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Homogenization of a viscous incompressible two-phase fluid flow through a random rough-walled filter

Mustapha El Jarroudi^a, Mustapha Er-Riani^a, Abdeluaab Lidouh^b, Youness Filali^c and Moussa El Jarroudi^d

^aDepartment of Mathematics, Abdelmalek Essaadi University, Tangier, Morocco; ^bDepartment of Mathematics and Computer Science, Laboratory LACSA, Faculty of Sciences, Mohammed 1st University, Oujda, Morocco; ^cISI Laboratory, ENSATE, Abdelmalek Essaâdi University, Tetouan, Morocco; ^dUniversity of Liège, Department of Environmental Sciences and Management, SPHERES Research Unit, Arlon, Belgium

ABSTRACT

We consider a viscous incompressible two-phase fluid flow, governed by a Stokes system, in a domain containing a free fluid, a random filter of finite thickness, and the interface between them. The filter consists of vertical small diameter pipes with random rough sections. We suppose that, within each tube, the interface fluid-fluid admits a random structure. We construct boundary layers describing the flow at the interface between the free fluid and the filter and, using Γ -convergence methods, we study the asymptotic behavior of the fluid as the thickness of the filter tends to zero. We derive a macroscopic model with a viscous incompressible two-phase fluid flow in the free domain coupled with a Darcy law, connecting two-phase velocities on the permeable interface, all combined with random interface conditions linking the velocity in the free domain and that in the porous interface.

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1. Introduction

The main objective of this work is to study the homogenization of a slow viscous two-phase incompressible flow in domain consisting of a free fluid domain, vertical small diameter pipes with random sections encrusted in a periodic network of rectangular parallelepiped of height $h > 0$ and width $\varepsilon \ll 1$, and the interface between them denoted by Σ and located in the plane $\{x_3 = 0\}$. Several methods exist in the literature to model randomly distributed surface roughness, such as Gaussian distributed roughness heights and the Mandelbrot–Weierstrass function involving random parameters (see, for instance, [1,2]). We suppose here that the pipe which is centred at the point $(\varepsilon i, \varepsilon j)$ for $(i, j) \in \mathbb{Z}^2$ has radius $\varepsilon \zeta(x_3 \chi_{ij}(\omega) \varepsilon^{-1/2}, \omega)$, where ζ is a stationary ergodic process and $\chi_{ij}(\omega)$ is a random variable. The topography of the rough surface of the pipe wall is characterized by using the random process ζ . Based on this characterization, the roughness on the pipe wall takes the form of small random asperities. Such a process with similar scaling was introduced

in [3] in order to study the passage of plane H-polarized electromagnetic waves through a perfectly conducting layer with random gaps.

We assume that the flow in each pipe is annular, which means that non wetting phase flows in the channel core and the wetting phase flows along the channel walls (see, for instance, [4–6]). Taking into account the influence of small diameter pipe surface roughness on the capillary surface (see, for instance, [7–9]), we suppose that the capillary surface inside the (i, j) -th pipe centred at the point $(\varepsilon i, \varepsilon j)$ is represented as graph $\varepsilon \varrho(x_3 \chi_{ij}(\omega) \varepsilon^{-1/2}, \omega)$, where ϱ is a stationary ergodic process.

Using Γ -convergence methods (see for instance [10,11]), we derive rigorously (as the parameter ε tends to zero) Equations (72)–(73) (see Corollary 7.3) governing the effective two-phase flow through the random filter. Due to the random scaling of the filter's roughness, the two-phase flow within the filter is asymptotically represented by a two-phase flow on the permeable interface Σ and is described by a Darcy law connecting two-phase velocities. The transition between the free-flow and the porous interface is described by the following conditions:

$$\begin{cases} \left. \frac{\partial u_\tau}{\partial x_3} \right|_{x_3=0} = \sum_{m=1,2} \frac{2\mu_\alpha}{\mu_o + \mu_w} (K_m^{\alpha,bl}(\omega))^t \cdot (u - v^\alpha) & \text{on } \Sigma, \\ \left. \frac{\partial u_3}{\partial x_3} \right|_{x_3=0} - \frac{p}{\mu_o + \mu_w} = \sum_{\alpha=o,w} \frac{2\mu_\alpha}{\mu_o + \mu_w} (K_3^{\alpha,bl}(\omega))^t \cdot (u - v^\alpha) & \text{on } \Sigma, \end{cases} \quad (1)$$

where u_τ is the tangential component of the free-flow velocity, u_3 is the normal component of the free-flow velocity, v_τ^α is the tangential component of the Darcy velocity of the α -phase, $\alpha = o, w$, where o and w denote the non-wetting phase and wetting phase respectively, μ_α is the kinematic viscosity of the α -phase, p is the free-flow pressure, and $K_l^{\alpha,bl}(\omega)$, $l = 1, 2, 3$, is the l -th row vector of the boundary layer random matrix $K^{\alpha,bl}(\omega)$ defined in (67) for $\alpha = o, w$.

Flow of fluids through permeable interfaces separating free flow regions from porous media has been extensively studied (see, for instance, [12–22]), starting from the original experimental work of Beavers and Joseph [13], who proposed an empirical slip boundary condition at the interface of the form

$$\frac{\partial u_\tau}{\partial n} = \frac{C_{BJ}}{\sqrt{K_p}} (u_\tau - v_\tau), \quad (2)$$

where n is the outer unit normal, v_τ is the tangential component of the Darcy velocity of the fluid in the porous medium, K_p is the permeability of the porous medium, and C_{BJ} is a dimensionless coefficient, which is strongly dependent on the local geometry of the interface. Saffman [22] justified the boundary condition of Beavers and Joseph, using statistical model, and derived an alternative condition

$$u_\tau = \frac{\sqrt{K_p}}{C_{BJ}} \frac{\partial u_\tau}{\partial n} + O(K_p), \quad (3)$$

arguing that v_τ is of lesser order than the other terms. The mathematical proof of the Saffman modification of the boundary condition of Beavers and Joseph was obtained by Jäger and Mikelić in [18] by applying the theory of homogenization. A generalization of the boundary condition of Beavers and Joseph is proposed for curved surfaces in

[20]. Alternatives to the Beavers-Joseph condition have been proposed in [12,16,17,21]. In [19] Jäger and Mikelić derived rigorously the laws governing the flow at the interface by combining homogenization and boundary layer approach and compare their results with those obtained in [13,16,21,22]. The work [19] has been extended to curved contact interfaces in [14]. Using the standard methods of homogenization, the authors in [15] derived a macroscopic boundary conditions of Beavers and Joseph type at the interface between a two-phase free fluid and a two-phase fluid-filled porous medium. Note that here only the first equation of the transmission condition (1) above seems to be similar to a Beavers-Joseph condition with random coefficients.

A viscous incompressible and nonstationary single phase fluid flow through a filter, consisting of ε -periodically distributed small diameter pipes separating two reservoir, has been considered by Jäger and Mikelić in [23]. Assuming that the flow is generated by the difference of the pressures imposed at the two sides of the filter, they proved that the effective velocity in the filter part is a constant vector in the axial direction, the effective pressure obeys the fourth-power law, and the effective flow is determined through the stabilization constants of boundary layers.

The present work can be extended to the case where the height of the pipes is also equal to ε . This can be done by a new scaling of the parameters, for instance, by replacing the radius of the pipe centred at the point $(\varepsilon i, \varepsilon j)$ for $(i, j) \in \mathbb{Z}^2$ by $\varepsilon \zeta(x_3 \chi_{ij}(\omega) \varepsilon^{-3/2}, \omega)$, the capillary surface by $\varepsilon \varrho(x_3 \chi_{ij}(\omega) \varepsilon^{-3/2}, \omega)$, and performing some appropriate new scaling of the viscosities of the two fluids in the pipe.

Due to the highly coupled interaction between the two-phase flow through the random filter and the free-fluid domain, including the coupling of different length scales and the presence of interfacial boundary layers phenomena, simulations at narrow cylinder scale are not feasible. Numerical simulations can be applied to the macroscale problem obtained in this paper. Several numerical methods exist in the literature; we mention here the multiscale numerical methods, which enjoy several similarities with both domain decomposition approaches and multigrid methods (see, for instance, [24, Chap. 5]) and the numerical volume averaging method, which is based on the volume averaging method (see, for instance, [25]). The latter, combined with the Monte Carlo method, can be used here to compute the effective coefficients of the macroscopic model.

Two-phase annular flows in vertical pipes play a significant role in many industrial and environmental applications, including air conditioning, enhanced oil recovery, groundwater flow, nuclear reactors, and refrigeration systems, see, for instance, [26] for a comprehensive bibliographical and historical survey. Due to the influence of surface tension on two phase processes, vertical small diameter pipes produce flow regimes different from those in conventional pipes and are of particular importance in practice (see, for instance, [27,28]). Surface roughness play an important role in solving industrial problem insofar as several studies indicate that surface roughness affect the velocity profile and consequently increase the pressure drop and flow resistance (see, for instance, [27–29]).

An important field to which the present investigation is closely related is the behavior of the transport of a class of organic groundwater contaminants such as organic pesticides, for instance a class of fungicides called weedkillers. Organic contaminants exist as non-aqueous phase liquids and their transport in underground environments constitutes a two-phase liquid-liquid flow through fractured porous media. Their presence on agricultural soils can lead to their infiltration through small, random cracks into shallow

groundwater. The present work can help to determine the characteristics of the mobility of organic contaminants present in agricultural fields to the groundwater.

This paper is organized as follows. The statement of the problem is presented in Section 2. In Section 2, we formulate and prove a priori estimates for the velocity and pressure fields. Section 3 is devoted to compactness results for the convergence of velocity and pressure fields. The two-phase relative permeability is computed in Section 4. In Section 5 we construct boundary layer problems. In Section 6 we formulate the main result of this work. Section 7 is consecrated to the proof of the main result.

2. Statement of the problem

Nomenclature

(Π, Υ, P)	probability space
$\mathcal{B}(\mathbb{R})$	the Borel σ -algebra on $\mathbb{R}$
$(\mathcal{T}(s))_{s \in \mathbb{R}}$	a group of transformations
ζ, ϱ	stationary ergodic processes on $\mathbb{R} \times \Pi$
ω	arbitrary realization from the set of full measure on Π
$(\chi_{ij}(\omega))_{i,j \in \mathbb{Z}}$	a sequence of random variables
$0 < \varepsilon \ll 1$	small parameter
$F_{ij}^\varepsilon(\omega)$	the (i, j) -th pipe centred at the point $(\varepsilon i, \varepsilon j)$
h	the height of the pipe
$\varepsilon \zeta(x_3 \chi_{ij}(\omega) \varepsilon^{-1/2}, \omega)$	the radius of the pipe $F_{ij}^\varepsilon(\omega)$
w	the wetting phase respectively
o	the non-wetting phase
$F_{w,ij}^\varepsilon(\omega)$	the part of $F_{ij}^\varepsilon(\omega)$ containing the wetting phase
$F_{o,ij}^\varepsilon(\omega)$	the part of $F_{ij}^\varepsilon(\omega)$ containing the non-wetting phase
$\mathcal{A}_{ow,ij}^\varepsilon(\omega)$	the capillary surface between $F_{w,ij}^\varepsilon(\omega)$ and $F_{o,ij}^\varepsilon(\omega)$
$\Omega_h^\varepsilon(\omega)$	the random filter
$\Omega_{w,h}^\varepsilon(\omega)$	the part of $\Omega_h^\varepsilon(\omega)$ containing the wetting phase
$\Omega_{o,h}^\varepsilon(\omega)$	the part of $\Omega_h^\varepsilon(\omega)$ containing the non-wetting phase
$\varepsilon \varrho(x_3 \chi_{ij}(\omega) \varepsilon^{-1/2}, \omega)$	the radius of $F_{o,ij}^\varepsilon(\omega)$
$\mathcal{A}_{ow}^\varepsilon(\omega)$	the capillary surface between $\Omega_{w,h}^\varepsilon(\omega)$ and $\Omega_{o,h}^\varepsilon(\omega)$
H^ε	the mean curvature of the capillary surface $\mathcal{A}_{ow}^\varepsilon$
$n^\varepsilon(x_3 \varepsilon^{-1/2}, \omega)$	the unit normal on $\mathcal{A}_{ow}^\varepsilon$
κ	the surface tension
μ_α	the kinematic viscosity of the phase $\alpha = o, w$
Re	characteristic Reynolds number
Fr	characteristic Froude number
Eu	characteristic Euler number
We	characteristic Weber number
Ω	free fluid domain
Ω_w	the part of Ω containing the wetting phase
Ω_o	the part of Ω containing the non-wetting phase
$\mathcal{S}_{ow}$	the capillary surface between Ω_w and Ω_o
H	the mean curvature of the capillary surface $\mathcal{S}_{ow}$

In this Section we formulate a boundary value problem modeling a slow viscous incompressible two-phase fluid flow through a random rough-walled filter and address the existence of solutions. We start with determining the geometry of the problem. Let (Π, Υ, P) be a probability space and $(\mathcal{T}(s))_{s \in \mathbb{R}}$ be a group of transformations on (Π, Υ) , that is satisfying

$$\begin{cases} \mathcal{T}(0) &= Id_{\Pi} \text{ (} Id_{\Pi} \text{ being the identical mapping),} \\ \mathcal{T}(s_1 + s_2) &= \mathcal{T}(s_1) \circ \mathcal{T}(s_2), \quad \forall s_1, s_2 \in \mathbb{R}, \\ P(\mathcal{T}^{-1}(s)A) &= P(A), \quad \forall A \in \Upsilon, \quad \forall s \in \mathbb{R}, \end{cases} \quad (4)$$

and such that, for every $A \in \Upsilon$, the set $\{(s, \omega) \in \mathbb{R} \times \Pi : \mathcal{T}(s)\omega \in A\}$ is $ds \otimes dP$ measurable. We suppose that $\mathcal{T}$ is ergodic, which means that every $A \in \Upsilon$ such that $\mathcal{T}(s)A = A$, for every $s \in \mathbb{R}$, has a probability $P(A)$ equal to 0 or 1.

Let ζ be a random process defined on $\mathbb{R} \times \Pi$ and satisfying the following conditions:

- (1) $\zeta(t, \omega)$ is a stationary ergodic process, that is

$$\forall s \in \mathbb{R}, \forall B \in \mathcal{B}(\mathbb{R}), P(\{\omega \mid \zeta(t + s, \omega) \in B\}) = P(\{\omega \mid \zeta(s, \mathcal{T}(t)\omega) \in B\}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on $\mathbb{R}$. Since $\mathcal{T}$ preserves the measure P , the above equality implies that $\zeta(s, \cdot)$ and $\zeta(t + s, \cdot)$ have the same probability distributions.

- (2) The derivative $\frac{d^m \zeta}{dt^m}$ exists, for $m = 1, 2, 3$, and there exist non-random constants c_1 and c_2 , such that, with probability 1,

$$0 < c_1 \leq \zeta(t, \omega) < 1/2, \left| \frac{d^m \zeta}{dt^m} \right| \leq c_2. \quad (5)$$

From the properties of $\mathcal{T}$ and ζ , we derive the ergodicity property (see, for instance, [30])

$$\forall n \in \mathbb{Z}^*, \langle \zeta^n(0) \rangle = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L \zeta^n(t, \omega) ds, \quad (6)$$

almost surely, where the symbol $\langle \cdot \rangle$ stands for the mathematical expectation with respect to the measure P .

Let $(\chi_{ij}(\omega))_{i,j \in \mathbb{Z}}$ be a sequence of random variables satisfying, with probability 1,

$$|\chi_{ij}(\omega)| \leq c_3, \quad \forall i, j \in \mathbb{Z}, \quad (7)$$

where c_3 is a non-random constant. We define, for every $(i, j) \in \mathbb{Z}^2$,

$$\zeta_{ij}(t, \omega) = \zeta(t\chi_{ij}(\omega), \omega). \quad (8)$$

Let ϱ be a stationary ergodic process on $\mathbb{R} \times \Pi$ such that

$$0 < \delta_0 \leq \varrho(t, \omega) < \zeta(t, \omega)/2, \quad (9)$$

where δ_0 is a non-random constant. We suppose that, for $m = 1, 2, 3$, the derivative $\frac{d^m \varrho}{dt^m}$ is a stationary ergodic process and there exists a non-random constant δ_1 , such that, with

probability 1,

$$\left| \frac{d^m \varrho}{dt^m} \right| \leq \delta_1, \quad (10)$$

and define, for every $(i, j) \in \mathbb{Z}^2$,

$$\varrho_{ij}(t, \omega) = \varrho(t\chi_{ij}(\omega), \omega). \quad (11)$$

Let $\varepsilon \in (0, 1)$ be a small parameter and h be a fixed positive constant. We take ω an arbitrary realization from the set of full measure on Π for which the conditions (5), (7), (9), and (10) are satisfied, and define, for every $i, j \in \mathbb{Z}$,

$$F_{ij}^\varepsilon(\omega) = \{x \in \mathbb{R}^3 \mid R_{ij}^\varepsilon(x_1, x_2) < \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega), x_3 \in (-h, 0)\}, \quad (12)$$

where $R_{ij}^\varepsilon(x_1, x_2) = \sqrt{(x_1 - i\varepsilon)^2 + (x_2 - j\varepsilon)^2}$. We suppose that

$$F_{ij}^\varepsilon(\omega) = F_{o,ij}^\varepsilon(\omega) \cup \mathcal{A}_{ow,ij}^\varepsilon(\omega) \cup F_{w,ij}^\varepsilon(\omega),$$

where the subscripts o and w denote the oil (or non-wetting) phase and wetting phase respectively (see Figure 1) and where the disjoint subsets $F_{o,ij}^\varepsilon(\omega)$, $\mathcal{A}_{ow,ij}^\varepsilon(\omega)$, $F_{w,ij}^\varepsilon(\omega)$ of

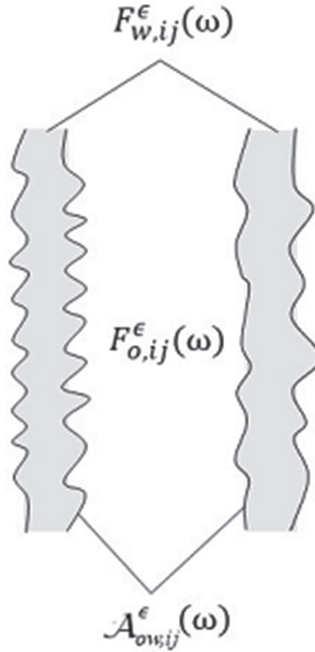


Figure 1. Representation of the pipe centred at the point $(\varepsilon i, \varepsilon j)$, for $(i, j) \in \mathbb{Z}^2$, with the two phases $F_{o,ij}^\varepsilon(\omega)$, $F_{w,ij}^\varepsilon(\omega)$, and the capillary surface $\mathcal{A}_{ow,ij}^\varepsilon(\omega)$ between them.

