

A Functorial Approach to Multi-Space Interpolation with Function Parameters

T. Lamby* and Samuel Nicolay†

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Abstract

We develop a functorial interpolation theory for finite families of Banach spaces associated with Boyd-function parameters. This framework unifies two classical extensions of real interpolation—interpolation with function parameters and interpolation of multiple spaces—by providing a common functorial framework encompassing both interpolation with function parameters and interpolation of multiple spaces.

Within this unified setting, we establish the equivalence between the associated K - and J -methods and investigate the resulting interpolation functors. In particular, we prove a power theorem together with several stability and reiteration results. Finally, we illustrate the scope of the theory by identifying interpolation spaces for generalized Sobolev and Lorentz spaces, obtaining generalized Besov spaces as interpolation spaces and deriving reiteration formulas for interpolation of multiple spaces.

Keywords: Real interpolation; multi-space interpolation; Boyd functions; Sobolev spaces; Besov spaces; Lorentz spaces

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1 Introduction

The theory of interpolation of Banach spaces originated with an observation of Marcinkiewicz [29], which ultimately led to the Riesz–Thorin theorem [35, 37]. Informally, if a linear operator acts continuously on two spaces L^p and L^q , then it also acts continuously on every intermediate space L^r . Real interpolation methods [3, 8] provide a systematic framework for this principle. Over the past decades, real interpolation theory has evolved along two complementary directions.

The first replaces the scalar interpolation parameter by a function parameter. Initiated by Peetre [33] and subsequently developed by Gustavsson, Brudnyi, Cwikel, Janson, Ovchinnikov and many others [14, 7, 10, 17, 32, 31, 28, 34, 23], this approach considerably enlarges the class of interpolation spaces while preserving the operator-theoretic nature of interpolation. Boyd functions, in particular, provide a flexible framework that naturally includes limiting interpolation

*University of Luxembourg, Department of Mathematics – Maison du Nombre, 6, Avenue de la Fonte, 4364 Esch-sur-Alzette, Luxembourg. thomas.lamby@uni.lu

†Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège. Orcid ID: 0000-0003-0549-0566. S.Nicolay@uliege.be

scales. Alternative formulations have been proposed in [20, 26], while the relationships between these approaches have recently been clarified in [22].

The second direction concerns the interpolation of more than two spaces. Beginning with the pioneering work of Foias [13], this theory has been further developed in [18, 39, 36, 38, 2]. Beyond extending the classical interpolation framework, these constructions have led to a rich theory of structural properties, notably reiteration, with important applications to scales of function spaces.

Although these two directions are rooted in the same interpolation principles, they have largely evolved independently. The aim of the present paper is to develop a functorial interpolation theory for finite families of Banach spaces associated with Boyd-function parameters. Our framework unifies interpolation with function parameters and interpolation of multiple spaces, showing that they are complementary manifestations of the same interpolation principle. It recovers the classical theory, naturally incorporates limiting interpolation parameters, and provides a common setting for structural results such as equivalence, power theorems and reiteration.

The interest of this approach is not merely one of generality. By separating structural properties from the particular realization of interpolation spaces, the functorial point of view allows fundamental questions to be treated in a unified manner. In particular, classical results such as the equivalence between the K - and J -methods, power theorems and reiteration formulas naturally extend to the present setting. The applications developed in the final sections should therefore be viewed as illustrations of the general theory rather than as isolated examples.

The paper is organized as follows. Section 2 recalls the properties of Boyd functions and interpolation functors required throughout the paper. Section 3 introduces the interpolation construction for finite families of Banach spaces. Section 4 establishes the equivalence between the associated K - and J -methods. Section 6 investigates the structural properties of the resulting interpolation functors, including the power theorem and stability results. Finally, Sections 8 and 9 demonstrate how the abstract framework naturally applies to generalized Sobolev and Lorentz spaces and leads to reiteration formulas for interpolation of multiple spaces.

2 Boyd functions

Boyd functions constitute the parameter class underlying the interpolation theory developed in this paper. Their stability properties and associated indices will play a central role in the construction of the interpolation functors and in the analysis of their structural properties. We briefly recall the definitions and results that will be needed throughout the paper; for a more detailed account, we refer to [22, 31] and the references therein. We denote by $L^q(E, dt/t)$ the space of measurable functions f defined on E , satisfying the condition that $(\int_E |f|^q dt/t)^{1/q}$ is finite; with the conventional adjustment made for the case when $q = \infty$.

Definition 2.1. A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is classified as a Boyd function if it satisfies the following conditions:

- $\phi(1) = 1$,

- ϕ is continuous,
- for any $t > 0$,

$$\bar{\phi}(t) = \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty.$$

The set of Boyd functions is denoted by $\mathcal{B}$.

