Mixed-integer sets from two rows of two adjacent simplex bases

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Abstract

In [1] we studied a mixed-integer set arising from two rows of a simplex tableau. We showed that facets of such a set can be obtained from lattice point free triangles and quadrilaterals associated with either three or four variables. In this paper we generalize our findings and show that, when upper bounds on the non-basic variables are also considered, further classes of facets arise that cannot be obtained from triangles and quadrilaterals. Specifically, when exactly one upper bound on a non-basic variable is introduced, stronger inequalities that can be derived from pentagons involving up to six variables also appear.

Keywords Mixed Integer Programming, Valid Inequalities, Two Rows, Lattice-Point-Free Polyhedra

1 Introduction

The mixed-integer set considered in this paper is given by

$$P_{\mathrm{I}} := \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j, \text{ and } s_j \leq u_j \text{ for } j \in N\},$$

where $N := \{1, 2, \ldots, n\}$, $f \in \mathbb{Q}^2$, $r^j \in \mathbb{Q}^2$ for $j \in N$ and $u_j \in \mathbb{R}_+ \cup \{+\infty\}$ for $j \in N$. We partition N into $N = B \cup U$ where U is the index set for variables s_j with $u_j = +\infty$. The set P_{LP} denotes the LP relaxation of P_I and the j^{th} unit vector in $\mathbb{R}^n$ is denoted e_j. We call the vectors $\{r^j\}_{j \in N}$ for *rays*, and we assume $r^j \neq 0$ for all $j \in N$.

Various attempts have been made to understand the polyhedral structure of $\operatorname{conv}(P_{\mathrm{I}})$. Gomory's mixed integer cuts [6], mixed integer rounding cuts [10], lift-and-project cuts and split cuts [3, 4] are all valid for $\operatorname{conv}(P_{\mathrm{I}})$. However, these classes of inequalities do not suffice to describe $\operatorname{conv}(P_{\mathrm{I}})$. The reason is

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that all these inequalities can be derived from a one-row relaxation of $\text{conv}(P_{\text{I}})$, and this is not sufficient in order to characterize all valid inequalities for $\text{conv}(P_{\text{I}})$ [4].

The polyhedron $\text{conv}(P_{\text{I}})$ was introduced in [1] in the special case when $u_j = \infty$ for all $j \in N$, and all facets of $\text{conv}(P_{\text{I}})$ were characterized geometrically: It was shown that all facets of $\text{conv}(P_{\text{I}})$ could be derived from either one-row relaxations or lattice point free triangles and quadrilaterals associated with three or four non-basic variables respectively. Cornuéjols and Margot later characterized exactly which lattice point free triangles and quadrilaterals give rise to facets of $\text{conv}(P_{\text{I}})$ [5]. Cornuéjols and Margot also related $\text{conv}(P_{\text{I}})$ to the corner polyhedron introduced by Gomory [7, 8] by associating a variable s_r with every vector $r \in \mathbb{Z}^2$, and then imposing finite support on the set of variables that are positive in a feasible solution. The key result in both papers [1, 5] is to provide a bijective map between the facets of $\text{conv}(P_{\text{I}})$ and certain two-dimensional lattice point free triangles and quadrilaterals.

In this paper we explore the geometric structure of $\text{conv}(P_{\text{I}})$ when upper bounds are present, *i.e.*, when we have $B \neq \emptyset$. It turns out that upper bounds substantially complicate the structure of $\text{conv}(P_{\text{I}})$. In the special case of exactly one upper bound on a non-basic variable, we provide a complete description of the lattice point free polygons associated with the facet defining inequalities for $\text{conv}(P_{\text{I}})$. Specifically, in this case, we show that a complete description of $\text{conv}(P_{\text{I}})$ is available if pentagons obtained from up to six variables are considered in addition to triangles and quadrilaterals. Furthermore, in the obtained inequalities, the coefficient of the bounded variable is strictly stronger. If we wanted to obtain such an inequality using a standard triangle or quadrilateral, it would contain at least an integer point in its interior. This case is however interesting, since relaxations of this type can be obtained from two adjacent bases of the LP relaxation of a mixed integer program. Such relaxations might therefore be interesting computationally. Specifically, consider two rows of a simplex tableau with basic variables $(x_1, x_2) \in \mathbb{Z}^2$ and nonbasic variables $s \in \mathbb{R}^n_+$, and assume $(x_1^{\text{B}}, x_2^{\text{B}}) \notin \mathbb{Z}^2$, where x^{B} denotes the value of x in the basic solution associated with the simplex tableau. Choose any edge in the polyhedron associated with the linear programming relaxation that connects the vertex x^{B} with a second vertex $x^{\text{B'}}$, where B and B' are adjacent bases. We can now write $(x_1^{\text{B'}}, x_2^{\text{B'}}) = (x_1^{\text{B}}, x_2^{\text{B}}) + u_i r^i$, where $(x_1, x_2) = (x_1^{\text{B}}, x_2^{\text{B}}) + \sum_{j \in N} s_j r^j$ denotes the two rows of the simplex tableau associated with B and $i \in N$. A natural mixed integer programming relaxation associated with the pivot along the edge from x^{B} to $x^{\text{B'}}$ is now the set $\{(x_1, x_2) \in \mathbb{Z}^2 : (x_1, x_2) = (x_1^{\text{B}}, x_2^{\text{B}}) + \sum_{j \in N} s_j r^j, s \geq 0 \text{ and } s_i \leq u_i\}$.

In the general case when several upper bounds are present, we have not been able to characterize all lattice point free polygons that arise from facet defining inequalities for $\text{conv}(P_{\text{I}})$. An explicit and geometric construction of these polygons is an interesting open problem for future research.

The remainder of the paper is organized as follows. In Sect. 2 we give some basic polyhedral properties of $\text{conv}(P_{\text{I}})$. In particular, we derive a general form of a facet defining inequalities for $\text{conv}(P_{\text{I}})$. Representations of integer points

$x \in \mathbb{Z}^2$ in terms of the non-basic variables are considered in Sect. 3. Finally we characterize the structure of the facets of $\text{conv}(P_\text{I})$ in Sect. 4, where we show that pentagons suffice to derive facets for $\text{conv}(P_\text{I})$ when only one upper bound is present.

2 Basic polyhedral properties of $\text{conv}(P_\text{I})$

We now describe a number of structural properties of $\text{conv}(P_\text{I})$. Many of these properties are generalizations of results in [1].

Observation 1 *The set* $\text{conv}(P_I)$ *has the following properties.*

(i) If $\dim(\text{cone}(\{r^j\}_{j\in U})) = 2$*, then the dimension of* $\text{conv}(P_I)$ *is* n*.*

(ii) The extreme rays of $\text{conv}(P_I)$ *are* (r^j, e_j) *for* $j \in U$*.*

(iii) The vertices (x, s) *of* $\text{conv}(P_I)$ *are such that the number of indices for which* $0 < s_j < u_j$ *is at most two.*

In the following, we study properties of the valid inequalities for $\text{conv}(P_\text{I})$. We are interested in *non-trivial* valid inequalities, *i.e.*, valid inequalities that are tight for at least a point $(\bar{x}, \bar{s}) \in P_\text{I}$, and inequalities that are not conic combinations of the upper and lower bounds.

Lemma 1 *Every non-trivial valid inequality for* P_I *can be written as*

$$\sum_{i\in B_-} \alpha_i(u_i - s_i) + \sum_{i\in B_+} \alpha_i s_i + \sum_{i\in U} \alpha_i s_i \geq 1,$$

where $\alpha_i \geq 0$ *for all* i *and* (B_-, B_+) *is a partitioning of* B*.*

Proof: Let $\sum_{j\in N} \alpha'' s_j \geq \beta''$ be a non-trivial valid inequality for $\text{conv}(P_\text{I})$ and let $(\bar{x}, \bar{s}) \in P_\text{I}$ be a tight feasible point. From the fact that the vectors (r^j, e_j) for $j \in U$ are the extreme rays of $\text{conv}(P_\text{I})$, we conclude that $\alpha''_j \geq 0$ for all $j \in U$ (these non-negativity constraints are explicitly part of the inequality set that defines the polar of $\text{conv}(P_\text{I})$).

Define $B_- := \{j \in B \mid \alpha''_j < 0\}$ and $B_+ := B \setminus B_-$. Also define $\alpha'_j := -\alpha''_j$ for $j \in B_-$, $\alpha'_j := \alpha''_j$ for $j \in N \setminus B_-$ and $\beta' := \beta'' - \sum_{j\in B_-} u_j\alpha''_j$. Then $\sum_{j\in B_-} \alpha'_j(u_j - s_j) + \sum_{j\in N\setminus B_-} \alpha'_j s_j \geq \beta'$ is equivalent to $\sum_{j\in N} \alpha''_j s_j \geq \beta''$, and $\alpha'_j \geq 0$ for all $j \in N$. Inserting the tight point $(\bar{x}, \bar{s})$ into this inequality, and observing that $\sum_{j\in B_-} \alpha'_j(u_j - \bar{s}_j) + \sum_{j\in N\setminus B_-} \alpha'_j\bar{s}_j \geq 0$, we can therefore not have $\beta' < 0$. Furthermore, if $\beta' = 0$, then the inequality is a trivial conic combination of the non-negativity constraints and the upper bounds which contradicts the assumption that the inequality is non-trivial. Hence, if we let $\alpha_j := \frac{\alpha'_j}{\beta'}$, we obtain the desired form. ∎

We now interpret Lemma 1 in terms of the following sets that are isomorphic to P_I. Given $B_- \subseteq B$, let $B_+ := B \setminus B_-$ and let $f(B_-) := f + \sum_{j \in B_-} u_j r^j$. The set

$$P_{\mathrm{I}}(B_-) := \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : \\ x = f(B_-) + \sum_{i \in U} r^i s_i - \sum_{j \in B_-} r^j s_j + \sum_{j \in B_+} r^j s_j \text{ and} \\ s_j \le u_j \text{ for } j \in B\} \tag{1}$$

is isomorphic to P_I. Indeed, given any $(x,s) \in P_I$, the point (x, s') with $s'_j = s_j$ for $j \in B_+ \cup U$ and $s'_j = u_j - s_j$ for $j \in B_-$ is in $P_I(B_-)$, and this mapping is one-to-one. Furthermore, from Lemma 1 it follows that an inequality $\sum_{j \in B_-} \alpha_i (u_i - s_i) + \sum_{j \in B_+} \alpha_j s_j + \sum_{i \in U} \alpha_i s_i \ge 1$ with $\alpha_i \ge 0$ for $i \in B \cup U$ is valid for P_I if and only if the inequality

$$\sum_{i \in U} \alpha_i s_i + \sum_{j \in B} \alpha_j s_j \ge 1, \tag{2}$$

is valid for $P_I(B_-)$. Since the purpose in the remainder of this paper is to study the structure of an arbitrary non-trivial facet defining inequality for P_I, we may assume without loss of generality that this inequality is of the form (2). We call valid inequalities for P_I of the form (2) in *standard form*, and we are interested in characterizing all non-trivial facet defining inequalities for $\mathrm{conv}(P_{\mathrm{I}})$ that are in standard form. Observe, however, that to obtain *all* of the valid inequalities for $\mathrm{conv}(P_{\mathrm{I}})$, *every* set $B_- \subseteq B$ must be considered. In other words, every basic feasible solution of P_{LP} needs to be examined.

We now associate a two-dimensional lattice point free polyhedron with a valid inequality $\sum_{j \in N} \alpha_j s_j \ge 1$ for $\mathrm{conv}(P_{\mathrm{I}})$ in standard form. This polyhedron gives a two-dimensional geometric representation of the facets of $\mathrm{conv}(P_I)$.

Lemma 2 *Let $\sum_{j \in N} \alpha_j s_j \ge 1$ be a valid inequality for $\mathrm{conv}(P_I)$ that is in standard form. Consider the following convex polyhedron in $\mathbb{R}^2$*

$$L_\alpha = \{x \in \mathbb{R}^2 : \text{ there exists } s \in \mathbb{R}^n \text{ s.t. } (x,s) \in P_{LP} \text{ and } \sum_{j \in N} \alpha_j s_j \le 1\}.$$

The interior of L_α does not contain any integer point.

