Geometric study of mixed-integer sets coming from two adjacent simplex bases

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Joint work with Kent Andersen and Robert Weismantel (Magdeburg)

Outline

- Cuts from two rows of the simplex tableau
- Upper bounds on the continuous variables
- Determining a facet with several upper bounds
- Geometry of facets in the case of one bound

Relaxation of MIP

Simplex Tableau

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$r_{m,1}x_1$	$\cdots +$	$r_{m,k}x_k$	$+$	$r_{m,k+1}s_{k+1}$	$\cdots +$	$r_{m,n}s_n$
$s_{B_{m+1}}$	$=$	f_{m+1}	$+$	$r_{m+1,1}x_1$	$\cdots +$	$r_{m+1,k}x_k$	$+$	$r_{m+1,k+1}s_{k+1}$	$\cdots +$	$r_{m+1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
s_{B_p}	$=$	f_p	$+$	$r_{p,1}x_1$	$\cdots +$	$r_{p,k}x_k$	$+$	$r_{p,k+1}s_{k+1}$	$\cdots +$	$r_{pn}s_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $s_{B_{m+1}}, \ldots, s_{B_p} \in \mathbb{R}_+$
3. $x_1, \ldots, x_k \in \mathbb{Z}_+$
4. $s_{k+1}, \ldots, s_n \in \mathbb{R}_+$

Solution is 'fractional', i.e. $f_1, \ldots, f_m$ are not all integer.

Relaxation of MIP

Relaxation Step 1 : Drop Some Constraints

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$r_{m,1}x_1$	$\cdots +$	$r_{m,k}x_k$	$+$	$r_{m,k+1}s_{k+1}$	$\cdots +$	$r_{m,n}s_n$
$s_{B_{m+1}}$	$=$	f_{m+1}	$+$	$r_{m+1,1}x_1$	$\cdots +$	$r_{m+1,k}x_k$	$+$	$r_{m+1,k+1}s_{k+1}$	$\cdots +$	$r_{m+1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
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Relaxation of MIP

Relaxation of Simplex Tableau

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x_{B_2}	$=$	f_2	$+$	$r_{2,1}x_1$	$\cdots +$	$r_{2,k}x_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+$
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$(f_1, f_2) \notin \mathbb{Z}^2$.

Relaxation of MIP

Relaxation Step 2 : Drop Integrality Requirement

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots+$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots+$	$r_{1,n}s_n$
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1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+$
2. $x_1, \ldots, x_k \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_1, \ldots, x_k \in \mathbb{R}_+$
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Relaxation of MIP

Relaxation Step 2 : Drop Integrality Requirement

Basic Variable		rhs				Columns Corresponding to Continous Variables				
x_{B_1}	$=$	f_1	$+$	$r_{1,1}s_1$	$\cdots +$	$r_{1,k}s_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots +$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+$
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Continuous Group Relaxation

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x_{B_1}	$=$	f_1	$+$	$r_{1,1}s_1$	$\cdots +$	$r_{1,k}s_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots +$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_{B_1}, x_{B_2} \in \mathbb{Z}$
2. $s_1, \ldots, s_k,\ s_{k+1}, \ldots, s_n \in \mathbb{R}_+$

$(f_1, f_2) \notin \mathbb{Z}^2$.

The valid inequalities for the above are valid for the original simplex tableau !

Model studied in Andersen, Louveaux, Weismantel, Wolsey, IPCO2007 (for the finite case), Cornuéjols and Margot, 2009.

Related to Group Relaxation of Gomory and Johnson (1972), Johnson (1974).

The 2 row-model

The model

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j, \qquad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+$$

Model studied in [Andersen, Louveaux, Weismantel, Wolsey, IPCO2007] (for the finite case) and [Cornuéjols, Margot, 2009] (for the infinite case).