$F_{ij}^\varepsilon(\omega)$ are defined by

$$\begin{aligned} F_{o,ij}^\varepsilon(\omega) &= \{x \in \mathbb{R}^3 \mid R_{ij}^\varepsilon(x_1, x_2) < \varepsilon \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega), x_3 \in (-h, 0)\}, \\ F_{w,ij}^\varepsilon(\omega) &= \left\{ x \in \mathbb{R}^3 \mid \varepsilon \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega) < R_{ij}^\varepsilon(x_1, x_2) < \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega), \right. \\ &\quad \left. x_3 \in (-h, 0) \right\}, \\ \mathcal{A}_{ow,ij}^\varepsilon(\omega) &= \{R_{ij}^\varepsilon(x_1, x_2) = \varepsilon \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)\}. \end{aligned} \quad (13)$$

Let a, b be fixed positive constants. We set

$$\begin{aligned} \Omega &= (0, a)^2 \times (0, b), \quad \Omega_h = (0, a)^2 \times (-h, 0), \\ \Sigma &= (0, a)^2 \times \{0\}, \quad \Sigma_b = (0, a)^2 \times \{b\}, \\ \Sigma_h &= (0, a)^2 \times \{-h\}, \quad \Gamma = \partial\Omega \setminus \Sigma. \end{aligned} \quad (14)$$

We assume that

$$\Omega = \Omega_o \cup \mathcal{S}_{ow} \cup \Omega_w, \quad (15)$$

where Ω_o and Ω_w are disjoint open sets and $\mathcal{S}_{ow} = \partial\Omega_o \cap \partial\Omega_w$. Let us set

$$I_\varepsilon(\omega) = \{(i, j) \in \mathbb{Z}^2 \mid F_{ij}^\varepsilon(\omega) \subset \Omega_h\}.$$

We define

$$\begin{cases} \Omega_h^\varepsilon(\omega) &= \cup_{(i,j) \in I_\varepsilon(\omega)} F_{ij}^\varepsilon(\omega), \quad \Omega_{\alpha,h}^\varepsilon(\omega) = \cup_{(i,j) \in I_\varepsilon(\omega)} F_{\alpha,ij}^\varepsilon(\omega), \quad \alpha = o, w, \\ \mathcal{A}_{ow}^\varepsilon(\omega) &= \cup_{(i,j) \in I_\varepsilon(\omega)} \mathcal{A}_{ow,ij}^\varepsilon(\omega), \quad \Sigma_\alpha^\varepsilon(\omega) = \overline{\Omega_{\alpha,h}^\varepsilon(\omega)} \cap \Sigma, \quad \alpha = o, w, \\ \Sigma^\varepsilon(\omega) &= \overline{\Omega_h^\varepsilon(\omega)} \cap \Sigma, \quad \Sigma_h^\varepsilon(\omega) = \overline{\Omega_h^\varepsilon(\omega)} \cap \Sigma_h, \end{cases} \quad (16)$$

and

$$\Omega^\varepsilon(\omega) = \Omega \cup \Sigma^\varepsilon(\omega) \cup \Omega_h^\varepsilon(\omega), \quad \Gamma^\varepsilon(\omega) = \partial\Omega^\varepsilon(\omega) \setminus (\Sigma_b \cup \Sigma_h^\varepsilon(\omega)). \quad (17)$$

Let ω be a fixed event for which the conditions (5), (7), (9), and (10) are satisfied. We suppose that $\Omega^\varepsilon(\omega)$ is filled with a two-phase viscous incompressible fluid. The difference between the characteristics of the flows in Ω and $\Omega_h^\varepsilon(\omega)$ leads to a different scaling in the two media. Taking into account the typical pipe diameter in the filter, which is small compared to the macroscopic characteristic length, we suppose, according to [31, page 98], that the Froude number Fr and the Reynolds numbers Re are low in $\Omega_h^\varepsilon(\omega)$, and that the ratio $\frac{Fr}{Re}$ behaves as ε^2 . This implies, using, for instance, [32, Paragraph 5.2], that the Euler number is high and that the inertia effects can be neglected in the filter. Thereby, assuming that the Euler number Eu in $\Omega_h^\varepsilon(\omega)$ behaves as $1/Fr$, we suppose that the two-phase fluid flow in $\Omega^\varepsilon(\omega)$ is governed by the following Stokes equations supplied with suitable boundary

and transmission conditions:

$$\left\{ \begin{array}{ll} -\mu_\alpha \Delta u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega_\alpha, \quad \alpha = o, w, \\ -\mu_\alpha \varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon = f & \text{in } \Omega_{\alpha,h}^\varepsilon(\omega), \quad \alpha = o, w, \\ \operatorname{div}(u^\varepsilon) = 0 & \text{in } \Omega^\varepsilon(\omega), \\ \left[\mu_* \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right) n_j - p^\varepsilon n_i \right]_{ow} & = 2\kappa H n_i \quad \text{on } \mathcal{S}_{ow}, i = 1, 2, 3, \\ \left[\mu_* \varepsilon^2 \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right) n_j^\varepsilon - p^\varepsilon n_i^\varepsilon \right]_{ow} & = 2\kappa H^\varepsilon n_i^\varepsilon \quad \text{on } \mathcal{A}_{ow}^{h,\varepsilon}(\omega), \\ [u^\varepsilon]_{\Sigma^\varepsilon(\omega)} & = 0 \quad \text{on } \Sigma^\varepsilon(\omega), \\ u^\varepsilon & = 0 \quad \text{on } \Gamma^\varepsilon(\omega), \\ \mu_\alpha \varepsilon^2 \frac{\partial u^\varepsilon}{\partial x_3} \Big|_{x_3=-h} & = p^\varepsilon e_3 \quad \text{on } \Sigma_b \cup \Sigma_h^\varepsilon(\omega), \quad \alpha = o, w, \end{array} \right. \quad (18)$$

where u^ε and p^ε are, respectively, the velocity and the pressure of the flow in $\Omega^\varepsilon(\omega)$, μ_α , $\alpha = o, w$, is the kinematic viscosity of the phase α , which is a positive constant, $f \in L^2(\Omega \cup \Sigma \cup \Omega_h, \mathbb{R}^3)$ is the density of an external body force, $[\phi]_{ow}$ is the jump $\phi_o - \phi_w$ of a function ϕ through the capillary surface $\mathcal{S}_{ow} \cup \mathcal{A}_{ow}^\varepsilon$, n and n^ε are, respectively, the outward unit normals on $\mathcal{S}_{ow}$ and $\mathcal{A}_{ow}^{h,\varepsilon}(\omega)$ directed to the non wetting phase, $e_3 = (0, 0, 1)^t$, $\mu_* = \mu_o$ on the non wetting phase side and $\mu_* = \mu_w$ on the wetting phase side, κ is the surface tension, H is the mean curvature of the capillary surface $\mathcal{S}_{ow}$ and H^ε is the mean curvature of the capillary surface $\mathcal{A}_{ow}^\varepsilon$.

Assuming that the ratio of the Froude number Fr to the Weber number We in the filter is very small, which means that surface tension forces are more dominant than gravitational forces, we define, using Equation (13)₃, the mean curvature on each capillary surface $\mathcal{A}_{ow,ij}^\varepsilon(\omega)$ by $H^\varepsilon = \varepsilon H_{ij}^\varepsilon$, where H_{ij}^ε is given by

$$H_{ij}^\varepsilon(x_3, \omega) = -\chi_{ij}^2(\omega) \frac{\partial^2 \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\partial x_3^2} \left(1 + \varepsilon \left| \chi_{ij}(\omega) \frac{\partial \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\partial x_3} \right|^2 \right)^{-3/2}. \quad (19)$$

The unit normal n^ε is given in cylindrical coordinates on each $\mathcal{A}_{ow,ij}^\varepsilon(\omega)$ by

$$\begin{aligned} n^\varepsilon(x_3 \varepsilon^{-1/2}, \omega) &= \left(1 + \varepsilon \left| \chi_{ij}(\omega) \frac{\partial \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\partial x_3} \right|^2 \right)^{-1/2} e_r \\ &\quad - \varepsilon^{1/2} \chi_{ij}(\omega) \frac{\partial \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\partial x_3} \left(1 + \varepsilon \left| \chi_{ij}(\omega) \frac{\partial \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\partial x_3} \right|^2 \right)^{-1/2} e_z. \end{aligned} \quad (20)$$

Let ω be a fixed event for which the conditions (5), (7), (9), and (10) are satisfied. Let us define the space

$$V^\varepsilon(\omega) = \{u \in H^1(\Omega^\varepsilon(\omega), \mathbb{R}^3); \quad \operatorname{div} u = 0, \quad u = 0 \text{ on } \Gamma^\varepsilon(\omega)\}. \quad (21)$$

The variational formulation of problem (18) reads as follows.

$$\begin{aligned}
 & 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u^\varepsilon) D_{ij}(\varphi) \, dx + 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} D_{ij}(u^\varepsilon) D_{ij}(\varphi) \, dx \\
 & - 2 \int_{S_{ow}} \kappa H n \cdot \varphi \, ds - 2 \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} \kappa H^\varepsilon n^\varepsilon \cdot \varphi \, ds_\varepsilon = \int_{\Omega^\varepsilon(\omega)} f \cdot \varphi \, dx,
 \end{aligned} \tag{22}$$

for every $\varphi \in V^\varepsilon(\omega)$, where $D_{ij}(u^\varepsilon) = \frac{1}{2}(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i})$ and where the summation convention with respect to repeated indices has been used and will be used in the sequel except in certain cases where the summation symbol will be used explicitly. Then, using, for instance, [33, Theorem 5.1], we deduce that problem (18) has a unique solution $(u^\varepsilon, p^\varepsilon) \in V^\varepsilon(\omega) \times L^2(\Omega^\varepsilon(\omega))/\mathbb{R}$.

Let us define the functional $\mathcal{F}_\varepsilon$ by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u) D_{ij}(u) \, dx \\ \quad + 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} D_{ij}(u) D_{ij}(u) \, dx & \text{if } u \in V^\varepsilon(\omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{23}$$

Then, the solution $(u^\varepsilon, p^\varepsilon) \in V^\varepsilon(\omega) \times L^2(\Omega^\varepsilon(\omega))/\mathbb{R}$ of problem (18) can be also obtained from the minimization problem

$$\inf_{u \in V^\varepsilon} \left\{ \mathcal{F}_\varepsilon(u) - 2 \int_{S_{ow}} \kappa H n \cdot u \, ds - 2 \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} \kappa H^\varepsilon n^\varepsilon \cdot u \, ds_\varepsilon - \int_{\Omega^\varepsilon(\omega)} f \cdot u \, dx \right\}. \tag{24}$$

3. A priori estimates

In this Section we formulate and prove fundamental a priori estimates for the velocity and pressure fields.

Lemma 3.1: *Let $u^\varepsilon \in V^\varepsilon(\omega)$. There exists a non-random constant C independent of ε , such that*

$$\begin{aligned}
 (1) \quad & \int_{\Omega_h^\varepsilon(\omega)} |u^\varepsilon|^2 \, dx \leq C \varepsilon^2 \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 \, dx, \quad \int_{\Omega} |u^\varepsilon|^2 \, dx \leq C \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx, \\
 (2) \quad & \left| \int_{S_{ow}} 2\kappa H n \cdot u^\varepsilon \, ds \right| \leq C \|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)}, \quad \left| \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot u^\varepsilon \, ds_\varepsilon \right| \leq C \varepsilon \\
 & \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)}.
 \end{aligned}$$

Proof: 1. Let $w \in H^1(D(0,1) \times (-h,0))$, where $D(0,1)$ is the unit disk centred at 0. We suppose that $w = 0$ on $\partial D(0,1) \times (-h,0)$. Using the Poincaré inequality in the disk

$D(0, 1)$, we have that

$$\int_{D(0,1)} w^2 \, dy_1 \, dy_2 \leq C \int_{D(0,1)} |\nabla w|^2 \, dy_1 \, dy_2,$$

where C is a positive constant. Then, rescaling by $(x_1 - i\varepsilon) = \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega) y_1$ and $(x_2 - j\varepsilon) = \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega) y_2$, we obtain for u^ε the inequality

$$\begin{aligned} & \int_{D((i\varepsilon, j\varepsilon), \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega))} |u^\varepsilon|^2 \, dx_1 \, dx_2 \\ & \leq C \varepsilon^2 \zeta_{ij}^2(x_3 \varepsilon^{-1/2}, \omega) \int_{D((i\varepsilon, j\varepsilon), \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega))} |\nabla u^\varepsilon|^2 \, dx_1 \, dx_2, \end{aligned}$$

where $D((i\varepsilon, j\varepsilon), \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega))$ is the disk of radius $\varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega)$ centred at the point $(i\varepsilon, j\varepsilon)$. Then, integrating over $(-h, 0)$ by taking into account hypothesis (5), we deduce that

$$\int_{F_{ij}^\varepsilon(\omega)} |u^\varepsilon|^2 \, dx \leq C \varepsilon^2 \int_{F_{ij}^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 \, dx,$$

where C is a non-random constant independent of ε . Therefore, summing over all $(i, j) \in I_\varepsilon(\omega)$, we obtain the estimate

$$\int_{\Omega_h^\varepsilon(\omega)} |u^\varepsilon|^2 \, dx \leq C \varepsilon^2 \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 \, dx. \quad (25)$$

On the other hand, since $u^\varepsilon = 0$ on $\partial\Omega \setminus \Sigma_b \cup \Sigma_h^\varepsilon(\omega)$, we have, using the Poincaré inequality,

$$\int_{\Omega} |u^\varepsilon|^2 \, dx \leq C \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx,$$

where C is a constant independent of ε .

2. Using the trace theorem, we immediately obtain that

$$\left| \int_{S_{ow}} 2\kappa H n \cdot u^\varepsilon \, ds \right| \leq C \|H\|_{L^2(S_{ow})} \|\nabla u^\varepsilon\|_{L^2(\Omega; \mathbb{R}^9)}.$$

Let $(i, j) \in I_\varepsilon(\omega)$, $x_3 \in (-h, 0)$, and $r \in (\varepsilon \varrho_{ij}^\varepsilon/2, \varepsilon \varrho_{ij}^\varepsilon)$, where $\varrho_{ij}^\varepsilon \equiv \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)$. Observing that

$$u^\varepsilon(\varepsilon \varrho_{ij}^\varepsilon, \theta, x_3) - u^\varepsilon(r, \theta, x_3) = \int_r^{\varepsilon \varrho_{ij}^\varepsilon} \frac{\partial u^\varepsilon}{\partial s} \, ds,$$

we deduce that

$$\begin{aligned} |u^\varepsilon(\varepsilon \varrho_{ij}^\varepsilon, \theta, x_3)|^2 & \leq 2|u^\varepsilon(r, \theta, x_3)|^2 + 2 \ln(\varepsilon \varrho_{ij}^\varepsilon/r) \int_r^{\varepsilon \varrho_{ij}^\varepsilon} \left| \frac{\partial u^\varepsilon}{\partial s} \right|^2 s \, ds \\ & \leq 2|u^\varepsilon(r, \theta, x_3)|^2 + 2 \ln(2) \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 \, dx, \end{aligned}$$

then, multiplying both sides of the inequality by r and integrating between 0 and ϱ_{ij}^ε , we infer, using (25), that

$$\begin{aligned} \int_{\mathcal{A}_{ow}^\varepsilon(\omega)} |u^\varepsilon(\varepsilon \varrho_{ij}^\varepsilon, \theta, x_3)|^2 ds_\varepsilon &\leq C \left(\int_{\Omega_h^\varepsilon(\omega)} |u^\varepsilon|^2 dx + \varepsilon \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx \right) \\ &\leq C \left(\varepsilon^2 \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx \right) \\ &\leq C\varepsilon \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx. \end{aligned} \quad (26)$$

Furthermore, using conditions (7) and (10), we have

$$\begin{aligned} \left(\int_{\mathcal{A}_{ow}^\varepsilon(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot u^\varepsilon ds_\varepsilon \right)^2 &\leq 4\kappa^2 \left(\int_{\mathcal{A}_{ow}^\varepsilon(\omega)} |H^\varepsilon n^\varepsilon|^2 ds_\varepsilon \right) \int_{\mathcal{A}_{ow}^\varepsilon(\omega)} |u^\varepsilon|^2 ds_\varepsilon \\ &= 4\kappa^2 \left(\sum_{(i,j) \in I_\varepsilon(\omega)} \int_{\mathcal{A}_{ow,ij}^\varepsilon(\omega)} \varepsilon^2 |H_{ij}^\varepsilon n^\varepsilon|^2 ds_\varepsilon \right) \|u^\varepsilon\|_{L^2(\mathcal{A}_{ow}^\varepsilon(\omega), \mathbb{R}^3)}^2 \\ &\leq C\varepsilon \|u^\varepsilon\|_{L^2(\mathcal{A}_{ow}^\varepsilon(\omega), \mathbb{R}^3)}^2. \end{aligned} \quad (27)$$

Thus, combining (26) and (27), we get

$$\left(\int_{\mathcal{A}_{ow}^\varepsilon(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot u^\varepsilon ds_\varepsilon \right)^2 \leq C\varepsilon^2 \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx. \quad \blacksquare$$

We have the following estimates:

Lemma 3.2: *Let $u^\varepsilon \in V^\varepsilon(\omega)$, such that $\sup \mathcal{F}_\varepsilon(u^\varepsilon) < +\infty$. Then*

- (1) $\sup_\varepsilon \int_\Omega |\nabla u^\varepsilon|^2 dx < +\infty$, $\sup_\varepsilon \varepsilon^2 \int_{\Omega_h^\varepsilon(\omega)} |\nabla u^\varepsilon|^2 dx < +\infty$,
- (2) $\sup_\varepsilon \int_\Omega |u^\varepsilon|^2 dx < +\infty$, $\sup_\varepsilon \int_{\Omega_h^\varepsilon(\omega)} |u^\varepsilon|^2 dx < +\infty$.