Proof: If $\bar{x} \in \mathrm{interior}(L_\alpha)$, then there exists $\bar{s} \in \mathbb{R}^n_+$ such that $(\bar{x}, \bar{s}) \in P_{LP}$ and $\sum_{j \in N} \alpha_j \bar{s}_j < 1$. Since $\sum_{j \in N} \alpha_j s_j \ge 1$ is valid for P_I, we can not have that $\bar{x}$ is integer. ■

Example 1: Consider the set

$$P_{\mathrm{I}} = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^4_+ : 0 \le s_e \le 1 \text{ and} \\ x = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} + \begin{pmatrix} 0.7 \\ 0 \end{pmatrix} s_e + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_3\}, \tag{3}$$

and the inequality

$$\frac{7}{30}s_e + s_1 + \frac{5}{3}s_2 + 2s_3 \geq 1. \tag{4}$$

The corresponding set L_α is shown in Fig. 1. As seen from the figure, L_α does not contain integer points in its interior. Hence (4) is valid for $\operatorname{conv}(P_{\rm I})$.

Note that, conversely, the coefficients α_j for $j = 1, \ldots, 3$ can be obtained from the polygon L_α as follows: α_j is the ratio between the length of r^j and the distance between f and the intersection of $\{f + \lambda r^j : \lambda \geq 0\}$ with L_α. The coefficient α_e is obtained by the ratio of the length of r^e and the distance between f and the intersection of the dotted lines on Fig. 1. We will see in the last section how to find the coefficients from the geometry in general. ■

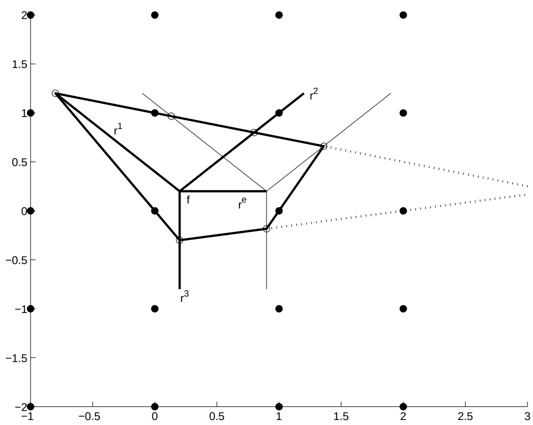


Figure 1: The set L_α for a valid inequality for $\operatorname{conv}(P_{\rm I})$

The interior of L_α gives a two-dimensional representation of the points $x \in \mathbb{R}^2$ affected by adding the inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ to the LP relaxation $P_{\rm LP}$ of $P_{\rm I}$. In other words, for any $(x, s) \in P_{\rm LP}$ satisfying $\sum_{j\in N} \alpha_j s_j < 1$, we have $x \in \operatorname{interior}(L_\alpha)$. Furthermore, for a facet defining inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ for $\operatorname{conv}(P_{\rm I})$, there exist n affinely independent points $(x^i, s^i) \in P_{\rm I}$, $i = 1, 2, \ldots, n$, such that $\sum_{j\in N} \alpha_j s^i_j = 1$. The integer points $\{x^i\}_{i\in N}$ are on the boundary of L_α, *i.e.*, they belong to the integer set

$$X_\alpha := \{x \in \mathbb{Z}^2 : \exists s \in \mathbb{R}^n_+ \text{ s.t. } (x, s) \in P_{\rm LP} \text{ and } \sum_{j\in N} \alpha_j s_j = 1\}.$$

We have $X_\alpha = L_\alpha \cap \mathbb{Z}^2$, and $X_\alpha \neq \emptyset$ whenever $\sum_{j\in N} \alpha_j s_j \geq 1$ defines a facet of $\operatorname{conv}(P_{\rm I})$.

In the remainder of the paper we only consider inequalities for which $\alpha_j > 0$ for all $j \in U$. The reason is the following result.

Lemma 3 *Any facet defining inequality* $\sum_{j\in N}\alpha_j s_j \geq 1$ *for* $\text{conv}(P_I)$ *of the form (2) with a zero coefficient on some unbounded variable is a split cut. In other words, if* $\alpha_j = 0$ *for some* $j \in U$*, then there exists* $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ *such that* $L_\alpha \subseteq \{(x_1, x_2) : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$.

Lemma 3 was proven in [1] in the case when $B = \emptyset$. This proof also applies when bounded variables are present, so we do not repeat it here. We will only sketch the main ideas. The key observation is that, if $\alpha_k = 0$ for some $k \in U$, then the line $\{f + \mu r^k : \mu \in \mathbb{R}\}$ is lattice point free and strictly contained in L_α. Hence r^k and $-r^k$ are extreme rays of L_α, and therefore L_α is of the form $\{x \in \mathbb{R}^2 : \pi_0^1 \leq (\pi')^T x \leq \pi_0^2\}$ for some $\pi' \in \mathbb{Z}^2$ and $\pi_0^1 < \pi_0^2$. Since L_α is lattice point free, this implies there exists $(\pi, \pi_0) \in \mathbb{Z}^3$ such that L_α is contained in $\{x \in \mathbb{R}^2 : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, which shows that $\sum_{j\in N}\alpha_j s_j \geq 1$ must be the split cut derived from (π, π_0).

In the following we therefore assume $\alpha_j > 0$ for all $j \in U$. Clearly $\text{conv}(X_\alpha)$ is a convex polygon with only integer vertices, and since $X_\alpha \subseteq L_\alpha$, $\text{conv}(X_\alpha)$ does not have any integer point in its interior. The following lemma shows that $\text{conv}(X_\alpha)$ can have at most four vertices.

Lemma 4 *[2, 9] Let* $P \subset \mathbb{R}^2$ *be a convex polygon with integer vertices that has no integer points in its interior.*

(i) P *has at most four vertices*

(ii) If P *has four vertices, then at least two of its four facets are parallel.*

(iii) If P *is not a triangle with integer points in the interior of all three facets, then there exist parallel lines* $\pi x = \pi_0$ *and* $\pi x = \pi_0 + 1$*,* $(\pi, \pi_0) \in \mathbb{Z}^3$*, such that* P *is contained in the corresponding split set, i.e.,* $P \subseteq \{x \in \mathbb{R}^2 : \pi_0 \leq \pi x \leq \pi_0 + 1\}$.

3 Representations of integer points

In order to characterize the geometry of the facet defining inequalities for $\text{conv}(P_\text{I})$, we exploit properties of the set of valid inequalities for $\text{conv}(P_\text{I})$. An inequality $\sum_{j\in N}\alpha_j s_j \geq \alpha_0$ is facet defining for $\text{conv}(P_\text{I})$ if and only if (α, α_0) is an extreme ray of the following polyhedral cone

$$V(P_I) = \{(\alpha, \alpha_0) \in \mathbb{R}^{n+1} : \alpha_j \geq 0, j \in U \text{ and } \sum_{j\in N}\alpha_j s_j \geq \alpha_0, s \in S^v\}, \quad (5)$$

where $S^v := \{s \in \mathbb{R}^n : \exists x \in \mathbb{Z}^2 \text{ s.t. } (x, s) \text{ is a vertex of } \text{conv}(P_\text{I})\}$.

The set $V(P_I)$, also known as the *polar* of $\text{conv}(P_I)$, describes the set of valid inequalities for $\text{conv}(P_\text{I})$. Recall that we are only interested in valid inequalities in standard form, *i.e.*, valid inequalities for $\text{conv}(P_\text{I})$ of the form $\sum_{j\in N}\alpha_j s_j \geq 1$, where $\alpha_j \geq 0$ for $j \in N$. To understand these inequalities, we investigate different representations of an integer point in terms of the variables s.

Definition 1 *Let $\sum_{j\in N}\alpha_j s_j \geq 1$ be valid for* $\operatorname{conv}(P_I)$ *and in standard form. Suppose $\bar{x} = f + \sum_{j\in N} s_j r^j$ with $0 \leq s_j \leq u_j$ for all $j \in N$.*

(a) We call s a representation *of $\bar{x}$.*

(b) The representation s of $\bar{x}$ is tight *if $\sum_{j\in N}\alpha_j s_j = 1$.*

(c) The set $T_\alpha(\bar{x}) := \{s \in \mathbb{R}^n : \bar{x} = f + \sum_{j\in N} s_j r^j, \sum_{j\in N}\alpha_j s_j = 1\}$ denotes the set of tight representations of $\bar{x}$.

(d) The representation s of $\bar{x}$ induces a partitioning of N into the sets $S_0 := \{j \in N : s_j = 0\}$, $S_u := \{j \in N : s_j = u_j\}$, $S_{strict} := \{j \in N : 0 < s_j < u_j\}$.

(e) The dimension *of s is the dimension of the set* $\operatorname{span}\{r^j : j \in S_{strict}\}$.

Example 1 (continued): Consider again the set (3)

$$P_I = \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^4_+ : 0 \leq s_e \leq 1 \text{ and}$$
$$x = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} + \begin{pmatrix} 0.7 \\ 0 \end{pmatrix} s_e + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_3\}$$

and the inequality (4) given by $\frac{7}{30}s_e + s_1 + \frac{5}{3}s_2 + 2s_3 \geq 1$.

The point $\bar{x} = (0, 1)$ is on the boundary of L_α (see Fig. 1). We have that $\bar{x}$ can be written in any of the following forms

$$\bar{x} = f \qquad +0.5\, r^1 + 0.3\, r^2, \tag{6}$$
$$\bar{x} = f + \frac{2}{7} r^e + 0.6\, r^1 + 0.2\, r^2, \tag{7}$$
$$\bar{x} = f + \ r^e + 0.9\, r^1 \qquad + 0.1\, r^3.$$

The representations $s^1 = (0, 0.5, 0.3, 0)$ and $s^2 = (\frac{2}{7}, 0.6, 0.2, 0)$ give tight representations of $\bar{x}$ with respect to inequality (4) whereas the representation $s^3 = (1, 0.9, 0, 0.1)$ does not. The partitionings of N induced by the representations s^2 and s^3 are given by the sets $S^2_{\mathrm{u}} = \emptyset$, $S^2_{\mathrm{strict}} = \{e, 1, 2\}$, $S^2_0 = \{3\}$, $S^3_{\mathrm{u}} = \{e\}$, $S^3_{\mathrm{strict}} = \{1, 3\}$, and $S^3_0 = \{2\}$. ■

The following concept will be key in the following.

Definition 2 *A subset $J \subseteq N$ has* the linear dependence property *with respect to $\alpha \in \mathbb{R}^n$ if for all $\lambda \in \mathbb{R}^{|J|}$,*

$$\sum_{j\in J}\lambda_j r^j = 0 \Rightarrow \sum_{j\in J}\lambda_j\alpha_j = 0.$$

The linear dependence property means, geometrically, that the boundary of L_α follows a straight line through the cone formed by the rays in J.