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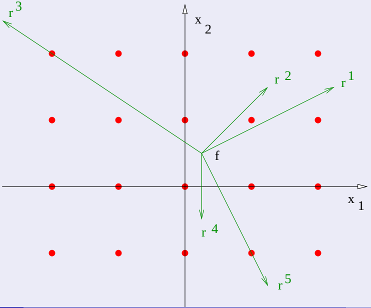
The geometry

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5$$



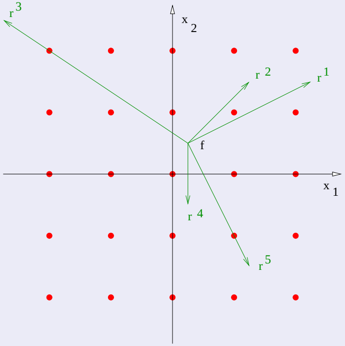
The geometry

The projection picture

$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1$$

- We project the $n+2$-dim space onto the x-space
- The facet is represented by a polygon L_α
- There is no integer point in the interior of L_α
- The coefficients are a ratio of distances on the figure

  α_1 α_3
- The polygon is either a triangle or a quadrilateral



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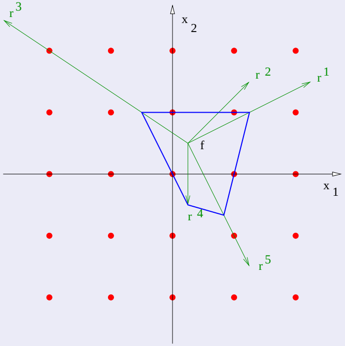

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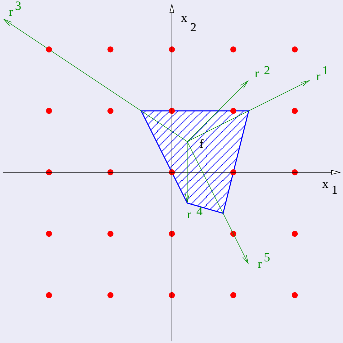

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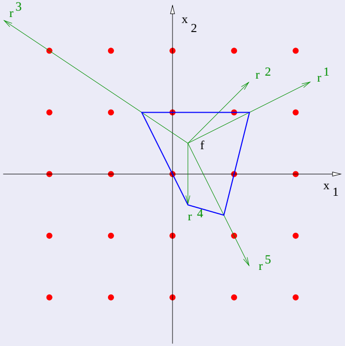

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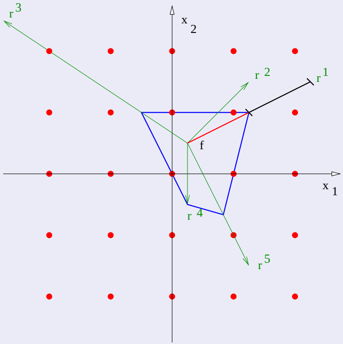

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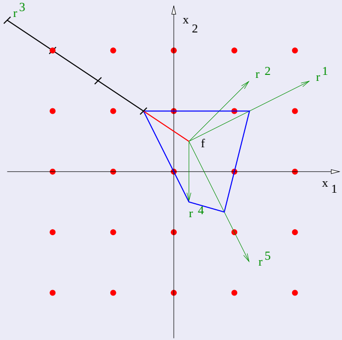

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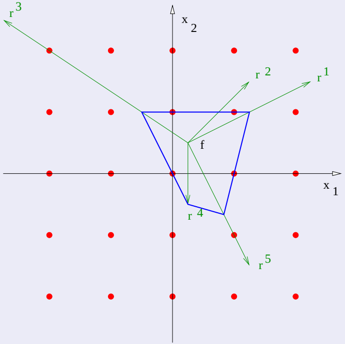

Looking at two fractional vertices at once

Consider the model

$$\begin{array}{ccccccccccc} x_{B_1} & = & f_1 & + & r_{1,1}s_1 & \cdots + & r_{1,k}s_k & + & r_{1,k+1}s_{k+1} & \cdots + & r_{1,n}s_n \\ x_{B_2} & = & f_2 & + & r_{2,1}s_1 & \cdots + & r_{2,k}s_k & + & r_{2,k+1}s_{k+1} & \cdots + & r_{2,n}s_n. \end{array}$$

We may consider a potential pivot.
For example : s_1 enters the basis
In the original full tableau, x_{B_m} must leave the basis as s_1 takes the value $-f_m/r_{m,1}$.

