poles goes to $-\infty$ along the negative real axis, the remaining $r-1$ closed loop poles tend to the zeros of $\sum_{i=0}^{r-1} p_i s^i = 0$. Thus, a necessary condition for (B.1) being Hurwitz is that $\sum_{i=0}^{r-1} p_i s^i$ is Hurwitz. ■

Proof (of Lemma B.2 b)) The companion matrix A_c corresponding to the polynomial $p_{\underline{g}}(\cdot)$ can be partitioned as

$$A_{\underline{g}} = J - \underline{g}\boldsymbol{b}\boldsymbol{p}^T$$

with

$$J = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{r \times r}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

As A_g is assumed to be Hurwitz for all $g \geq \underline{g}$ there exists a symmetric, positive definite matrix P such that for any $g \geq \underline{g}$

$$A_g^T \cdot P + P \cdot A_g < 0 \quad \text{for all } g \geq \underline{g}. \tag{B.3}$$

Multiplication by $P^{-1} =: X$ from left and right yields

$$XA_g^T + A_g X < 0 \quad \text{for all } g \geq \underline{g}. \tag{B.4}$$

Introducing $\boldsymbol{y}$ by $\boldsymbol{p} = P\boldsymbol{y}$ results in

$$XJ^T + JX - g\left(\boldsymbol{b}\boldsymbol{y}^T + \boldsymbol{y}^T\boldsymbol{b}\right) < 0 \quad \text{for all } g \geq \underline{g} \tag{B.5}$$

$$XJ^T + JX < (\underline{g} + (g - \underline{g}))\left(\boldsymbol{b}\boldsymbol{y}^T + \boldsymbol{y}^T\boldsymbol{b}\right) \quad \text{for all } g \geq \underline{g}. \tag{B.6}$$

It is therefore necessary that

$$XJ^T + JX < \underline{g}\left(\boldsymbol{b}\boldsymbol{y}^T + \boldsymbol{y}^T\boldsymbol{b}\right). \tag{B.7}$$

As P is positive definite, X needs to be it,too. A necessary condition for $X > 0$ is that

$$\boldsymbol{b}\boldsymbol{y}^T + \boldsymbol{y}^T\boldsymbol{b} = \begin{bmatrix} \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \\ \boldsymbol{y}^T \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{y} \end{bmatrix} \geq 0.$$

Therefore, $\boldsymbol{y}$ needs to be of the following form:

$$\boldsymbol{y} = m\boldsymbol{b} \text{ with } m > 0.$$

Taking $m = 1$ completes the proof. ■

Lemma B.3 (Matrix Inequalities) *For the matrix inequalities*

$$A^T P + PA \leq -2\mu P \tag{B.8a}$$
$$DP + PD \geq -2\epsilon P \tag{B.8b}$$
$$P > 0, \tag{B.8c}$$

where

$$\begin{aligned} &D = -\operatorname{diag}(d_1, d_2, \ldots, d_n), \quad d_i \geq 0,\ i = 1, 2, \ldots, n, \\ &P = P^T \in \mathbb{R}^{n\times n} \\ &A = J - \underline{g}\boldsymbol{b}\boldsymbol{q}^T; \quad A \in \mathbb{R}^{n\times n}; \quad \boldsymbol{b},\ \boldsymbol{q} \in \mathbb{R}^n \end{aligned}$$

with $\{J, \boldsymbol{b}\}$ *is a prime tuple, i.e.*

$$J = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & \ldots & \ldots & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

the following statements are true.

a) *For any Hurwitz polynomial* $p(\cdot)$ *and constant* $\underline{g} > 0$ *there exist arbitrarily many* ϵ *and* μ *such that the inequalities* (B.8) *holds.*

b) *For any* $\underline{g} > 0$, $\epsilon > 0$ *and* $\mu > 0$ *there exists arbitrarily many polynomials* $p(\cdot)$ *such that* (B.8) *holds.*

c) *If* $p_g(\cdot)$ *is Hurwitz for all* $g \geq \underline{g} > 0$ *and if* (B.8) *holds for* $\underline{g}$ *and some* $\epsilon > 0$ *and* $\mu > 0$, *then* (B.8) *holds for all* $g \geq \underline{g} > 0$.

Proof (of Lemma B.3 a)) As $p(\cdot)$ is Hurwitz, there exists a symmetric, positive matrix P such that

$$A^T P + PA = -2I.$$

Define $\mu = \frac{1}{\sigma_{\max}(P)}$, the largest singular value of P, then

$$A^T P + PA \leq -\tilde{\mu} P \quad \text{for all } \tilde{\mu} \leq \mu$$

For ϵ sufficiently large, (B.8b) clearly holds. Define ϵ by

$$\underline{\epsilon} \text{ such that } DP + PD \geq -2\underline{\epsilon} P.$$

Then for all $\epsilon \geq \underline{\epsilon}$

$$DP + PD \geq -2\epsilon P.$$

This proves part a) of Lemma B.3. ∎

Proof (of Lemma B.3 b)) Define $X = P^{-1}$ and restrict $\boldsymbol{q}$ to $\boldsymbol{q} = P\boldsymbol{b}$. Then (B.8) can be rewritten as

$$JX + XJ^T - 2\underline{g}\boldsymbol{b}\boldsymbol{b}^T \leq -2\mu X \tag{B.9a}$$

$$DX + XD \geq -2\epsilon X \tag{B.9b}$$

$$X > 0. \tag{B.9c}$$

Inequality (B.9a) is equivalent to

$$(J + \mu I)\, X + X(J + \mu I)^T \leq 2\underline{g}\boldsymbol{b}\boldsymbol{b}^T. \tag{B.10}$$

The equation corresponding to (B.10),

$$(J + \mu I)\, X + X(J + \mu I)^T = 2\bar{g}\boldsymbol{b}\boldsymbol{b}^T. \tag{B.11}$$

is a Lyapunov equation.

We want to show that the solution X of (B.11) is positive definite and use (Horn and Johnson, 1991, Theorem 2.4.7). As $2\bar{g}\boldsymbol{b}\boldsymbol{b}^T \geq 0$, a necessary condition is that $\{J + \mu I, \boldsymbol{b}\boldsymbol{b}^T\}$ is reachable. This can be shown via the Hautus-test which consists on testing that the matrix $[\lambda I - (J + \mu I)|\boldsymbol{b}\boldsymbol{b}^T]$ has full column rank for all eigenvalues λ of $(J + \mu I)$. As the eigenvalues of $J + \mu I$ are all μ this reduces to testing the rank of

$$[J|\boldsymbol{b}\boldsymbol{b}^T] = \left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & 0 \\ & & \ddots & & & & & \vdots \\ & & & 1 & & & & 0 \\ 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 & 1 \end{array}\right].$$

This matrix has clearly full column rank. Thus $\{J + \mu I, \boldsymbol{b}\boldsymbol{b}^T\}$ is reachable. From reachability, positive semi-definiteness of the right hand side and the fact that $\sigma(J + \mu I) \in \mathbb{C}^+$ it follows that the solution X of (B.11) is positive definite.

Scaling the solution X of equation (B.11) by a factor $\delta \in (0, 1]$, still solves (B.10). Thus, there exist arbitrarily many $\boldsymbol{q} = \delta X^{-1}\boldsymbol{b}$ satisfying (B.8). ■

Proof (of Lemma B.3 c)) From Lemma B.2 b) it follows that (B.8) is equivalent to (B.9). Combining $g \geq \underline{g}$ with Inequality (B.10) yields that

$$(J + \mu I)\, X + X(J + \mu I)^T \leq 2\underline{g}\boldsymbol{b}\boldsymbol{b}^T \leq 2g\boldsymbol{b}\boldsymbol{b}^T,$$

which completes the proof. ■

Remark B.1 *Combining* $g \geq \bar{g}$ *with* (B.10) *yields*

$$JX + XJ^T - 2\bar{g}\boldsymbol{b}\boldsymbol{b}^T \leq -2\mu X \tag{B.12a}$$

$$DX + XD \geq -2\epsilon X \tag{B.12b}$$

$$X > 0. \tag{B.12c}$$

The matrix inequalities (B.12) *form a generalized eigenvalue problem. This can efficiently be solved for a minimizing* ϵ *or for a maximizing* μ *by interior-point methods(Gahinet et al., 1995).* ◇

Remark B.2 *Does there exist a solution to* (B.8) *for* $\epsilon = 0$*? In the two-dimensional case, we show that no solution exists if the diagonal elements of* D *are not strictly positive. We look at* (B.8) *and, without loss of generality take* $d_1 = 0$ *and* $d_2 = 1$*. Thus, we have*

$$\begin{bmatrix} 0 & -q_1 \\ 1 & -q_2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -q_1 & -q_2 \end{bmatrix} =$$

$$\begin{bmatrix} 2p_2 & p_3 - q_1p_1 - q_2p_2 \\ p_3 - q_1p_1 - q_2p_2 & -2(q_1p_2 + q_2p_3) \end{bmatrix} \leq -2 \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & p_2 \\ p_2 & 2p_3 \end{bmatrix} \geq 0$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0$$

Necessary conditions for these inequalities to hold are

$$-2p_1 \geq 2p_2, \text{ (top left element of first equation)}$$
$$p_2 = 0 \text{ (second inequality) and}$$
$$p_1 > 0,\ p_3 > 0 \text{ (third inequality).}$$

As these conditions are contradicting, no solution exists. ◇

B.3 High-gain lemmata

In this section, four lemmata are presented which are applicable to systems of the following form:

$$\dot{\boldsymbol{\xi}} = J\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\psi}^T(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\phi}^T(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\eta} + \boldsymbol{b}v(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{b}g(\boldsymbol{\xi},\boldsymbol{\eta})u$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{w}(\boldsymbol{\xi},\boldsymbol{\eta})$$

where $\{J, \boldsymbol{b}, \boldsymbol{c}^T\}$ is a prime triple, $\boldsymbol{\xi}(t) \in \mathbb{R}^r$ and $\boldsymbol{\eta}(t) \in \mathbb{R}^{n-r}$. Furthermore, for all $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the following bounds hold for some $M > 0$, $\underline{g} > 0$:

- $\|\boldsymbol{\psi}(\boldsymbol{\xi},\boldsymbol{\eta})\|_2 \leq M$, $\|\boldsymbol{\phi}(\boldsymbol{\xi},\boldsymbol{\eta})\|_2 \leq M$, $\|\boldsymbol{\chi}(\boldsymbol{\xi},\boldsymbol{\eta})\|_2 \leq M$,
- $|v(\boldsymbol{\xi},\boldsymbol{\eta})| \leq M$, $\|\boldsymbol{w}(\boldsymbol{\xi},\boldsymbol{\eta})\|_2 \leq M$,
- $\underline{g} \leq |g(\boldsymbol{\xi},\boldsymbol{\eta})| \leq M$.