Example 1 (continued) Consider again the set (3) and the valid inequality (4). Observe that $r^2 = \frac{20}{7} r^e + r^1$, where we have

$$r^e = \begin{pmatrix} 0.7 \\ 0 \end{pmatrix}, r^1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } r^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Furthermore $\alpha_2 = \frac{20}{7}\alpha_e + \alpha_1$ since $\alpha_e = \frac{7}{30}, \alpha_1 = 1, \alpha_2 = \frac{5}{3}$. Hence $\{e, 1, 2\}$ satisfies the linear dependence property wrt. α. Observe that in Fig. 1, the border of L_α follows a straight line through the cone spanned by r^e and r^1. ∎

Lemma 5 *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet defining inequality for* $\text{conv}(P_I)$ *in standard form. Also let s be a tight representation of $x \in \mathbb{Z}^2$. Then S_{strict} satisfies the linear dependence property wrt. α.*

Proof: Since s is a tight representation of x, we have $x = f + \sum_{i \in N} r^i s_i$, and

$$\sum_{i \in N} \alpha_i s_i = 1. \tag{8}$$

Consider multipliers σ_k for $k \in S_{\text{strict}}$ (not all zero) such that $\sum_{k \in S_{\text{strict}}} \sigma_k r^k = 0$. There exists $\epsilon > 0$ such that we have the following representations of x

$$x = f + \sum_{i \in N \setminus S_{\text{strict}}} r^i s_i + \sum_{k \in S_{\text{strict}}} r^k (s_k + \sigma_k \epsilon),$$
$$x = f + \sum_{i \in N \setminus S_{\text{strict}}} r^i s_i + \sum_{k \in S_{\text{strict}}} r^k (s_k - \sigma_k \epsilon).$$

We therefore have

$$\sum_{i \in N \setminus S_{\text{strict}}} \alpha_i s_i + \sum_{k \in S_{\text{strict}}} \alpha_k (s_k + \sigma_k \epsilon) \geq 1, \tag{9}$$
$$\sum_{i \in N \setminus S_{\text{strict}}} \alpha_i s_i + \sum_{k \in S_{\text{strict}}} \alpha_k (s_k - \sigma_k \epsilon) \geq 1. \tag{10}$$

Using (8) and (9) we obtain $\sum_{k \in S_{\text{strict}}} \sigma_k \alpha_k \epsilon \geq 0$, and using (8) and (10) we obtain $-\sum_{k \in S_{\text{strict}}} \sigma_k \alpha_k \epsilon \geq 0$. It follows that $\sum_{k \in S_{\text{strict}}} \sigma_k \alpha_k = 0$. ∎

The linear dependence property can be used to create additional tight representations from a given tight representation. Specifically, given a valid inequality $\sum_{j \in N} \alpha_j s_j \geq 1$ for $\text{conv}(P_\text{I})$ that is in standard form, if $J \subseteq N$ has the linear dependence property wrt. α, if $\{\lambda_j\}_{j \in J}$ satisfies $\sum_{j \in J} \lambda_j r^j = 0$ and if $s \in \mathbb{R}^n_+$ is a tight representation of some $x \in X_\alpha$ that satisfies $0 \leq s_j + \lambda_j \leq u_j$ for all $j \in J$, then $t \in \mathbb{R}^n_+$ is also a tight representation of x, where $t_j := s_j$ for $j \in N \setminus J$ and $t_j := s_j + \lambda_j$ for $j \in J$. For example, in Fig. 2, the set $\{1, 2, 3, 4\}$ satisfies the linear dependence property wrt. α. Any representation of x that use rays from $\{1, 2, 3, 4\}$ is tight. We formalize this construction in the following lemma.

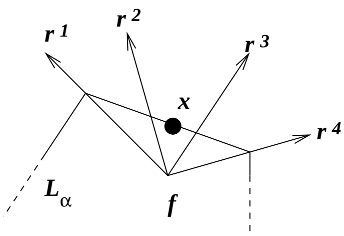


Figure 2: Example of rays that satisfy the linear dependence property

Lemma 6 *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a valid inequality for* conv(P_I) *that is in standard form. Suppose s is a tight representation of $x \in \mathbb{Z}^2$, and let t be a representation of x satisfying $S_u \subseteq T_u$ and $T_{strict} \cup (T_u \setminus S_u) \subseteq S_{strict}$. Then*

(i) t is tight.

(ii) For any $\beta \in \mathbb{R}^n$, if

(a) $\sum_{j \in N} \beta_j s_j = 1$ and

(b) S_{strict} has the linear dependence property wrt. β,

then $\sum_{j \in N} \beta_j t_j = 1$.

Proof: (i) Since s and t are representations of x we have $\sum_{j \in N} s_j r^j = \sum_{j \in N} t_j r^j$. We can rewrite this as

$$\begin{aligned}\sum_{j \in S_{\text{strict}}} s_j r^j + \sum_{j \in S_{\text{u}}} u_j r^j &= \sum_{j \in T_{\text{strict}}} t_j r^j + \sum_{j \in T_{\text{u}}} u_j r^j \\ &= \sum_{j \in T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}})} t_j r^j + \sum_{j \in S_{\text{u}}} u_j r^j.\end{aligned}$$

This implies $\sum_{j \in S_{\text{strict}}} s_j r^j = \sum_{j \in T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}})} t_j r^j$. Since $T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}}) \subseteq S_{\text{strict}}$, the linear dependence property of S_{strict} wrt. α implies $\sum_{j \in S_{\text{strict}}} \alpha_j s_j = \sum_{j \in T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}})} \alpha_j t_j$. Since s is a tight representation, t is tight as well.
(ii) We need to prove that (a) and (b) imply $\sum_{j \in N} \beta_j t_j = 1$. From what was shown in (i), we have $\sum_{j \in S_{\text{strict}}} s_j r^j = \sum_{j \in T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}})} t_j r^j$. By using (b), it follows that $\sum_{j \in S_{\text{strict}}} \beta_j s_j = \sum_{j \in T_{\text{strict}} \cup (T_{\text{u}} \setminus S_{\text{u}})} \beta_j t_j$. From (a) we now conclude that $\sum_{j \in N} \beta_j t_j = 1$. ■

Example 1 (continued) Consider again the set P_I from (3) and the valid inequality (4). We have seen earlier that $\{e, 1, 2\}$ satisfies the linear dependence property wrt. α. We also have representations s^1 and s^2 of $\bar{x} = (0, 1)$ given by (6) and (7) that use rays from $\{e, 1, 2\}$. Since s^1 is tight, it follows from Lemma 6 that s^2 is also tight. ■

Given an integer point $x \in \mathbb{Z}^2$ and a tight representation $s \in T_\alpha(x)$ of x, the set $S_{\text{strict}} \cup S_{\text{u}}$ denotes the relevant rays in this representation of x. An important question is whether the cone $\text{cone}\{r^j : j \in S_{\text{strict}} \cup S_{\text{u}}\}$ obtained from these rays cover $\mathbb{R}^2$ or not, since this shows whether two or three rays are needed to describe $\text{cone}\{r^j : j \in S_{\text{strict}} \cup S_{\text{u}}\}$.

Lemma 7 *Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a facet defining inequality for* $\text{conv}(P_I)$ *that is in standard form. Also let $s \in \mathbb{R}^n$ be a tight representation of $x \in \mathbb{Z}^2$. Then* $\text{cone}\{r^j : j \in S_{strict} \cup S_u\} \neq \mathbb{R}^2$.

Proof: If $\text{cone}\{r^j : j \in S_{\text{strict}} \cup S_{\text{u}}\} = \mathbb{R}^2$, there exists $\sigma_j > 0$ for $j \in S_{\text{strict}} \cup S_{\text{u}}$ such that $\sum_{j\in S_{\text{strict}} \cup S_{\text{u}}} \sigma_j r^j = 0$. This implies there exists $\epsilon > 0$ such that we have the following representation of x

$$x = f + \sum_{j\in S_{\text{strict}} \cup S_{\text{u}}} (s_j - \epsilon\sigma_j) r^j.$$

Hence $\sum_{j\in S_{\text{strict}} \cup S_{\text{u}}} \alpha_j (s_j - \epsilon\sigma_j) \geq 1$, and therefore $\sum_{j\in S_{\text{strict}} \cup S_{\text{u}}} \sigma_j \alpha_j \leq 0$. However, since $\alpha_j \geq 0$ and $\sigma_j > 0$ for $j \in S_{\text{strict}} \cup S_{\text{u}}$, this means $\alpha_j = 0$ for $j \in S_{\text{strict}} \cup S_{\text{u}}$, and this contradicts that s is a tight representation of x. ■

For any $x \in \mathbb{Z}^2$ and valid inequality $\sum_{i\in N} \alpha_i s_i \geq 1$ for $\text{conv}(P_{\text{I}})$ in standard form, $T_\alpha(x)$ is a polyhedron. If $T_\alpha(x)$ is not full dimensional, there exists $\mathcal{S}^\alpha_{\text{u}}(x) \subseteq B$ and $\mathcal{S}^\alpha_0(x) \subseteq N$ s.t. $s_j = u_j$ and $s_k = 0$ for all $s \in T_\alpha(x)$, $j \in \mathcal{S}^\alpha_{\text{u}}(x)$ and $k \in \mathcal{S}^\alpha_0(x)$. Furthermore, $T_\alpha(x)$ has an inner point, *i.e.*, a point $\bar{s} \in T_\alpha(x)$ that satisfies $0 < \bar{s}_j < u_j$ for all $j \in N \setminus (\mathcal{S}^\alpha_0(x) \cup \mathcal{S}^\alpha_{\text{u}}(x))$.

Definition 3 *Let $\sum_{i\in N} \alpha_i s_i \geq 1$ be valid for* $\text{conv}(P_I)$ *and in standard form, and let $x \in \mathbb{Z}^2$ satisfy $T_\alpha(x) \neq \emptyset$. Define the sets*

$$\mathcal{S}^\alpha_0(x) := \bigcap_{s\in T_\alpha(x)} S_0, \quad \mathcal{S}^\alpha_{strict}(x) := \bigcup_{s\in T_\alpha(x)} S_{strict} \ \text{ and } \ \mathcal{S}^\alpha_u(x) := \bigcap_{s\in T_\alpha(x)} S_u$$

of coordinates on lower bound in all *tight representations of x, coordinates between bounds in* some *tight representation of x, and coordinates on upper bound in* all *tight representations of x respectively.*

In the following, $s^* \in T_\alpha(x)$ denotes a representation of x satisfying

$$S^*_0 = \mathcal{S}^\alpha_0(x), \qquad S^*_{\text{strict}} = \mathcal{S}^\alpha_{\text{strict}}(x) \qquad \text{and} \qquad S^*_{\text{u}} = \mathcal{S}^\alpha_{\text{u}}(x). \tag{11}$$

The coefficients $\alpha \in \mathbb{R}^n$ in a facet defining inequality $\sum_{i\in N} \alpha_i s_i \geq 1$ for $\text{conv}(P_{\text{I}})$ in standard form is the unique solution to following equality system in variables β.

$$\sum_{j\in N} \beta_j s_j = 1 \quad \text{for all } x \in X_\alpha \text{ and } s \in T_\alpha(x) \tag{12}$$

It is clear that the system (12) contains many redundant equalities. In the remainder of this section, we construct a system equivalent to (12) that is significantly smaller.

Lemma 8 *Let* $\sum_{i\in N}\alpha_i s_i \geq 1$ *be valid for* $\operatorname{conv}(P_I)$ *and in standard form. Suppose* $\{x^k\}_{k=1}^3 \subset \mathbb{Z}^2$ *satisfy* $x^3 = \lambda x^1 + (1-\lambda)x^2$*, where* $x^1 \neq x^2$ *and* $\lambda \in]0,1[$*. If there exists a tight 2D representation (see Definition 1.(e)) of* either x^1 or x^2*, then the following equalities in variables* β

$$\sum_{j\in N}\beta_j s_j = 1 \quad \text{for all } s \in T_\alpha(x^3) \tag{13}$$

are implied by the following equalities in variables β

$$\sum_{j\in N}\beta_j s_j = 1 \quad \text{for all } s \in T_\alpha(x^1)\cup T_\alpha(x^2). \tag{14}$$

Proof: Suppose $\{x^k\}_{k=1}^3 \subset \mathbb{Z}^2$ satisfy $x^3 = \lambda x^1 + (1-\lambda)x^2$, where $x^1 \neq x^2$ and $\lambda \in]0,1[$. Let $\bar{\beta} \in \mathbb{R}^n$ satisfy (14), and let $\bar{t} \in T_\alpha(x^3)$ be an arbitrary tight representation of x^3. We will show that $\sum_{j\in N}\bar{\beta}_j\bar{t}_j = 1$.

Let s^* and t^* be tight representations of x^1 and x^2 that satisfy (11) for x^1 and x^2 respectively. Observe that $\bar{s} := \lambda s^* + (1-\lambda)t^*$ is a tight representation of x^3 satisfying

$$\bar{S}_0 = \bigcap_{s\in T_\alpha(x^1)\cup T_\alpha(x^2)} S_0 \qquad \text{and} \qquad \bar{S}_u = \bigcap_{s\in T_\alpha(x^1)\cup T_\alpha(x^2)} S_u. \tag{15}$$

In other words, the representation $\bar{s}$ of x^3 only has a coordinate which is on lower bound (on upper bound) if *all* representations of x^1 and x^2 are on lower bound (on upper bound) on this coordinate.

To finish the proof we show that the representations $\bar{s}$ and $\bar{t}$ of x^3 satisfy

(i) $\bar{S}_{\text{strict}}$ satisfies the linear dependence property wrt. $\bar{\beta}$.