Proof: Observing that

$$\mathcal{F}_\varepsilon(u^\varepsilon) \geq \min(\mu_o, \mu_w) \{ \|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)}^2 + \varepsilon^2 \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)}^2 \}, \quad (28)$$

we deduce that

$$\sup_\varepsilon \{ \|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)} \} < +\infty.$$

Then, using Lemma 3.1, we deduce the second point of the Lemma. \blacksquare

Remark 3.3: Let u^ε be the (velocity) solution of problem (24). Then

$$\mathcal{F}_\varepsilon(u^\varepsilon) - 2\kappa \int_{S_{ow}} H n \cdot u^\varepsilon ds$$

$$- 2\kappa \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} H^\varepsilon n^\varepsilon \cdot u^\varepsilon \, ds_\varepsilon - \int_{\Omega^\varepsilon(\omega)} f \cdot u^\varepsilon \, dx \leq 0,$$

from which we deduce that

$$\begin{aligned} \mathcal{F}_\varepsilon(u^\varepsilon) &\leq \left| 2\kappa \int_{S_{ow}} H n \cdot u^\varepsilon \, ds \right| + \left| 2\kappa \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} H^\varepsilon n^\varepsilon \cdot u^\varepsilon \, ds_\varepsilon \right| \\ &\quad + \|f\|_{L^2(\Omega \cup \Sigma \cup \Omega_h, \mathbb{R}^3)} \|u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^3)} \\ &\quad + \|f\|_{L^2(\Omega \cup \Sigma \cup \Omega_h, \mathbb{R}^3)} \|u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^3)}. \end{aligned}$$

Then, using Lemma 3.1, we deduce that

$$\mathcal{F}_\varepsilon(u^\varepsilon) \leq C\{\|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)}\},$$

and, using inequality (28), we deduce that $\sup_\varepsilon \mathcal{F}_\varepsilon(u^\varepsilon) < +\infty$ and

$$\sup_\varepsilon \{\|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)}\} < +\infty.$$

We now deal with the pressure. We start with the following useful result due to Bogovskii [34]:

Lemma 3.4: *Let $D \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary ∂D . There exists a linear operator $\mathcal{B} : L^2(D) \rightarrow H_0^1(D, \mathbb{R}^3)$ such that, for every $g \in L^2(D)$ satisfying $\int_D g \, dx = 0$,*

$$\begin{cases} \operatorname{div}(\mathcal{B}(g)) &= g \quad \text{in } D, \\ \|\nabla \mathcal{B}(g)\|_{L^2(D, \mathbb{R}^9)} &\leq C(D) \|g\|_{L^2(D)}, \end{cases}$$

where $C(D)$ is a constant which only depends on D .

Let $(u^\varepsilon, p^\varepsilon) \in V^\varepsilon(\omega) \times L^2(\Omega^\varepsilon(\omega))/\mathbb{R}$ be the solution of problem (18). We define the zero mean-value pressure

$$\begin{cases} \tilde{p}^\varepsilon &= p^\varepsilon - \frac{1}{|\Omega|} \int_\Omega p^\varepsilon \, dx \quad \text{in } \Omega, \\ \bar{p}^\varepsilon &= p^\varepsilon - \frac{1}{|\Omega_{\alpha,h}^\varepsilon(\omega)|} \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} p^\varepsilon \, dx \quad \text{in } \Omega_{\alpha,h}^\varepsilon(\omega). \end{cases} \quad (29)$$

We have the following estimates:

Lemma 3.5: (1) $\sup_\varepsilon \|\tilde{p}^\varepsilon\|_{L^2(\Omega)} < +\infty$,
 (2) $\sup_\varepsilon \|\bar{p}^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega))} < +\infty$.

Proof: According to Lemma 3.4, there exists $\varphi^\varepsilon \in H_0^1(\Omega, \mathbb{R}^3)$ such that

$$\begin{cases} -\operatorname{div} \varphi^\varepsilon &= \tilde{p}^\varepsilon & \text{in } \Omega, \\ \varphi^\varepsilon &= 0 & \text{on } \partial\Omega, \end{cases} \quad (30)$$

and $\|\nabla \varphi^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)} \leq C\|\tilde{p}^\varepsilon\|_{L^2(\Omega)}$, where C is a positive constant independent of ε . Then, using the variational formulation (22) with $\varphi := \varphi^\varepsilon$, we have that

$$\begin{aligned} & 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u^\varepsilon) D_{ij}(\varphi^\varepsilon) \, dx \\ & + \int_{\Omega} (\tilde{p}^\varepsilon)^2 \, dx - \int_{S_{\text{ow}}} 2\kappa H n \cdot \varphi^\varepsilon \, ds = \int_{\Omega} f \cdot \varphi^\varepsilon \, dx, \end{aligned}$$

which implies, using Lemma 3.1, that

$$\int_{\Omega} (\tilde{p}^\varepsilon)^2 \, dx \leq C\{\|f\|_{L^2(\Omega, \mathbb{R}^3)} + \|\nabla u^\varepsilon\|_{L^2(\Omega, \mathbb{R}^9)} + 1\}\|\tilde{p}^\varepsilon\|_{L^2(\Omega)},$$

from which we deduce, using (29) and Remark 3.3, that

$$\sup_{\varepsilon} \|\tilde{p}^\varepsilon\|_{L^2(\Omega)} < +\infty. \quad (31)$$

Let us now define the cylinder

$$G_{ij}^\varepsilon(\omega) = \{(y_1, y_2, x_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 < \zeta_{ij}^2(x_3 \varepsilon^{-1/2}, \omega), x_3 \in (-h, 0)\}. \quad (32)$$

We consider the following problem in $G_{ij}^\varepsilon(\omega)$:

$$\begin{cases} \operatorname{div}_y(\psi^\varepsilon) &= \bar{p}^\varepsilon(\varepsilon y_1 + i\varepsilon, \varepsilon y_2 + j\varepsilon, x_3) & \text{in } G_{ij}^\varepsilon(\omega), \\ \psi^\varepsilon &= 0 & \text{on } \partial G_{ij}^\varepsilon(\omega). \end{cases} \quad (33)$$

Then, according to Lemma 3.4, there exists a solution $\psi^\varepsilon \in H_0^1(G_{ij}^\varepsilon(\omega), \mathbb{R}^3)$ of problem (33), such that

$$\int_{G_{ij}^\varepsilon(\omega)} |\nabla \psi^\varepsilon|^2 \, dy \, dx_3 \leq C \int_{G_{ij}^\varepsilon(\omega)} (\bar{p}^\varepsilon)^2 \, dy \, dx_3, \quad (34)$$

where the constant C is non-random and independent of ε . Let us define

$$\phi^\varepsilon(x) = \begin{pmatrix} \varepsilon(\psi^\varepsilon)_1((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \\ \varepsilon(\psi^\varepsilon)_2((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \\ (\psi^\varepsilon)_3((x_1 - i\varepsilon)/\varepsilon, (x_2 - j\varepsilon)/\varepsilon, x_3) \end{pmatrix},$$

then ϕ^ε verifies

$$\begin{cases} \operatorname{div}(\phi^\varepsilon)(x) &= \operatorname{div}_y(\psi^\varepsilon) \begin{pmatrix} (x_1 - i\varepsilon)/\varepsilon \\ (x_2 - j\varepsilon)/\varepsilon \\ x_3 \end{pmatrix} = \bar{p}^\varepsilon(x) & \text{in } F_{ij}^\varepsilon(\omega), \\ \phi^\varepsilon &= 0 & \text{on } \partial F_{ij}^\varepsilon(\omega), \end{cases} \quad (35)$$

and, according to (34),

$$\int_{F_{ij}^\varepsilon(\omega)} |\nabla \phi^\varepsilon|^2 \, dx \leq \varepsilon^2 \int_{G_{ij}^\varepsilon(\omega)} |\nabla \psi^\varepsilon|^2 \, dy \, dx_3$$

$$\begin{aligned}
&\leq \varepsilon^2 C \int_{G_{ij}^\varepsilon(\omega)} (\bar{p}^\varepsilon)^2 \, dy \, dx_3 \\
&= C \int_{F_{ij}^\varepsilon(\omega)} (\bar{p}^\varepsilon)^2 \, dx.
\end{aligned} \tag{36}$$

Then, applying the variational formulation (22) for $\varphi := \phi^\varepsilon$, we get

$$\begin{aligned}
&2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} D_{ij}(u^\varepsilon) D_{ij}(\phi^\varepsilon) \, dx \\
&+ \int_{\Omega_h^\varepsilon(\omega)} (\bar{p}^\varepsilon)^2 \, dx - \int_{\mathcal{A}_{\text{ow}}^{h,\varepsilon}(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot \phi^\varepsilon \, ds = 0,
\end{aligned}$$

from which we deduce, using inequality (36) and Remark 3.3, that

$$\begin{aligned}
\int_{\Omega_h^\varepsilon(\omega)} (\bar{p}^\varepsilon)^2 \, dx &\leq C\{\varepsilon^2 \|\nabla u^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)} + 1\} \|\nabla \phi^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega), \mathbb{R}^9)} \\
&\leq C\|\bar{p}^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega))},
\end{aligned}$$

hence

$$\sup_\varepsilon \|\bar{p}^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega))} < +\infty.$$

■

Remark 3.6: We define the mean of extension of the pressure in each phase by

$$\bar{p}_\alpha^\varepsilon(x) = \begin{cases} \bar{p}^\varepsilon - \frac{1}{|\Omega_{\alpha,h}^\varepsilon(\omega)|} \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} \bar{p}^\varepsilon(x) \, dx & \text{in } \Omega_{\alpha,h}^\varepsilon(\omega), \alpha = o, w, \\ \frac{1}{|\Omega_{\beta,h}^\varepsilon(\omega)|} \int_{\Omega_{\beta,h}^\varepsilon(\omega)} \bar{p}^\varepsilon(x) \, dx & \text{in } \Omega_{\beta,h}^\varepsilon(\omega), \beta \neq \alpha. \end{cases} \tag{37}$$

Then, $\bar{p}^\varepsilon = \bar{p}_o^\varepsilon + \bar{p}_w^\varepsilon$ and, using Lemma 3.5₂, we have

$$\sup_\varepsilon \|\bar{p}_\alpha^\varepsilon\|_{L^2(\Omega_h^\varepsilon(\omega))} < +\infty, \quad \alpha = o, w. \tag{38}$$

4. Compactness results

In this Section we establish compactness results for the convergence of velocity and pressure fields. Let ω be a fixed event for which the conditions (5), (7), (9), and (10) are satisfied. We prove the following convergence in the filter:

Lemma 4.1: (1) For every $\varphi \in C_c^1(\Omega_h)$, we have

$$\lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \int_{F_{ij}^\varepsilon(\omega)} \varphi(x_1, x_2, x_3) \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma \varphi(x', 0) \, dx',$$

where $x' = (x_1, x_2)$.

- (2) Let $w_\varepsilon \in L^2(\Omega_h^\varepsilon(\omega))$, such that $\sup_\varepsilon \int_{\Omega_h^\varepsilon(\omega)} (w_\varepsilon)^2 dx < +\infty$. There exists a subsequence of the sequence $(w_\varepsilon)_\varepsilon$, still denoted in the same way, and $w \in L^2(\Sigma)$, such that, for every $\varphi \in C_0(\mathbb{R}^3)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} w_\varepsilon \varphi dx = h \langle \zeta^2(0) \rangle \int_\Sigma w(x') \varphi(x', 0) dx'.$$

Proof: 1. Let us introduce the change of variables $y_1 = \frac{(x_1 - i\varepsilon)}{\varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega)}$ and $y_2 = \frac{(x_2 - j\varepsilon)}{\varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega)}$. Then, using conditions (5), the smoothness of φ , and the ergodicity property (6), we infer that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \int_{F_{ij}^\varepsilon(\omega)} \varphi(x_1, x_2, x_3) dx \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \varepsilon^2 \int_{D(0,1)} \int_{-h}^0 \varphi \left(\frac{\varepsilon y_1 \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega) + i\varepsilon}{\varepsilon y_2 \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega) + j\varepsilon}, x_3 \right) \zeta_{ij}^2(x_3 \varepsilon^{-1/2}, \omega) dy dx_3 \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \varepsilon^2 \int_{-h}^0 \varphi(\varepsilon i, \varepsilon j, x_3) \zeta_{ij}^2(x_3 \varepsilon^{-1/2}, \omega) dx_3 \\ &= \lim_{\varepsilon \rightarrow 0} \frac{h}{h\varepsilon^{-1/2}} \int_{-h\varepsilon^{-1/2}}^0 \left(\int_\Sigma \varphi(x_1, x_2, \varepsilon^{1/2} z) dx_1 dx_2 \right) \zeta^2(z) dz \\ &= h \langle \zeta^2(0) \rangle \int_\Sigma \varphi(x', 0) dx'. \end{aligned}$$

2. Let us introduce the measure $\nu_\varepsilon = \mathbf{1}_{\Omega_h^\varepsilon(\omega)} dx$, where $\mathbf{1}_A$ is the characteristic function of the set A . We can easily check that, as ε goes to 0, the sequence $(\nu_\varepsilon)_\varepsilon$ converges in the weak sense of measures to the measure $\nu = h \langle \zeta^2(0) \rangle \mathbf{1}_\Sigma(x') dx'$. By Hölder inequality

$$\int_{\mathbb{R}^3} |w_\varepsilon| d\nu_\varepsilon \leq \left\{ \int_{\Omega_h^\varepsilon(\omega)} (w_\varepsilon)^2 dx \right\}^{1/2} (\nu_\varepsilon(\mathbb{R}^3))^{1/2},$$

thus, using the hypothesis on $(w_\varepsilon)_\varepsilon$, the sequence $(\nu_\varepsilon w_\varepsilon)_\varepsilon$ is uniformly bounded in variation, hence $*$ -weakly relatively compact. We deduce from this that, up to some subsequence, the sequence $(\nu_\varepsilon w_\varepsilon)_\varepsilon$ $*$ -weakly converges to some measure χ . Let $\varphi \in C_0(\mathbb{R}^3)$. We have, using Fenchel's inequality

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} (w_\varepsilon)^2 d\nu_\varepsilon &\geq \liminf_{\varepsilon \rightarrow 0} \left(2 \int_{\mathbb{R}^3} w_\varepsilon \varphi d\nu_\varepsilon - \int_{\mathbb{R}^3} \varphi^2 d\nu_\varepsilon \right) \\ &\geq 2 \langle \chi, \varphi \rangle - \int_{\mathbb{R}^3} \varphi^2 d\nu. \end{aligned} \tag{39}$$

As the left hand side of (39) is bounded, we infer that

$$\sup \left\{ \langle \chi, \varphi \rangle \mid \varphi \in C_0(\mathbb{R}^3), \int_{\mathbb{R}^2} \varphi^2(x', 0) \, dv \leq 1 \right\} < +\infty.$$

Therefore, according to Riesz' representation Theorem, there exists $w \in L_v^2(\mathbb{R}^3)$, such that $\chi = wv$. ■

We have the following convergences:

Lemma 4.2: *Let $u^\varepsilon \in V^\varepsilon(\omega)$, such that $\sup \mathcal{F}_\varepsilon(u^\varepsilon) < +\infty$. Then, there exists a subsequence of the sequence $(u^\varepsilon)_\varepsilon$, still denoted in the same way, $u \in H^1(\Omega, \mathbb{R}^3)$, and $v \in L^2(\Sigma, \mathbb{R}^3)$, such that*

$$\begin{aligned} u^\varepsilon|_\Omega &\rightharpoonup u \quad H^1(\Omega, \mathbb{R}^3)\text{-weak}, \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \varphi \, dx &= h \langle \zeta^2(0) \rangle \int_\Sigma v(x') \cdot \varphi(x', 0) \, dx', \quad \forall \varphi \in C_0(\mathbb{R}^3, \mathbb{R}^3), \end{aligned}$$

$\operatorname{div} u = 0$ in Ω , $\operatorname{div} v_\tau = 0$ in Σ , where $v_\tau = (v_1, v_2)$, and $v_3 = \frac{u_3|_\Sigma}{\langle \zeta^2(0) \rangle}$.

Proof: 1. Due to Lemma 3.2, there exists a subsequence, still denoted in the same way, and $u \in H^1(\Omega, \mathbb{R}^3)$, such that

$$u^\varepsilon|_\Omega \rightharpoonup u \quad H^1(\Omega, \mathbb{R}^3)\text{-weak}.$$

Let $\varphi \in C_c(\Omega)$. Then

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \operatorname{div} u^\varepsilon \varphi \, dx = - \lim_{\varepsilon \rightarrow 0} \int_\Omega u^\varepsilon \cdot \nabla \varphi \, dx \\ &= - \int_\Sigma u \cdot \nabla \varphi \, dx \\ &= \int_\Sigma \operatorname{div} u \varphi \, dx, \end{aligned}$$

from which we deduce that $\operatorname{div} u = 0$ in Ω .