The unperturbed zero-dynamics, i.e.

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta})$$

is assumed to be exponentially stable. Therefore, there exist a function $W \in \mathbb{R}^{n-r} \to \mathbb{R}$ and positive constants m_1, m_2, m_3, m_4 satisfying (Vidyasagar, 1993)

$$\begin{aligned} m_1\|\boldsymbol{\eta}\|_2^2 \le W(\boldsymbol{\eta}) &\le m_2\|\boldsymbol{\eta}\|_2^2 \\ \|\frac{\partial W}{\partial \boldsymbol{\eta}}\| &\le m_3\|\boldsymbol{\eta}\|_2 \\ \frac{\partial W}{\partial \boldsymbol{\eta}}\boldsymbol{\theta}(\boldsymbol{\eta}) &\ge -m_4\|\boldsymbol{\eta}\|_2^2. \end{aligned}$$

The system will be controlled by

$$u = -\sum_{i=0} q_i k^{r-i}\zeta_{i+1} = -k^r \boldsymbol{q}^T K^{-1}\boldsymbol{\xi}$$

for some fixed $k \in \mathbb{R}$. $K = \mathrm{diag}\{1, k, \dots, k^{r-1}\}$.

Lemma B.4 (High-gain stabilization) *Assume that $\boldsymbol{q}$ achieves that $J - g\boldsymbol{b}\boldsymbol{q}^T$ is Hurwitz for all $g \ge \underline{g}$. Then*

$$\dot{\boldsymbol{\xi}} = J\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\psi}(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}g(\boldsymbol{\xi},\boldsymbol{\eta})u$$

is high-gain stabilizable by

$$u = -k^r\boldsymbol{q}^T K^{-1}\boldsymbol{\xi}.$$

Proof (of B.4) By Lemma B.2, there exist $\mu > 0$ and $P = P^T > 0$ satisfying for any $g \ge \underline{g}$

$$P(J - g\boldsymbol{b}\boldsymbol{q}^T) + (J - g\boldsymbol{b}\boldsymbol{q}^T)^T P \le -2\mu P.$$

Define the coordinates

$$\boldsymbol{\zeta} = K^{-1}\boldsymbol{\xi}.$$

In these coordinates, the system can be described by

$$\dot{\boldsymbol{\zeta}} = k\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right)\boldsymbol{\zeta} + \boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta}.$$

Choose $V = \boldsymbol{\zeta}^T P\boldsymbol{\zeta}$ as a Lyapunov function candidate. Its derivative along a trajectory of the system is

$$\begin{aligned} \dot{V} &= 2k\boldsymbol{\zeta}^T P\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right) + 2\boldsymbol{\zeta}^T P\boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} \\ &\le -2k\mu V + 2\|P\boldsymbol{b}\|_2\|\boldsymbol{\psi}(\cdot)\|_2\,\|k^{-r+1}K\|_2\,\|\boldsymbol{\zeta}\|_2^2. \end{aligned}$$

By the assumptions, there exists an $m > 0$ s.t.

$$\dot{V} \leq -2(k\mu - m)V.$$

Thus, for any sufficiently large k, the closed-loop is exponentially stable. ∎

Lemma B.5 (High-gain stabilization with internal dynamics) *Assume that the controller*

$$u = -k^r \boldsymbol{q}^T K^{-1} \boldsymbol{\xi}$$

high-gain stabilizes the system

$$\dot{\boldsymbol{\xi}} = J\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\psi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}g(\boldsymbol{\xi}, \boldsymbol{\eta})u.$$

Then the same controller high-gain stabilizes the system

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= J\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\psi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\phi}^T(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{\eta} + \boldsymbol{b}g(\boldsymbol{\xi}, \boldsymbol{\eta})u \\
\dot{\boldsymbol{\eta}} &= \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta}).
\end{aligned}$$

Proof (of B.5) Choose, as in Lemma B.4, $P = P^T > 0$ s.t.

$$P(J - g\boldsymbol{b}\boldsymbol{q}) + (J - g\boldsymbol{b}\boldsymbol{q})^T P \leq -2\mu P$$

and define the coordinates

$$\boldsymbol{\zeta} = K^{-1}\boldsymbol{\xi},$$

in which the closed-loop can be described by

$$\begin{aligned}
\dot{\boldsymbol{\zeta}} &= k\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{q}^T\right)\boldsymbol{\zeta} + k^{-r+1}\boldsymbol{b}\boldsymbol{\phi}^T(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{\eta} \\
&\quad + \boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta}, \boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} \qquad \text{(B.13a)}
\end{aligned}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\zeta} + \boldsymbol{\theta}(\boldsymbol{\eta}). \qquad \text{(B.13b)}$$

The function $\tilde{V} = V + W$ where V is the Lyapunov function of Lemma B.4 and W the one of the zero-dynamics is a Lyapunov function candidate for (B.13). Its derivative along a trajectory of the system is

$$\begin{aligned}
\dot{\tilde{V}} &= 2k\boldsymbol{\zeta}^T P\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{q}^T\right) + 2\boldsymbol{\zeta}^T P\boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta}, \boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} \\
&\quad + 2\boldsymbol{\zeta}^T P k^{-r+1}\boldsymbol{b}\boldsymbol{\phi}^T(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{\eta} + \frac{\partial W}{\partial \boldsymbol{\eta}}\left(\boldsymbol{\chi}(K\boldsymbol{\zeta}, \boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta})\right) \\
&\leq -2k\mu V + 2\|P\boldsymbol{b}\|_2 \|\boldsymbol{\psi}(\cdot)\|_2 \|k^{-r+1}K\|_2 \|\boldsymbol{\zeta}\|_2^2 \\
&\quad + 2k^{-r+1}\|P\boldsymbol{b}\|_2 \|\boldsymbol{\phi}(\cdot)\|_2 \|\boldsymbol{\zeta}\|_2 \|\boldsymbol{\eta}\|_2.
\end{aligned}$$

By Lemma B.1 there exist positive constants m and $\tilde{\mu}$ s.t.

$$\dot{\tilde{V}} \leq -2(k\tilde{\mu} - m)V - \tilde{\mu}W.$$

Thus, for any sufficiently large k, the closed-loop is exponentially stable. ∎

Lemma B.6 (High-gain $\boldsymbol{\lambda}$-stabilization) *Assume that $\boldsymbol{q}$ achieves that $J - g\boldsymbol{b}\boldsymbol{q}^T$ is Hurwitz for all $g \geq \underline{g}$. Then*

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= J\boldsymbol{\xi} + \boldsymbol{b}\psi(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}v(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{b}g(\boldsymbol{\xi},\boldsymbol{\eta})u \\ y &= \boldsymbol{c}^T\boldsymbol{\xi}\end{aligned}$$

is high-gain λ-stabilizable for any $\lambda > 0$ by

$$u = -k^r\boldsymbol{q}^T K^{-1}\boldsymbol{\xi}.$$

Proof (of B.6) Choose, as in Lemma B.6, $P = P^T > 0$ s.t.

$$P(J - g\boldsymbol{b}\boldsymbol{q}) + (J - g\boldsymbol{b}\boldsymbol{q})^T P \leq -2\mu P$$

and define the coordinates

$$\boldsymbol{\zeta} = K^{-1}\boldsymbol{\xi},$$

in which the closed-loop can be described by

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= k\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right)\boldsymbol{\zeta} + \boldsymbol{b}\psi(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} + \boldsymbol{b}v(K\boldsymbol{\zeta},\boldsymbol{\eta}) \\ y &= \boldsymbol{c}^T\boldsymbol{\zeta}.\end{aligned}$$

Choose $V = \boldsymbol{\zeta}^T P\boldsymbol{\zeta}$ as a Lyapunov function candidate. Its derivative along a trajectory of the system is

$$\begin{aligned}\dot{V} &= 2k\boldsymbol{\zeta}^T P\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right) \\ &\quad + 2\boldsymbol{\zeta}^T P\boldsymbol{b}\left(\psi(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} + v(K\boldsymbol{\zeta},\boldsymbol{\eta})\right) \\ &\leq -2k\mu V + 2\|P\boldsymbol{b}\|_2\left(\|\psi(\cdot)\|_2\,\|k^{-r+1}K\|_2\,\|\boldsymbol{\zeta}\|_2^2 + \|v(\cdot)\|_2\,\|\boldsymbol{\zeta}\|_2\right).\end{aligned}$$

There exist positive constants $m > 0$ and $\tilde{\mu}$ s.t.

$$\dot{V} \leq -2(k\tilde{\mu} - m)V + m.$$

Therefore, V converges to the set $[0, \frac{m}{2(k\tilde{\mu}-m)}]$ and

$$\|\boldsymbol{\zeta}\|_2^2 \to [0, \frac{m}{2(k\tilde{\mu} - m)\sigma_{\min}(P)}].$$

As $y = \xi_1 = \zeta_1$ and $|\zeta_1| \leq \|\boldsymbol{\zeta}\|_2$,

$$y^2 \to [0, \frac{m}{2(k\tilde{\mu} - m)\sigma_{\min}(P)}].$$

Thus, for any sufficiently large k,

$$y \to [0, \lambda]. \qquad \blacksquare$$

Lemma B.7 (High-gain $\boldsymbol{\lambda}$-stabilization with internal dynamics) *Assume that for $\boldsymbol{\eta} \equiv \mathbf{0}$, the system*

$$\dot{\boldsymbol{\xi}} = J\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\psi}^T(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\xi} + \boldsymbol{b}\boldsymbol{\phi}^T(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{\eta} + \boldsymbol{b}v(\boldsymbol{\xi},\boldsymbol{\eta}) + \boldsymbol{b}g(\boldsymbol{\xi},\boldsymbol{\eta})u \tag{B.14a}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(\boldsymbol{\xi},\boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{w}(\boldsymbol{\xi},\boldsymbol{\eta}) \tag{B.14b}$$

is high-gain λ-stabilizable for any $\lambda > 0$ by

$$u = -k^r \boldsymbol{q}^T K^{-1}\boldsymbol{\xi}.$$

Then the same controller high-gain λ-stabilizes the system (B.14)*, possibly with a larger minimum gain k.*

Proof (of B.7) Choose, as in Lemma B.6, $P = P^T > 0$ s.t.