The function $\bar{\phi}$ is sub-multiplicative, Lebesgue-measurable and it holds that $1/\bar{\phi}(1/t) \leq \phi(t) \leq \bar{\phi}(t)$ and $1/\phi(t) \leq \bar{\phi}(1/t)$.

Definition 2.2. For a given $\phi \in \mathcal{B}$, the lower and upper Boyd indices [30, 6] are defined as follows:

$$\underline{b}(\phi) = \sup_{t \in (0,1)} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t \rightarrow 0^+} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\bar{b}(\phi) = \inf_{t \in (1,\infty)} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t \rightarrow \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

For instance, considering $\gamma \geq 0$ and $\theta \in \mathbb{R}$, if we set

$$\phi(t) = t^\theta (1 + |\log t|)^\gamma,$$

for $t > 0$, it can be readily verified that $\underline{b}(\phi) = \bar{b}(\phi) = \theta$. If $\phi \in \mathcal{B}$, for $\varepsilon > 0$ and $R > 0$, there exists a constant $C > 0$ such that

$$C^{-1}r^{\bar{b}(\phi)+\varepsilon} \leq \phi(r) \leq Cr^{\underline{b}(\phi)-\varepsilon},$$

for any $r \in (0, R)$ and a constant $C > 0$ such that

$$C^{-1}r^{\underline{b}(\phi)-\varepsilon} \leq \phi(r) \leq Cr^{\bar{b}(\phi)+\varepsilon},$$

for any $r \geq R$. Furthermore, for such a function ϕ , the following properties hold:

- $\underline{b}(\phi) > 0 \Leftrightarrow \bar{\phi} \in L^1((0,1), dt/t) \Leftrightarrow \lim_{t \rightarrow 0^+} \bar{\phi}(t) = 0$,
- $\bar{b}(\phi) < 0 \Leftrightarrow \bar{\phi} \in L^1((1,\infty), dt/t) \Leftrightarrow \lim_{t \rightarrow \infty} \bar{\phi}(t) = 0$,
- $\underline{b}(\phi) > 0 \Rightarrow \phi \in L^\infty(0,1)$,
- $\bar{b}(\phi) < 0 \Rightarrow \phi \in L^\infty(1,\infty)$.

Definition 2.3. A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is considered Boyd-regular if it satisfies the following conditions:

- $\phi(1) = 1$,
- $\phi \in C^1(0, \infty)$,

- the inequalities

$$0 < \inf_{t>0} t \frac{|D\phi(t)|}{\phi(t)} \leq \sup_{t>0} t \frac{|D\phi(t)|}{\phi(t)} < \infty$$

hold.

The set of Boyd-regular functions is denoted by $\mathcal{B}^*$.

It is known that $\mathcal{B}^* \subset \mathcal{B}$. The subset of functions $\phi \in \mathcal{B}^*$ that are strictly increasing (resp. strictly decreasing) is denoted by $\mathcal{B}_+^*$ (resp. $\mathcal{B}_-^*$).

Given two functions f and g defined on $(0, \infty)$, the notation $f \asymp g$ is used when there exists a constant $C > 0$ such that $C^{-1}g(t) \leq f(t) \leq Cg(t)$ for any $t > 0$. If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ (resp. $\bar{b}(\phi) < 0$), then there exists $\xi \in \mathcal{B}_+^*$ (resp. $\xi \in \mathcal{B}_-^*$) such that $\xi^{-1} \in \mathcal{B}_+^*$ (resp. $\xi^{-1} \in \mathcal{B}_-^*$) and $\phi \asymp \xi$ (see [31]).

3 Interpolation

The construction developed in this paper is based on extending the functorial viewpoint of real interpolation from couples of Banach spaces to finite families of spaces. While the passage from two to several spaces does not always preserve the full structure of the classical theory, a large class of interpolation functors can still be constructed by combining the K - and J -functionals with suitable parameters. This section introduces the general framework in which these constructions will be carried out.

Definition 3.1. We denote $\mathbf{A}$ (resp. $\mathbf{B}$) as a $(n+1)$ -tuples of Banach spaces $A_0, \dots, A_n$ (resp. $B_0, \dots, B_n$) such that each A_j (resp. each B_j) can be linearly and continuously imbedded in a Hausdorff topological vector space $\mathcal{A}$ (resp. $\mathcal{B}$): $\mathbf{A} = (A_0, \dots, A_n)$. Given two such sets $\mathbf{A}$ and $\mathbf{B}$, morphisms $T : \mathbf{A} \rightarrow \mathbf{B}$ are homomorphisms $T : \mathcal{A} \rightarrow \mathcal{B}$ such that $T : A_j \rightarrow B_j$ are continuous homomorphisms.