Due to Lemmas 3.2 and 4.1, there exists a subsequence, still denoted in the same way, and $v \in L^2(\Sigma, \mathbb{R}^3)$, such that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \varphi \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma v(x') \cdot \varphi(x', 0) \, dx', \quad \forall \varphi \in C_0(\mathbb{R}^3, \mathbb{R}^3).$$

Let $\varphi \in C_c^1(\Sigma)$. Then, since $\operatorname{div} u^\varepsilon = 0$, we have, using Lemmas 3.2 and 4.1,

$$0 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \operatorname{div} u^\varepsilon \varphi \, dx = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ 0 \end{pmatrix} \, dx$$

$$\begin{aligned}
 &= -h\langle \zeta^2(0) \rangle \int_{\Sigma} v \cdot \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ 0 \end{pmatrix} dx' \\
 &= h\langle \zeta^2(0) \rangle \int_{\Sigma} \varphi \operatorname{div} v_{\tau} dx',
 \end{aligned}$$

from which we deduce that $\operatorname{div} v_{\tau} = 0$ in Σ .

Let $\varphi \in C_c(\Omega \times \Sigma \times \Omega_h)$, then

$$\begin{aligned}
 0 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \operatorname{div} u^{\varepsilon} \varphi dx = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u^{\varepsilon} \cdot \nabla \varphi dx - \lim_{\varepsilon \rightarrow 0} \int_{\Sigma^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx \\
 &= - \int_{\Omega} u \cdot \nabla \varphi dx - \lim_{\varepsilon \rightarrow 0} \int_{\Sigma^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx' \\
 &= \int_{\Sigma} u_3(x') \varphi(x', 0) dx' - \lim_{\varepsilon \rightarrow 0} \int_{\Sigma^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx',
 \end{aligned}$$

from which we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx' = \int_{\Sigma} u_3(x') \varphi(x', 0) dx'. \quad (40)$$

Let us now take $\varphi \in C_c(\Sigma)$. Then, using (40) and the fact that $\operatorname{div} v_{\tau} = 0$ in Σ , we have

$$\begin{aligned}
 0 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^{\varepsilon}(\omega)} (h + x_3) \operatorname{div} u^{\varepsilon} \varphi dx = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^{\varepsilon}(\omega)} (h + x_3) u_{\tau}^{\varepsilon} \cdot \nabla \varphi dx \\
 &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx + \lim_{\varepsilon \rightarrow 0} h \int_{\Sigma^{\varepsilon}(\omega)} u_3^{\varepsilon} \varphi dx' \\
 &= -h\langle \zeta^2(0) \rangle \int_{\Sigma} v_3 \varphi dx' + h \int_{\Sigma} u_3 \varphi dx',
 \end{aligned}$$

from which we deduce that $v_3 = \frac{u_3|_{\Sigma}}{\langle \zeta^2(0) \rangle}$. ■

Remark 4.3: Let $(u^{\varepsilon}, p^{\varepsilon}) \in V^{\varepsilon}(\omega) \times L^2(\Omega^{\varepsilon}(\omega))/\mathbb{R}$ be the solution of problem (18). Then, using Remark 3.3, Lemmas 3.5, and 4.2, we deduce that there exists a subsequence of the sequence $(u^{\varepsilon}, p^{\varepsilon})_{\varepsilon}$, still denoted in the same way, $u \in H^1(\Omega, \mathbb{R}^3)$, $p \in L^2(\Omega)$, $v \in L^2(\Sigma)$, and $q \in L^2(\Sigma)$, such that

$$(1) \quad u^{\varepsilon} \rightharpoonup_{\varepsilon \rightarrow 0} u \text{ in } H^1(\Omega, \mathbb{R}^3)\text{-weak}, \quad \tilde{p}^{\varepsilon} \rightharpoonup_{\varepsilon \rightarrow 0} p \text{ in } L^2(\Omega)\text{-weak}, \quad \text{and } \operatorname{div} u = 0 \text{ in } \Omega,$$

(2) for every $\varphi \in C_0(\mathbb{R}^3, \mathbb{R}^3)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \varphi \, dx = h \langle \zeta^2(0) \rangle \int_{\Sigma} v(x') \cdot \varphi(x', 0) \, dx',$$

for every $\varphi \in C_0(\mathbb{R}^3)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \bar{P}^\varepsilon \varphi \, dx = h \langle \zeta^2(0) \rangle \int_{\Sigma} q(x') \varphi(x', 0) \, dx',$$

$$\operatorname{div} v_\tau = 0 \text{ in } \Sigma, \text{ and } v_3 = \frac{u_3|_\Sigma}{\langle \zeta^2(0) \rangle}.$$

5. Local problems determining the two-phase relative permeability

In this Section we introduce and study the auxiliary problems determining the two-phase relative permeability in the filter. Let $D(0, 1/2)$ and $D(0, 1/4)$ be the disks centred at the origin with radii $1/2$ and $1/4$ respectively. We set

$$\begin{aligned} D_o &= D(0, 1/4), \\ D_w &= D(0, 1/2) \setminus \overline{D(0, 1/4)}, \\ \mathcal{C}_{ow} &= \{(y_1, y_2) \in \mathbb{R}^2; \sqrt{y_1^2 + y_2^2} = 1/4\}. \end{aligned}$$

For $m = 1, 2$, we consider the following two-phase viscous incompressible fluid problem:

$$\left\{ \begin{array}{ll} -\mu_\alpha \Delta_y w^m + \nabla_y \pi^m &= e_m \quad \text{in } D_\alpha, \alpha = o, w, \\ \operatorname{div}(w^m) &= 0 \quad \text{in } D(0, 1/2), \\ [w^m]_{ow} &= 0 \quad \text{on } \mathcal{C}_{ow}, \\ \left[\mu_* \left(\frac{\partial w_i^m}{\partial y_j} + \frac{\partial w_j^m}{\partial y_i} \right) n_j - \pi^m n_i \right]_{ow} &= 0 \quad \text{on } \mathcal{C}_{ow}, \\ w^m &= 0 \quad \text{on } \partial D(0, 1/2), \end{array} \right. \quad (41)$$

where $e_m = (\delta_{1m}, \delta_{2m})$ and n is the outward unit normal directed to the non wetting phase. Problem (41) has a unique solution (w^m, π^m) , such that $w^m \in H_0^1(D(0, 1/2), \mathbb{R}^2)$ and $\pi^m \in L^2(D(0, 1/2))/\mathbb{R}$ (see for instance [33, Theorem 5.1]). Let us now consider the following problem related to a two-phase axisymmetric Poiseuille flow in the circular cylinder $D(0, 1/2) \times (-h, 0)$:

$$\left\{ \begin{array}{ll} \mu_\alpha \Delta_y \eta &= 1 \quad \text{in } D_\alpha, \alpha = w, o, \\ [\eta]_{ow} &= 0 \quad \text{on } \mathcal{C}_{ow}, \\ \eta &= 0 \quad \text{on } \partial D(0, 1/2). \end{array} \right. \quad (42)$$

The solution of problem (42) is given explicitly (see, for instance, [5]) by a radial function $\eta(r)$, $r = \sqrt{y_1^2 + y_2^2}$, defined by

$$\eta(r) = \begin{cases} \frac{1}{4\mu_o\mu_w} \left(\mu_w r^2 - \frac{(3\mu_o + \mu_w)}{16} \right) & \text{in } D_o, \\ -\frac{4^3\mu_w}{3} & \text{on } C_{ow}, \\ \frac{1}{4\mu_w} \left(r^2 - \frac{1}{4} \right) & \text{in } D_w. \end{cases} \quad (43)$$

Let ω be a fixed event for which the conditions (5), (7), (9), and (10) are satisfied. Let us introduce the following intermediate random functions:

$$\begin{aligned} \kappa_{ij}(x_3\varepsilon^{-1/2}, \omega) &= \frac{1}{4(\zeta_{ij}(x_3\varepsilon^{-1/2}, \omega) - \varrho_{ij}(x_3\varepsilon^{-1/2}, \omega))}, \\ \lambda_{ij}(x_3\varepsilon^{-1/2}, \omega) &= \frac{\zeta_{ij}(x_3\varepsilon^{-1/2}, \omega) - 2\varrho_{ij}(x_3\varepsilon^{-1/2}, \omega)}{4(\zeta_{ij}(x_3\varepsilon^{-1/2}, \omega) - \varrho_{ij}(x_3\varepsilon^{-1/2}, \omega))}, \end{aligned} \quad (44)$$

for every $(i, j) \in I_\varepsilon(\omega)$, and set

$$\sigma_{ij}^\varepsilon(x, \omega) = \kappa_{ij}(x_3\varepsilon^{-1/2}, \omega)r_{ij}^\varepsilon(x') + \lambda_{ij}(x_3\varepsilon^{-1/2}, \omega), \quad (45)$$

where

$$r_{ij}^\varepsilon(x') = \sqrt{\left(\frac{x_1 - i\varepsilon}{\varepsilon}\right)^2 + \left(\frac{x_2 - j\varepsilon}{\varepsilon}\right)^2}.$$

We define the sequence of functions $(w^{\varepsilon, m})_\varepsilon$, $m = 1, 2, 3$, by

$$\begin{aligned} w^{\varepsilon, m} &= \begin{pmatrix} w_{1ij}^{\varepsilon, m} \\ w_{2ij}^{\varepsilon, m} \\ 0 \end{pmatrix} \quad \text{in } F_{ij}^\varepsilon(\omega), \forall (i, j) \in I_\varepsilon(\omega), \\ w^{\varepsilon, 3} &= \begin{pmatrix} 0 \\ 0 \\ w_{3ij}^{\varepsilon, 3} \end{pmatrix} \quad \text{in } F_{ij}^\varepsilon(\omega), \quad \forall (i, j) \in I_\varepsilon(\omega), \end{aligned} \quad (46)$$

where, using the polar components w_r^m and w_θ^m of w^m for $m = 1, 2$,

$$\begin{cases} w_{1ij}^{\varepsilon, m}(x) = \begin{cases} (w_r^m \cos \theta - w_\theta^m \sin \theta) \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3\varepsilon^{-1/2}, \omega)}, \theta \right) & \text{for } x \in F_{o,ij}^\varepsilon(\omega), \\ (w_r^m \cos \theta - w_\theta^m \sin \theta) (\sigma_{ij}^\varepsilon(x, \omega), \theta) & \text{for } x \in F_{w,ij}^\varepsilon(\omega), \end{cases} \\ w_{2ij}^{\varepsilon, m}(x) = \begin{cases} (w_r^m \sin \theta + w_\theta^m \cos \theta) \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3\varepsilon^{-1/2}, \omega)}, \theta \right) & \text{for } x \in F_{o,ij}^\varepsilon(\omega), \\ (w_r^m \sin \theta + w_\theta^m \cos \theta) (\sigma_{ij}^\varepsilon(x, \omega), \theta) & \text{for } x \in F_{w,ij}^\varepsilon(\omega), \end{cases} \end{cases} \quad (47)$$

and where

$$w_{3ij}^{\varepsilon,3}(x) = \begin{cases} \eta(r_{ij}^{\varepsilon}(x')/4\varrho_{ij}(x_3\varepsilon^{-1/2},\omega)) & \text{for } x \in F_{o,ij}^{\varepsilon}(\omega), \\ \eta(\sigma_{ij}^{\varepsilon}(x,\omega)) & \text{for } x \in F_{w,ij}^{\varepsilon}(\omega). \end{cases} \quad (48)$$

Let us introduce the 2×2 matrix $K = \sum_{\alpha=o,w} K^{\alpha}$, where $K^{\alpha} = (K_{lm}^{\alpha})_{l,m=1,2}$, $\alpha = o, w$, with

$$\begin{aligned} K_{lm}^o &= 16\langle \varrho(0)^2 \rangle \int_{D_o} w_l^m(y) \, dy, \\ K_{lm}^w &= 16\langle (\zeta(0) - \varrho(0))^2 \rangle \int_{D_w} w_l^m(y) \, dy \\ &\quad - 4 \frac{\langle (\zeta(0) - \varrho(0))(\zeta(0) - 2\varrho(0)) \rangle}{|D_w|} \int_{D_w} \frac{w_l^m(y)}{|y|} \, dy, \end{aligned} \quad (49)$$

and the constant $\gamma = \gamma_o + \gamma_w$ with

$$\begin{aligned} \gamma_o &= -\frac{2\pi(6\mu_o + \mu_w)\langle \varrho^2(0) \rangle}{4^4\mu_o\mu_w}, \\ \gamma_w &= \frac{2\pi\langle (\zeta(0) - \varrho(0))(7\varrho(0) - 5\zeta(0)) \rangle}{4^4\mu_w}. \end{aligned} \quad (50)$$

Let $v \in C^1(\Sigma, \mathbb{R}^3)$. We set

$$\begin{aligned} v_{\tau} &= (v_1, v_2), \\ v_{\tau}^{\alpha} &= K^{-1}K^{\alpha}v_{\tau}, \quad \alpha = o, w, \\ v_3^{\alpha} &= \frac{\gamma_{\alpha}}{\gamma}v_3, \quad \alpha = o, w. \end{aligned} \quad (51)$$

define the sequence of functions $(v^{\alpha\varepsilon})_{\varepsilon}$, $\alpha = o, w$, by

$$\begin{aligned} v_{\tau}^{\alpha\varepsilon}(x') &= \frac{\langle \zeta^2(0) \rangle}{h} \sum_{(i,j) \in I_{\varepsilon}(\omega)} v_m^{\alpha}(i\varepsilon, j\varepsilon) K^{-\alpha} \int_{-h}^0 w_{ij}^{\varepsilon,m}(x) \, dx_3, \\ v_3^{\alpha\varepsilon}(x') &= \frac{\langle \zeta^2(0) \rangle}{h\gamma_{\alpha}} \sum_{(i,j) \in I_{\varepsilon}(\omega)} v_3^{\alpha}(i\varepsilon, j\varepsilon) \left(\int_{-h}^0 w_{3ij}^{\varepsilon,3}(x) \, dx_3 \right), \end{aligned} \quad (52)$$

and the sequence of test-functions $(v^{\varepsilon})_{\varepsilon}$ by

$$v^{\varepsilon} = \begin{cases} v^{o\varepsilon} & \text{in } \Omega_{oh}^{\varepsilon}(\omega), \\ v^{w\varepsilon} & \text{in } \Omega_{wh}^{\varepsilon}(\omega). \end{cases} \quad (53)$$

We define the matrix $\Lambda = \sum_{\alpha=o,w} \Lambda^{\alpha}$, $\alpha = o, w$, by

$$\Lambda_{lm}^o = \int_{D_o} w_l^m(y) \, dy,$$

$$\begin{aligned}\Lambda_{lm}^w &= \int_{D_w} w_l^m(y) dy \\ &\quad - \frac{1}{4} \left\langle \frac{\zeta(0) - 2\varrho(0)}{\zeta(0) - \varrho(0)} \right\rangle \int_{D_w} \frac{\nabla w^m(y) : \nabla w^l(y)}{|y|} dy.\end{aligned}\quad (54)$$

The sequence $(v^\varepsilon)_\varepsilon$ verifies the following convergences:

Lemma 5.1:

- (1) $\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} v^\varepsilon \cdot \varphi \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma v \cdot \varphi \, dx'$, for every $\varphi \in C_0(\Sigma, \mathbb{R}^3)$.
- (2) $\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \mu_\alpha \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} |\nabla v_\tau^{\alpha\varepsilon}|^2 \, dx = 2\mu_\alpha \langle \zeta^2(0) \rangle^2 \int_\Sigma (K^{*\alpha})^{-1} v_\tau^\alpha \cdot v_\tau^\alpha \, dx'$, where

$$K^{*\alpha} = \Lambda^{-\alpha} (K^\alpha)^2.$$

- (3) $\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \mu_\alpha \int_{\Omega_{\alpha,h}^\varepsilon(\omega)} |\nabla v_3^{\alpha\varepsilon}|^2 \, dx = 2\mu_\alpha \langle \zeta^2(0) \rangle^2 \gamma_\alpha^* \gamma_\alpha^{-2} \int_\Sigma (v_3^\alpha)^2 \, dx'$, where

$$\begin{aligned}\gamma_o^* &= \frac{2\pi}{\mu_o^2 4^6}, \\ \gamma_w^* &= \frac{2\pi}{\mu_w^2 4^4} \left(\frac{15}{16} - \frac{7}{3} \left\langle \frac{\zeta(0) - 2\varrho(0)}{\zeta(0) - \varrho(0)} \right\rangle \right).\end{aligned}$$

Proof: 1. We have

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} v_\tau^\varepsilon \cdot \varphi \, dx \\ = \lim_{\varepsilon \rightarrow 0} \langle \zeta^2(0) \rangle \sum_{\substack{m=1,2 \\ \alpha=o,w \\ (i,j) \in I_\varepsilon(\omega)}} (v_m^\alpha \varphi)(i\varepsilon, j\varepsilon) K^{-\alpha} \int_{F_{\alpha ij}^\varepsilon(\omega)} w_{ij}^{\varepsilon,m}(x) \, dx_3 \, dx.\end{aligned}\quad (55)$$

for every $\varphi \in C_0(\Sigma, \mathbb{R}^3)$. Let us set, for every $x_3 \in (-h, 0)$,

$$\begin{aligned}D_{o,ij}^\varepsilon(x_3, \omega) &= \{x \in \mathbb{R}^3 \mid R_{ij}^\varepsilon(x_1, x_2) < \varepsilon \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)\}, \\ D_{w,ij}^\varepsilon(x_3, \omega) &= \{x \in \mathbb{R}^3 \mid \varepsilon \varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega) < R_{ij}^\varepsilon(x_1, x_2) < \varepsilon \zeta_{ij}(x_3 \varepsilon^{-1/2}, \omega)\}.\end{aligned}\quad (56)$$