$$P(J - g\boldsymbol{b}\boldsymbol{q}) + (J - g\boldsymbol{b}\boldsymbol{q})^T P \leq -2\mu P$$

and define the coordinates

$$\boldsymbol{\zeta} = K^{-1}\boldsymbol{\xi},$$

in which the closed-loop can be described by

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= k\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right)\boldsymbol{\zeta} + k^{-r+1}\boldsymbol{b}\boldsymbol{\phi}^T(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{\eta} \\ &\quad + \boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta}\end{aligned} \tag{B.15a}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\chi}(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta}). \tag{B.15b}$$

The function $\tilde{V} = V + W$ where V is the Lyapunov function of Lemma B.6 and W the one of the zero-dynamics is a Lyapunov function candidate for (B.15). Its derivative along a trajectory of the system is

$$\begin{aligned}\dot{\tilde{V}} &= 2k\boldsymbol{\zeta}^T P\left(J - \boldsymbol{b}g(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{q}^T\right) + 2\boldsymbol{\zeta}^T P\boldsymbol{b}\boldsymbol{\psi}(K\boldsymbol{\zeta},\boldsymbol{\eta})k^{-r+1}K\boldsymbol{\zeta} \\ &\quad + 2\boldsymbol{\zeta}^T P k^{-r+1}\boldsymbol{b}\boldsymbol{\phi}^T(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{\eta} + \frac{\partial W}{\partial \boldsymbol{\eta}}\left(\boldsymbol{\chi}(K\boldsymbol{\zeta},\boldsymbol{\eta})\boldsymbol{c}^T\boldsymbol{\xi} + \boldsymbol{\theta}(\boldsymbol{\eta})\right) \\ &\leq -2k\mu V + 2\|P\boldsymbol{b}\|_2\|\boldsymbol{\psi}(\cdot)\|_2\,\|k^{-r+1}K\|_2\,\|\boldsymbol{\zeta}\|_2^2 \\ &\quad + 2k^{-r+1}\|P\boldsymbol{b}\|_2\,\|\boldsymbol{\phi}(\cdot)\|_2\,\|\boldsymbol{\zeta}\|_2\,\|\boldsymbol{\eta}\|_2.\end{aligned}$$

By Lemma B.1 there exist positive constants m and $\tilde{\mu}$ s.t.

$$\dot{\tilde{V}} \leq -2(k\tilde{\mu} - m)V - \tilde{\mu}W + m.$$

With the same argument as in Lemma B.6, the output y is λ-stabilized for any sufficiently large k. ∎

B.4 Variation of constants

Lemma B.8 (Variation of constants) *Assume that*

$$\dot{\boldsymbol{\zeta}}(t) = H(\boldsymbol{\zeta}(t), t)$$

is globally exponentially stable, i.e. that there exist positive constants m_1*,* m_2*,* m_3 *and* m_4 *and a function* $\zeta \mapsto V(\zeta)$ *such that*

$$\begin{aligned} m_1\|\zeta\|^2 \leq V(\zeta) &\leq m_2\|\zeta\|^2, \\ \frac{\partial}{\partial\zeta}V(\zeta)H(\zeta,t) &\leq -m_3\|\zeta\|^2, \\ \|\frac{\partial}{\partial\zeta}V(\zeta)\| &\leq m_4\|\zeta\| \end{aligned}$$

Then the solution $\boldsymbol{\zeta}(\cdot)$ *of the differential equation*

$$\dot{\boldsymbol{\zeta}}(t) = H(\boldsymbol{\zeta}(t), t) + \hat{A}_1(t)\boldsymbol{\zeta}(t) + \hat{A}_2(t)\boldsymbol{\zeta}(t) + \hat{\boldsymbol{b}}(t)e(t) \tag{B.16}$$

is bounded if for some constants $t_0 \geq 0$*,* $M_1 > 0$*,* $M_2 > 0$*,* $M_{b_1} > 0$ *and* $M_{b_2} > 0$*, the following bounds hold:*

$$\begin{aligned} \|\hat{A}_1(t)\| &\leq M_1 \text{ for all } t \geq t_0, \text{ with } M_1 \leq \frac{m_3}{2m_4} \\ \|\hat{\boldsymbol{b}}(t)\| &\leq M_b, \\ e(\cdot) &\in \mathcal{L}_\infty\big([t_0, \omega)\big). \end{aligned}$$

Proof (of B.8) Analyzing the function $V(\zeta)$ along trajectories of (B.16) reveals:

$$\begin{aligned} \dot{V} &= \frac{\partial}{\partial\zeta}\Big(H(\zeta(t),t) + \hat{A}_1(t)\boldsymbol{\zeta}(t) + \hat{A}_2(t)\boldsymbol{\zeta}(t) + \hat{\boldsymbol{b}}(t)e(t)\Big) \\ &\leq -m_3\|\zeta\|^2 + m_4\|\zeta\|\,(M_1\|\zeta\| + M_b|e|) \\ &\leq -(m_3 - m_4M_1) + m_4\|\zeta\|\,M_b|e|\,. \end{aligned}$$

Completing the squares gives

$$\begin{aligned} \dot{V} &\leq -\frac{m_3}{2} + m_4\|\zeta\|\,M_b|e| + \frac{m_3}{4}\left(\|\zeta\| - \frac{2m_4M_b|e|}{m_3}\right)^2 \\ &\leq -\frac{m_3}{4}V(\zeta) + \frac{(m_4M_b|e|)^2}{m_3} \end{aligned}$$

Integrating from $t_0 \geq 0$ to t yields

$$V(t) - V(t_0) \leq \int_{t_0}^{t} e^{\tau - t_0}\left(\frac{(m_4M_b|e|)^2}{m_3}\right)d\tau.$$

As $e(\cdot) \in \mathcal{L}_\infty\big([t_0, \omega)\big)$ there exist a constant $\tilde{\mu} > 0$ such that

$$V(t) - V(t_0) \le \tilde{\mu} \int_{t_0}^{t} e^{\tau - t_0} d\tau \le \tilde{\mu}.$$

Thus $V(\cdot)$ and therefore also $\zeta(\cdot)$ are bounded. ■

B.5 Scaling coefficients

Lemma B.9 (Scaling functions in the Proof of Theorem 3.1) *For any $\tilde{\gamma}$ satisfying* (3.11c), (3.11d) *there exist k-dependent functions c_1, c_2 and c_3 satisfying the inequalities* (3.16).

Proof (of B.9)

B.12.a) Solvability for $\tilde{c}_2$. Combining (3.16a) and (3.16d) yields the inequality

$$-(r - \frac{1}{2})\beta < \tilde{c}_2 - \tilde{c}_1 < 0,$$

or, equivalently

$$\tilde{c}_2 - \tilde{c}_1 = \delta, \quad \delta \in I_\delta = (-(r - \frac{1}{2})\beta, 0). \tag{B.17}$$

Inequality (3.16b) can be written in the following way:

$$\tilde{c}_1 - \tilde{c}_3 = -(r - \frac{1}{2})(\alpha - \beta) + \gamma, \quad \gamma \in I_\gamma = (0, \alpha - \beta) \tag{B.18}$$

Adding (B.17) and (B.18) results in

$$\begin{aligned} \tilde{c}_2 - \tilde{c}_3 &= -(r - \frac{1}{2})(\alpha - \beta) + \delta + \gamma \\ &> -(r - \frac{1}{2})(\alpha - \beta) - (r - \frac{1}{2})\beta. \end{aligned}$$

Thus,

$$\tilde{c}_2 - \tilde{c}_3 > -(r - \frac{1}{2})\alpha,$$

which is the same as (3.16c). As I_δ is nonempty for positive β, the inequalities (3.16) are solvable for c_2 for any given c_1 and c_3.

B.12.b) Solvability for $\tilde{c}_1$ and $\tilde{c}_3$.
Case: $\tilde{c}_1 + \beta\epsilon \geq \tilde{c}_3 + \alpha\epsilon$**:**
As $\alpha \geq \beta$, it follows from (3.16b) that

$$0 \leq \tilde{c}_1 - \tilde{c}_3 < -(r - \frac{3}{2})(\alpha - \beta).$$

This inequality is solvable only if $r = 1$. Therefore, $\tilde{c}_1 + \beta\epsilon \geq \tilde{c}_3 + \alpha\epsilon$ implies that $r = 1$.

Using $\tilde{c}_1 + \beta\epsilon \geq \tilde{c}_3 + \alpha\epsilon$, (3.16f) is simplified to

$$0 \geq \beta\epsilon + \frac{1}{2}\tilde{c}_1$$

which is equivalent to

$$\tilde{c}_1 \leq -2\beta\epsilon. \tag{B.19}$$

Case: $\tilde{c}_1 + \beta\epsilon < \tilde{c}_3 + \alpha\epsilon$**:**

$$0 \geq \alpha\epsilon + \tilde{c}_3 - \frac{1}{2}\tilde{c}_1.$$

Using (B.18) this inequality can be solved for c_1:

$$0 \geq \alpha\epsilon + \tilde{c}_1 + (r - \frac{1}{2})(\alpha - \beta) - \gamma - \frac{1}{2}\tilde{c}_1,$$

or, equivalently to

$$\tilde{c}_1 < -2\alpha\epsilon - 2(r - \frac{1}{2})(\alpha - \beta). \tag{B.20}$$

Case: arbitrary $\tilde{c}_3$**:**
Combining (B.19) and (B.20), the following inequality has to be satisfied by c_1 independently of c_3:

$$\begin{aligned} &\tilde{c}_1 \leq -2\beta\epsilon \quad \text{for } r = 1, \\ &\tilde{c}_1 < -2\alpha\epsilon - (\alpha - \beta)(2r - 1) \quad \text{for } r \geq 1. \end{aligned} \tag{B.21}$$

By (3.16g),

$$\tilde{c}_1 \geq -\tilde{\gamma} - \frac{1}{2}. \tag{B.22}$$

Combining (B.21) with (B.22) yields

$$\begin{aligned} &-\tilde{\gamma} \leq \tilde{c}_1 < -2\alpha\epsilon - 2(\alpha - \beta)(r - \frac{1}{2}) \quad \text{for } r > 1, \\ &-\tilde{\gamma} \leq \tilde{c}_1 \leq -2\alpha\epsilon \quad \text{for } r = 1. \end{aligned} \tag{B.23}$$

By (3.11c), (3.11d), these inequalities are solvable for $\tilde{c}_1$.