In the model with 2 rows, we may consider

$$0 \leq s_1 \leq u_1$$

(and in this case, $u_1 = -f_m/r_{m,1}$.)

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Updated Two Row Model

The model with bounds

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j \in U} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j + \sum_{j \in B} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j$$

$$x_1, x_2 \in \mathbb{Z},$$

$$s_j \in \mathbb{R}_+ \text{ for } j \in U,$$

$$0 \leq s_j \leq u_j \text{ for } j \in B.$$

Remark : we denote by $N := U \cup B$

The model with bounds on the integer variables has been studied by Basu, Conforti, Cornuéjols, Zambelli [2009], Dey, Wolsey [2009] and Fukasawa, Günlük [2009]

Characterizing the facets of the model with bounds

Let $\sum_{j \in N} \beta_j s_j \geq 1$ be a facet-defining inequality for the convex hull of the model.

- All coefficients $\beta_j \not\geq 0$
- We can complement continuous variables that have a negative coefficient and write

$$\sum_{j \in B_-} \alpha_j (u_j - s_j) + \sum_{j \in B_+} \alpha_j s_j + \sum_{j \in U} \alpha_j s_j \geq 1$$

  with $\alpha_j \geq 0$ for all j

- It is as seeing the problem from the vertex $f + \sum_{j \in B_-} \alpha_j u_j$ and reversing the direction of the rays in B_-

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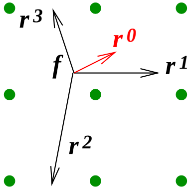

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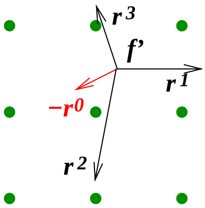

Facet-defining inequality in standard form

In the following, we assume that every facet considered from the right vertex is in standard form and can be written

$$\sum_{j \in N} \alpha_j s_j \geq 1$$

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We can also consider the projection picture similar to the unbounded case !

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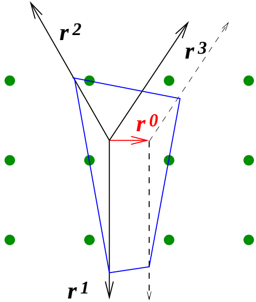
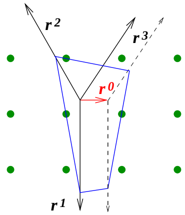


What does carry over from the unbounded to the bounded case ?



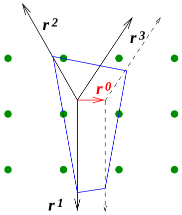
What does carry over

- $L_\alpha = \{x \in \mathbb{R}^2 \mid \exists s \text{ with } x = f + \sum_{j \in N} s_j r^j, 0 \leq s_j \leq u_j, \sum_j \alpha_j s_j \leq 1\}$ is a polygon
- The interior of L_α has no integer point
- $X_\alpha = L_\alpha \cap \mathbb{Z}^2$ is a triangle or a quadrilateral

What does not carry over

- L_α is a triangle or quadrilateral
- The vertices of L_α are on the rays $(f + \lambda r^j)$ with $\lambda \geq 0$
- α can be determined by a system with as many variables as the number of sides of the polygon (L_α or X_α)
- Each integer point has exactly one relevant "vertex"-representation

What does carry over from the unbounded to the bounded case ?



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The polar system

A facet-defining inequality in standard form is a nonnegative basic feasible solution of the system in unknowns β

$$\sum_{j \in N} \bar{s}_j \beta_j \geq 1 \qquad \text{for all } (\bar{x}, \bar{s}_j) \in P_I$$

$$\beta_j \geq 0 \qquad j \in U,$$

where $\bar{x} = \sum_{j \in N} \bar{s}_j r^j, \bar{x} \in \mathbb{Z}^2, 0 \leq \bar{s}_j \leq u_j$.