Definition 3.2. The standard norm on $\Sigma(\mathbf{A}) = A_0 + \dots + A_n$ is defined by

$$a \mapsto \|a\|_{\Sigma(\mathbf{A})} = \inf\{\|a_0\|_{A_0} + \dots + \|a_n\|_{A_n}\},$$

where the infimum is taken over all decompositions $a = a_0 + \dots + a_n$, with $a_j \in A_j$.

Definition 3.3. The standard norm on $\Delta(\mathbf{A}) = A_0 \cap \dots \cap A_n$ is defined by

$$a \mapsto \|a\|_{\Delta(\mathbf{A})} = \max\{\|a\|_{A_0}, \dots, \|a\|_{A_n}\}.$$

Let $\mathcal{B}_n$ be the collection of the elements $\mathbf{A}$; if $\mathcal{B}$ denotes the category of Banach spaces, it is evident that Σ and Δ are functors from $\mathcal{B}_n$ to $\mathcal{B}$.

Definition 3.4. An interpolation functor of order n is a functor $F : \mathcal{B}_n \rightarrow \mathcal{B}$ such that

- (i) $\Delta(\mathbf{A}) \hookrightarrow F(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A})$ for any $\mathbf{A}$ in $\mathcal{B}_n$,
- (ii) $F(T) = T|_{F(\mathbf{A})}$ for any morphism $T : \mathbf{A} \rightarrow \mathbf{B}$.

Definition 3.5. Let f be a function from $(0, \infty)^{n+1}$ to $(0, \infty)$; an interpolation functor F is of type f if there exists a constant $C \geq 1$ such that

$$\|T\|_{F(\mathbf{A}), F(\mathbf{B})} \leq Cf(\|T\|_{A_0, B_0}, \dots, \|T\|_{A_n, B_n}),$$

for any morphism $T : \mathbf{A} \rightarrow \mathbf{B}$. If, moreover,

- $f(t_0, \dots, t_n) = \max\{t_0, \dots, t_n\}$, we say that F is bounded,
- $C = 1$, we say that F is exact.

Definition 3.6. Given $\phi_1, \dots, \phi_n \in \mathcal{B}$, $p \in [1, \infty]$, we will say that condition $(\mathcal{H}_1)_p$ is satisfied when

$$\left(\int_{(0, \infty)^n} \left(\frac{1}{\phi_1(t_1) \cdots \phi_n(t_n)} \min\{1, t_1, \dots, t_n\} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p} < \infty,$$

with the usual modification in the case $p = \infty$.

Definition 3.7. Given $\phi_1, \dots, \phi_n \in \mathcal{B}$, we say that condition $(\mathcal{H}_2)$ is satisfied if

$$\min\{\underline{b}(\phi_1), \dots, \underline{b}(\phi_n)\} > 0 \quad \text{and} \quad \max\{\bar{b}(\phi_1), \dots, \bar{b}(\phi_n)\} < 1$$

and that condition $\overline{(\mathcal{H}_2)}$ is satisfied if

$$\min\{\underline{b}(\phi_1), \dots, \underline{b}(\phi_n)\} \geq 0 \quad \text{and} \quad \max\{\bar{b}(\phi_1), \dots, \bar{b}(\phi_n)\} \leq 1.$$

Remark 3.8. Using the properties of Boyd functions, one can readily show that $(\mathcal{H}_2)$ implies $(\mathcal{H}_1)_p$ for all $p \in [1, \infty)$ and that $\overline{(\mathcal{H}_2)}$ implies $(\mathcal{H}_1)_\infty$.

Let $\sim$ be the equivalence relation defined by $(t_0, \dots, t_n) \sim (s_0, \dots, s_n)$ if and only if there exists $\lambda > 0$ such that $t_j = \lambda s_j$ for all $j \in \{0, \dots, n\}$.

Definition 3.9. Let $p \in [1, \infty]$, $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $\overline{(\mathcal{H}_2)}$ and f be a function from $(0, \infty)^{n+1}$ to $(0, \infty)$; one defines

$$\Phi_p^{\phi_1, \dots, \phi_n}(f) = \left(\int_{\mathbb{P}_+^n} \left(\frac{\phi_1(t_0) \cdots \phi_n(t_0)}{t_0 \phi_1(t_1) \cdots \phi_n(t_n)} f(t_0, t_1, \dots, t_n) \right)^p d\omega(t) \right)^{1/p},$$

with the usual modification in the case $p = \infty$, where ω is the Haar measure on $\mathbb{P}_+^n = (0, \infty)^{n+1} / \sim$.

Remark 3.10. For the Haar measure ω in the previous definition, we have

$$d\omega(t) = \sum_{j=0}^n (-1)^j \frac{dt_0}{t_0} \wedge \cdots \wedge \frac{dt_j}{t_j} \wedge \cdots \wedge \frac{dt_n}{t_n},$$

where the symbol $\sim$ indicates a missing factor. Consequently,

$$\Phi_p^{\phi_1, \dots, \phi_n}(f) = \left(\int_{(0, \infty)^n} \left(\frac{f(1, t_1, \dots, t_n)}{\phi_1(t_1) \cdots \phi_n(t_n)} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p}, \quad (1)$$

with the usual modification in the case $p = \infty$.