Rewriting inequality (3.16b) as

$$\tilde{c}_3 - \tilde{c}_1 < (r - \frac{1}{2})(\alpha - \beta) = (r - \frac{1}{2})\alpha - \underbrace{(r - \frac{1}{2})\beta}_{>0}$$

reveals that (3.16e) is contained in (3.16b). Thus, given a $\tilde{c}_1$ satisfying (B.23), any $\tilde{c}_3$ satisfying (3.16b) and any $\tilde{c}_2$ satisfying (3.16a) is a solution of the system of inequalities (3.16).■

Appendix C

Higher-Order Root Locus

The root-locus technique is a well-known tool developed by Evans (1950) (see for example Evans, 1954) for analyzing the qualitative behavior of the zeros of a polynomial in s that is linearly dependent on a parameter k, i.e.

$$\sum_{i=1}^{n} \alpha_i s^i + k \sum_{i=1}^{m} \beta_i s^i = 0. \tag{C.1}$$

An extension for polynomials depending nonlinearly on k has been proposed by Hahn (1981). This enables to analyze graphically for small and large k polynomials of the form:

$$q(s,k) = \sum_{i=0}^{n} \sum_{j} \alpha_{ij} s^i k^j = 0, \tag{C.2}$$

where j does not need to take integer values. In the following, a short description of this method is given. It is used it Sections C.3 and C.4 in the context of high-gain controllers with reduced-order observers, see Section 1.3.2 page 20 and (Mareels, 1984; Mudgett and Morse, 1989; Bullinger et al., 2001). Then, in Section C.5 the technique is applied to the full-order high-gain controllers, see 3. In both cases, the adaptation parameter is fixed.

C.1 Construction of the exponent diagram

1. Each term $t(s,k)$ of (C.2) is mapped to the point $(\mathrm{ord}_s(t), \mathrm{ord}_k(t))$. These are denoted by crosses in Figure C.1.

2. Find the minimal convex hull of theses points.

3. Define for each edge of the convex hull the corresponding supporting polynomial $p_i(s)$ by taking the sum of the term mapped to this edge modulo a common factor and

setting $k = 1$. For example, the supporting polynomial for a line whose generators are $\alpha s k^3$, $\beta s^2 k^2$ and $\gamma s^3 k$ lie on the line $k(\alpha k^2 + \beta s k + \gamma s^2) = 0$. The supporting polynomial is $\alpha + \beta s + \gamma s^2 = 0$

4. Label the vertical edges III and VI, the horizontal edges II, and VII, the remaining ones by I, IV, V and VIII, if necessary with subscripts as for edge I in Figure C.1.

5. Denote by α_i the slope angle of edge i and by l_0 the distance between edge III and the ord_k-axis. The length of edge i projected onto the ord_s-direction is called e_i, projected onto the ord_k-direction $\bar{e}$, see Figure C.2.

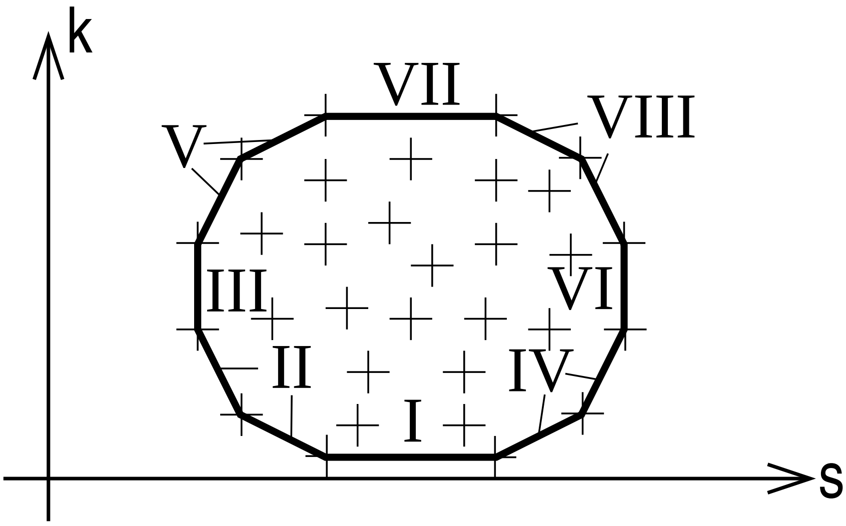


Figure C.1: Edges of the Exponent Diagram.

C.2 Analysis of the exponent diagram

Intuitively, it is clear that the lower edges describe the solutions of (C.2) for small values of k.

0. The equation (C.2) has l_0 roots at 0.

I. The edges I stand for the e_{I} roots of the form

$$s_i = \phi_j k_i^{\beta}$$

where $\{\phi_j\}$ are the roots of the supporting polynomial $p_{\mathrm{I}_i}(s)$ and $\beta_{\mathrm{I}_i} = \tan \alpha_{\mathrm{I}_i}$.

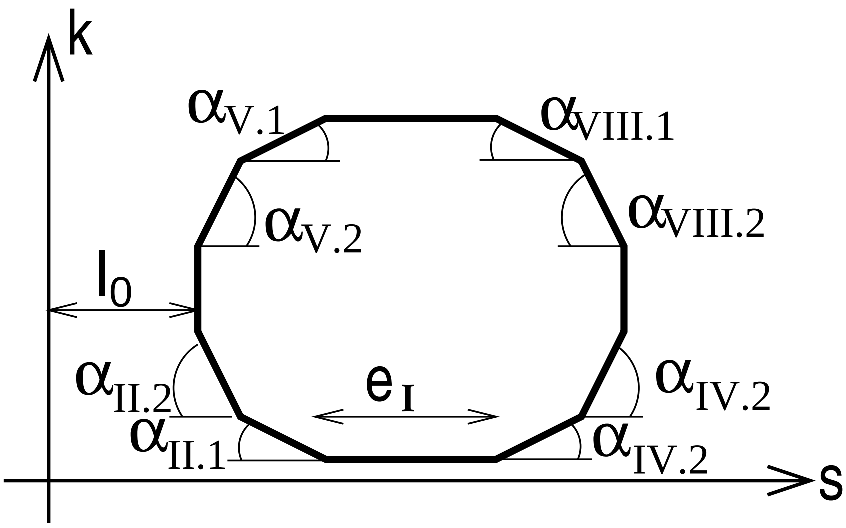


Figure C.2: Angles of the Exponent Diagram.

II. The roots ϕ_j of the supporting polynomial $p_{\mathrm{II}}(s)$ are the non-vanishing roots of $q(s,0)$ which, for small k, are approximately

$$s_j = \phi_j k^{\beta_j}$$

where $\beta_{\mathrm{II}_i} = \tan \alpha_{\mathrm{II}_i}$.

III. There are e_{III} possibly complex values of k such that $s = 0$ is a root of $s^{-l_0} q(s,k)$.

IV. Exactly e_{IV} roots of $q(s,k)$ start at infinity for $k = 0$.

V. Edge Vstands for the vanishing sinks of $q(s,k) : q(s,\infty)$ has $l_0 + e_{\mathrm{V}}$ zeros at $s = 0$.

VI. There are e_{VI} possibly complex values of k, k_j, such that $q(s,k_j)$ is infinite.

VII. Edge VII: Finite non-vanishing sinks for $k \to \infty$: For $k \to \infty$ e_{VII} roots of $q(s,k)$ go to the roots of the supporting polynomial $p_{\mathrm{VII}}(s)$.

VIII. The edges VIII$_i$ stand for the e_{VIII} roots of the form

$$s_j = \phi_j k^{\beta_j}$$

where $\{\phi_j\}$ are the roots of the supporting polynomial $p_{\mathrm{VIII}_i}(s)$ and $\beta_{\mathrm{VIII}_i} = \tan \alpha_{\mathrm{VIII}_i}$.

C.3 Reduced-order high-gain controller

In the following, the closed-loop poles are analyzed in the case of a linear plant of dimension n and relative degree r controlled by a high-gain controller with reduced-order observer of dimension ρ for a fixed high-gain parameter k, see Section 1.3.2 page 20. For simplicity, it is assumed that the observer is in controllability normal form, i.e.

$$\hat{A} = \begin{bmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -p_0 & \cdots & \cdots & \cdots & -p_{\rho-1} \end{bmatrix} = \hat{J} - \hat{\boldsymbol{b}}\boldsymbol{p}^T \tag{C.3}$$

Then,

$$\begin{aligned} u &= -\hat{q}_0 y - \hat{\boldsymbol{q}}^T \hat{\boldsymbol{x}} \\ u &= -\hat{q}_0 y + \hat{\boldsymbol{q}}^T \left(sI - \hat{A} + \hat{\boldsymbol{b}}\boldsymbol{p}^T \right)^{-1} \hat{\boldsymbol{b}} \\ &= -q_0 k^{\alpha\rho+\beta} \left(1 - \frac{\sum_{i=1}^{\rho} q_i k^{\alpha(\rho+1-i)} s^{i-1}}{s^\rho + \sum_{i=1}^{\rho} (p_{i-1} + q_i k^{\alpha(\rho+1-i)} s^{i-1})} \right) y \\ &= -q_0 k^{\alpha\rho+\beta} \frac{s^\rho + \sum_{i=1}^{\rho} p_{i-1} s^{i-1}}{s^\rho + \sum_{i=1}^{\rho} (p_{i-1} + q_i k^{\alpha(\rho+1-i)}) s^{i-1}} y \end{aligned} \tag{C.4}$$

The general frequency-domain representation of a linear system with relative degree r is

$$y = \frac{\sum_{i=0}^{n-r} b_i s^i}{s^n + \sum_{i=0}^{n} a_i s^i} u, \tag{C.5}$$

where b_{n-r} is the high-frequency gain, i.e. $g = b_{n-r}$.

The corner points of the exponent diagram corresponding to the open loop of (C.4) and (C.5) are

① $(0, 0)$,

② $(n + \rho, 0)$,

③ $(n, \alpha\rho)$,

④ $(n - r + \rho, \alpha\rho + \beta)$,

⑤ $(0, \alpha\rho + \beta)$,

where the point ③ is a corner point only for $\alpha > \beta$, see Figure C.3 to C.6.

For the different edges, the supporting polynomials are

Edge I: $(s^n + \sum_{i=0}^{n} a_i s^i)(s^\rho + \sum_{i=0}^{\rho-1} p_i s^i)$,

Edge III: $p_0 + q_0 k^{\beta\rho} + p_0 q_0 k^{\alpha\rho+\beta}$, Edge VII: $s^\rho + \sum_{i=0}^{\rho-1} p_i s^i$,

Edge VIII: Several cases have to be distinguished:

(a) $\rho \geq r$: The point ③ lies above the point ② (Figure C.6) of to the right of it (Figure C.7). The edge$VIII$ is therefore generated by $s^r + gq_0$.

(b) $\alpha(r-\rho) = \beta$: The point ③ lies on the line connecting the points ② and ④, see Figure C.5. $s^{\rho+r} + \sum_{i=0}^{\rho-1} q_{i+1}s^{i+r} + q_0 g$ with $\tan \alpha_{VIII} = \alpha$.