Then, using the expression (47) of $w_{ij}^{\varepsilon,m}$ in polar coordinates, we can write

$$\int_{F_{w,ij}^\varepsilon(\omega)} w_{ij}^{\varepsilon,m}(x) \, dx = \int_{-h}^0 \int_{D_{w,ij}^\varepsilon(x_3, \omega)} w^m(\sigma_{ij}^\varepsilon(x, \omega) \cos \theta, \sigma_{ij}^\varepsilon(x, \omega) \sin \theta) r \, dr \, d\theta \, dx_3,$$

and

$$\int_{F_{o,ij}^\varepsilon(\omega)} w_{ij}^{\varepsilon,m}(x) \, dx$$

$$= \int_{-h}^0 \int_{D_{0,ij}^\varepsilon(x_3, \omega)} w^m \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \cos \theta, \frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \sin \theta \right) r \, dr \, d\theta \, dx_3,$$

we obtain, introducing first the change of variable

$$s = \sigma_{ij}^\varepsilon(x, \omega) \quad \text{in } D_{w,ij}^\varepsilon(x_3, \omega),$$

and then passing to Cartesian coordinates, that

$$\begin{aligned} & \int_{-h}^0 \int_{D_{w,ij}^\varepsilon(x_3, \omega)} w^m(\sigma_{ij}^\varepsilon(x, \omega) \cos \theta, \sigma_{ij}^\varepsilon(x, \omega) \sin \theta) r \, dr \, d\theta \, dx_3 \\ &= 16\varepsilon^2 \int_{-h}^0 ((\zeta_{ij} - \varrho_{ij})^2(x_3 \varepsilon^{-1/2}, \omega)) \, dx_3 \left(\int_{D_w} w^m(y) \, dy \right) \\ & \quad - 4\varepsilon^2 \int_{-h}^0 ((\zeta_{ij} - \varrho_{ij})(\zeta_{ij} - 2\varrho_{ij}))(x_3 \varepsilon^{-1/2}, \omega) \, dx_3 \left(\int_{D_w} \frac{w^m(y)}{|y|} \, dy \right), \end{aligned} \quad (57)$$

and similarly, by first introducing the change of variable

$$s = \frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \quad \text{in } D_{o,ij}^\varepsilon(x_3, \omega),$$

we have

$$\begin{aligned} & \int_{-h}^0 \int_{D_{o,ij}^\varepsilon(x_3, \omega)} w^m \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \cos \theta, \frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \sin \theta \right) r \, dr \, d\theta \, dx_3 \\ &= 16\varepsilon^2 \int_{-h}^0 \varrho_{ij}^2(x_3 \varepsilon^{-1/2}, \omega) \, dx_3 \left(\int_{D_o} w^m(y) \, dy \right), \end{aligned} \quad (58)$$

hence, inserting (57) and (58) into (55), we obtain, after some computations and using the ergodicity property (6),

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} v_\tau^\varepsilon \cdot \varphi \, dx &= h \langle \zeta^2(0) \rangle \sum_{\alpha=o,w} \int_{\Sigma} K^{-\alpha} K^\alpha v_\tau^\alpha \cdot \varphi \, dx' \\ &= h \langle \zeta^2(0) \rangle \sum_{\alpha=o,w} \int_{\Sigma} v_\tau^\alpha \cdot \varphi \, dx' \\ &= h \langle \zeta^2(0) \rangle \int_{\Sigma} v_\tau \cdot \varphi \, dx'. \end{aligned}$$

Similarly, using the expression (48) of $w_{ij}^{\varepsilon,3}$ in polar coordinates, we have, for every $\varphi \in C_0(\Sigma)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} v_3^\varepsilon \varphi \, dx = \lim_{\varepsilon \rightarrow 0} \langle \zeta^2(0) \rangle \sum_{\substack{\alpha=o,w \\ (i,j) \in I_\varepsilon(\omega)}} \frac{(v_3^\alpha \varphi)(i\varepsilon, j\varepsilon)}{\gamma_\alpha} \int_{F_{\alpha,ij}^\varepsilon(\omega)} w_{3ij}^{\varepsilon,3}(x) \, dx_3 \, dx$$

$$\begin{aligned}
 &= \lim_{\varepsilon \rightarrow 0} \langle \zeta^2(0) \rangle \sum_{(i,j) \in I_\varepsilon(\omega)} 8\pi \varepsilon^2 \frac{(v_3^o \varphi)(i\varepsilon, j\varepsilon)}{\gamma_o} A_\varepsilon(\omega) \\
 &\quad + \lim_{\varepsilon \rightarrow 0} \langle \zeta^2(0) \rangle \sum_{(i,j) \in I_\varepsilon(\omega)} 8\pi \varepsilon^2 \frac{(v_3^w \varphi)(i\varepsilon, j\varepsilon)}{\gamma_w} (B_\varepsilon(\omega) + C_\varepsilon(\omega)),
 \end{aligned}$$

where

$$\begin{aligned}
 A_\varepsilon(\omega) &= 4 \int_{-h}^0 ((\zeta_{ij} - \varrho_{ij})^2(x_3 \varepsilon^{-1/2}, \omega)) \, dx_3 \left(\int_{1/4}^{1/2} \eta(s) s \, ds \right), \\
 B_\varepsilon(\omega) &= - \int_{-h}^0 ((\zeta_{ij} - \varrho_{ij})(q_{ij} - 2\varrho_{ij}))(x_3 \varepsilon^{-1/2}, \omega) \, dx_3 \left(\int_{1/4}^{1/2} \eta(s) \, ds \right), \\
 C_\varepsilon(\omega) &= 4 \int_{-h}^0 \varrho_{ij}^2(x_3 \varepsilon^{-1/2}, \omega) \, dx_3 \left(\int_0^{1/4} \eta(s) s \, ds \right),
 \end{aligned}$$

from which we deduce, using the ergodicity property (6), that

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} v_3^\varepsilon \varphi \, dx &= h \langle \zeta(0) \rangle \int_{\Sigma} v_3^o \varphi \, dx' + h \langle \zeta^2(0) \rangle \int_{\Sigma} v_3^w \varphi \, dx' \\
 &= h \langle \zeta^2(0) \rangle \int_{\Sigma} v_3 \varphi \, dx'.
 \end{aligned}$$

2. We have

$$\begin{aligned}
 &\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \mu_\alpha \int_{\Omega_{ah}^\varepsilon(\omega)} |\nabla v_\tau^{\alpha\varepsilon}|^2 \, dx \\
 &= 2 \frac{\langle \zeta^2(0) \rangle^2}{h^2} \lim_{\varepsilon \rightarrow 0} \sum_{\substack{m=1,2 \\ (i,j) \in I_\varepsilon(\omega)}} \mu_\alpha \varepsilon^2 v_m^\alpha(i\varepsilon, j\varepsilon) v_l^\alpha(i\varepsilon, j\varepsilon) \Pi_\varepsilon^{\alpha, l, m}(\omega),
 \end{aligned}$$

where, for $\alpha = o, w$,

$$\Pi_\varepsilon^{\alpha, l, m}(\omega) = \int_{F_{\alpha, ij}^\varepsilon(\omega)} \left(\int_{-h}^0 K^{-\alpha} \nabla w_{ij}^{\varepsilon, m}(x) : K^{-\alpha} \nabla w_{ij}^{\varepsilon, l}(x) \, dx_3 \right) \, dx.$$

Passing to polar coordinates by setting $s = \frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3 \varepsilon^{-1/2}, \omega)}$ in $F_{o, ij}^\varepsilon(\omega)$, we have

$$\Pi_\varepsilon^{o, l, m}(\omega) = h \int_{-h}^0 \int_0^{1/4} K^{-o} \nabla w_{ij}^{\varepsilon, m}(s, \theta) : K^{-o} \nabla w_{ij}^{\varepsilon, l}(s, \theta) s \, ds \, d\theta \, dx_3,$$

and by setting $s = \kappa_{ij}(x_3 \varepsilon^{-1/2}, \omega) r_{ij}^\varepsilon(x')$ in $F_{w, ij}^\varepsilon(\omega)$, we have

$$\Pi_\varepsilon^{w, l, m}(\omega) = h \int_{-h}^0 \int_{\kappa_{ij} \varrho_{ij}}^{\kappa_{ij} q_{ij}} (K^{-w} \nabla w_{ij}^{\varepsilon, m} : K^{-w} \nabla w_{ij}^{\varepsilon, l})(s + \lambda_{ij}(x_3 \varepsilon^{-1/2}, \omega), \theta) s \, ds \, d\theta \, dx_3.$$

We deduce, after some computations, using the ergodicity property (6), that

$$\lim_{\varepsilon \rightarrow 0} \sum_{\substack{m=1,2 \\ (i,j) \in I_\varepsilon(\omega)}} 2\mu_\alpha \varepsilon^2 v_m^\alpha(i\varepsilon, j\varepsilon) v_l^\alpha(i\varepsilon, j\varepsilon) \Pi_\varepsilon^{\alpha,l,m}(\omega) = 2h^2 \mu_\alpha \int_{\Sigma} (K^{-\alpha})^2 \Lambda^\alpha v_\tau^\alpha \cdot v_\tau^\alpha dx',$$

and consequently

$$\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \mu_\alpha \int_{\Omega_{ah}^\varepsilon(\omega)} |\nabla v_\tau^\varepsilon|^2 dx = 2\mu_\alpha \langle \zeta^2(0) \rangle^2 \int_{\Sigma} (\Lambda^\alpha (K^\alpha)^2)^{-1} v_\tau^\alpha \cdot v_\tau^\alpha dx'.$$

3. In similar way, we have

$$2\varepsilon^2 \mu_o \int_{\Omega_{ah}^\varepsilon(\omega)} |\nabla v_3^{o\varepsilon}|^2 dx = \frac{4\pi \mu_o \varepsilon^2 \langle \zeta^2(0) \rangle^2}{\gamma_o^2} \int_0^{1/4} (\eta'(s))^2 ds \sum_{(i,j) \in I_\varepsilon(\omega)} (v_3^o)^2(i\varepsilon, j\varepsilon),$$

and

$$\begin{aligned} & 2\varepsilon^2 \mu_w \int_{\Omega_{ah}^\varepsilon(\omega)} |\nabla v_3^{w\varepsilon}|^2 dx \\ &= \frac{4\pi \mu_w \varepsilon^2 \langle \zeta^2(0) \rangle^2}{\gamma_w^2 h} \sum_{(i,j) \in I_\varepsilon(\omega)} (v_3^w(i\varepsilon, j\varepsilon))^2 \int_{-h}^0 \int_{1/4}^{1/2} (\eta'(s))^2 (s + \lambda_{ij}(x_3 \varepsilon^{-1/2}, \omega)) ds dx_3, \end{aligned}$$

from which we deduce, using the ergodicity property (6), that

$$\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \mu_\alpha \int_{\Omega_{ah}^\varepsilon(\omega)} |\nabla v_3^{\alpha\varepsilon}|^2 dx = 2\mu_\alpha \langle \zeta^2(0) \rangle^2 \gamma_\alpha^* \gamma_\alpha^{-2} \int_{\Sigma} (v_3^\alpha)^2 dx'.$$

■

6. Boundary layer problems

In this Section we introduce and study auxiliary problems connected with the boundary layers in a small zone at the inlet of the filter. Let ω be a fixed event for which the conditions (5), (7), (9), and (10) are satisfied. Let $Y = (-1/2, 1/2)^2$, such that

$$Y = Y_o \cup \mathcal{A}_{ow}(\omega) \cup Y_w, \quad (59)$$

where Y_o and Y_w are disjoint open subsets of Y and $\mathcal{A}_{ow}(\omega) = \{\sqrt{y_1^2 + y_2^2} = \varrho(0, \omega)\}$. Let us set

$$D^o(\omega) = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < \sqrt{y_1^2 + y_2^2} < \varrho(0, \omega)\},$$

$$D^w(\omega) = \{(y_1, y_2) \in \mathbb{R}^2 \mid \varrho(0, \omega) < \sqrt{y_1^2 + y_2^2} < \zeta(0, \omega)\},$$

$$D(\omega) = D^o(\omega) \cup \mathcal{A}_{ow}(\omega) \cup D^w(\omega),$$

and

$$Z_o = Y_o \times (0, +\infty),$$

$$Z_w = Y_w \times (0, +\infty),$$

$$Z = Y \times (0, +\infty).$$

Let us define for $m = 1, 2$

$$w^{0,m}(y_1, y_2) = \begin{cases} \begin{pmatrix} w_1^m \left(\frac{\sqrt{y_1^2 + y_2^2}}{4\varrho(0, \omega)}, \theta \right) \\ w_2^m \left(\frac{\sqrt{y_1^2 + y_2^2}}{4\varrho(0, \omega)}, \theta \right) \\ 0 \end{pmatrix} & \text{in } D^o(\omega), \\ \begin{pmatrix} w_1^m(\sigma(y, \omega), \theta) \\ w_2^m(\sigma(y, \omega), \theta) \\ 0 \end{pmatrix} & \text{in } D^w(\omega), \end{cases} \quad (60)$$

where

$$\sigma(y, \omega) = \kappa(0, \omega) \sqrt{y_1^2 + y_2^2} + \lambda(0, \omega),$$

$$\kappa(0, \omega) = \frac{1}{4(\zeta(0, \omega) - \varrho(0, \omega))},$$

$$\lambda(0, \omega) = \frac{\zeta(0, \omega) - 2\varrho(0, \omega)}{4(\zeta(0, \omega) - \varrho(0, \omega))},$$

and for $m = 3$

$$w^{0,3}(y_1, y_2) = \begin{cases} \eta(\sqrt{y_1^2 + y_2^2}/4\varrho(0, \omega))e_3 & \text{in } D^o(\omega), \\ \eta(\sigma(y, \omega))e_3 & \text{in } D^w(\omega). \end{cases}$$

We make the hypothesis that

$$\mathcal{S}_{ow} \cap (D(\omega) \times \{0\}) = \mathcal{A}_{ow}(\omega), \quad (61)$$

and consider the following boundary layer problems for $m = 1, 2$:

$$\begin{cases} -\mu_\alpha \Delta w^{m,bl} + \nabla \pi^{m,bl} = 0 & \text{in } Z, \alpha = o, w, \\ \operatorname{div} w^{m,bl} = 0 & \text{in } Z, \alpha = o, w, \\ [w^{m,bl}]_{ow} = 0 & \text{on } \mathcal{S}_{ow}, \\ \left[\mu_* \left(\frac{\partial w_i^{m,bl}}{\partial y_j} + \frac{\partial w_j^{m,bl}}{\partial y_i} \right) n_j - \pi^m n_i \right]_{ow} = 0 & \text{on } \mathcal{S}_{ow}, \\ w^{m,bl}(y_1, y_2, 0) = \begin{pmatrix} w_1^{0,m}(y_1, y_2) \\ w_2^{0,m}(y_1, y_2) \\ 0 \end{pmatrix} & \text{on } D(\omega), \\ w^{m,bl}(y_1, y_2, 0) = 0 & \text{on } Y \setminus D(\omega), \\ (w^{m,bl}, \pi^{m,bl}) \text{ is } (y_1, y_2)\text{-periodic,} \end{cases} \quad (62)$$

and the following for $m = 3$:

$$\begin{cases} -\mu_\alpha \Delta w^{3,bl} + \nabla \pi^{3,bl} = 0 & \text{in } Z, \quad \alpha = o, w, \\ \operatorname{div} w^{3,bl} = 0 & \text{in } Z, \quad \alpha = o, w, \\ [w^{3,bl}]_{ow} = 0 & \text{on } \mathcal{S}_{ow}, \\ \left[\mu_* \left(\frac{\partial w_i^{3,bl}}{\partial y_j} + \frac{\partial w_j^{3,bl}}{\partial y_i} \right) n_j - \pi^3 n_i \right]_{ow} = 0 & \text{on } \mathcal{S}_{ow}, \\ w^{3,bl}(y_1, y_2, 0) = \bar{w}^{0,3}(y_1, y_2) & \text{on } D(\omega), \\ w^{3,bl}(y_1, y_2, 0) = 0 & \text{on } Y \setminus D(\omega), \\ (w^{m,bl}, \pi^{m,bl}) \text{ is } (y_1, y_2)\text{-periodic,} \end{cases} \quad (63)$$

where

$$\bar{w}^{0,3} = \left(w^{0,3}(y_1, y_2) - \frac{1}{|D(\omega)|} \int_{D(\omega)} w^{0,3}(y_1, y_2) \right) e_3.$$

Let us define the space

$$V^{bl}(\omega) = \left\{ w \in L^2_{loc}(Z, \mathbb{R}^3) \mid \nabla w \in L^2(Z, \mathbb{R}^9), \right. \\ \left. \begin{array}{ll} \operatorname{div}(w) = 0 & \text{in } Z, \\ w(y_1, y_2, 0) = 0 & \text{on } Y \setminus D(\omega) \end{array} \right\}. \quad (64)$$

Then, using for instance [19, Section 3], one can prove that, for $m = 1, 2, 3$, problems (62) and (63) each have a unique solution $(w^{m,bl}, \pi^{m,bl})$, which depends on $\omega \in \Pi$ and belongs to the space $(V^{bl}(\omega) \cap C^\infty_{loc}(Z, \mathbb{R}^3)) \times C^\infty_{loc}(Z)$. For every $a, b \in \mathbb{R}$, $b > a > 0$, we define

$$Z(a, b) = \{y \in Z \mid a < y_3 < b\}.$$

The following Phragmen-Lindelöf type estimates can be obtained for $(w^{m,bl}, \pi^{m,bl})$ (see, for instance, [35,36]):

$$\begin{aligned} \|w^{m,bl} - C^m\|_{L^2(Z(s,s+1), \mathbb{R}^3)} &\leq C_0 e^{-C_1 s} \quad \forall s \in \mathbb{N}, \\ \|\pi^{m,bl} - C^m_\pi\|_{L^2(Z(s,s+1), \mathbb{R}^3)} &\leq C_0 e^{-C_1 s} \quad \forall s \in \mathbb{N}, \\ \|\nabla w^{m,bl}\|_{L^2(Z(s,s+1), \mathbb{R}^3)} &\leq C_0 e^{-C_1 s} \quad \forall s \in \mathbb{N}, \end{aligned} \quad (65)$$

where C_0, C_1 , and C^m_π are random constants with $C_0, C_1 > 0$, and C^m is a random constant vector of $\mathbb{R}^3$. We introduce the 3×3 matrix $K^{bl}(\omega) = \sum_{\alpha=o,w} K^{\alpha,bl}(\omega)$, where $K^{\alpha,bl}(\omega) = (K^{\alpha,bl}_{lm}(\omega))_{l,m=1,2}$, $\alpha = o, w$, with

$$K^{\alpha,bl}_{lm}(\omega) = \int_{Z_\alpha} \nabla w^{m,bl}(z)(\omega) : \nabla w^{l,bl}(z)(\omega) \, dz. \quad (66)$$