(ii) $\bar{S}_{\text{u}} \subseteq \bar{T}_{\text{u}}$ and

(iii) $\bar{T}_{\text{strict}} \cup (\bar{T}_{\text{u}} \setminus \bar{S}_{\text{u}}) \subseteq \bar{S}_{\text{strict}}$

Lemma 6.(ii) then shows $\sum_{j\in N}\bar{\beta}_j\bar{t}_j = 1$ which proves the lemma.

By assumption either x^1 or x^2 have a tight 2D representation. Without loss of generality suppose x^1 has a tight 2D representation. Let s^1 be a tight 2D representation of x^1, and let s^2 be an arbitrary tight representation of x^2

$$x^1 = f + \sum_{j\in N} s^1_j r^j \text{ and } x^2 = f + \sum_{j\in N} s^2_j r^j.$$

Define the tight representation $\bar{z} := \frac{\lambda}{2}s^1 + \frac{1-\lambda}{2}s^2 + \frac{1}{2}\bar{t}$ of x^3. Observe that, if $j \in \bar{S}_0$, then (15) shows $s_j = 0$ for every representation $s \in T_\alpha(x^1)\cup T_\alpha(x^2)$, and therefore $s^1_j = s^2_j = 0$. Similarly if $j \in \bar{S}_{\text{u}}$, then $s^1_j = s^2_j = u_j$.

Since s^1 is a 2D representation of x^1, there exists linearly independent vectors r^{l_1} and r^{l_2} with $l_1, l_2 \in S^1_{\text{strict}}$. Therefore, for every $j \in N$, there exists $\sigma^j_{l_1}, \sigma^j_{l_2} \in \mathbb{R}$ such that $r^j = \sigma^j_{l_1} r^{l_1} + \sigma^j_{l_2} r^{l_2}$. We can now prove (i)-(iii).

(i) Let $\lambda' \in \mathbb{R}^n$ satisfy $\sum_{j \in \bar{S}_{\text{strict}}} \lambda'_j r^j = 0$. Since $\bar{s} = \lambda s^* + (1-\lambda)t^*$ with $s^* \in T_\alpha(x^1)$ and $t^* \in T_\alpha(x^2)$, and $\bar{\beta}$ satisfies (14), we have $\sum_{j \in N} \bar{\beta}_j s^*_j = 1$ and $\sum_{j \in N} \bar{\beta}_j t^*_j = 1$. Now partition the set $\{j \in \bar{S}_{\text{strict}} : \lambda'_j \neq 0\}$ into the sets $S^+ := \{j \in \bar{S}_{\text{strict}} : \lambda'_j > 0, s^*_j < u_j\}$, $S^- := \{j \in \bar{S}_{\text{strict}} : \lambda'_j < 0, s^*_j > 0\}$, $T^+ := \{j \in \bar{S}_{\text{strict}} : \lambda'_j > 0, j \notin S^+\}$ and $T^- := \{j \in \bar{S}_{\text{strict}} : \lambda'_j < 0, j \notin S^-\}$. This implies there exists $\epsilon > 0$ s.t. $x^1 = f + \sum_{j \in N} \bar{s}^*_j r^j + \epsilon \sum_{j \in S^+ \cup S^-} \lambda'_j r^j$ and $x^2 = f + \sum_{j \in N} \bar{t}^*_j r^j + \epsilon \sum_{j \in T^+ \cup T^-} \lambda'_j r^j$ are tight representations of x^1 and x^2 respectively. Hence from (14) we have $\sum_{j \in N} \bar{\beta}_j \bar{s}^*_j + \epsilon \sum_{j \in S^+ \cup S^-} \bar{\beta}_j \lambda'_j = 1$ and $\sum_{j \in N} \bar{\beta}_j \bar{t}^*_j + \epsilon \sum_{j \in T^+ \cup T^-} \bar{\beta}_j \lambda'_j = 1$, which implies $\sum_{j \in \bar{S}_{\text{strict}}} \bar{\beta} \lambda'_j = 0$.

(ii) Suppose, for a contradiction, that $\bar{j} \in \bar{S}_{\text{u}}$ and $\bar{j} \notin \bar{T}_{\text{u}}$. Since $\bar{j} \in \bar{S}_{\text{u}}$, (15) shows $s_{\bar{j}} = u_{\bar{j}}$ for all tight representations $s \in T_\alpha(x^1) \cup T_\alpha(x^2)$ of x^1 and x^2. Hence $\bar{j} \in S^1_{\text{u}} \cap S^2_{\text{u}}$, and therefore $\bar{j} \notin \bar{T}_{\text{u}}$ implies $\bar{j} \in \bar{Z}_{\text{strict}}$. We have $\bar{j} \neq l_1$ and $\bar{j} \neq l_2$ since $\bar{j} \notin S^1_{\text{strict}}$ and $l_1, l_2 \in S^1_{\text{strict}}$. Furthermore, since $\{l_1, l_2, \bar{j}\} \subseteq \bar{Z}_{\text{strict}}$, $\{l_1, l_2, \bar{j}\}$ satisfies the linear dependence property wrt. α. Hence there exists $\epsilon > 0$ such that $x^1 = f + \sum_{j \in N} s^1_j r^j + \epsilon(\sigma^{\bar{j}}_{l_1} r^{l_1} + \sigma^{\bar{j}}_{l_2} r^{l_2} - r^{\bar{j}})$ is a tight representation of x^1. However, this contradicts that $s_{\bar{j}} = u_{\bar{j}}$ for all tight representations s of x^1.

(iii) Observe that (ii) implies $\bar{T}_{\text{strict}} \cap \bar{S}_{\text{u}} = \emptyset$: If $\bar{j} \in \bar{S}_{\text{u}}$, then (ii) implies $\bar{j} \in \bar{T}_{\text{u}}$, and therefore $\bar{j} \notin \bar{T}_{\text{strict}}$. Hence, to show (iii), it suffices to show $\bar{T}_{\text{u}} \cup \bar{T}_{\text{strict}} \subseteq \bar{S}_{\text{u}} \cup \bar{S}_{\text{strict}}$. Suppose for a contradiction that $\bar{j} \in \bar{T}_{\text{u}} \cup \bar{T}_{\text{strict}}$ and $\bar{j} \in \bar{S}_0$. Since $\bar{j} \in \bar{S}_0$, (15) shows that $s_{\bar{j}} = 0$ for every tight representation $s \in T_\alpha(x^1) \cup T_\alpha(x^2)$ of x^1 and x^2. We therefore have $s^1_{\bar{j}} = s^2_{\bar{j}} = 0$. Since $\bar{j} \in \bar{T}_{\text{u}} \cup \bar{T}_{\text{strict}}$, this implies $\bar{j} \in \bar{Z}_{\text{strict}}$. Furthermore we have $\bar{j} \neq l_1$ and $\bar{j} \neq l_2$ since $\bar{j} \notin S^1_{\text{strict}}$ and $l_1, l_2 \in S^1_{\text{strict}}$. Since $\{l_1, l_2, \bar{j}\} \subseteq \bar{Z}_{\text{strict}}$, Lemma 5 shows $\{l_1, l_2, \bar{j}\}$ satisfies the linear dependence property wrt. α. Hence there exists $\epsilon > 0$ such that $x^1 = f + \sum_{i \in N} s^1_i r^i + \epsilon(r^{\bar{j}} - \sigma^{\bar{j}}_{l_1} r^{l_1} - \sigma^{\bar{j}}_{l_2} r^{l_2})$ is a tight representation of x^1. However, this contradicts that $s_{\bar{j}} = 0$ for all tight representations s of x^1. ∎

We now identify "important rays".

Definition 4 *Given $x^i \in X_\alpha \cap \mathbb{Z}^2$, define*

(i) $I^U(x^i)$ to be s.t. $\text{cone}(\{r^j\}_{j \in I^U(x^i)}) = \text{cone}(\{r^j\}_{j \in \mathcal{S}^\alpha_{strict}(x^i) \cap U})$,

(ii) $I(x^i)$ to be $I^U(x^i)$ if $|I^U(x^i)| = 2$. Otherwise $I(x^i)$ is defined to be s.t. $\text{cone}(\{r^j\}_{j \in I(x^i)}) = \text{cone}(\{r^j\}_{j \in \mathcal{S}^\alpha_{strict}(x^i)})$.

Observe that Lemma 7 implies that the cardinality of these two sets is at most 2. Using this notation, we now reformulate system (12) as follows.

Lemma 9 *Let $\sum_{i \in N} \alpha_i s_i \geq 1$ be valid for* $\text{conv}(P_I)$ *and in standard form, let $\bar{x} \in \mathbb{Z}^2$, and suppose $\bar{x}$ has a tight 2D representation. Also, given $j \in \mathcal{S}^\alpha_{strict}(\bar{x}) \cap U$ and $k \in I^U(\bar{x})$, let $\sigma^j_k \geq 0$ be defined such that $r^j = \sum_{k \in I^U(\bar{x})} \sigma^j_k r^k$. Finally, given $j \in \mathcal{S}^\alpha_{strict}(\bar{x}) \cap B$ and $k \in I(\bar{x})$, let σ^j_k be defined such that*

$r^j = \sum_{k \in I(\bar{x})} \sigma^j_k r^k$. *The two linear systems in unknowns* β

$$\sum_{k \in N} \beta_k s_k = 1 \qquad \textit{for all } s \in T_\alpha(\bar{x}) \tag{16}$$

and

$$\sum_{k \in N} \beta_k \bar{t}_k = 1, \tag{17}$$

$$\beta_j = \sum_{k \in I^U(\bar{x})} \sigma^j_k \beta_k \qquad \textit{for all } j \in (\mathcal{S}^\alpha_{strict}(\bar{x}) \cap U) \setminus I^U(\bar{x}), \tag{18}$$

$$\beta_j = \sum_{k \in I(\bar{x})} \sigma^j_k \beta_k \qquad \textit{for all } j \in (\mathcal{S}^\alpha_{strict}(\bar{x}) \cap B) \setminus I(\bar{x}) \tag{19}$$

have the same solution set.

Proof: We call the solution sets of (16) and (17)-(19) for X_1 and X_2 respectively. Let $\bar{t}$ be the representation of $\bar{x}$ that satisfies (11).

We first prove $X_1 \subseteq X_2$. Therefore suppose $\beta \in X_1$. We prove that (18) holds. The proof for (19) is similar. Consider $l \in (\mathcal{S}^\alpha_{\text{strict}}(\bar{x}) \cap U) \setminus I^U(\bar{x})$. Observe that, since $I^U(\bar{x}) \subseteq \mathcal{S}^\alpha_{\text{strict}}(\bar{x})$, we have $I^U(\bar{x}) \cup \{l\} \subseteq \bar{T}_{\text{strict}}$ and therefore Lemma 5 shows that $I^U(\bar{x}) \cup \{l\}$ satisfies the linear dependence property wrt. α. Hence $\alpha_l = \sum_{i \in I^U(\bar{x})} \alpha_i \sigma^l_i$. We need to show $\beta_l = \sum_{i \in I^U(\bar{x})} \beta_i \sigma^l_i$. Since $I^U(\bar{x}) \cup \{l\}$ satisfies the linear dependence property wrt. α, there exists $\epsilon > 0$ s.t. the representation $\bar{x} = f + \sum_{j \in N} \bar{t}_j r^j + \epsilon(r^l - \sum_{i \in I^U(\bar{x})} \sigma^l_i r^i)$ of $\bar{x}$ is tight. Since there is an equality of (16) for every tight representation of $\bar{x}$, we have $\sum_{j \in N} \beta_j \bar{t}_j + \epsilon(\beta_l - \sum_{i \in I^U(\bar{x})} \beta_i \sigma^l_i) = 1$. Therefore, since $\sum_{j \in N} \beta_j \bar{t}_j = 1$, we have $\beta_l - \sum_{i \in I^U(\bar{x})} \beta_i \sigma^l_i = 0$ which shows $\beta \in X_2$.