(c) $\rho < r$ and $\alpha > \beta$: $VIIIa$: $s^\rho + \sum_{i=0}^{\rho-1} q_{i+1}s^i$ with $\tan \alpha_{VIIIa} = \alpha$ and $VIIIb$: $s^{r-\rho}q_1 + q_0 g$ with $\tan \alpha_{VIIIa} = \beta$, see Figure C.3.

(d) $\rho < r$ and $\alpha \leq \beta$: $VIII$: $s^r + q_0 g$, see Figure C.4.

By analyzing the exponent diagram, it can be seen that for large k, n-1 poles go to the zeros of $\hat{A}$. The remaining ones go to infinity. To guarantee stability, the following conditions have to hold

- $r = 1$: $q_0 > 0$ and an arbitrary ρ is sufficient.
- $r > 1$: $r = \rho + 1$ and

  for $\alpha > \beta$: $s^\rho + \sum_{i=0}^{\rho-1} q_{i+1}s^i$ is Hurwitz,

  for $\alpha = \beta$: $s^r + \sum_{i=0}^{\rho-1} q_{i+1}s^{i+(r-\rho)} + q_0 g$ is Hurwitz for any allowable value of g.

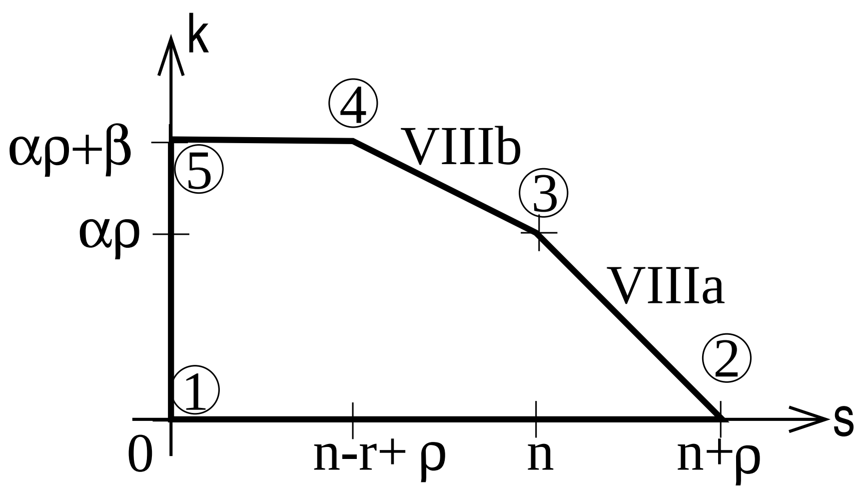


Figure C.3: Exponent Diagram for the high-gain controller with reduced-order observer where $r > \rho$ and $r\alpha(r-\rho) > \beta$.

It is interesting to note that a relative degree one system can be stabilized by a high-gain controller of arbitrary dimension.

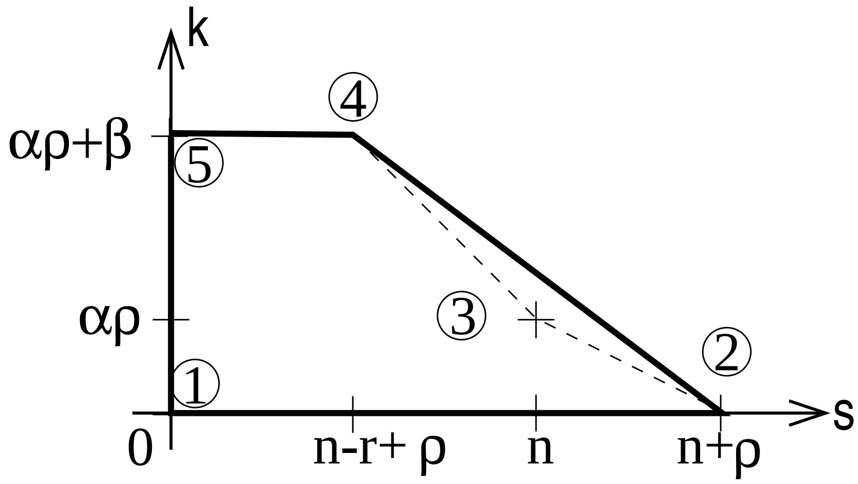


Figure C.4: Exponent Diagram for the high-gain controller with reduced-order observer where $r > \rho$ and $\alpha(r - \rho) < \beta$.

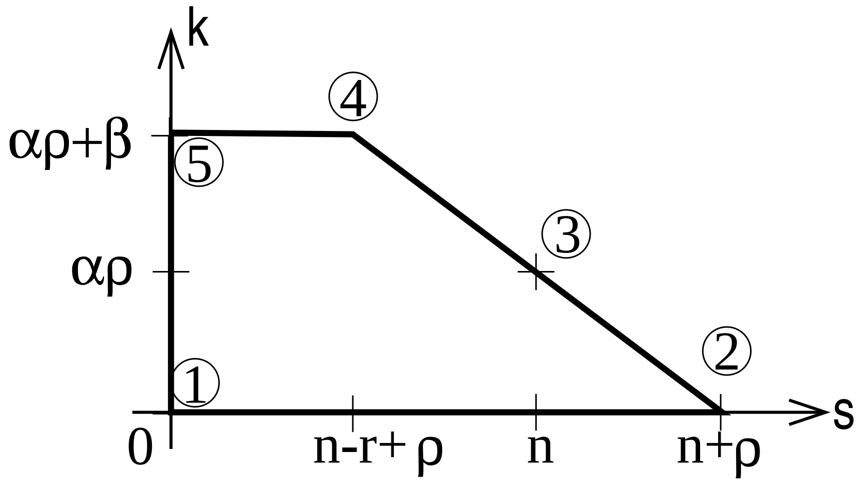


Figure C.5: Exponent Diagram for the high-gain controller with reduced-order observer where $r > \rho$ and $\alpha(r - \rho) = \beta$.

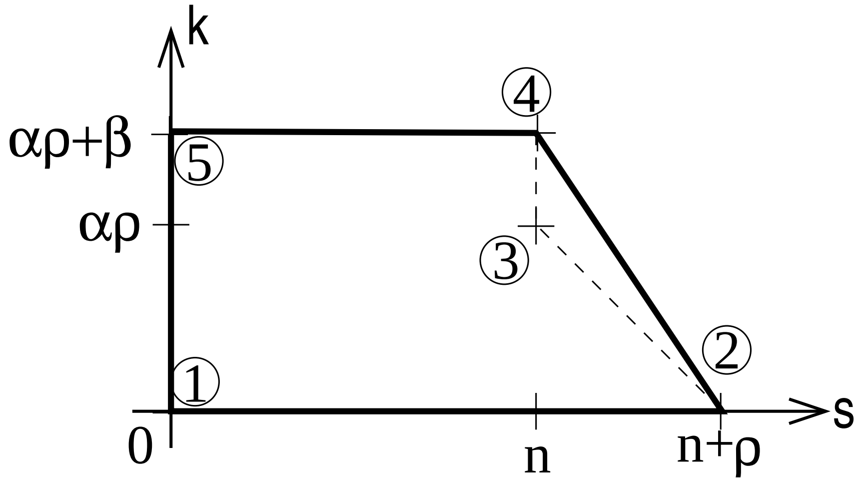


Figure C.6: Exponent Diagram for the high-gain controller with reduced-order observer where $r = \rho$.

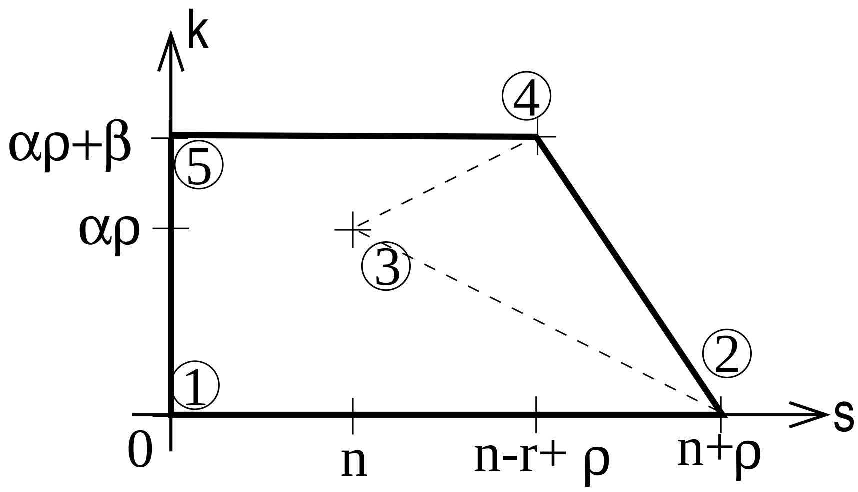


Figure C.7: Exponent Diagram for the high-gain controller with reduced-order observer where $r < \rho$.

C.4 Low-gain result

In Section 2.1.3 the following result is needed.

Lemma C.1 (Low-gain Lemma) *A polynomial*

$$s^r + \sum_{i=1}^{r-1} s^i q_i + q_0 k \tag{C.6}$$

is Hurwitz if

1. *k is sufficiently small and*
2. *$s^{r-1} + \sum_{i=0}^{r-1} s^i q_{i+1}$.*

Proof (of Lemma C.1) The exponent diagram of (C.6) is the convex hull of the following three points in the (s, k)-plane (see Figure C.8)

1. $(1, 0)$,
2. $(r, 0)$ and
3. $(0, 1)$.

For $k \to \infty$, the roots of the polynomial (C.6) tend to

- the roots of $s^{r-1} + \sum_{i=0}^{r-1} s^i q_{i+1} = 0$, the generating polynomial of edge II,
- the last root goes asymptotically to zero along the negative real axis as $s = -k\frac{q_0}{q_1}$. ■

C.5 High-gain controller with full-order observer

Again, assume that the system is linear, has relative degree r and is represented by:

$$y = \frac{\sum_{i=0}^{n-r} b_i s^i}{s^n + \sum_{i=0}^{n} a_i s^i} u,$$

where b_{n-r} is the high-frequency gain, i.e. $g = b_{n-r}$.