Let

$$\begin{aligned} Z^{o,\varepsilon}_{ij} &= \varepsilon Y^o + (i\varepsilon, j\varepsilon), \\ Z^{w,\varepsilon}_{ij} &= \varepsilon Y^w + (i\varepsilon, j\varepsilon), \\ \mathcal{S}^{\varepsilon}_{ow,ij} &= \varepsilon \mathcal{S}_{ow} + (i\varepsilon, j\varepsilon), \\ Z^{\varepsilon}_{ij} &= \varepsilon Z + (i\varepsilon, j\varepsilon). \end{aligned} \quad (67)$$

We define, for every $(i, j) \in I_\varepsilon(\omega)$, the sequence of functions $w^{m,bl,\varepsilon}$, $m = 1, 2, 3$, by

$$w^{m,bl,\varepsilon}(x) = w^{m,bl}\left(\frac{x_1 - i\varepsilon}{\varepsilon}, \frac{x_2 - j\varepsilon}{\varepsilon}, \frac{x_3}{\varepsilon}\right) \text{in } Z_{ij}^\varepsilon, \forall (i, j) \in I_\varepsilon(\omega). \quad (68)$$

We have the following

Lemma 6.1: *Let $\varkappa \in (0, 1)$. For every $\varphi \in C^1(\Omega, \mathbb{R}^3)$, we have*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} 2\varepsilon \mu_\alpha \int_{\Omega_\alpha \cap (\Sigma \times (0, \varepsilon^\varkappa))} \nabla w^{m,bl,\varepsilon} \varphi_m(x) : \nabla w^{l,bl,\varepsilon} \varphi_l(x) \, dx \\ = 2\mu_\alpha \int_{\Sigma} K^{\alpha,bl}(\omega) \varphi(x') \cdot \varphi(x') \, dx'. \end{aligned}$$

Proof: Observe that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} 2\varepsilon \mu_\alpha \int_{\Omega_\alpha \cap (\Sigma \times (0, \varepsilon^\varkappa))} \nabla w^{m,bl,\varepsilon} \varphi_m(x) : \nabla w^{l,bl,\varepsilon} \varphi_l(x) \, dx \, dx \\ = \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} 2\varepsilon \mu_\alpha \int_{Z_{ij}^{\alpha,\varepsilon}(\omega) \cap (\Sigma \times (0, \varepsilon^\varkappa))} (\varphi_m \varphi_l)(i\varepsilon, j\varepsilon) \nabla w^{m,bl,\varepsilon}(x) : \nabla w^{l,bl,\varepsilon}(x) \, dx. \end{aligned}$$

Then, introducing the change of variables $y_1 = \frac{x_1 - i\varepsilon}{\varepsilon}$, $y_2 = \frac{x_2 - j\varepsilon}{\varepsilon}$, and $y_3 = \frac{x_3}{\varepsilon}$, we deduce, using the expression (68) defining $w^{m,bl,\varepsilon}(x)$, that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} 2\varepsilon \mu_\alpha \int_{Z_{ij}^{\alpha,\varepsilon}(\omega) \cap (\Sigma \times (0, \varepsilon^\varkappa))} (\varphi_m \varphi_l)(i\varepsilon, j\varepsilon) \nabla w^{m,bl,\varepsilon}(x) : \nabla w^{l,bl,\varepsilon}(x) \, dx \\ = \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} 2\mu_\alpha \varepsilon^2 (\varphi_m \varphi_l)(i\varepsilon, j\varepsilon) \int_{Y_\alpha} \int_0^{\varepsilon^{\varkappa-1}} (\nabla w_{ij}^{m,bl} : \nabla w_{ij}^{l,bl})(y) \, dy \\ = 2\mu_\alpha \left(\int_{Z_\alpha} \nabla w^{m,bl}(y) : \nabla w^{l,bl}(y) \, dy \right) \int_{\Sigma} (\varphi_m \varphi_l)(x', 0) \, dx' \\ = 2\mu_\alpha \int_{\Sigma} K^{\alpha,bl}(\omega) \varphi(x') \cdot \varphi(x') \, dx'. \end{aligned}$$

■

7. The main result

The purpose of this Section is to state our result on asymptotic behavior of problem (18). According to Lemmas 4.1 and 4.3, we are now able to introduce the following topology τ_0 , which will be used to describe the asymptotic behavior of the sequence $(\mathcal{F}_\varepsilon)_\varepsilon$ of functionals defined in (23):

Definition 7.1: A sequence $(u^\varepsilon)_\varepsilon$, such that $u^\varepsilon \in V^\varepsilon(\omega)$, τ_0 -converges to the couple (u, v) if

$$u^\varepsilon|_\Omega \rightharpoonup u \text{ } H^1(\Omega, \mathbb{R}^3)\text{-weak,}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \varphi \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma v \cdot \varphi \, dx', \quad \forall \varphi \in C_0(\Sigma, \mathbb{R}^3).$$

We define the space V^0 by

$$V^0 = \left\{ (u, v) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Sigma, \mathbb{R}^3) \mid \operatorname{div} u = 0 \text{ in } \Omega, \right. \\ \left. \operatorname{div} v_\tau = 0 \text{ in } \Sigma, \quad u_3 = \langle \zeta^2(0) \rangle v_3 \text{ on } \Sigma, \quad u = 0 \text{ on } \partial\Omega \setminus \Sigma \right\}, \quad (69)$$

and the functional $\mathcal{F}_0$ on $L^2(\Omega, \mathbb{R}^3) \times L^2(\Sigma, \mathbb{R}^3)$ by

$$\mathcal{F}_0(u, v) = \begin{cases} 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u) D_{ij}(u) \, dx + 2\mu_\alpha \langle \zeta^2(0) \rangle^2 \int_\Sigma (K^{*\alpha})^{-1} v_\tau^\alpha \cdot v_\tau^\alpha \, dx' \\ + 2\mu_\alpha (\langle \zeta^2(0) \rangle)^2 \gamma_\alpha^* \gamma^{-2} \int_\Sigma (v_3^\alpha)^2 \, dx' \\ + 2\mu_\alpha \int_\Sigma K^{\alpha, bl}(\omega) (u - v) \cdot (u - v) \, dx' \\ + \infty \end{cases} \quad \begin{array}{l} \text{if } (u, v) \in V^0, \\ \text{otherwise.} \end{array} \quad (70)$$

Our main result reads as follows.

Theorem 7.2: *We have*

- (1) (*lim sup inequality*). For every $(u, v) \in V^0$ there exists a sequence $(u^{0,\varepsilon})_\varepsilon$, such that $u^{0,\varepsilon} \in V^\varepsilon(\omega)$, $(u^{0,\varepsilon})_\varepsilon$ τ_0 -converges to (u, v) , and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) \leq \mathcal{F}_0(u, v).$$

- (2) (*lim inf inequality*). For every $u^\varepsilon \in V^\varepsilon(\omega)$, such that $(u^\varepsilon)_\varepsilon$ τ_0 -converges to (u, v) , we have $(u, v) \in V^0$ and

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) \geq \mathcal{F}_0(u, v).$$

Before proving this main result, let us write the problem associated to the limit energy functional (70).

Corollary 7.3: *Let $(u^\varepsilon, p^\varepsilon)$ be the solution of problem (18). Then, the sequence $(u^\varepsilon)_\varepsilon$ τ_0 -converges to $(u, v) \in V^0$ and*

$$\tilde{p}^\varepsilon \rightharpoonup p \text{ } L^2(\Omega)\text{-weak,}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \tilde{p}^\varepsilon \varphi \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma q(x') \varphi(x', 0) \, dx', \quad \forall \varphi \in C_0(\mathbb{R}^3),$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \bar{p}_\alpha^\varepsilon \varphi \, dx = h \langle \zeta^2(0) \rangle \int_\Sigma q_\alpha(x') \varphi(x', 0) \, dx', \quad \forall \varphi \in C_0(\mathbb{R}^3),$$

$$q = q_o + q_w,$$

where (u, v, p, q_o, q_w) is the solution of the problem

$$\begin{cases} -\mu_\alpha \Delta u + \nabla p = f & \text{in } \Omega_\alpha, \alpha = o, w, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ [2\mu D_{ij}(u)n_j - pn_i]_{ow} = 2\kappa H n_i & \text{in } S_{ow}, i = 1, 2, 3, \\ v = v^o + v^w & \text{in } \Sigma, \\ v_\tau^\alpha = \frac{K^{*\alpha}(\nabla q_\alpha - f_\tau)}{2\mu_\alpha \langle \zeta^2(0) \rangle} \\ \quad + \frac{1}{2\langle \zeta^2(0) \rangle^2} \sum_{m=1,2} (K^{*\alpha} K_m^{\alpha,bl}(\omega))^t \cdot (u - v^\alpha) & \text{in } \Sigma, \alpha = o, w, \\ v_3 = \frac{u_3}{\langle \zeta^2(0) \rangle} & \text{in } \Sigma, \\ \operatorname{div} v_\tau^\alpha = 0 & \text{in } \Sigma, \alpha = o, w, \end{cases} \quad (71)$$

with the following boundary conditions for u on Σ and q_α on $\partial \Sigma$

$$\begin{cases} u_3 + \sum_{\alpha=o,w} \frac{\gamma_\alpha^2}{2\langle \zeta^2(0) \rangle \gamma_\alpha^*} K_3^{\alpha,bl}(\omega) \cdot (u - v^\alpha) = f_3 \sum_{\alpha=o,w} \frac{\gamma_\alpha^2}{2\mu_\alpha \gamma_\alpha^*} & \text{on } \Sigma, \\ \frac{\partial u_\tau}{\partial x_3} \Big|_{x_3=0} = \sum_{m=1,2} \frac{2\mu_\alpha}{\mu_o + \mu_w} (K_m^{\alpha,bl}(\omega))^t \cdot (u - v^\alpha) & \text{on } \Sigma, \\ \frac{\partial u_3}{\partial x_3} \Big|_{x_3=0} - \frac{p}{\mu_o + \mu_w} = \sum_{\alpha=o,w} \frac{2\mu_\alpha}{\mu_o + \mu_w} (K_3^{\alpha,bl}(\omega))^t \cdot (u - v^\alpha) & \text{on } \Sigma, \\ q_\alpha = 0 & \text{on } \partial \Sigma, \\ u = 0 & \text{on } \Omega \setminus \Sigma, \end{cases} \quad (72)$$

where $K^{*\alpha}$ is the permeability tensor of the α -phase in the filter defined by the relations (49) and (54), the coefficient γ_α and γ_α^* are defined, respectively, in (50) and Lemma 5.13, $K^{\alpha,bl}(\omega)$ is the boundary layer (random) tensor corresponding to the α -phase given in (66), and $K_l^{\alpha,bl}(\omega)$, $l = 1, 2, 3$, is the l -th row vector of $K^{\alpha,bl}(\omega)$.

Proof: Let us introduce the vectorial measure $\nu_\varepsilon = 2\kappa \varepsilon H^\varepsilon n^\varepsilon \mathbf{1}_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} \, dx$, where H^ε and n^ε are respectively the mean curvature and the unit normal given, respectively, by the relations (19) and (20). Then, using the law of averages and the ergodicity property (6), we have, for any $\varphi \in C_0(\mathbb{R}^3, \mathbb{R}^3)$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \varphi \, d\nu_\varepsilon(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot \varphi \, ds_\varepsilon \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \int_{\mathcal{A}_{ow,i,j}^\varepsilon(\omega)} 2\kappa \varepsilon H_{ij}^\varepsilon(x_3, \omega) n^\varepsilon \cdot \varphi \, ds_\varepsilon \\ &= 2\kappa h \lim_{\varepsilon \rightarrow 0} \sum_{(i,j) \in I_\varepsilon(\omega)} \varepsilon^2 \left(\frac{1}{h\varepsilon^{-1/2}} \int_{-h}^0 (\varrho_{ij} H_{ij}^*(z, \omega)) \varphi(i, j, z\varepsilon^{1/2}) \cdot \left(\int_0^{2\pi} n \, d\theta \right) dz \right) \end{aligned}$$

$$\begin{aligned}
 &= 2\kappa h \langle H^*(0) \rangle \int_{\Sigma} \int_0^{2\pi} (\varphi_1 \cos \theta + \varphi_2 \sin \theta)(x', 0) d\theta dx' \\
 &= 0,
 \end{aligned} \tag{73}$$

where $n = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and $H_{ij}^*(t, \omega) = -\chi_{ij}^2(\omega) \frac{\partial^2 \varrho_{ij}(t, \omega)}{\partial x_3^2}$. We deduce that $(v_\varepsilon)_\varepsilon$ $*$ -weakly converges in the sense of measures to the zero vectorial measure. Let $u^\varepsilon \in V^\varepsilon(\omega)$, such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u^\varepsilon) < +\infty$. We have, using Lemma 3.1₂ and Lemma 3.2₁,

$$\sup_\varepsilon \left| \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot u^\varepsilon ds_\varepsilon \right| < +\infty,$$

which implies that the sequence of measures $(u^\varepsilon \cdot v_\varepsilon)_\varepsilon$ has bounded variation, thus $*$ -weakly converges, up to some subsequence, to some measure ν . On the other hand, using Lemma 4.2, we have, up to some subsequence, for every $\varphi \in C_0(\mathbb{R}^3, \mathbb{R}^3)$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} u^\varepsilon \cdot \varphi dx = h \langle \zeta^2(0) \rangle \int_{\Sigma} \nu(x') \cdot \varphi(x', 0) dx',$$

As $(v_\varepsilon)_\varepsilon$ $*$ -weakly converges to the zero vectorial measure, we deduce that $\nu = 0$. We thus obtain that, up to some subsequence,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{A}_{ow}^{h,\varepsilon}(\omega)} 2\kappa H^\varepsilon n^\varepsilon \cdot u^\varepsilon \varphi ds_\varepsilon = 0. \tag{74}$$

Let $(u^\varepsilon, p^\varepsilon)$ be the solution of problem (18). Then, according to Remark 3.3, Theorem 7.2, and convergence (74), we infer, using [11, Theorem 7.8], that the sequence $(u^\varepsilon)_\varepsilon$ τ_0 -converges to $(u, \nu) \in V^0$ solution of the minimization problem

$$\inf_{(u, \nu) \in V^0} \left\{ \mathcal{F}_0(u, \nu) - 2 \int_{S_{ow}} \kappa H n_i u_i ds - \int_{\Omega} f \cdot u dx - h \langle \zeta^2(0) \rangle \int_{\Sigma} f \cdot \nu dx' \right\}, \tag{75}$$

with

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) = \mathcal{F}_0(u, \nu), \tag{76}$$

and, using Lemma 3.5, Remarks 3.6, and 4.3, we have that $\tilde{p}^\varepsilon \rightharpoonup_{\varepsilon \rightarrow 0} p$ $L^2(\Omega)$ -weak and, for every $\varphi \in C_0(\mathbb{R}^3)$,

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \bar{p}^\varepsilon \varphi dx &= h \langle \zeta^2(0) \rangle \int_{\Sigma} q(x') \varphi(x', 0) dx', \\
 \lim_{\varepsilon \rightarrow 0} \int_{\Omega_h^\varepsilon(\omega)} \bar{p}_\alpha^\varepsilon \varphi dx &= h \langle \zeta^2(0) \rangle \int_{\Sigma} q_\alpha(x') \varphi(x', 0) dx',
 \end{aligned} \tag{77}$$

with $q = q_o + q_w$. The variational formulation associated with the minimization problem (75) reads as follows.

$$\int_{\Omega_\alpha} (-\mu_\alpha \Delta u + \nabla p) \cdot \varphi dx + \int_{S_{ow}} [\mu_* D_{ij}(u) n_j - p n_i]_{ow} \varphi_i ds$$

$$\begin{aligned}
 & -2 \int_{S_{ow}} \kappa H n_i \varphi_i \, ds - \int_{\Sigma} (\mu_{\alpha} \nabla u - p Id_{\mathbb{R}^3}) e_3 \cdot \varphi \, dx' \\
 & + 2\mu_{\alpha} \int_{\Sigma} K^{\alpha, bl}(\omega)(u - v^{\alpha}) \cdot (\varphi - \psi^{\alpha}) \, dx' + 2\mu_{\alpha} \langle \zeta^2(0) \rangle^2 \int_{\Sigma} (K^{\alpha*})^{-1} v_{\tau}^{\alpha} \cdot \psi_{\tau}^{\alpha} \, dx \\
 & + 2\mu_{\alpha} \langle \zeta^2(0) \rangle^2 \gamma_{\alpha}^* \gamma_{\alpha}^{-2} \int_{\Sigma} v_3^{\alpha} \psi_3^{\alpha} \, dx' + h \langle \zeta^2(0) \rangle \int_{\Sigma} \nabla q_{\alpha} \cdot \psi_{\tau}^{\alpha} \, dx' \\
 & - h \langle \zeta^2(0) \rangle \int_{\partial \Sigma} q_{\alpha} n \cdot \psi^{\alpha} \, ds \\
 & = \int_{\Omega} f \cdot \varphi \, dx + h \langle \zeta^2(0) \rangle \sum_{\alpha=o,w} \int_{\Sigma} f \cdot \psi^{\alpha} \, dx', \tag{78}
 \end{aligned}$$

for every $(\varphi, \psi) \in V^0$, from which we deduce the homogenized problem (71)–(72). ■

8. Proof of the main result

This section is devoted to the proof of Theorem 7.2. This proof is split into two steps.