We now prove that $X_2 \subseteq X_1$. Let $\beta \in X_2$ and $\bar{w} \in T_\alpha(\bar{x})$ be arbitrary. We must prove $\sum_{l \in N} \beta_l \bar{w}_l = 1$. Define

$$X^u := \bar{T}_{\text{u}} \cap \bar{W}_{\text{u}}, \qquad Y := (\bar{T}_u \setminus X^u) \cup \bar{T}_{\text{strict}} \qquad Z := (\bar{W}_u \setminus X^u) \cup \bar{W}_{\text{strict}}.$$

Observe that X^u and Y form a partitioning of $\bar{T}_{\text{strict}} \cup \bar{T}_{\text{u}}$, and that X^u and Z form a partitioning of $\bar{W}_{\text{strict}} \cup \bar{W}_{\text{u}}$. Also observe that, since $\bar{T}_{\text{u}} = \bigcap_{s \in T_\alpha(\bar{x})} S_{\text{u}}$, and $\bar{w} \in T_\alpha(\bar{x})$, we have $X^u = \bar{T}_{\text{u}}$ and $Y = \bar{T}_{\text{strict}}$. We may write

$$\bar{x} = f + \sum_{i \in Y} \bar{t}_i r^i + \sum_{j \in X^u} u_j r^j \quad \text{and} \tag{20}$$

$$\bar{x} = f + \sum_{i \in Z} \bar{w}_i r^i + \sum_{j \in X^u} u_j r^j. \tag{21}$$

Both $\bar{t}$ and $\bar{w}$ are tight representations of $\bar{x}$ wrt. $\sum_{i \in N} \alpha_i s_i \geq 1$. Therefore, since $\mathcal{S}^\alpha_{\text{strict}}(\bar{x})$ consists of those coordinates that are between bounds in *some* tight representation of $\bar{x}$ wrt. $\sum_{i \in N} \alpha_i s_i \geq 1$, we have $\bar{T}_{\text{strict}} \cup \bar{W}_{\text{strict}} \subseteq \mathcal{S}^\alpha_{\text{strict}}(\bar{x})$. Also, for $l \in \bar{W}_u \setminus X^u$, we have $0 < \bar{t}_l < u_l$ and $\bar{w}_l = u_l$, and therefore the tight representation $\frac{1}{2}\bar{t}+\frac{1}{2}\bar{w}$ of $\bar{x}$ satisfies $0<\frac{1}{2}\bar{t}_l+\frac{1}{2}\bar{w}_l<u_l$. It follows that $\bar{W}_u\setminus X^u\subseteq\mathcal{S}^\alpha_{\text{strict}}(\bar{x})$, and therefore $Z\subseteq\mathcal{S}^\alpha_{\text{strict}}(\bar{x})$. Assume that $I(\bar{x})=I^U(\bar{x})$ (otherwise the proof applies with $I(\bar{x})$). From (18) it follows that

$$\beta_l=\sum_{i\in I^U(\bar{x})}\sigma^l_i\beta_i \qquad l\in(Y\cup Z)\setminus I^U(\bar{x}). \tag{22}$$

For $i\in I^U(\bar{x})$, define

$$\bar{t}^i_Y:=\bar{t}_i+\sum_{l\in Y\setminus I^U(\bar{x})}\sigma^l_i\bar{t}_l \qquad \text{and} \qquad \bar{w}^i_Z:=\bar{w}_i+\sum_{l\in Z\setminus I^U(\bar{x})}\sigma^l_i\bar{w}_l.$$

Adding $\sum_{l\in Y\setminus I^U(\bar{x})}\bar{t}_l(-r^l+\sum_{i\in I^U(\bar{x})}\sigma^l_ir^i)(=0)$ to the right-hand-side of (20), and adding $\sum_{l\in Z\setminus I^U(\bar{x})}\bar{w}_l(-r^l+\sum_{i\in I^U(\bar{x})}\sigma^l_ir^i)(=0)$ to the right-hand-side of (21), we obtain

$$\bar{x}=f+\sum_{i\in I^U(\bar{x})}\bar{t}^i_Yr^i+\sum_{i\in X^u}u_ir^i \tag{23}$$

$$\bar{x}=f+\sum_{i\in I^U(\bar{x})\cap Z}\bar{w}^i_Zr^i+\sum_{i\in I^U(\bar{x})\setminus Z}(\bar{w}^i_Z-\bar{w}_i)r^i+\sum_{i\in X^u}u_ir^i \tag{24}$$

Both (23) and (24) give an expression for the vector $\bar{x}-f-\sum_{i\in X^U}u_ir^i$ as an element of $\text{cone}(\{r^i\}_{i\in I^U(\bar{x})})$. Since the non-negative numbers involved in this expression are unique, we have $\bar{t}^i_Y=\bar{w}^i_Z$ for $i\in I^U(\bar{x})\cap Z$ and $\bar{t}^i_Y=\bar{w}^i_Z-\bar{w}_i$ for $i\in I^U(\bar{x})\setminus Z$. We may now write, denoting $I:=I^U(\bar{x})$,

$$\begin{aligned}
1&=\sum_{i\in Y}\bar{t}_i\beta_i+\sum_{i\in X^u}u_i\beta_i\\
&=\sum_{i\in Y}\bar{t}_i\beta_i+\sum_{i\in X^u}u_i\beta_i+\sum_{l\in Y\setminus I}\bar{t}_l(-\beta_l+\sum_{i\in I}\sigma^l_i\beta_i))\\
&=\sum_{i\in I}\beta_i\bar{t}^i_Y+\sum_{i\in X^u}u_i\beta_i\\
&=\sum_{i\in I\cap Z}\beta_i\bar{w}^i_Z+\sum_{i\in I\setminus Z}\beta_i(\bar{w}^i_Z-\bar{w}_i)+\sum_{i\in X^u}u_i\beta_i-\sum_{l\in Z\setminus I}\bar{w}_l(-\beta_l+\sum_{i\in I}\sigma^l_i\beta_i))\\
&=\sum_{i\in N}\bar{w}_i\beta_i+\sum_{i\in I\cap Z}\beta_i(\bar{w}^i_Z-\bar{w}_i)+\sum_{i\in I\setminus Z}\beta_i(\bar{w}^i_Z-\bar{w}_i)-\sum_{i\in I}\beta_i(\bar{w}^i_Z-\bar{w}_i)\\
&=\sum_{i\in N}\bar{w}_i\beta_i.
\end{aligned}$$

■

The next step is to consider possible interactions between the sets $I^U(x^i)$ and $I(x^i)$ for different vertices of $\text{conv}(X_\alpha)$.

Observation 2 *Let* $\sum_{j\in N}\alpha_j s_j \geq 1$ *be facet defining for* $\mathrm{conv}(P_I)$ *and in standard form. If* $\mathcal{S}^{\alpha}_{strict}(x^2) \subseteq \mathcal{S}^{\alpha}_{strict}(x^1)$*, then the equalities*

$$\beta_j = \sum_{i\in I(x^2)} \sigma_i^j \beta_i \qquad \textit{for all } j \in (\mathcal{S}^{\alpha}_{strict}(x^2)\cap B)\setminus I(x^2),$$

are implied by the equalities

$$\beta_j = \sum_{i\in I(x^1)} \sigma_i^j \beta_i \qquad \textit{for all } j \in (\mathcal{S}^{\alpha}_{strict}(x^1)\cap B)\setminus I(x^1).$$

If $(\mathcal{S}^{\alpha}_{strict}(x^2)\cap U) \subseteq (\mathcal{S}^{\alpha}_{strict}(x^1)\cap U)$*, then the equalities*

$$\beta_j = \sum_{i\in I^U(x^2)} \sigma_i^j \beta_i \qquad \textit{for all } j \in (\mathcal{S}^{\alpha}_{strict}(x^2)\cap U)\setminus I^U(x^2),$$

are implied by the equalities

$$\beta_j = \sum_{i\in I^U(x^1)} \sigma_i^j \beta_i \qquad \textit{for all } j \in (\mathcal{S}^{\alpha}_{strict}(x^1)\cap U)\setminus I^U(x^1).$$

Lemma 10 *Let* $\sum_{j\in N}\alpha_j s_j \geq 1$ *be facet defining for* $\mathrm{conv}(P_I)$ *and in standard form. If* $j\in U$ *satisfies* $j \in \mathcal{S}^{\alpha}_{strict}(x^1)\setminus I^U(x^1)$*, then* $j \notin I^U(x^2)$.

Proof: Assume, for a contradiction, that $j\in U$, $j \in \mathcal{S}^{\alpha}_{\mathrm{strict}}(x^1)\setminus I^U(x^1)$ and $j \in I^U(x^2)$. Let s^* be a tight representation of x^2 satisfying (11), and let $k,l \in I^U(x^1)$, which by definition means $k,l\in U$. Lemma 5 implies that there exist $\sigma_k,\sigma_l \geq 0$ satisfying $r^j = \sigma_k r^k + \sigma_l r^l$ and $\alpha_j = \sigma_k r^k + \sigma_l \alpha_l$. Hence there exists $\epsilon > 0$ such that $x^2 = f + \sum_{i\in N} s_i^* r^i + \epsilon(-r^j + \sigma_k r^k + \sigma_l r^l)$ is a valid and tight representation of x^2. This implies $k,l \in \mathcal{S}^{\alpha}_{\mathrm{strict}}(x^2)$, which is a contradiction since $j\in I^U(x^2)$ and $r^j \in \mathrm{cone}\{r^k, r^l\}$. ■

We can now prove the main result of this section, namely that for a facet $\sum_{j\in N}\alpha_j s_j \geq 1$ of $\mathrm{conv}(P_I)$ in standard form, α is the unique solution of a system consisting of one equation expressing the tightness of each *vertex* of $\mathrm{conv}(X_\alpha)$, at most one linear dependence property for each unbounded ray and possibly some linear dependence properties for bounded rays.

Theorem 1 *Consider the set* (5)*. Let* X^v *denote the set of vertices of* $\mathrm{conv}(X_\alpha)$*. If* α *is an extreme point of* (5)*, then* α *is the unique solution to the following system in variables* β

$$\sum_{j=1}^{n} \beta_j s_j^x = 1 \qquad \textit{for all } x\in X^v, \tag{25}$$

$$\beta_j = \sum_{k\in I(x)} \sigma_j^k \beta_k \qquad \textit{for all } x\in X_\alpha \textit{ and } j \in (\mathcal{S}^{\alpha}_{strict}(x)\cap B)\setminus I(x), \tag{26}$$

$$\beta_j = \sum_{k\in I^U(x)} \sigma_j^k \beta_k \qquad \textit{for all } x\in X_\alpha \textit{ and } j \in (\mathcal{S}^{\alpha}_{strict}(x)\cap U)\setminus I^U(x), \tag{27}$$

where $s^x \in T_\alpha(x)$ *satisfies* $s^x_i > 0$ *for all* $i \in I^U(x)$ *and* $s^x_j = 0$ *for all* $j \in U \setminus I^U(x)$. *Furthermore, the system* (25)-(26) *is also uniquely solvable.*

Proof: Consider the set (5). An extreme point satisfies a subset of the inequalities with equality. These we call the *tightness equalities*. Using Lemma 8, we know that we only need to consider equations corresponding to *vertices* of $\text{conv}(X_\alpha)$ - except for possibly points $x^3 \in X_\alpha$ that are true convex combinations of vertices x^1 and x^2 of $\text{conv}(X_\alpha)$ that do not have 2D representations. We will deal with the latter case later in the proof. Then using Lemma 9, we can write the system of tightness equations equivalently as a system consisting of one tightness equation (25) per vertex x of $\text{conv}(X_\alpha)$, and equations of type (26)-(27). Observe that, $\bar{s}^x_i$ may be nonzero for $i \in U \setminus I^U(x)$. However we can create a new tight representation (using $r^i = \sum_{k \in I^U(x)} \sigma^k_i r^k$ together with $\alpha_i = \sum_{k \in I^U(x)} \sigma^k_i \alpha_k$) that satisfies $s^x_i = 0$ for $i \in U \setminus I^U(x)$.

We now classify the unbounded rays in two sets. Let

$$U_1 := \{i \in U \mid \text{ there exists a vertex } x \text{ of } \text{conv}(X_\alpha) \text{ with } i \in I^U(x)\},$$
$$U_2 := \{i \in U \mid \text{ there exists a vertex } x \text{ of } \text{conv}(X_\alpha) \text{ with } i \in \mathcal{S}^\alpha_{\text{strict}}(x) \setminus I^U(x)\}.$$

In Lemma 10 we have proved that $U_1 \cap U_2 = \emptyset$. Observe that if $i \in U_2$, then β_i only appears once in the equations (27). Deleting these equalities from (27) therefore leaves a system that remains uniquely solvable.