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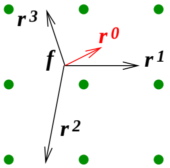

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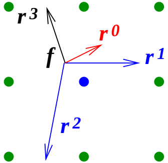

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + \frac{3}{10} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{20} \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\tfrac{3}{10}\beta_1 + \tfrac{1}{20}\beta_2 \geq 1$$

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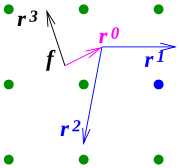

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix} + \frac{17}{20}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

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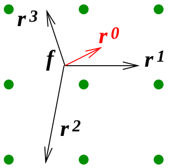

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Obtaining a subsystem from the polar

A facet $\alpha \in \mathbb{R}^n_+$ is a basic feasible solution of the polar and therefore satisfies at least n linearly independent constraints with equality.

Our goal

For a given facet $\sum_{j=1}^{n} \alpha_j s_j \geq 1$, we want to find the smallest subsystem of equations from the polar (or induced by the polar) for which the unique solution is α.

The linear dependence property

Definition

A subset $J \subseteq N$ has the linear dependence property with respect to $\alpha \in \mathbb{R}^n$ if for all $\lambda \in \mathbb{R}^{|J|}$,

$$\sum_{j \in J} \lambda_j r^j = 0 \Rightarrow \sum_{j \in J} \lambda_j \alpha_j = 0.$$

Geometrically

Implication about representations

If $\bar{x} = f + \sum_{j \in J} s_j r^j$ and $\sum_{j \in J} s_j \alpha_j = 1$

implies

if $\bar{x} = f + \sum_{j \in J} t_j r^j$ then $\sum_{j \in J} t_j \alpha_j = 1$

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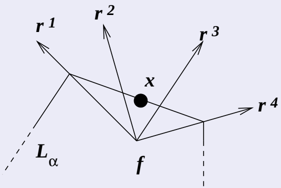

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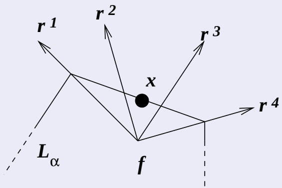

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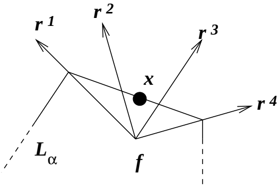
Linear dependence property allows us to get rid of some rays



Here $J = \{1, 2, 3, 4\}$ satisfies the linear dependence property wrt. α.
Assume $\{1, 2, 3, 4\} \subseteq U$
Any equation of the type

$$\alpha_2 \bar{s}_2 + \alpha_i \bar{s}_i = 1$$

involving r^2 or r^3 can be equivalently replaced by

$$\alpha_2 = \sigma_1^2 \alpha_1 + \sigma_4^2 \alpha_4$$

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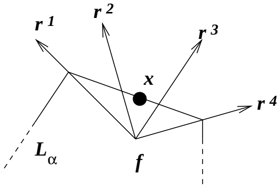

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Main theorem about reduction of the polar

Theorem

Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet in standard form.
Let X^v be the set of extreme points of X_α. α is a nonnegative extreme point of the polar and is the unique solution of the system in unknowns β

$$\sum_{j \in N} \beta_j s_j^x = 1 \quad \text{for all } x \in X^v \tag{1}$$

One equation corresponding to one representation for each x

$$\sum_{k \in I(X)} \sigma_j^k \beta_k = \beta_j \qquad j \in B \tag{2}$$

linear dependence relations for bounded rays j

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Furthermore the system (1)-(2) (on less variables) is also uniquely solvable

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Geometric study in the case of one upper bound

Henceforth we assume that $|B| = 1$, and that $\text{cone}_{j \in U}\, r^j = \mathbb{R}^2$ (simplifying assumption). We denote $B := \{e\}$.

Further restrictions in the size of the minimal uniquely solvable system

Restriction on the number of equations of type (1)

There are two main cases : either 3 or 4 integer points have one equation corresponding to a tight representation.