8.1. First step: verification of the lim sup property

Let $\varkappa \in (0, 1/3)$. We set

$$\begin{aligned}
 \Sigma_{\varepsilon}^{1,\varkappa} &= \Sigma \times (0, \varepsilon^{\varkappa}), \\
 \Sigma_{\varepsilon}^{2,\varkappa} &= \Sigma \times (-\varepsilon^{\varkappa}, 0). \tag{79}
 \end{aligned}$$

We introduce the following smooth functions $\psi^{\varepsilon}(x)$ and $\psi_h^{\varepsilon}(x)$ defined by

$$\psi_1^{\varepsilon}(x) = \begin{cases} 1 & \text{on } \Sigma, \\ 0 & \text{in } \Omega \setminus \Sigma_{\varepsilon}^{1,\varkappa}, \end{cases} \quad \text{and} \quad \psi_2^{\varepsilon}(x) = \begin{cases} 1 & \text{on } \Sigma, \\ 0 & \text{in } \Omega_h \setminus \Sigma_{\varepsilon}^{2,\varkappa}. \end{cases} \tag{80}$$

Let $u \in C^1(\overline{\Omega}, \mathbb{R}^3)$, such that $u = 0$ on $\partial\Omega \setminus (\Sigma \cup \Sigma_b)$ and $\operatorname{div} u = 0$ in Ω . There exists $\tilde{u}^{\varepsilon}$ in $C_c^{\infty}(\Omega, \mathbb{R}^3)$ such that $\operatorname{div}(\tilde{u}^{\varepsilon}) = 0$ and $\|u - \tilde{u}^{\varepsilon}\|_{L^2(\Omega)} < \varepsilon$. Since $\operatorname{div}(u - \tilde{u}^{\varepsilon}) = 0$ in Ω , there exists $\widehat{u - \tilde{u}^{\varepsilon}} \in H^1(\Omega, \mathbb{R}^3)$, such that (see, for instance, [37, Section 2] and [38])

$$\begin{cases} \widehat{\operatorname{div} u - \tilde{u}^{\varepsilon}} &= 0 & \text{in } \Omega, \\ \widehat{\operatorname{curl} u - \tilde{u}^{\varepsilon}} &= u - \tilde{u}^{\varepsilon} & \text{in } \Omega, \\ \widehat{u - \tilde{u}^{\varepsilon}} \cdot n &= 0 & \text{on } \partial\Omega, \\ \|\widehat{u - \tilde{u}^{\varepsilon}}\|_{H^1(\Omega, \mathbb{R}^3)} &\leq C(\Omega) \|u - \tilde{u}^{\varepsilon}\|_{L^2(\Omega, \mathbb{R}^3)}. \end{cases} \tag{81}$$

Let $v \in C^1(\Sigma, \mathbb{R}^3)$, such that $v_3 = \frac{u_3|_{\Sigma}}{\langle \zeta^2(0) \rangle}$. We define the sequence $\tilde{u}^{0,\varepsilon}$ as

$$\tilde{u}^{0,\varepsilon} = \begin{cases} u & \text{in } \Omega \setminus \Sigma_{\varepsilon}^{1,\varkappa}, \\ u - \operatorname{div}((\widehat{u - \tilde{u}^{\varepsilon}}) \psi^{\varepsilon}) & \\ \quad + \sqrt{\varepsilon} (w^{i,bl,\varepsilon} + a \delta_{i3} e_3) (u_i - v_i^{\alpha}) \psi_1^{\varepsilon} & \text{in } \Sigma_{\varepsilon}^{1,\varkappa} \cap \Omega_{\alpha}, \quad \alpha = o, w, \\ \sqrt{\varepsilon} w^{\varepsilon,i} (u_i - v_i^{\alpha}) \psi_2^{\varepsilon} + v^{\alpha\varepsilon} (1 - \psi_2^{\varepsilon}) & \text{in } \Sigma_{\varepsilon}^{2,\varkappa} \cap \Omega_{\alpha h}^{\varepsilon}(\omega), \alpha = o, w, \\ v^{\alpha\varepsilon} & \text{in } \Omega_{\alpha h}^{\varepsilon}(\omega) \setminus \Sigma_{\varepsilon}^{2,\varkappa}, \quad \alpha = o, w, \end{cases} \tag{82}$$

where $(v^{\alpha\epsilon})_\epsilon$ is the sequence of functions associated with v by the relations (52) and $a = \frac{4}{\pi} \int_{D(0,1/2)} \eta(\sqrt{y_1^2 + y_2^2}) dy$. The function $\tilde{u}^{0,\epsilon}$ belongs to $H^1(\Omega^\epsilon(\omega), \mathbb{R}^3)$ with $\tilde{u}^{0,\epsilon} = 0$ on $\Gamma^\epsilon(\omega)$ but is not divergence-free. In order to satisfy this property, we first introduce, for $\alpha = o, w$, the functions

$$\begin{cases} \Psi_{\alpha,1}^\epsilon &= \sqrt{\epsilon}(w^{i,bl,\epsilon} + a^i \delta_{i3} e_3)(u_i - v_i^\alpha) \psi_1^\epsilon, \\ F_{\alpha,1}^\epsilon &= -\operatorname{div}(\Psi_{\alpha,1}^\epsilon) + \int_{\Sigma_\alpha^\epsilon(\omega)} \Psi_{\alpha,1}^\epsilon \cdot e_3 \, ds, \end{cases}$$

and consider the solution $\Theta_{\alpha,1}^\epsilon$ of the problem

$$\begin{cases} \operatorname{div} \Theta_{\alpha,1}^\epsilon &= F_{\alpha,1}^\epsilon \quad \text{in } \Sigma_\epsilon^{1,\mathcal{K}} \cap \Omega_\alpha, \\ \Theta_{\alpha,1}^\epsilon &= 0 \quad \text{on } \partial(\Sigma_\epsilon^{1,\mathcal{K}} \cap \Omega_\alpha). \end{cases} \quad (83)$$

Then we introduce the functions

$$\begin{cases} \Psi_{\alpha,2}^\epsilon &= \sqrt{\epsilon} w^{\epsilon,i} (u_i - v_i^\alpha) \psi_2^\epsilon + v^{\alpha\epsilon} (1 - \psi_2^\epsilon) \quad \text{in } \Sigma_\epsilon^{2,\mathcal{K}} \cap \Omega_{\alpha,h}^\epsilon(\omega), \\ F_{\alpha,2}^\epsilon &= -\operatorname{div}(\Psi_{\alpha,2}^\epsilon) - \int_{\Sigma_\alpha^\epsilon(\omega)} \Psi_{\alpha,2}^\epsilon \cdot e_3 \, ds, \end{cases}$$

and consider the solution $\Theta_{\alpha,2}^\epsilon$ of the problem

$$\begin{cases} \operatorname{div} \Theta_{\alpha,2}^\epsilon &= F_{\alpha,2}^\epsilon \quad \text{in } \Sigma_\epsilon^{2,\mathcal{K}} \cap \Omega_{\alpha,h}^\epsilon(\omega), \\ \Theta_{\alpha,2}^\epsilon &= 0 \quad \text{on } \partial(\Sigma_\epsilon^{2,\mathcal{K}} \cap \Omega_h). \end{cases} \quad (84)$$

The following statement is a consequence of Lemma 3.4 (one can see [15, Lemma 2] for the proof):

Lemma 8.1: *Let $U \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary ∂U and $\epsilon > 0$. Let $U_\epsilon = U \times (0, \epsilon)$. There exists a linear operator $\mathcal{B}_\epsilon : L^2(U_\epsilon) \rightarrow H_0^1(U_\epsilon, \mathbb{R}^3)$ such that, for every $g \in L^2(U_\epsilon)$ satisfying $\int_{U_\epsilon} g \, dx = 0$*

$$\begin{cases} \operatorname{div}(\mathcal{B}_\epsilon(g)) &= g \quad \text{in } U_\epsilon, \\ \|\nabla \mathcal{B}_\epsilon(g)\|_{L^2(U_\epsilon, \mathbb{R}^9)} &\leq \frac{C(U)}{\epsilon} \|g\|_{L^2(U_\epsilon)}, \end{cases}$$

where $C(U)$ is a constant which still only depends on U .

We have the following

Lemma 8.2: *We have*

$$\begin{aligned} (1) \quad & \lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon^{1,\mathcal{K}} \cap \Omega_\alpha} |\Theta_{\alpha,1}^\epsilon|^2 \, dx = \lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon^{1,\mathcal{K}} \cap \Omega_\alpha} |\nabla \Theta_{\alpha,1}^\epsilon|^2 \, dx = 0, \\ (2) \quad & \lim_{\epsilon \rightarrow 0} \int_{\Sigma_\epsilon^{2,\mathcal{K}} \cap \Omega_{\alpha,h}^\epsilon(\omega)} |\Theta_{\alpha,2}^\epsilon|^2 \, dx = \lim_{\epsilon \rightarrow 0} \epsilon^2 \int_{\Sigma_\epsilon^{2,\mathcal{K}} \cap \Omega_{\alpha,h}^\epsilon(\omega)} |\nabla \Theta_{\alpha,2}^\epsilon|^2 \, dx = 0. \end{aligned}$$

Proof: 1. According to Lemma 8.1, there exists a constant C independent of ε such that

$$\|\nabla \Theta_{\alpha,1}^\varepsilon\|_{L^2(\Sigma_\varepsilon^{1,\varkappa} \cap \Omega, \mathbb{R}^9)} \leq \frac{C}{\varepsilon^\varkappa} \|\operatorname{div}(\Psi_{\alpha,1}^\varepsilon)\|_{L^2(\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha)}. \quad (85)$$

Taking into account the fact that $\operatorname{div} w^{\varepsilon,i,bl}(x) = \frac{1}{\varepsilon} \operatorname{div} w^{i,bl}(y) = 0$, where $y_1 = \frac{x_1 - i\varepsilon}{\varepsilon}$, $y_2 = \frac{x_2 - j\varepsilon}{\varepsilon}$, and $y_3 = \frac{x_3}{\varepsilon}$, we have

$$\begin{aligned} \operatorname{div}(\Psi_{\alpha,1}^\varepsilon) &= \sqrt{\varepsilon}(w^{\varepsilon,i,bl} + a\delta_{i3}e_3)(u_i - v_i^\alpha) \cdot \nabla \psi_1^\varepsilon \\ &\quad + \sqrt{\varepsilon}(w^{\varepsilon,i,bl} + a\delta_{i3}e_3) \frac{\partial(u_i - v_i^\alpha)}{\partial x_i} \psi_1^\varepsilon \\ &= \sqrt{\varepsilon}(w^{\varepsilon,i,bl} - C^i)(u_i - v_i^\alpha) \cdot \nabla \psi_1^\varepsilon \\ &\quad + \sqrt{\varepsilon}(w^{\varepsilon,i,bl} - C^i) \frac{\partial(u_i - v_i^\alpha)}{\partial x_i} \psi_1^\varepsilon \\ &\quad + \sqrt{\varepsilon}(a\delta_{i3}e_3 + C^i)(u_i - v_i^\alpha) \cdot \nabla \psi_1^\varepsilon \\ &\quad + \sqrt{\varepsilon}(a\delta_{i3}e_3 + C^i) \frac{\partial(u_i - v_i^\alpha)}{\partial x_j} \psi_1^\varepsilon, \end{aligned}$$

where C^i is the constant vector in the Phragmen-Lindelöf inequalities (65). Observing that

$$\begin{aligned} &\int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} |\sqrt{\varepsilon}(a\delta_{i3}e_3 + C^i)(u_i - v_i^\alpha) \cdot \nabla \psi_1^\varepsilon|^2 dx \\ &\quad + \int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} \left| \sqrt{\varepsilon}(a\delta_{i3}e_3 + C^i) \frac{\partial(u_i - v_i^\alpha)}{\partial x_j} \psi_1^\varepsilon \right|^2 dx \leq C\varepsilon^{1-\varkappa}, \end{aligned}$$

and by using inequalities (65)

$$\begin{aligned} &\int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} |\sqrt{\varepsilon}(w^{\varepsilon,i,bl} - C^i)(u_i - v_i^\alpha) \cdot \nabla \psi_1^\varepsilon|^2 dx \\ &\quad + \int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} \left| \sqrt{\varepsilon}(w^{\varepsilon,i,bl} - C^i) \frac{\partial(u_i - v_i^\alpha)}{\partial x_i} \psi_1^\varepsilon \right|^2 dx \leq C\varepsilon^{1-\varkappa}, \end{aligned}$$

we deduce, according to (85), that

$$\int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} |\nabla \Theta_{\alpha,1}^\varepsilon|^2 dx \leq C\varepsilon^{1-3\varkappa},$$

from which we deduce, using the fact that $\varkappa \in (0, 1/3)$, that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} |\nabla \Theta_{\alpha,1}^\varepsilon|^2 dx = 0,$$

and, using Poincaré's inequality,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon^{1,\varkappa} \cap \Omega_\alpha} |\Theta_{\alpha,1}^\varepsilon|^2 dx = 0.$$

2. Likewise, we have

$$\|\nabla \Theta_{\alpha,2}^\varepsilon\|_{L^2(\Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_h, \mathbb{R}^9)} \leq \frac{C}{\varepsilon^{\mathcal{K}}} \|\operatorname{div}(\Psi_{\alpha,2}^\varepsilon)\|_{L^2(\Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_h)}.$$

Passing to polar coordinates by setting

$$\begin{aligned} s &= \kappa_{ij}(x_3 \varepsilon^{-1/2}, \omega) r_{ij}^\varepsilon(x') \quad \text{for } x \in F_{w,ij}^\varepsilon(\omega), \\ s &= \frac{r_{ij}^\varepsilon(x')}{4\rho_{ij}(x_3 \varepsilon^{-1/2}, \omega)} \quad \text{for } x \in F_{0,ij}^\varepsilon(\omega), \end{aligned}$$

we deduce that, for every $(i, j) \in I_\varepsilon(\omega)$,

$$\begin{aligned} \operatorname{div}(w^{\varepsilon,m}) &= \operatorname{div}_s(w^{\varepsilon,m}) \\ &= \frac{\kappa_{ij}(x_3 \varepsilon^{-1/2}, \omega)}{\varepsilon} \left(\frac{\partial w_s^m}{\partial s} + \frac{w_s^m}{s} + \frac{1}{s} \frac{\partial w_\theta^m}{\partial \theta} \right) (s + \lambda_{ij}, \theta) \\ &= 0, \end{aligned}$$

and $\operatorname{div} v^{\alpha\varepsilon} = 0$. Hence

$$\begin{aligned} \operatorname{div}(\Psi_{\alpha,2}^\varepsilon) &= \sqrt{\varepsilon} w^{\varepsilon,i} (u_i - v_i^\alpha) \cdot \nabla \psi_2^\varepsilon \\ &\quad + \sqrt{\varepsilon} w^{\varepsilon,i} \frac{\partial (u_i - v_i^\alpha)}{\partial x_i} \psi_2^\varepsilon - v^{\alpha\varepsilon} \cdot \nabla \psi_2^\varepsilon, \end{aligned}$$

from which we deduce, using expressions (47) of $w^{\varepsilon,i}$, that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_{\Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_h} |\nabla \Theta_{\alpha,2}^\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_h} |\Theta_{\alpha,2}^\varepsilon|^2 dx = 0.$$

■

We now define the test-function $u^{0,\varepsilon}$ by

$$u^{0,\varepsilon} = \begin{cases} u & \text{in } \Omega \setminus \Sigma_\varepsilon^{1,\mathcal{K}}, \\ u - \operatorname{div}((\widehat{u - \tilde{u}^\varepsilon}) \psi_1^\varepsilon) + \Theta_{\alpha,1}^\varepsilon & \text{in } \Sigma_\varepsilon^{1,\mathcal{K}} \cap \Omega_\alpha, \alpha = o, w, \\ \quad + \sqrt{\varepsilon} (w^{i,bl,\varepsilon} + a\delta_{i3}e_3)(u_i - v_i^\alpha) \psi_1^\varepsilon & \text{in } \Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_{ah}^\varepsilon(\omega), \alpha = o, w, \\ \sqrt{\varepsilon} w^{\varepsilon,i} (u_i - v_i^\alpha) \psi_2^\varepsilon + v^{\alpha\varepsilon} (1 - \psi_2^\varepsilon) + \Theta_{\alpha,2}^\varepsilon & \text{in } \Omega_{ah}^\varepsilon(\omega) \setminus \Sigma_\varepsilon^{2,\mathcal{K}}, \alpha = o, w. \end{cases} \quad (86)$$

The sequence $(u^{0,\varepsilon})_\varepsilon$ verify the following

Lemma 8.3:

- (1) $u^{0,\varepsilon} \in V^\varepsilon(\omega)$.
- (2) $(u^{0,\varepsilon})_\varepsilon$ τ_0 -converges to $(u, v) \in V^0$ and $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) = \mathcal{F}_0(u, v)$.

Proof: 1. Observe that $u^{0,\varepsilon} = 0$ on $\Gamma^\varepsilon(\omega)$ and

$$\begin{aligned} [u^{0,\varepsilon}]_\Sigma &= \sqrt{\varepsilon}(w^{i,bl,\varepsilon} + a\delta_{i3}e_3)(u_i - v_i^\alpha)\psi_1^\varepsilon \\ &\quad - \sqrt{\varepsilon}w^{\varepsilon,i}(u_i - v_i^\alpha)\psi_2^\varepsilon \\ &= 0. \end{aligned}$$

On the other hand

$$\begin{aligned} \operatorname{div} u^{0,\varepsilon} &= \operatorname{div} u - \operatorname{div}\{\operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon)\} \\ &\quad + \operatorname{div}\{\sqrt{\varepsilon}(w^{i,bl,\varepsilon} + a\delta_{i3}e_3)(u_i - v_i^\alpha)\psi_1^\varepsilon\} + \operatorname{div}\Theta_{\alpha,1}^\varepsilon \\ &\quad + \operatorname{div}\{\sqrt{\varepsilon}w^{\varepsilon,i}(u_i - v_i^\alpha)\psi_2^\varepsilon + v^{\alpha\varepsilon}(1 - \psi_2^\varepsilon)\} + \operatorname{div}(\Theta_{\alpha,2}^\varepsilon). \end{aligned}$$

As

$$\operatorname{div} u = \operatorname{div}\{\operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon)\} = 0,$$

we have, according to Equations (83) and (84), and to hypothesis (61), that

$$\begin{aligned} \operatorname{div} u^{0,\varepsilon} &= \int_{\Sigma_\alpha^\varepsilon(\omega)} \Psi_{\alpha,1}^\varepsilon \cdot e_3 \, ds - \int_{\Sigma_\alpha^\varepsilon(\omega)} \Psi_{\alpha,2}^\varepsilon \cdot e_3 \, ds \\ &= 0. \end{aligned}$$

Therefore $\operatorname{div} u^{0,\varepsilon} = 0$ and $u^{0,\varepsilon} \in V^\varepsilon(\omega)$.