It remains to check the case when there exist $x^1, x^2, x^3 \in X_\alpha$ and $\lambda \in]0,1[$ such that $x^3 = \lambda x^1 + (1-\lambda)x^2$ and neither x^1 nor x^2 admit a 2D representation. This implies that x^1 and x^2 admit exactly one tight representation, say $T_\alpha(x^1) = \{s^1\}$ and $T_\alpha(x^2) = \{s^2\}$. Consider a tight representation $s^3 \in T_\alpha(x^3)$ of x^3. If $S^3_{strict} \cup S^3_u \subseteq S^1_{strict} \cup S^1_u \cup S^2_{strict} \cup S^2_u$, the equation derived from the tightness of the representation s^3 is a convex combination of those obtained from s^1 and s^2 (see the proof of Lemma 8). Assume now there exists $J \subseteq (S^3_{strict} \cup S^3_u)$, $J \cap (S^1_{strict} \cup S^1_u \cup S^2_{strict} \cup S^2_u) = \emptyset$ with $|J| \geq 1$. Now consider the tight representation $t^3 := \frac{\lambda}{2}s^1 + \frac{1-\lambda}{2}s^2 + \frac{1}{2}s^3$ of x^3. Let $K \subseteq N$ denote those coordinates of s^1, s^2 and s^3 for which not all representations have identical values, *i.e.*, $s^1_i = s^2_i = s^3_i$ for $i \notin K$. Observe that s^1 and s^2 must differ on at least two coordinates, since otherwise we would have either $\sum_{i \in N} \alpha_i s^1_i \neq 1$ or $\sum_{i \in N} \alpha_i s^2_i \neq 1$. Furthermore, since $|J| \geq 1$, we have $|K| \geq 3$. Also note that $K \subseteq T^3_{strict}$, and therefore K satisfies the linear dependence property wrt. α. We now consider $\tau := s^3 - \lambda s^1 - (1-\lambda)s^2$ which satisfies $0 = \sum_{j \in K} \tau_j r^j$. Observe that the equation $0 = \sum_{j \in K} \tau_j \beta_j$ must be a linear combination of equations of type (26)-(27). It follows that the equation $\sum_{j \in N} s^3_j \beta_j = 1$ is the linear combination of equations of type (26)-(27) obtained by $\lambda(\sum_{j \in N} s^1_j \beta_j = 1) + (1-\lambda)(\sum_{j \in N} s^2_j \beta_j = 1)$. ■

4 A characterization of the facets of $\text{conv}(P_\mathbf{I})$

In this section we focus on the set L_α. We assume $\alpha_j > 0$ for all $j \in U$. Due to the direct correspondence between the set L_α and a facet defining inequality

$\sum_{j\in N}\alpha_j s_j \geq 1$ for $\text{conv}(P_\text{I})$ in standard form, this gives a characterization of the facets of $\text{conv}(P_\text{I})$. We first provide some general results on the structure of L_α in Sect. 4.1. We then review the main results in [1] in the case where $B=\emptyset$ in Sect. 4.2. Finally, in Sect. 4.3, we characterize L_α when exactly one upper bound is present, *i.e.*, when $|B|=1$. The presence of an upper bound might seem to be only a minor extension. However, as we will demonstrate later, the addition of an upper bound on a variable substantially complicates the geometry of L_α. Indeed, whereas L_α is either a triangle or a quadrilateral when no upper bounds are present, L_α can also be a pentagon when an upper bound on a variable is present. The following theorem was proved in [1].

Theorem 2 *Suppose $B=\emptyset$. Let $\sum_{j\in N}\alpha_j s_j \geq 1$ be a facet defining inequality for* $\text{conv}(P_I)$ *that satisfies $\alpha_j>0$ for all $j\in N$. Then L_α is a polygon with at most four vertices.*

Theorem 2 shows that there exists a set $S\subseteq N$ such that $|S|\leq 4$ and $\sum_{j\in S}\alpha_j s_j\geq 1$ is facet defining for $\text{conv}(P_\text{I}(S))$, where

$$P_\text{I}(S) := \{(x,s)\in\mathbb{Z}^2\times\mathbb{R}^{|S|}_+ : x = f + \sum_{j\in S} s_j r^j\}.$$

The main theorem in this section is the following.

Theorem 3 *Suppose $|B|=1$. Let $\sum_{j\in N}\alpha_j s_j\geq 1$ be a facet defining inequality in standard form for* $\text{conv}(P_I)$ *that satisfies $\alpha_j>0$ for all $j\in U$. Then L_α is a polygon with at most five vertices.*

Throughout this section we assume that no two rays point in the same direction.

4.1 General geometric statements about L_α

The set L_α is the projection of a polyhedron onto the 2-dimensional plane. It is therefore a polygon. First we characterize all points that are candidates for being vertices of the polygon.

Assumption 1 *All upper bounds u_j for $j\in B$ are equal to one, i.e., we have $u_j=1$ for all $j\in B$.*

Lemma 11 *Let $\sum_{j\in N}\alpha_j s_j\geq 1$ be facet defining for* $\text{conv}(P_I)$ *and in standard form. Assume $\alpha_j>0$ for all $j\in U$. All vertices of L_α are of one of the following two forms*

$$(i)\quad f+\sum_{j\in I} r^j, \qquad \textit{where } I\subseteq B \tag{28}$$

$$(ii)\quad f+\sum_{j\in I} r^j + \frac{1-\sum_{j\in I}\alpha_j}{\alpha_k} r^k, \qquad \textit{where } I\subseteq B \textit{ and } k\in N\setminus I. \tag{29}$$

Furthermore, if $\text{cone}_{i\in U} r^i = \mathbb{R}^2$*, then all vertices of L_α are of the form* (29)*.*

Proof: By definition L_α is the projection of the polyhedron

$$L_\alpha^s := \{(x, s) \in P : \sum_{j \in N} \alpha_j s_j \leq 1\}$$

onto the space of the x variables. From LP theory, every vertex of L_α is the projection of a vertex of L_α^s. Let $(\bar{x}, \bar{s})$ be a vertex of L_α^s. We first distinguish two cases based on whether $\sum_{j=1}^n \alpha_j \bar{s}_j \leq 1$ is at equality or not.

Case 1 $\sum_{j=1}^n \alpha_j \bar{s}_j < 1$.

If there exists $j \in B$ such that $0 < \bar{s}_j < u_j$, then there exists $\epsilon > 0$ such that $(\bar{x} - \epsilon r^j, \bar{s} - \epsilon e_j), (\bar{x} + \epsilon r^j, \bar{s} + \epsilon e_j) \in L_\alpha^s$, which proves that $(\bar{x}, \bar{s})$ is not a vertex of L_α^s, and therefore $\bar{x}$ is not a vertex of L_α. Hence it follows that, in this case, every vertex of L_α^s must be of the form (28).

Case 2 $\sum_{j=1}^n \alpha_j \bar{s}_j = 1$.

A vertex v of L_α^s is determined by setting n linearly independent inequalities from the description of L_α^s to equality. A vertex of L_α^s is therefore determined by $n-1$ lower and upper bound constraints and the equality $\sum_{j \in N} \alpha_j s_j = 1$.

Finally, if $\text{cone}_{i \in U} r^i = \mathbb{R}^2$, then there exist $i_1, i_2, i_3 \in U$ such that $\sigma_{i_1} r^{i_1} + \sigma_{i_2} r^{i_2} + \sigma_{i_3} r^{i_3} = 0$, with $\sigma_{i_k} \geq 0$ for $k = 1, 2, 3$. We claim that this implies that Case 1 cannot occur. Indeed, consider $(\bar{x}, \bar{s}) \in L_\alpha^s$ with $\sum_{j=1}^n \alpha_j \bar{s}_j < 1$. Then there exists $\epsilon > 0$ such that $(\bar{x} + \epsilon \sigma_{i_k} r^{i_k}, \bar{s} + \epsilon \sigma_{i_k} e_{i_k}) \in L_\alpha^s$ for $k = 1, 2, 3$, which shows that $(\bar{x}, \bar{s})$ is a convex combination of points in L_α^s. ∎

In the following we assume $\text{cone}_{i \in U} r^i = \mathbb{R}^2$ in order to reduce the number of cases to consider. We therefore only consider vertices of L_α of the form (29). We next prove that, for vertices of L_α of the form (29) generated from an unbounded ray r^k with $k \in U$, we only need to consider$k \in I^U(x)$ for some vertex x of $\text{conv}(X_\alpha)$.

Lemma 12 *Let* $\sum_{j \in N} \alpha_j s_j \geq 1$ *be facet defining for* $\text{conv}(P_I)$ *and in standard form. Assume* $\alpha_j > 0$ *for* $j \in U$. *Let* $I \subseteq B$ *satisfy* $0 \leq \sum_{i \in I} \alpha_i < 1$, *and let* $\{j, k, l\} \subseteq N$ *be such that* $k, l \in U$ *and* $r^j \in \text{cone}\{r^k, r^l\}$. *If* $\{j, k, l\}$ *satisfies the linear dependence property wrt.* α, *then* $f + \sum_{i \in I} r^i + \frac{1 - \sum_{i \in I} \alpha_i}{\alpha_j} r^j$ *is not a vertex of* L_α.

Proof: From linear dependence property there exist $\sigma_k, \sigma_l \geq 0$ such that $r^j = \sigma_k r^k + \sigma_l r^l$ together with $\alpha_j = \sigma_k \alpha_k + \sigma_l \alpha_l$. Hence we have

$$\begin{aligned} f + \sum_{i \in I} r^i + \frac{1 - \sum_{i \in I} \alpha_i}{\alpha_j} r^j &= f + \sum_{i \in I} r^i + \frac{1 - \sum_{i \in I} \alpha_i}{\sigma_k \alpha_k + \sigma_l \alpha_l} (\sigma_k r^k + \sigma_l r^l) \\ &= \frac{\sigma_k \alpha_k}{\sigma_k \alpha_k + \sigma_l \alpha_l} (f + \sum_{i \in I} r^i + \frac{1 - \sum_{i \in I} \alpha_i}{\alpha_k} r^k) \\ &\quad + \frac{\sigma_l \alpha_l}{\sigma_k \alpha_k + \sigma_l \alpha_l} (f + \sum_{i \in I} r^i + \frac{1 - \sum_{i \in I} \alpha_i}{\alpha_l} r^l) \end{aligned}$$

which proves that it is the convex combination of two points of L_α. ∎

Lemma 12 shows that rays r^k with $k \in U$ that are in cone$\{r^i, r^j\}$ where $i, j \in U$, and with $\{i, j, k\}$ satisfying the linear dependence property wrt. α, do not generate vertices of L_α of the form (29). From this lemma we conclude that it is enough to consider unbounded rays involved in maximal representations (rays in some $I^U(x)$) and the bounded rays to construct the vertices of L_α. Note that this corresponds to the indices appearing in the simplified polar (25)-(26) in Theorem 1.

4.2 The unbounded case

In the unbounded case, there is no linear dependence property (26) involving a bounded ray, and therefore the simplified polar only consists of (25), and this gives the same number of equations as the number of integer points with at least one tight representation. We conclude that the simplified polar, which is uniquely solvable, contains either three rows and three rays or four rows and four rays. The vertices of L_α are given by $f + \frac{1}{\alpha_i} r^i$ as stated by Lemma 11. This leads to either a triangle or a quadrilateral. We can therefore obtain the coefficients of a facet from the polygon L_α. In the unbounded case, it is explicitly given by the ratio of the norm of a ray i divided by the distance to which the ray i intersects L_α. See [1] for more details on the geometry of the unbounded case.

4.3 The one edge case

In the remainder of this section, we consider the case when $B = \{e\}$, and we call the only bounded ray r^e for the *edge*. The situation is slightly different in this case. There is still one maximal tight representation (25) for each integer point, and some linear dependence equations (27) (that we discard as in Theorem 1) and (26) (that we keep).

The difference comes from the fact that the edge may occur in several distinct linear dependencies (26). This number is however limited by two as we will prove in the following lemmas.