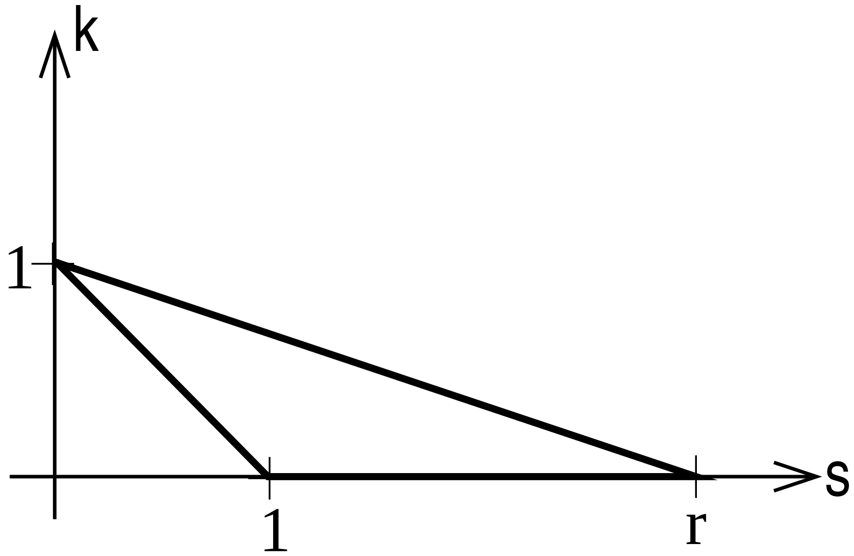


Figure C.8: Higher-order root locus for the low gain lemma.

The controller has a full-order observer of dimension ρ. For simplicity, the observer is assumed to be in observability normal form as in Section 3.3.1. Then, the states of the full-order observer are

$$\hat{\boldsymbol{x}}_i(s) = s^{i-1} \frac{s^{\rho-i} k^{\alpha i} p_{\rho-i} + \ldots + k^{\alpha\rho} p_0}{s^\rho + s^{\rho-1} k^\alpha p_{\rho-1} + \ldots + k^{\alpha\rho} p_0} y(s).$$

The state-feedback is

$$u(s) = -\sum_{i=1}^{\rho} q_{i-1} \hat{x}_i k^{\beta(\rho-i+1)}.$$

The possible corner points of the exponent diagram are

① $(\rho, 0)$

② $(n + \rho, 0)$

③ $(n, \alpha\rho)$

④ $(n - r, (\alpha + \beta)\rho)$

⑤ $(0, (\alpha + \beta)\rho)$

⑥ $(0, \alpha\rho)$

and possibly

⑦ $(n-r+\rho-1, \alpha+\beta\rho)$

⑧ $(n-r+\rho-1, \alpha\rho+\beta)$

For the different edges, the supporting polynomials are

Edge I: $p_{\mathrm{I}}(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$.

Edge II: $p_{\mathrm{II}}(s) = s^\rho + \sum_{i=0}^{n-1} p_i s^i$ with angle $\tan\alpha$.

Edge III: $p_{\mathrm{III}}(s) = k^{\beta\rho} + a_0$.

Edge VII: $p_{\mathrm{VII}}(s) = \sum_{i=0}^{n-r} b_i s^i$.

For the edges VIII, it is necessary to consider several cases.

- $r = \rho$, $\alpha > \beta$: There are two edges VIII, see Figure C.9. The supporting polynomials are

$$p_{\mathrm{VIIIa}}(s) = s^\rho + \sum_{i=0}^{r-1} p_i s^i \text{ with angle } \tan\alpha,$$

$$p_{\mathrm{VIIIb}}(s) = s^r + b_{n-r} \sum_{i=0}^{r-1} q_i s_i \text{ with angle } \tan\beta.$$

- $r = \rho$, $\alpha = \beta$: There is only one supporting polynomial:

$$p_{\mathrm{VIII}}(s) = s^\rho \left(s^\rho + \sum_{i=0}^{r-1} p_i s^i \right) + p_0 b_{n-r} \sum_{i=0}^{r-1} q_i s_i \tag{C.7}$$

  with angle $\tan\alpha$.

- $\alpha < \beta$: There is only one supporting polynomial, see Figure C.10

$$p_{\mathrm{VIII}}(s) = s^{r+\rho} + p_0 b_{n-r}.$$

- $r < \rho$: This makes the point ⑦ lie further to the right of the line connecting ③ and ④. In Figure C.11, the case of $\rho = r+1$ and $\alpha > \beta$ is shown. If $\alpha < \beta$, then ⑦ should be replaced by ⑧. In general, the supporting polynomials are:

$$p_{\mathrm{VIIIa}}(s) = s^{r+1} + b_{n-r} p_0,$$

$$p_{\mathrm{VIIIb}}(s) = \sum_{i=0}^{r-1} q_i s_i.$$

- $r > \rho$: This makes the point ⑦ lie further to the left of the line connecting ③ and ④, see Figure C.12. If $\alpha < \beta$, then ⑦ should be replaced by ⑧. The supporting polynomials are therefore

$$p_{\text{VIIIa}}(s) = s^{\rho} + \sum_{i=0}^{r-1} p_i s^i,$$
$$p_{\text{VIIIb}}(s) = s^r + b_{n-r} q_0.$$

For large k, the closed loop poles are in the left half of the complex plane if $\sum_{i=0}^{n-r} b_i s^i$ is Hurwitz. Furthermore, the following conditions need to be satisfied for

- $r = \rho$, $\alpha > \beta$: $s^{\rho} + \sum_{i=0}^{r-1} p_i s^i$ and $s^r + b_{n-r} \sum_{i=0}^{r-1} q_i s_i$ are Hurwitz. The parameter b_{n-r} is the high-frequency gain of the system.
- $r = \rho$, $\alpha = \beta$: (C.7) is Hurwitz. This condition is rather difficult to check, as is the state-space approach, see Section 3.5.
- $\alpha < \beta$: $r = 1$ and $\rho = 0$ or $r = 0$ and $\rho \leq 1$, as $r + \rho$ needs to be not larger than one.
- $r > \rho$: Only if $r = 1$ can the roots of the supporting polynomial VIIIb lie in the left half of the complex plane.
- $r < \rho$: Only if $r = 0$ can the roots of the supporting polynomial VIIIa lie in the left half of the complex plane.

Therefore, for a relative degree r larger than one, it is necessary that $\rho = r$ and $\alpha \geq \beta$.

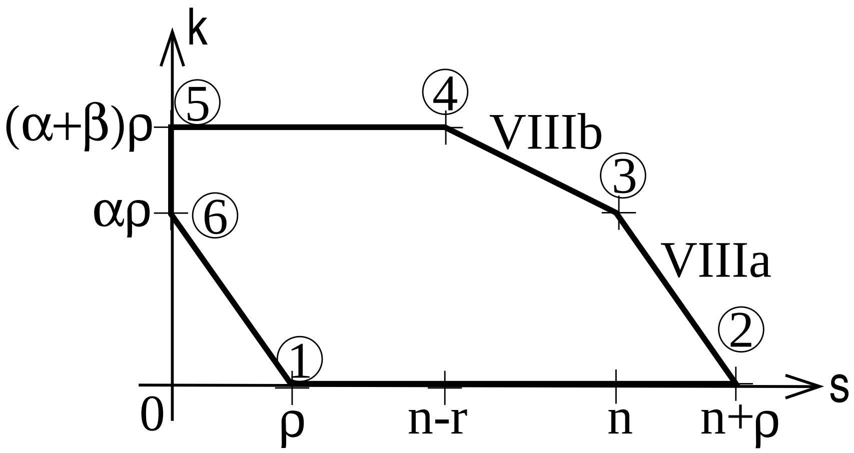


Figure C.9: Exponent Diagram for the high-gain controller with full-order observer where $r = \rho$ and $\alpha > \beta$.

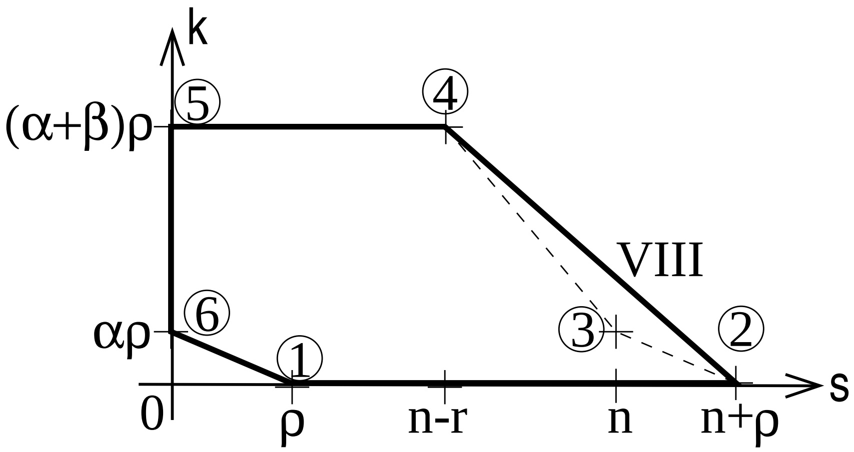


Figure C.10: Exponent Diagram for the high-gain controller with full-order observer where $\alpha < \beta$.

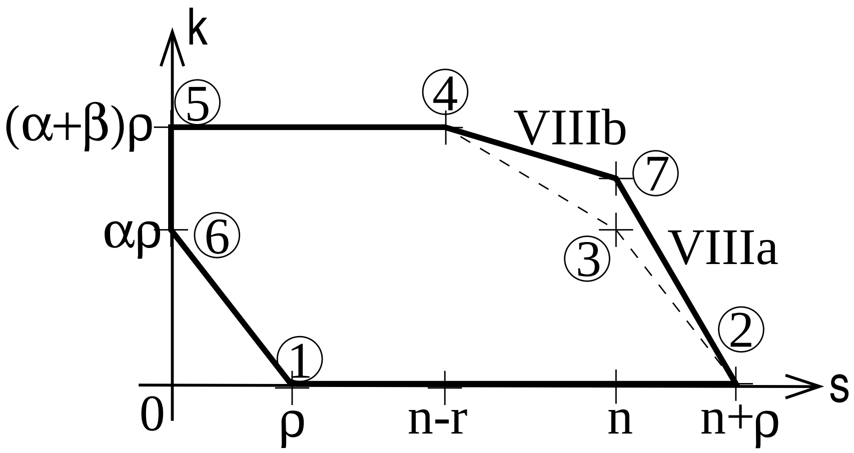


Figure C.11: Exponent Diagram for the high-gain controller with full-order observer where $\alpha > \beta$ and $\rho = r + 1$.