2. Using Lemma 8.2, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) &= 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u)D_{ij}(u) \, dx \\ &\quad + 2 \lim_{\varepsilon \rightarrow 0} \mu_\alpha \int_{\Sigma_\varepsilon^{1,\varepsilon} \cap \Omega_\alpha} \left\{ D_{ij}(\operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon)) \right. \\ &\quad \left. \times D_{ij} \operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon) \right\} \, dx \\ &\quad + 2 \lim_{\varepsilon \rightarrow 0} \mu_\alpha \int_{\Sigma_\varepsilon^{1,\varepsilon} \cap \Omega_\alpha} \left\{ D_{ij} \left(\begin{aligned} &\sqrt{\varepsilon}(w^{i,bl,\varepsilon} + a^i\delta_{i3}e_3) \\ &\times (u_i - v_i^\alpha)\psi_1^\varepsilon \end{aligned} \right) \right. \\ &\quad \left. \times D_{ij} \left(\begin{aligned} &\sqrt{\varepsilon}(w^{i,bl,\varepsilon} + a^i\delta_{i3}e_3) \\ &\times (u_i - v_i^\alpha)\psi_1^\varepsilon \end{aligned} \right) \right\} \, dx \\ &\quad + 2 \lim_{\varepsilon \rightarrow 0} \mu_\alpha \varepsilon^2 \int_{\Sigma_\varepsilon^{2,\varepsilon} \cap \Omega_{ah}^\varepsilon(\omega)} \left\{ D_{ij}(\sqrt{\varepsilon}w^{\varepsilon,i}(u_i - v_i^\alpha)\psi_2^\varepsilon) \right. \\ &\quad \left. \times D_{ij}(\sqrt{\varepsilon}w^{\varepsilon,i}(u_i - v_i^\alpha)\psi_2^\varepsilon) \right\} \, dx \\ &\quad + 2 \lim_{\varepsilon \rightarrow 0} \mu_\alpha \varepsilon^2 \int_{\Omega_{ah}^\varepsilon(\omega)} \left\{ D_{ij}(v^{\alpha\varepsilon}(1 - \psi_2^\varepsilon)) \right. \\ &\quad \left. \times D_{ij}(v^{\alpha\varepsilon}(1 - \psi_2^\varepsilon)) \right\} \, dx. \end{aligned} \quad (87)$$

As $\|u - \tilde{u}^\varepsilon\|_{L^2(\Omega_1)} < \varepsilon$, we have, using (81),

$$\lim_{\varepsilon \rightarrow 0} \mu_\alpha \int_{\Sigma_\varepsilon^{1,\varepsilon} \cap \Omega_\alpha} \left\{ D_{ij}(\operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon)) \right. \\ \left. \times D_{ij} \operatorname{div}(\widehat{(u - \tilde{u}^\varepsilon)}\psi_1^\varepsilon) \right\} \, dx = 0. \quad (88)$$

Using Lemma 6.1, we deduce that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \int_{\Sigma_\varepsilon^{1,\mathcal{K}} \cap \Omega_\alpha} \left\{ \begin{aligned} & D_{ij} \left(\sqrt{\varepsilon} (w^{i,bl,\varepsilon} + a^i \delta_{i3} e_3) \right) \\ & \times (u_i - v_i^\alpha) \psi_1^\varepsilon \end{aligned} \right\} \\ & \times D_{ij} \left(\sqrt{\varepsilon} (w^{i,bl,\varepsilon} + a^i \delta_{i3} e_3) \right) \\ & \times (u_i - v_i^\alpha) \psi_1^\varepsilon \Big\} dx \\ & = 2\mu_\alpha \int_{\Sigma} K^{\alpha,bl}(\omega) (u - v^\alpha) \cdot (u - v^\alpha) dx'. \end{aligned} \quad (89)$$

Using Lemma 5.1, we deduce that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \varepsilon^2 \int_{\Omega_{ah}^\varepsilon(\omega)} \left\{ \begin{aligned} & D_{ij} (v^{\alpha\varepsilon} (1 - \psi_2^\varepsilon)) \\ & \times D_{ij} (v^{\alpha\varepsilon} (1 - \psi_2^\varepsilon)) \end{aligned} \right\} dx \\ & = 2\mu_\alpha \langle \zeta^{-2}(0) \rangle^2 \int_{\Sigma} (K^{*\alpha})^{-1} v_\tau^\alpha \cdot v_\tau^\alpha dx' \\ & \quad + 2\mu_\alpha \langle \zeta^{-2}(0) \rangle^2 \gamma_\alpha^* \gamma_\alpha^{-2} \int_{\Sigma} (v_3^\alpha)^2 dx', \end{aligned} \quad (90)$$

and

$$\lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \varepsilon^2 \int_{\Sigma_\varepsilon^{2,\mathcal{K}} \cap \Omega_{ah}^\varepsilon(\omega)} \left\{ \begin{aligned} & D_{ij} (\sqrt{\varepsilon} w^{\varepsilon,i} (u_i - v_i^\alpha) \psi_2^\varepsilon) \\ & \times D_{ij} (\sqrt{\varepsilon} w^{\varepsilon,i} (u_i - v_i^\alpha) \psi_2^\varepsilon) \end{aligned} \right\} dx = 0. \quad (91)$$

We deduce from (88), (89), and (91) that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) = \mathcal{F}_0(u, v).$$

■

We are now in a position to prove the first assertion of Theorem 7.2.

Proposition 8.4: *For every $(u, v) \in V^0$ there exists $u^{0,\varepsilon} \in V^\varepsilon(\omega)$, such that $(u^{0,\varepsilon})_\varepsilon \tau_0$ -converges to (u, v) and*

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) \leq \mathcal{F}_0(u, v).$$

Proof: For every $(u, v) \in V^0$, there exists a sequence

$$(u_n, v_n)_n \subset (C^1(\overline{\Omega}, \mathbb{R}^3) \times C_c^1(\Sigma, \mathbb{R}^3)) \cap V^0,$$

which converges to (u, v) in the strong topology of $H^1(\Omega, \mathbb{R}^3) \times L^2(\Sigma, \mathbb{R}^3)$. We built a test-function $u_n^{0,\varepsilon} \in V^\varepsilon(\omega)$ by replacing (u, v) in (86) by (u_n, v_n) . Then, according to Lemma 8.3, the sequence $(u_n^{0,\varepsilon})_\varepsilon \tau_0$ -converges to (u_n, v_n) and $\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_n^{0,\varepsilon}) = \mathcal{F}_0(u_n, v_n)$. The continuity of the functional $\mathcal{F}_0$ with respect to the strong topology of $H^1(\Omega, \mathbb{R}^3) \times L^2(\Sigma, \mathbb{R}^3)$ implies that $\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_n^{0,\varepsilon}) = \mathcal{F}_0(u, v)$. The topology τ_0 being metrizable, the diagonalization argument of [10, Corollary 1.18] proves the existence of a subsequence $(u_{n(\varepsilon)}^0)_\varepsilon$ such that $(u_{n(\varepsilon)}^0)_\varepsilon \tau_0$ -converges to (u, v) and $\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_{n(\varepsilon)}^0) = \mathcal{F}_0(u, v)$. Thus, taking $(u^{0,\varepsilon})_\varepsilon = (u_{n(\varepsilon)}^0)_\varepsilon$, we have that $(u^{0,\varepsilon})_\varepsilon \tau_0$ -converges to (u, v) and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^{0,\varepsilon}) = \mathcal{F}_0(u, v).$$

■

8.2. Second step: verification of the lim inf property

Proposition 8.5: For every $u^\varepsilon \in V^\varepsilon(\omega)$, such that $(u^\varepsilon)_\varepsilon$ τ_0 -converges to (u, v) , we have $(u, v) \in V^0$ and

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) \geq \mathcal{F}_0(u, v).$$

Proof: Let $u^\varepsilon \in V^\varepsilon(\omega)$, such that $(u^\varepsilon)_\varepsilon$ τ_0 -converges to (u, v) . We may suppose that $\sup_\varepsilon \mathcal{F}_\varepsilon(u^\varepsilon) < +\infty$, otherwise the verification of lim inf property is trivial. Then, using Lemma 4.2, we have $(u, v) \in V^0$. Let us consider a sequence

$$(u_n, v_n)_n \subset (C^1(\overline{\Omega}, \mathbb{R}^3) \times C_c^1(\Sigma, \mathbb{R}^3)) \cap V^0,$$

which converges to (u, v) in the strong topology of $H^1(\Omega, \mathbb{R}^3) \times L^2(\Sigma, \mathbb{R}^3)$, and a test-function $u_n^{0,\varepsilon} \in V^\varepsilon(\omega)$ constructed for each n by replacing (u, v) in (86) by (u_n, v_n) . Using the definition of the subdifferentiability of convex functionals, we have that

$$\begin{aligned} \mathcal{F}_\varepsilon(u^\varepsilon) &\geq \mathcal{F}_\varepsilon(u_n^{0,\varepsilon}) + 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u_n^{0,\varepsilon}) D_{ij}(u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\ &\quad + 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha h}^\varepsilon(\omega)} D_{ij}(u_n^{0,\varepsilon}) D_{ij}(u^\varepsilon - u_n^{0,\varepsilon}) \, dx. \end{aligned} \quad (92)$$

We immediately obtain that

$$\liminf_{\varepsilon \rightarrow 0} 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u_n^{0,\varepsilon}) D_{ij}(u^\varepsilon - u_n^{0,\varepsilon}) \, dx \geq 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u_n) D_{ij}(u - u_n) \, dx. \quad (93)$$

On the other hand, using Green's formula, we get

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha h}^\varepsilon(\omega)} D_{ij}(u_n^{0,\varepsilon}) D_{ij}(u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha h}^\varepsilon(\omega)} \nabla(v^{\alpha\varepsilon}(1 - \psi_2^\varepsilon)) : \nabla(u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} 2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha h}^\varepsilon(\omega)} \nabla v^{\alpha\varepsilon} : \nabla(u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\ &= \lim_{\varepsilon \rightarrow 0} -2\mu_\alpha \varepsilon^2 \int_{\Omega_{\alpha h}^\varepsilon(\omega)} \Delta v^{\alpha\varepsilon} : (u^\varepsilon - u_n^{0,\varepsilon}) \, dx. \end{aligned} \quad (94)$$

Passing to polar coordinates, we can easily check that

$$\begin{aligned} \Delta_{r,\theta} v_\tau^{w\varepsilon} &= \frac{\langle \zeta^2(0) \rangle}{h} \sum_{(i,j) \in I_\varepsilon(\omega)} v_m^w(i\varepsilon, j\varepsilon) K^{-w} \int_{-h}^0 \frac{(\kappa_{ij}^\varepsilon)^2}{\varepsilon^2} \Delta_{r,\theta} w^m(\sigma_{ij}^\varepsilon(x, \omega), \theta) \, dx_3, \\ \Delta_{r,\theta} v_3^{w\varepsilon} &= \frac{\langle \zeta^2(0) \rangle}{h\gamma_w} \sum_{(i,j) \in I_\varepsilon(\omega)} v_3^w(i\varepsilon, j\varepsilon) \left(\int_{-h}^0 \frac{(\kappa_{ij}^\varepsilon)^2}{\varepsilon^2} \Delta_{r,\theta} \eta(\sigma_{ij}^\varepsilon(x, \omega), \theta) \, dx_3 \right), \end{aligned} \quad (95)$$

where $\kappa_{ij}^\varepsilon = \kappa_{ij}^\varepsilon(x_3\varepsilon^{-1/2}, \omega)$, and

$$\begin{aligned}\Delta_{r,\theta} v_\tau^{o\varepsilon} &= \frac{\langle \zeta^2(0) \rangle}{h} \sum_{\substack{m=1,2 \\ (i,j) \in I_\varepsilon(\omega)}} v_m^o(i\varepsilon, j\varepsilon) K^{-o} \int_{-h}^0 \frac{1}{16\varepsilon^2 \varrho_{ij}^\varepsilon} \Delta_{r,\theta} w^m \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}^\varepsilon}, \theta \right) dx_3, \\ \Delta_{r,\theta} v_3^{o\varepsilon} &= \frac{\langle \zeta^2(0) \rangle}{h\gamma_o} \sum_{(i,j) \in I_\varepsilon(\omega)} v_3^o(i\varepsilon, j\varepsilon) \int_{-h}^0 \frac{1}{16\varepsilon^2 \varrho_{ij}^\varepsilon} \Delta_{r,\theta} \eta \left(\frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}^\varepsilon}, \theta \right) dx_3,\end{aligned}\tag{96}$$

where $\varrho_{ij}^\varepsilon = \varrho_{ij}(x_3\varepsilon^{-1/2}, \omega)$. Let us introduce the following vectorial measure

$$m_\varepsilon^\alpha = \mu_\alpha \varepsilon^2 \begin{pmatrix} \Delta v_\tau^{\alpha\varepsilon}(x) \\ \Delta v_3^{\alpha\varepsilon}(x) \end{pmatrix} \mathbf{1}_{\Omega_{ah}^\varepsilon(\omega)}(x) dx, \quad \alpha = o, w.\tag{97}$$

Then, according to (95) and (96), we deduce, by introducing the change of variables $s = \frac{r_{ij}^\varepsilon(x')}{4\varrho_{ij}(x_3\varepsilon^{-1/2}, \omega)}$ in $F_{o,ij}^\varepsilon(\omega)$ and $s = \kappa_{ij}(x_3\varepsilon^{-1/2}, \omega)r_{ij}^\varepsilon(x')$ in $F_{w,ij}^\varepsilon(\omega)$, that the sequence $(m_\varepsilon^\alpha)_\varepsilon$ $*$ -weakly converges in the sense of measures to the measure m^α defined by

$$m^\alpha = \mu_\alpha \begin{pmatrix} v_1^\alpha A_\alpha^1 \\ v_2^\alpha A_\alpha^2 \\ v_3^w A_\alpha^3 \end{pmatrix} \mathbf{1}_\Sigma(x') dx',\tag{98}$$

where

$$\begin{aligned}A_w^m &= \int_{D_w} \Delta w^m(y) dy - \frac{1}{4} \left\langle \frac{\zeta(0) - 2\varrho(0)}{\zeta(0) - \varrho(0)} \right\rangle \int_{D_w} \frac{\Delta w^m(y)}{|y|} dy, \quad m = 1, 2, \\ A_w^3 &= \int_{D_w} \Delta \eta(y) dy - \frac{1}{4} \left\langle \frac{\zeta(0) - 2\varrho(0)}{\zeta(0) - \varrho(0)} \right\rangle \int_{D_w} \frac{\Delta \eta(y)}{|y|} dy, \\ A_o^m &= \int_{D_o} \Delta w^m(y) dy, \quad m = 1, 2, \\ A_o^3 &= \int_{D_o} \Delta \eta(y) dy.\end{aligned}\tag{99}$$

Using (97)–(99) we deduce that

$$\left| \int_{\mathbb{R}^3} (u^\varepsilon - u_n^{0,\varepsilon}) \cdot dm_\varepsilon^\alpha \right| \leq C_\alpha \left(\int_{\Omega_{ah}^\varepsilon(\omega)} |u^\varepsilon - u_n^{0,\varepsilon}|^2 dx \right)^{1/2},\tag{100}$$

where C_α is a positive constant. As $\sup_\varepsilon (\int_{\Omega_{ah}^\varepsilon(\omega)} |u^\varepsilon - u_n^{0,\varepsilon}|^2 dx)^2 dx < +\infty$, we deduce from (100) that the sequence $((u^\varepsilon - u_n^{0,\varepsilon}) \cdot dm_\varepsilon^\alpha)_\varepsilon$ of measures has bounded variation.

Thus, using Fenchel's inequality, we infer that

$$\begin{aligned}
 & \liminf_{\varepsilon \rightarrow 0} \mu_\alpha \varepsilon^2 \int_{\Omega_{ah}^\varepsilon(\omega)} \Delta v^{\alpha\varepsilon} \cdot (u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\
 &= \liminf_{\varepsilon \rightarrow 0} \mu_\alpha \varepsilon^2 \int_{\Omega_{ah}^\varepsilon(\omega)} \Delta v^{\alpha\varepsilon} \cdot (u^\varepsilon - u_n^{0,\varepsilon}) \, dx \\
 &\geq \mu_\alpha h \langle \zeta^2(0) \rangle A_\alpha^m \int_\Sigma v_m^\alpha (v - v_n)_m \, dx.
 \end{aligned} \tag{101}$$

Combining (92), (93), and (101), we deduce that

$$\begin{aligned}
 \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) &\geq \mathcal{F}_0(u_n, v_n) + 2\mu_\alpha \int_{\Omega_\alpha} D_{ij}(u_n) D_{ij}(u - u_n) \, dx \\
 &\quad - 2\mu_\alpha h \langle \zeta^2(0) \rangle A_\alpha^m \int_\Sigma v_m^\alpha (v - v_n)_m \, dx.
 \end{aligned} \tag{102}$$

Letting n tend to $+\infty$ in (102), we conclude that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u^\varepsilon) \geq \mathcal{F}_0(u, v).$$

■

Author contributions

CRediT: **Mustapha El Jarroudi**: Conceptualization, Formal analysis, Supervision, Writing – original draft; **Mustapha Er-Riani**: Conceptualization, Formal analysis, Writing – original draft; **Abdelouaah Lidouh**: Conceptualization, Formal analysis, Writing – original draft; **Youness Filali**: Conceptualization, Formal analysis, Writing – original draft; **Moussa El Jarroudi**: Conceptualization, Formal analysis, Writing – original draft

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