Lemma 13 *Let $\sum_{i \in N} \alpha_i s_i \geq 1$ be facet defining for* $\text{conv}(P_I)$ *and in standard form. Consider two points $x, y \in \mathbb{Z}^2$ such that $\mathcal{S}^\alpha_{strict}(x) \cap \mathcal{S}^\alpha_{strict}(y) \supseteq \{i, e\}$ with $i \in U$. Then $\mathcal{S}^\alpha_{strict}(x) = \mathcal{S}^\alpha_{strict}(y)$.*

Proof: We only prove $\mathcal{S}^\alpha_{\text{strict}}(x) \subseteq \mathcal{S}^\alpha_{\text{strict}}(y)$. The proof of the other inclusion is symmetric. Let $j \in \mathcal{S}^\alpha_{\text{strict}}(x)$. Hence $\{i, j, e\}$ satisfies the linear dependence property wrt. α, and therefore there exist $\sigma_i, \sigma_j, \sigma_e \in \mathbb{R}$ with $\sigma_j > 0$ such that $\sigma_i r^i + \sigma_j r^j + \sigma_e r^e = 0$ and $\sigma_i \alpha_i + \sigma_j \alpha_j + \sigma_e \alpha_e = 0$. Let $t^* \in T_\alpha(y)$ denote a representation of y that satisfies (11). We have $\{i, e\} \subseteq T^*_{strict}$. We therefore have $t^* + \epsilon(\sigma_i \mathbf{e}_i + \sigma_j \mathbf{e}_j + \sigma_e \mathbf{e}_e) \in T_\alpha(y)$ for $\epsilon > 0$ small enough, where $\mathbf{e}_i$, $\mathbf{e}_j$ and $\mathbf{e}_e$ are unit vectors. This implies $j \in \mathcal{S}^\alpha_{\text{strict}}(y)$. ■

The previous lemma implies that if e is involved in two linear dependence properties, then the vectors must be different. Furthermore, if e is involved in two linear dependence properties, then it cannot belong to the corresponding cones as the next lemma shows.

Lemma 14 *Let* $\sum_{i\in N}\alpha_i s_i \geq 1$ *be facet defining for* $\text{conv}(P_I)$ *and in standard form. Consider* $x, y \in \mathbb{Z}^2$ *such that* $\mathcal{S}^{\alpha}_{strict}(x) \cap \mathcal{S}^{\alpha}_{strict}(y) = \{e\}$*. Then* $r^e \notin \text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{strict}(x) \setminus \{e\}\}$ *and* $r^e \notin \text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{strict}(y) \setminus \{e\}\}$*.*

Proof: Assume for a contradiction that $r^e \in \text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{\text{strict}}(x) \setminus \{e\}\}$. Then there exists $k, l \in \mathcal{S}^{\alpha}_{\text{strict}}(x)$ and $\sigma_k, \sigma_l \geq 0$ such that $r^e = \sigma_k r^k + \sigma_l r^l$ and $\alpha_e = \sigma_k\alpha_k + \sigma_l\alpha_l$. Let $t^* \in T_\alpha(y)$ satisfy $0 < t^*_e < 1$. We have $t^* + \epsilon(-\mathbf{e}_e + \sigma_k\mathbf{e}_k + \sigma_l\mathbf{e}_l) \in T_\alpha(y)$ for $\epsilon > 0$ small enough, where $\mathbf{e}_e$, $\mathbf{e}_k$ and $\mathbf{e}_l$ are unit vectors. We conclude that $k, l \in \mathcal{S}^{\alpha}_{\text{strict}}(y)$ which is a contradiction with the hypothesis that $\mathcal{S}^{\alpha}_{\text{strict}}(x) \cap \mathcal{S}^{\alpha}_{\text{strict}}(y) = \{e\}$. ■

The next lemma shows that the edge cannot be involved in three linear dependence properties.

Lemma 15 *Let* $\sum_{i\in N}\alpha_i s_i \geq 1$ *be facet defining for* $\text{conv}(P_I)$ *and in standard form. Consider* $x^1, x^2, x^3 \in \mathbb{Z}^2$ *such that* $\mathcal{S}^{*}_{strict}(x^i, \alpha) \cap \mathcal{S}^{*}_{strict}(x^j, \alpha) \supseteq \{e\}$ *for all pairs* $\{i, j\} \subset \{1, 2, 3\}$*. Then there exists at least one pair* $\{i, j\} \subset \{1, 2, 3\}$ *with* $i \neq j$ *such that* $\mathcal{S}^{\alpha}_{strict}(x^i) = \mathcal{S}^{\alpha}_{strict}(x^j)$*.*

Proof: From Lemma 13, we know that if $\mathcal{S}^{\alpha}_{\text{strict}}(x^i) \cap \mathcal{S}^{\alpha}_{\text{strict}}(x^j) \supsetneq \{e\}$ for some $i \neq j$, then $\mathcal{S}^{\alpha}_{\text{strict}}(x^i) = \mathcal{S}^{\alpha}_{\text{strict}}(x^j)$. In this case the result therefore follows.

We may therefore assume $\mathcal{S}^{\alpha}_{\text{strict}}(x^i) \cap \mathcal{S}^{\alpha}_{\text{strict}}(x^j) = \{e\}$ for all $\{i, j\} \subset \{1, 2, 3\}$. Lemma 7 and Lemma 14 imply $-r^e, r^e \notin \text{cone}\{r^k : k \in \mathcal{S}^{\alpha}_{\text{strict}}(x^i)\}$ for $i = 1, 2, 3$. Therefore the line $\text{span}\{e\}$ separates $\mathbb{R}^2$ in two halfspaces such that there exists two indices $\{p, q\} \subset \{1, 2, 3\}$ with $\text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{\text{strict}}(x^p)\}$ and $\text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{\text{strict}}(x^q)\}$ belonging to the same halfspace. We claim $\mathcal{S}^{\alpha}_{\text{strict}}(x^p) = \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$. Since they belong to the same halfspace, we can write $\text{cone}\{r^j : j \in \mathcal{S}^{\alpha}_{\text{strict}}(x^p) \cup \mathcal{S}^{\alpha}_{\text{strict}}(x^q) \cup \{e\}\} = \text{cone}\{r^e, r^i\}$ with wlog $i \in \mathcal{S}^{\alpha}_{\text{strict}}(x^p)$. We claim that also $i \in \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$, which implies $\{i, e\} \subseteq \mathcal{S}^{\alpha}_{\text{strict}}(x^p) \cap \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$, and therefore Lemma 13 shows $\mathcal{S}^{\alpha}_{\text{strict}}(x^p) = \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$.

Let $t^* \in T_\alpha(x^q)$ and $v^* \in T_\alpha(x^p)$ be tight representations of x^q and x^p respectively that satisfy (11). For any $j \in \mathcal{S}^{\alpha}_{\text{strict}}(x^p) \cup \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$, there exists $\sigma_i, \sigma_e \geq 0$ such that $r^j = \sigma_i r^i + \sigma_e r^e$. Now, there exist $\delta, \epsilon > 0$ such that $t^* + \epsilon(-\mathbf{e}_j + \sigma_i\mathbf{e}_i + \sigma_e\mathbf{e}_e)$ is a valid representation of x^q and $v^* + \delta(\mathbf{e}_j - \sigma_i\mathbf{e}_i - \sigma_e\mathbf{e}_e)$ is a valid representation of x^p. Therefore we have $\sum_{\bar{i}\in N}\alpha_{\bar{i}}t^*_{\bar{i}} = 1$ and $\sum_{\bar{i}\in N}\alpha_{\bar{i}}t^*_{\bar{i}} + \epsilon(-\alpha_j + \sigma_i\alpha_i + \sigma_e\alpha_e) \geq 1$ from which we conclude $\alpha_j \leq \sigma_i\alpha_i + \sigma_e\alpha_e$. Similarly we have $\sum_{\bar{i}\in N}\alpha_{\bar{i}}v^*_{\bar{i}} = 1$ and $\sum_{\bar{i}\in N}\alpha_{\bar{i}}v^*_{\bar{i}} + \delta(\alpha_j - \sigma_i\alpha_i - \sigma_e\alpha_e) \geq 1$ from which we conclude $\alpha_j \geq \sigma_i\alpha_i + \sigma_e\alpha_e$. Therefore $\alpha_j = \sigma_i\alpha_i + \sigma_e\alpha_e$ from which we conclude that $t^* + \epsilon(-\mathbf{e}_j + \sigma_i\mathbf{e}_i + \sigma_e\mathbf{e}_e) \in T_\alpha(x^q)$ which proves that $i \in \mathcal{S}^{\alpha}_{\text{strict}}(x^q)$. ■

Corollary 4 *Let* $\sum_{i\in N}\alpha_i s_i \geq 1$ *be facet defining for* $\text{conv}(P_I)$ *and in standard form. If* $\{e, 1, 2\}$ *and* $\{e, 3, 4\}$ *satsify the linear dependence property wrt.* α *and* $\text{cone}\{r^e, r^1, r^2, r^3, r^4\} \neq \mathbb{R}^2$*, then* $\{e, 1, 2, 3, 4\}$ *satisifes the linear dependence property wrt.* α*.*

Proof: The proof is identical to the second part of the proof of Lemma 15. ■

In the unbounded case, the vertices of L_α are given by $f + \frac{1}{\alpha_i}$. When an edge is present, each ray r^i involved in maximal representations gives rise to two potential vertices $f + \frac{1}{\alpha_i} r^i$ and $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$. Observe that if $\alpha_e > 1$, each ray gives rise to one potential vertex. This is therefore essentially the same situation as in the unbounded case. Therefore, in this section, we only consider facet-defining inequalities in standard form with $\alpha_i > 0$ for all $i \in U$ and $\alpha_e \leq 1$. We next present results that enable us to rule out some potential vertices of L_α.

Lemma 16 *Let $\sum_{i \in N} \alpha_i s_i \geq 1$ be facet defining for* $\mathrm{conv}(P_I)$ *and in standard from. Suppose $\alpha_i > 0$ for all $i \in U$ and $\alpha_e < 1$. If $r^e \in \mathrm{cone}\{r^j, r^k\}$ and $\{e, j, k\}$ satisfies the linear dependence property wrt. α, then $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$ is not a vertex of L_α for any $i \in U$.*

Proof: There are two cases to analyze. We first prove the lemma for $i = j$ (the case $i = k$ is similar), and then for $i \neq j, k$.

(i) We first prove $f + r^e + \frac{1-\alpha_e}{\alpha_j} r^j$ is not a vertex when $i = j$.

We have $r^e = \sigma_j r^j + \sigma_k r^k$ with $\sigma_j, \sigma_k \geq 0$ and $\alpha_e = \sigma_j \alpha_j + \sigma_k \alpha_k$ by linear dependence property. Therefore

$$
\begin{aligned}
f + r^e + \frac{1-\alpha_e}{\alpha_j} r^j &= f + \sigma_j r^j + \sigma_k r^k + \frac{1 - \sigma_j \alpha_j - \sigma_k \alpha_k}{\alpha_j} r^j \\
&= f + (1 - \sigma_k \alpha_k) \frac{r^j}{\alpha_j} + (\sigma_k \alpha_k) \frac{r^k}{\alpha_k} \\
&= (1 - \sigma_k \alpha_k)(f + \frac{r^j}{\alpha_j}) + (\sigma_k \alpha_k)(f + \frac{r^k}{\alpha_k})
\end{aligned}
$$

which proves that $f + r^e + \frac{1-\alpha_e}{\alpha_j} r^j$ is not a vertex of L_α (observe that $0 \leq \sigma_k \alpha_k \leq 1$ since we assumed $\alpha_e \leq 1$).

(ii) We next prove $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$ is not a vertex when $i \neq j, k$.