Restriction on the number of equations of type (2)

- The bounded ray e is involved in either 0 or 1 or 2 linear dependence relations.
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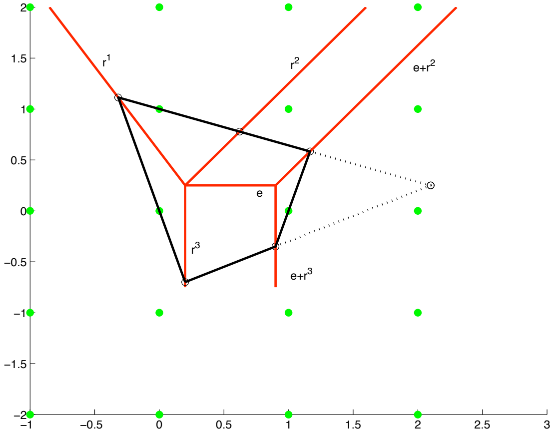
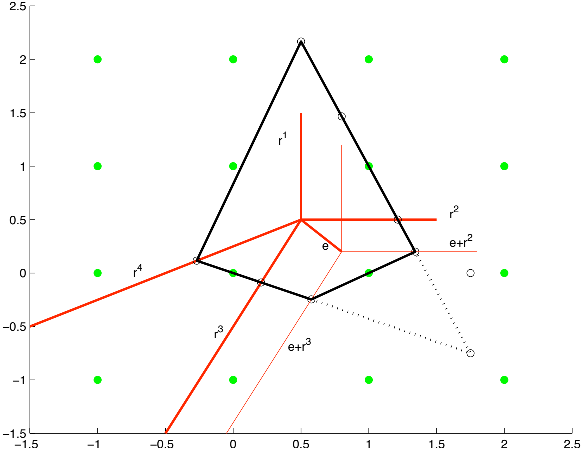
3 integer points and 1 linear dependence relation for the bounded ray

The minimal subsystem is 4×4
Variables : 3 unbounded rays and 1 bounded ray
Equations : 3 tight representations and 1 linear dependence relation



3 integer points and 2 linear dependence relation for the bounded ray

The minimal subsystem is 5×5
Variables : 4 unbounded rays and 1 bounded ray
Equations : 3 tight representations and 2 linear dependence relation



4 integer points and 0 linear dependence relation for the bounded ray

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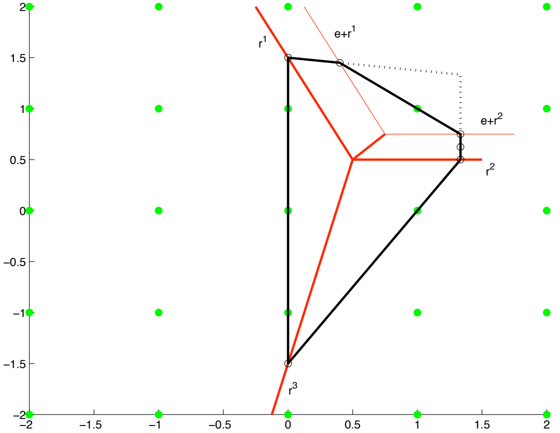
Variables : 3 unbounded rays and 1 bounded ray
Equations : 4 tight representations



L_α is now a pentagon !

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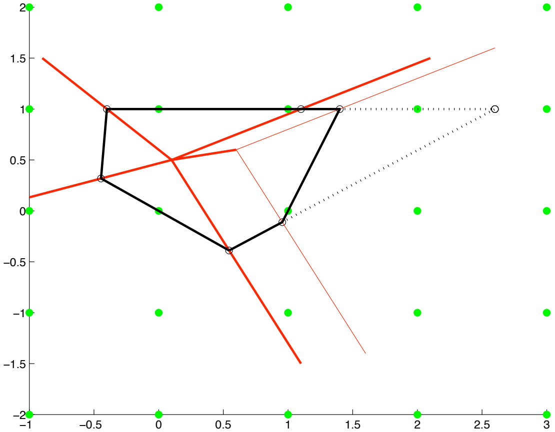

4 integer points and 2 linear dependence relation for the bounded ray

The minimal subsystem is 6×6
Variables : 5 unbounded rays and 1 bounded ray
Equations : 4 tight representations and 2 linear dependence relations