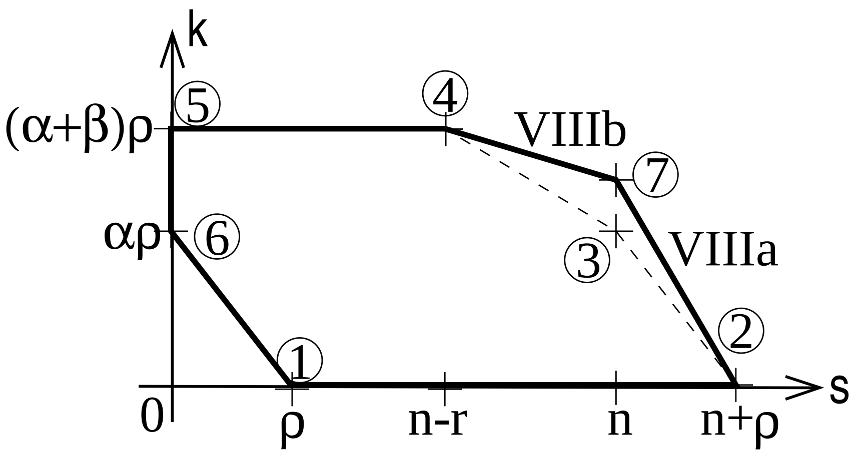


Figure C.12: Exponent Diagram for the high-gain controller with full-order observer where $\alpha > \beta$ and $\rho < r$.

Bibliography

Allgöwer, F. and Ilchmann, A.: 1995, Multivariable adaptive λ-tracking for nonlinear chemical processes, *Proc. of the 3rd European Control Conference, ECC'95, Roma, Italy*, pp. 3247–3252.

Allgöwer, F., Ashman, J. and Ilchmann, A.: 1997, High-gain adaptive λ-tracking for nonlinear systems, *Automatica* **33**(5), 881–888.

Åström, K. J. and Wittenmark, B.: 1973, On self-tuning regulators, *Automatica* **9**(2), 185–199.

Barbălat, I.: 1959, Systemes d'équations différentielles d'oscillations non linéaires, *Rev. Math. Pur. Appl.* **4**, 267–270. Acad. Republ. Popul. Roum.

Baum, J.: 1998, *Die Inhalationsnarkose mit niedrigem Frischgasfluß: Praxis der Low-flow- und der Minimal-flow-Anästhesie sowie der Narkose mit geschlossenem System*, third edn, Thieme, Stuttgart.

Bellman, R.: 1953, *Stability Theory of Differential Equations*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York.

Brega, R.: 1998, A real-time operating system designed for predictability and run-time safety, *Proc. of The Fourth International Conference on Motion and Vibration Control (MOVIC '98), Zurich, Switzerland, August 25-28*, Vol. 1, pp. 379–384.

Bullinger, E. and Allgöwer, F.: 1997, An adaptive high-gain observer for nonlinear systems, *Proc. of the 36th IEEE Conf. on Decision and Control, San Diego, USA*, pp. 4348–4353.

Bullinger, E. and Kraus, F.: 1999, Solvability of certain matrix inequalities, *Technical Report 99-06*, Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH), Zürich, Switzerland.

Bullinger, E., Findeisen, R. and Allgöwer, F.: 2001, Adaptiveλ-tracking of nonlinear systems with higher relative degree using reduced-order high gain control, *NOLCOS 2001*, St. Petersburg, Russia. accepted.

Bullinger, E., Findeisen, R., Kraus, F. and Allgöwer, F.: 2000a, Some further results on adaptive λ-tracking for linear systems with high relative degree, *Proc. of the 2000 American Control Conf.*, Chicago, Il, pp. 3655–3659.

Bullinger, E., Frei, C. W., Sieber, T. J., Glattfelder, A. H., Allgöwer, F. and Zbinden, A. M.: 2000b, Adaptive λ-tracking in anesthesia, *in* E. Carson and E. Salzsieder (eds), *Preprints of the 4th IFAC Symposium Modelling and Control in Biomedical Systems*, Pergamon, Oxford, UK, pp. 217–222.

Bullinger, E., Ilchmann, A. and Allgöwer, F.: 1998, A simple adaptive observer for nonlinear systems, *Proc. of the IFAC Nonlinear Control Systems Design, NOLCOS98, Enschede, The Netherlands*, Vol. 3, pp. 806–811. Preprints.

Bullinger, E., Ilchmann, A. and Allgöwer, F.: 1999, Piecewise constant high-gain adaptive λ-tracking for higher relative degree linear systems, *Proc. of the 14th IFAC World Congress, Beijing, China*, Vol. D, pp. 249–254.

Byrnes, C. I. and Isidori, A.: 1984, A frequency domain philosophy for nonlinear systems with applications to stabilization and to adaptive control, *Proc. of the 23rd IEEE Conf. on Decision and Control*, Vol. 3, pp. 1569–1573.

Byrnes, C. I. and Isidori, A.: 1985, Global feedback stabilization of nonlinear systems, *Proc. of the 24th IEEE Conf. on Decision and Control*, Vol. 2, pp. 1031–1037.

Byrnes, C. I. and Willems, J. C.: 1984, Adaptive stabilization of multivariable linear systems, *Proc. of the 23rd IEEE Conf. on Decision and Control*, Vol. 3, IEEE, New York, NY, USA, pp. 1574–1577.

Byrnes, C. I., Helmke, U. and Morse, A. S.: 1986, Necessary conditions in adaptive control, *in* C. Byrnes and A. Lindquist (eds), *Modeling, Identification, and Robust Control*, North-Holland, New York, pp. 2–14.

Corless, M.: 1991, Simple adaptive controllers for systems which are stabilizable via high gain feedback, *IMA J. of Mathematical Control & Information* **8**(4), 379–387.

Derighetti, M.: 1999, *Feedback Control in Anaesthesia*, PhD thesis, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland.

Desoer, C. A.: 1969, Slowly varying systems $\dot{x} = A(t)x$, *IEEE Trans. Automatic Control* **14**(6), 780–781.

Egardt, B.: 1979, *Stability of adaptive controllers*, number 20 in *Lecture notes in control and information sciences*, Springer-Verlag.

Esfandiari, F. and Khalil, H. K.: 1987, Observer-based design of uncertain systems: recovering state feedback robustness under matching conditions, *Proc. of the 25th Anual Allerton Conf. on Comunication, Control, and Computing*, Monticello, IL, pp. 97–106.

Esfandiari, F. and Khalil, H. K.: 1992, Output feedback stabilization of fully linearizable systems, *Int. J. Control* **56**(5), 1007–1037.

Evans, W. R.: 1950, Control synthesis by root locus method, *AIEE Transactions* **69**, 66–69.

Evans, W. R.: 1954, *Control-system dynamics*, McGraw-Hill, New York.

Feuer, A. and Morse, A. S.: 1978, Adaptive control of single-input, single-output linear systems, *IEEE Trans. Automatic Control* **23**(4), 557–69.

Föllinger, O.: 1969, *Nichtlineare Regelungen I*, Oldenbourg, München.

Francis, B. A. and Wonham, W. M.: 1975, The internal model principle for linear multivariable regulators, *J. Appl. Maths. Optimization* **2**(2), 170–194.

Frei, C.: 2000, *Fault Tolerant Control Concepts Applied to Anesthesia*, PhD thesis, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland.

Frei, C., Bullinger, E., Gentilini, A., Glattfelder, A. H., Sieber, T. and Zbinden, A. M.: 2000, Artifact-tolerant controllers for automatic drug delivery in anesthesia, *Crit. Rev. Biomed. Eng.* **28**(1-2), 187–192.

Frei, C., Bullinger, E., Sieber, T., Glattfelder, A. and Zbinden, A.: 1999, Artefakttolerante Regelungen für die Anästhesie, *Fortschritts-Berichte VDI, Reihe 17, Nr. 183, AUTOMED '99, Beiträge zum Workshop "Automatisierungstechnische Verfahren für die Medizin", 25./26. Februar, Darmstadt*, pp. 37–38.

Frei, C., Leibundgut, D., Branca, A., Glattfelder, A. and Zbinden, A.: 1998, Eine Plattform für Automatisierungssysteme in der Anästhesie, *Bulletin SEV/VSE* **17**, 27–33.

Fu, M. and Barmish, B. R.: 1986, Adaptive stabilization of linear systems via switching control, *IEEE Trans. Automatic Control* **31**(12), 1097–1103.

Gahinet, P., Nemirovski, A., Laub, A. J. and Chilali, M.: 1995, *LMI Control Handbook for Use with MATLAB*, The MathWorks, Natick, Mass.

Gentilini, A., Frei, C. W., Glattfelder, A. H., Morari, M., Sieber, T. J., Wymann, R., Schnider, T. W. and Zbinden, A. M.: 2001, Multitasked closed-loop control in anesthesia, *IEEE Engineering in Medicine and Biology* **20**(1), 39–53.

Gibson, J. E.: 1963, *Nonlinear Automatic Control*, McGraw-Hill, New York.

Hahn, H.: 1981, Higher order root-locus technique with applications in control system design, *in* I. Hartmann (ed.), *Advances in control systems and signal processing*, Vol. 2, Friedr. Vieweg & Sohn, Wiesbaden, Germany, pp. 1–274.

Helmke, U. Prätzel-Wolters, D.: 1988, Stability and robustness properties of universal adaptive controllers for first-order linear systems, *Int. J. Control* **48**(3), 1153–1182.

Horn, R. A. and Johnson, C. R.: 1991, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK.

Ilchmann, A.: 1991, Non-identifier-based adaptive control of dynamical systems: A survey, *IMA J. of Mathematical Control & Information* **8**(4), 321–366.

Ilchmann, A.: 1993, *Non-Identifier-Based High-Gain Adaptive Control*, number 189 in *Lecture notes in control and information sciences*, Springer-Verlag, Berlin.

Ilchmann, A.: 1997, Universal stabilization of nonlinear systems, *Dynamics and Control* **7**(3), 199–213.

Ilchmann, A. and Pahl, M.: 1998, Adaptive multivariable ph regulation of a biogas tower reactor, *European J. of Control* **4**(2), 116–131.

Ilchmann, A. and Ryan, E. P.: 1994, Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise, *Automatica* **30**(2), 337–346.

Ilchmann, A., Owens, D. H. and Prätzel-Wolters, D.: 1987, High-gain robust adaptive controllers for multivariable systems, *Syst. Control Lett.* **8**(5), 397–404.

Ilchmann, A., Weirig, M.-F. and Mareels, I. M. Y.: 1998, Modelling and adaptive control of biochemical processes, *Proc. of the IFAC Nonlinear Control Systems Design, NOLCOS98, Enschede, The Netherlands*, Vol. 2, pp. 465–470.

Ioannou, P. A. and Kokotović, P. V.: 1983, *Adaptive Systems with Reduced Models*, number 47 in *Lecture notes in control and information sciences*, Springer-Verlag, Berlin.

Isidori, A.: 1995, *Nonlinear Control Systems*, third edn, Springer, London.