We have

$$
\begin{aligned}
f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i &= f + \sigma_j r^j + \sigma_k r^k + \frac{1 - \sigma_j \alpha_j - \sigma_k \alpha_k}{\alpha_i} r^i \\
&= (\sigma_j \alpha_j)(f + \frac{r^j}{\alpha_j}) + (\sigma_k \alpha_k)(f + \frac{r^k}{\alpha_k}) + (1 - \sigma_j \alpha_j - \sigma_k \alpha_k)(f + \frac{r^i}{\alpha_i})
\end{aligned}
$$

which shows $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$ is a convex combination of three points of L_α. ■

Lemma 17 *Let $\sum_{i \in N} \alpha_i s_i \geq 1$ be facet defining for* $\mathrm{conv}(P_I)$ *and in standard form. Suppose $\alpha_i > 0$ for all $i \in U$ and $\alpha_e < 1$, and let j, k be such that $r^k \in \mathrm{cone}\{r^j, r^e\}$. Then* *(i) If $\{e,j,k\}$ satisfies the linear dependence property wrt. α, then $f+r^e+\frac{1-\alpha_e}{\alpha_j}r^j$ and $f+\frac{1}{\alpha_k}r^k$ are convex combinations of $f+r^e+\frac{1-\alpha_e}{\alpha_k}r^k$ and $f+\frac{1}{\alpha_j}r^j$.*

(ii) If $\{e,j,k\}$ does not satisfy the linear dependence property wrt. α, then either $f+r^e+\frac{1-\alpha_e}{\alpha_j}r^j$ or $f+\frac{1}{\alpha_k}r^k$ is not a vertex of L_α.

Proof: (i) We have $r^k=\sigma_j r^j+\sigma_e r^e$ with $\sigma_j,\sigma_e\geq 0$ and also $\alpha_k=\sigma_j\alpha_j+\sigma_e\alpha_e$. It can be verified that

$$f+\frac{r^k}{\alpha_k}=(1-\frac{\sigma_e}{\sigma_e+\sigma_j\alpha_j})(f+\frac{r^j}{\alpha_j})+(\frac{\sigma_e}{\sigma_e+\sigma_j\alpha_j})(f+r^e+\frac{1-\alpha_e}{\alpha_k}r^k).$$

Observe that the coefficients in the above combination are in the interval $[0,1]$. It can also be verified that

$$\begin{aligned}f+r^e+\frac{1-\alpha_e}{\alpha_j}r^j=(1-\frac{\sigma_j\alpha_j+\sigma_e\alpha_e}{\sigma_j\alpha_j+\sigma_e})(f+\frac{r^j}{\alpha_j})\\ +(\frac{\sigma_j\alpha_j+\sigma_e\alpha_e}{\sigma_j\alpha_j+\sigma_e})(f+r^e+\frac{1-\alpha_e}{\alpha_k}r^k).\end{aligned}$$

The coefficients in the above combination belong to $[0,1]$ since $\alpha_e\leq 1$.

(ii) Observe that $r^k=\sigma_j r^j+\sigma_e r^e$ and either

$$\alpha_k>\sigma_j\alpha_j+\sigma_e\alpha_e \quad \text{or} \tag{30}$$

$$\alpha_k<\sigma_j\alpha_j+\sigma_e\alpha_e \tag{31}$$

In the case of (30), $f+\frac{1}{\alpha_k}r^k$ is not a vertex of L_α. Indeed there exists $\epsilon>0$ such that $(\frac{1}{\alpha_k}-\epsilon)>0$ and

$$f+\frac{1}{\alpha_k}r^k=f+(\frac{1}{\alpha_k}-\epsilon)r^k+\epsilon\sigma_j r^j+\epsilon\sigma_e r^e$$

with

$$\alpha_k(\frac{1}{\alpha_k}-\epsilon)+\alpha_j\epsilon\sigma_j+\alpha_e\epsilon\sigma_e=1+\epsilon(\sigma_j\alpha_j+\sigma_e\alpha_e-\alpha_k)<1.$$

Therefore $f+\frac{1}{\alpha_k}r^k$ is not a vertex of L_α, since a point $f+\sum_{j\in N}s_j r^j$ satisfying $\sum_{j\in N}\alpha_j s_j<1$ cannot be a vertex of L_α.

The same kind of argument works for $f+r^e+\frac{1-\alpha_e}{\alpha_j}r^j$ in the case of (31).■

We are now able to classify the geometry of all facets occurring in a problem with one edge. To this end, we distinguish whether or not three or four integer points have tight representations. We then consider three subcases depending on the number of occurences of the edge in a linear dependence property.

Three integer points

(i) The edge is involved in no linear dependence property.
This case is impossible, because then there are three equations (involving α_e) that give a uniquely solvable system. Hence there are at most two rays remaining. This is a contradiction since they do not span $\mathbb{R}^2$.

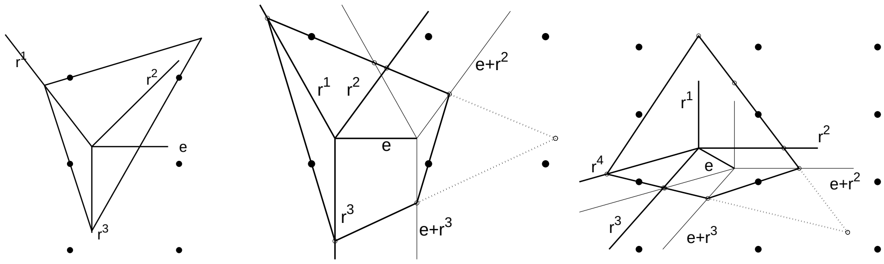
(ii) The edge is involved in one linear dependence property, e.g. $\{e, j, k\}$ satisfies the linear dependence property.
Two main cases occur. If $r^e \in \text{cone}\{r^j, r^k\}$, then by Lemma 16, all vertices are of the form $f + \frac{1}{\alpha_i} r^i$ which makes three vertices (see Fig. 3(a) for an example). If $r^e \notin \text{cone}\{r^j, r^k\}$, then $f + \frac{1}{\alpha_j} r^j$ and $f + r^e + \frac{1-\alpha_e}{\alpha_k} r^k$ are vertices of L_α by Lemma 17(i). The third ray r^i involved in the reduced polar induces a maximum of two vertices $f + \frac{1}{\alpha_i} r^i$ and $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$. We therefore have at most four vertices in total (see Fig. 3(b) for an example).

(iii) The edge is involved in two linear dependencies.
From Lemma 13, we know that this implies that four rays are involved in the maximal tight representations. In total, we therefore have 5 equations and 4 rays and one edge. Let r^i, r^j, r^k, r^l be the four rays involved in the linear dependence properties. We must have, from Lemma 14, wlog $r^j \in \text{cone}\{r^i, r^e\}$ and $r^l \in \text{cone}\{r^k, r^e\}$. From Lemma 17(i), there are four vertices $f + \frac{1}{\alpha_i} r^i$, $f + r^e + \frac{1-\alpha_e}{\alpha_j} r^j$, $f + \frac{1}{\alpha_i} r^k$, and $f + r^e + \frac{1-\alpha_e}{\alpha_l} r^l$ (see Fig. 3(c)).

Four integer points

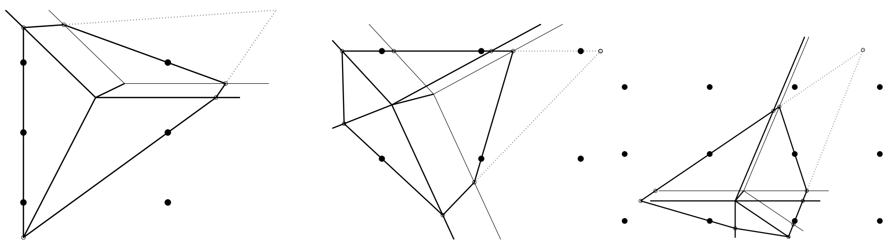
(i) The edge is involved in no linear dependence property
We therefore have four equations, one edge and three rays, r^i, r^j, r^k. There must exist a pair, say i, j, such that $r^j \in \text{cone}\{r^i, r^e\}$. By Lemma 17(ii), we have that either $f + r^e + \frac{1-\alpha_e}{\alpha_i} r^i$ or $f + \frac{1}{\alpha_j} r^j$ is not a vertex. We therefore have a maximum of five vertices (see Fig. 4(a)).

(ii) The edge is involved in one linear dependence property.
We therefore have five equations, one edge and four rays. Let $\{i, j, e\}$ satisfy the linear dependence property, and r^k, r^l be the two remaining rays. If $r^e \in \text{cone}\{r^i, r^j\}$, by Lemma 16, all vertices are of the form $f + \frac{1}{\alpha_t} r^t$.
Let us assume $r^j \in \text{cone}\{r^i, r^e\}$. Then $f + \frac{1-\alpha_e}{\alpha_i} r^i$ and $f + \frac{1}{\alpha_j} r^j$ are not vertices of L_α. If we consider the line $\lambda r^e, \lambda \in \mathbb{R}$, it separates the plane into two half-planes H_1, H_2. Let us assume that $r^i, r^j \in H_1$. At least one among r^k, r^l belongs to H_2. The other ray, say r^l, is either in H_1 or in H_2. If it is in H_2, then $r^k \in \text{cone}\{r^l, r^e\}$ or $r^l \in \text{cone}\{r^k, r^e\}$. In both cases, it creates at most three more vertices (by Lemma 17(ii)) and yields five vertices in total (see Fig. 4(b)). If $r^l \in H_1$, then we have in one case $r^i, r^j \in \text{cone}\{r^l, r^e\}$ which implies that either $f + r^e + \frac{1-\alpha_e}{\alpha_l} r^l$ is not a vertex (but makes 5 vertices in total) or $f + \frac{r^i}{\alpha_i}$ is not a vertex (5 vertices in total). We obtain similarly 5 vertices in the other case if $r^l \in \text{cone}\{r^j, r^e\}$.



(a) 3 points, one lin. dep. (b) 3 points, one lin. dep. (c) 3 points, two lin. dep.

Figure 3: The possible cases with three integer points



(a) 4 points, no lin. dep. (b) 4 points, one lin. dep. (c) 4 points, two lin. dep.

Figure 4: The possible cases with four integer points

(iii) The edge is involved in two linear dependence properties.
We then have six equations and five rays $\{r^i, r^j, r^k, r^l, r^m\}$ where $\{i, j, e\}$ and $\{k, l, e\}$ satisfy the linear dependence property. From Lemma 14 and Corollary 4, we have wlog $r^j \in \text{cone}\{r^i, r^e\}$ and $r^l \in \text{cone}\{r^k, r^e\}$. This implies four vertices involving i, j, k, l from Lemma 17(i). Now r^m is either on the same side as $\text{cone}\{r^i, r^j\}$ with respect to r^e or as $\text{cone}\{r^k, r^l\}$. Following the same reasoning as in the previous case, we also conclude to a maximum of five vertices (see Fig. 4(c)).

When an edge is present, the vertices of L_α are not necessarily located on $f + \lambda r^i$. They may also be located on $f + u_e r^e + \lambda r^i$. This observation allows us to determine the coefficients of an inequality from a polygon L_α. For every ray i such that a vertex of L_α lies on $f + \lambda r^i, \lambda \geq 0$, the coefficient α_i can be obtained as before.

The coefficient α_e can be obtained as usual if the edge intersects the polygon L_α (in this case the bound is irrelevant). If the edge does not intersect L_α, there exist two sides of the polygon that meet on the line $f + \lambda r^e, \lambda \geq 0$. This "hidden vertex" of L_α determines the coefficient α_e. It is shown by the intersection of two dotted lines on Fig. 3 and 4.

Finally, for every ray i such that a vertex of L_α is on $f + u_e r^e + \lambda r^i, \lambda \geq 0$, the coefficient α_i is given by $(1 - u_e \alpha_e)$ multiplied by the ratio of the norm of the ray r^i divided by λ where $f + u_e r^e + \lambda r^i$ is a vertex of L_α.

Some rays do not determine a vertex of L_α. The coefficient of such a ray i can be obtained by one of the two previous ways. It is essentially obtained through the maximum of the two previous methods.

4.4 Conclusion

In this paper, we have shown that the presence of finite upper bounds complicates the geometric description of the facet structure of $\text{conv}(P_I)$. For the case of one bounded variable, we managed to give a complete description of the mixed integer hull. We found that for most inequalities that have $\alpha_e < 1$, stronger inequalities can be obtained than in the unbounded case. Consider, for example, the inequality shown in Fig. 3 (b). Let us forget for a while that the variable s_e is bounded, but suppose that we can still obtain the coefficients from the geometry of the bounded problem. In that case, the natural polygon obtained is the large triangle supported by the solid and dotted lines. We see that this polygon includes the integer point $(1, 0)$ in its interior. This proves that the inequality could not be obtained from the unbounded relaxation. The fact that we consider the bound explicitly allows us to strengthen the coefficient α_e.

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