Khalil, H. K.: 1996, *Nonlinear Systems*, second edn, Prentice Hall, Upper Saddle River, NJ.

Khalil, H. K.: 1999, High-gain observers in nonlinear feedback control, *in* H. Nijmeijer and T. I. Fossen (eds), *New Directions in Nonlinear Observer Design*, number 244 in *Lecture notes in control and information sciences*, Springer-Verlag, London, pp. 249–268.

Khalil, H. K. and Saberi, A.: 1987, Adaptive stabilization of a class of nonlinear systems using high-gain feedback, *IEEE Trans. Automatic Control* **32**(11), 1031–1035.

Krener, A. J.: 1987, Normal forms for linear and nonlinear systems, *in* M. Luksic (ed.), *Differential geometry: the interface between pure and applied mathematics*, Vol. 68 of *Contemporary mathematics*, AMS, Providence, RI, pp. 157–189.

Krstić, M., Kanellakopoulos, I. and Kokotović, P.: 1995, *Nonlinear and Adaptive Control*, John Wiley, New York.

La Salle, J. and Lefschetz, S.: 1961, *Stability by Liapunov's Direct Method*, Vol. 4 of *Mathematics in Science and Engineering*, Academic Press, New York.

Logemann, H. and Townley, S.: 1997, Adaptive control of infinite-dimensioanl systems without parameter estimation: an overview, *IMA J. of Mathematical Control & Information* **14**(2), 175–206.

Luenberger, D. G.: 1964, Observing the state of a linear system, *IEEE Trans. on Military Electronics* **8**(2), 74–80.

Mareels, I. M. Y.: 1984, A simple selftuning controller for stably invertible systems, *Syst. Control Lett.* **4**(1), 5–16.

Mareels, I. M. Y. and Polderman, J. W.: 1996, *Adaptive Systems: An introduction*, Birkhäuser, Boston, Ma.

Mareels, I. M. Y., Van Gils, S., W., P. J. and Ilchmann, A.: 1999, Asymptotic dynamics in adaptive gain control, *in* P. M. Frank (ed.), *Advances in Control: Highlights of ECC'99*, Springer-Verlag, London, pp. 29–63.

Mårtensson, B.: 1985, The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization, *Syst. Control Lett.* **6**(2), 87–91.

Mårtensson, B.: 1986, *Adaptive Stabilization*, PhD thesis, Lund Institute of Technology, Lund, Sweden.

Mårtensson, B. and Polderman, J. W.: 1993, Correction and simplification to "the order of a stabilizing regulator is sufficient a priori information for adaptive stabilization", *Syst. Control Lett.* **20**(6), 465–470.

Mazur, A. and Hossa, R.: 1997, Universal adaptive λ-tracking controller for wheeled mobile robots, *Proc. of Symposium on Robot Control. Nantes, France*, Vol. 1, pp. 33–38.

Menold, P. H., Pearson, R. K. and Allgöwer, F.: 1999, Online outlier detection and removal, *Proc. of the 7th Mediterranean on Control and Automation (MED '99), Haifa, Israel*, pp. 1110–1133.

Miller, D. E.: 1994, Adaptive stabilization using a nonlinear time-varying controller, *IEEE Trans. Automatic Control* **39**(7), 1347–1359.

Miller, D. E.: 1998, On the order of nonlinear time-invariant stabilizing controllers, *Syst. Control Lett.* **35**(2), 119–129.

Miller, D. E. and Davison, E. J.: 1991a, An adaptive controller which provides an arbitrarily good transient and steady-state response, *IEEE Trans. Automatic Control* **36**(1), 68–81.

Miller, D. E. and Davison, E. J.: 1991b, An adaptive controller which provides Lyapunov stability, *IEEE Trans. Automatic Control* **34**(6), 599–609.

Miller, D. E., Chang, M. and Davison, E. J.: 1997, An approach to switching control: Theory and application, *in* A. Morse (ed.), *Control Using Logic-Based Switching*, Vol. 222 of *Lecture notes in control and information sciences*, Springer-Verlag, London, pp. 234–247.

Monopoli, R. A.: 1974, Model reference adaptive control with an augmented error signal, *IEEE Trans. Automatic Control* **19**(5), 474–484.

Morse, A. S.: 1973, Structural invariants of linear multivariable systems, *SIAM J. Control* **11**(3), 446–465.

Morse, A. S.: 1980, Global stability of parameter-adaptive control systems, *IEEE Trans. Automatic Control* **25**(3), 433–439.

Morse, A. S.: 1984, An adaptive control for globally stabilizing linear systems with unknown high-frequency gains, *in* A. Balakrishnan and M. Thoma (eds), *Analysis ond Optimization of Systems*, Vol. 62 of *Lecture notes in control and information sciences*, Springer-Verlag, pp. 58–68. Proc. of the Sixth Int. Conf. on Analysis and Optimization of Systems, Nice.

Morse, A. S.: 1987, High–gain feedback algorithms for adaptive stabilization, *Proc. of the 5th Yale Workshop on Appls. of Adaptive Systems Theory*, pp. 13–18.

Morse, A. S.: 1993, Recent problems in parameter adaptive control, *in* I. Landau (ed.), *Outils et modèles mathématiques pour l'automatique, l'analyse de systèmes et le traitement du signal*, Vol. 3, CNRS, pp. 733–740.

Morse, A. S.: 1995, Control using logic-based switching, *in* A. Isidori (ed.), *Trends in Control - A European Perspective*, Lecture notes in control and information sciences, Springer-Verlag, London, pp. 69–113.

Morse, A. S.: 1996, Overcoming the obstacle of high relative degree, *European J. of Control* **2**(1), 29–35.

Mudgett, D. R. and Morse, A. S.: 1989, A smooth, high-gain adaptive stabilizer for linear systems with known relative degree, *Proc. of the 1989 American Control Conf.*, Vol. 3, pp. 2315–2320.

Narendra, K. S., Lin, Y.-H. and S., V. L.: 1980, Stable adaptive controller design. ii. proof of stability, *IEEE Trans. Automatic Control* **25**(3), 440–448.

Nicosia, S. and Tornambè, A.: 1989, High-gain observers in the state and parameter estimation of robots having elastic joints, *Syst. Control Lett.* **13**(4), 331–337.

Nicosia, S., Tomei, P. and Tornambè, A.: 1986, Feedback control of elastic robots by pseudo-linearization techniques, *Proc. of the 25th IEEE Conf. on Decision and Control*, Athens, Greece.

Nicosia, S., Tomei, P. and Tornambè, A.: 1989, An approximate observer for a class of nonlinear systems, *Syst. Control Lett.* **13**(1), 43–51.

Nussbaum, R. D.: 1983, Some remarks on a conjecture in parameter adaptive control, *Syst. Control Lett.* **3**(5), 243–246.

Owens, D. H., Prätzel-Wolters, D. and Ilchmann, A.: 1987, Positive-real structures and high-gain adaptive stabilization, *IMA J. of Mathematical Control & Information* **4**(2), 167–181.

Prätzel-Wolters, D., Owens, D. H. and Ilchmann, A.: 1989, Robust adaptive stabilization by high-gain feedback and switching, *Int. J. Control* **49**(6), 1861–1868.

Ryan, E. P.: 1992, Universal $w^{1,\infty}$-tracking for a class of nonlinear systems, *Syst. Control Lett.* **18**(3), 201–210.

Ryan, E. P.: 1994, A nonlinear universal servomechanism, *IEEE Trans. Automatic Control* **39**(4), 753–761.

Saberi, A. and Lin, Z.: 1990, Adaptive high-gain stabilization of 'minimum-phase' nonlinear systems, *Proc. of the 29th IEEE Conf. on Decision and Control, Brighton, UK*, Honolulu, Hawaii, pp. 3417–3422.

Saberi, A. and Sannuti, P.: 1990, Observer design for loop tranfer recovery and for uncertain dynamical systems, *IEEE Trans. Aut. Control* **35**(8), 878–897.

Sannuti, P.: 1983, Direct singular perturbation analysis of high-gain and cheap control problems, *Automatica* **19**(1), 41–51.

Sastry, S.: 1999, *Nonlinear Systems: Analysis, Stability, and Control*, Springer-Verlag, New York.

Schwilden, H. and Stoeckel, H. (eds): 1995, *Control and Automation in Anaesthesia*, Springer.

Sieber, T. J., Frei, C. W., Derighetti, M., Feigenwinter, P., Leibundgut, D. and Zbinden, A. M.: 2000, Model-based automatic feedback control versus human control of end-tidal isoflurane concentration using low-flow anaesthesia, *British J. of Anaesthesia* **85**(6), 818–825.

Smith, N., Zwart, A. and Beneken, J.: 1972, Interaction between the circulatory effects and the uptake and distribution of halothane: Use of a multiple model, *Anesthesiology* **37**(1), 47–58.

Sussmann, H. J. and Kokotović, P. V.: 1991, The peaking phenomenon and the global stabilization of nonlinear systems, *IEEE Trans. Aut. Control* **36**(4), 424–440.

Teel, A. and Praly, L.: 1995, Tools for semiglobal stabilization by partial state and output feedback, *SIAM J. Control and Optimization* **33**(5), 1443–1488.

Tornambè, A.: 1992, High-gain observers for non-linear systems, *Int. J. of Systems Science* **23**(9), 1475–1489.

Unbehauen, H.: 1989, *Regelungstechnik I*, 6th edn, Friedr. Vieweg & Sohn, Braunschweig.

Vidyasagar, M.: 1993, *Nonlinear Systems Analysis*, second edn, Prentice Hall, Englewood Cliffs, NJ.

Weisstein, E. W.: 2000, *Eric Weisstein's World of Mathematics*, http://mathworld.wolfram.com.

Willems, J. C. and Byrnes, C. I.: 1984, Global adaptive stabilization in the absence of information on the sign of the high frequency gain, *Analysis and optimization of systems: Proc. of the Sixth Int. Conf. on Analysis and Optimization of Systems, Nice, 1984*, number 62 in *Lecture notes in control and information sciences*, Springer-Verlag, Berlin, pp. 49–57.

Ye, X.: 1999, Universal λ-tracking for nonlinearly-perturbed systems without restrictions on the relative degree, *Automatica* **35**(1), 109–119.

Zbinden, A., Frei, F., Westenskow, D. and Thomson, D.: 1986, Control of end-tidal halothane concentration, Part B: Verification in dogs, *British Journal of Anaesthesia* **58**, 563–571.

Zeitz, M.: 1989, Canonical forms for nonlinear systems, *in* A. Isidori (ed.), *Nonlinear Control System Design. Selected papers from the IFAC-Symposium, Capri, Italy*, Pergamon Press, Oxford, pp. 33–38.

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