Definition 3.11. Given $a \in \Sigma(\mathbf{A})$ and $t \in (0, \infty)^{n+1}$, let

$$K(t, a) = K(t_0, \dots, t_n, a) = \inf_a \{t_0 \|a_0\|_{A_0} + t_1 \|a_1\|_{A_1} + \dots + t_n \|a_n\|_{A_n}\},$$

where the infimum is taken over all the decompositions $a = a_0 + \dots + a_n$, $a_j \in A_j$. In the same way, for $a \in \Delta(\mathbf{A})$, we set

$$J(t, a) = J(t_0, \dots, t_n, a) = \max\{t_0 \|a\|_{A_0}, t_1 \|a\|_{A_1}, \dots, t_n \|a\|_{A_n}\}.$$

When the dependence on the couple $\mathbf{A}$ needs to be specified, we write $K^{\mathbf{A}}(t, a)$ (resp. $J^{\mathbf{A}}(t, a)$) instead of $K(t, a)$ (resp. $J(t, a)$).

It follows that, for $t, s \in (0, \infty)^{n+1}$,

$$K(t_0, \dots, t_n, a) \leq \max\left\{\frac{t_0}{s_0}, \dots, \frac{t_n}{s_n}\right\} K(s_0, \dots, s_n, a) \quad \text{for } a \in \Sigma(\mathbf{A}),$$

$$J(t_0, \dots, t_n, a) \leq \max\left\{\frac{t_0}{s_0}, \dots, \frac{t_n}{s_n}\right\} J(s_0, \dots, s_n, a) \quad \text{for } a \in \Delta(\mathbf{A})$$

and

$$K(t_0, \dots, t_n, a) \leq \min\left\{\frac{t_0}{s_0}, \dots, \frac{t_n}{s_n}\right\} J(s_0, \dots, s_n, a) \quad \text{for } a \in \Sigma(\mathbf{A}).$$

Definition 3.12. Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $\overline{(\mathcal{H}_2)}$; we define $K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ as the set of all $a \in \Sigma(\mathbf{A})$ for which

$$\|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} = \Phi_p^{\phi_1, \dots, \phi_n}(K(\cdot, a)) < \infty.$$

Proposition 3.13. Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_1)_p$; the functor $K_p^{\phi_1, \dots, \phi_n}$ is an exact interpolation functor of type f , where

$$f(t_0, \dots, t_n) = t_0 \overline{\phi_1}(t_1/t_0) \cdots \overline{\phi_n}(t_n/t_0).$$

Moreover, there exists $C > 0$ depending on $p, \phi_1, \dots, \phi_n$ such that for $a \in \Sigma(\mathbf{A})$, $t_1, \dots, t_n > 0$,

$$K(1, t_1, \dots, t_n, a) \leq C \phi_1(t_1) \cdots \phi_n(t_n) \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})}.$$

Proof. Using $(\mathcal{H}_1)_p$, for $s_1, \dots, s_n \in (0, \infty)$ and $a \in \Sigma(\mathbf{A})$, we have

$$\begin{aligned} & \left(\int_{(0, \infty)^n} \left(\frac{\min\{1, u_1, \dots, u_n\}}{\phi_1(u_1 s_1) \cdots \phi_n(u_n s_n)} \right)^p \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} \right)^{1/p} \\ & \quad K(1, s_1, \dots, s_n, a) \\ & \leq \left(\int_{(0, \infty)^n} \left(\frac{\min\{1, \frac{t_1}{s_1}, \dots, \frac{t_n}{s_n}\} K(1, s_1, \dots, s_n, a)}{\phi_1(t_1) \cdots \phi_n(t_n)} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p} \\ & \leq \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})}, \end{aligned}$$

so that

$$K(1, s_1, \dots, s_n, a) \leq C \phi_1(s_1) \cdots \phi_n(s_n) \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})}. \quad (2)$$

By using (2) with $s_1 = \dots = s_n = 1$, we get

$$\|a\|_{\Sigma(\mathbf{A})} = K(1, 1, \dots, 1, a) \leq C \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})},$$

so that $K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A})$.

Now, for $a \in \Delta(\mathbf{A})$, since

$$K(1, t_1, \dots, t_n, a) \leq \min\{1, t_1, \dots, t_n\} \|a\|_{\Delta(\mathbf{A})},$$

we have, using $(\mathcal{H}_1)_p$,

$$\|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} \leq C \|a\|_{\Delta(\mathbf{A})},$$

and thus, using $(\mathcal{H}_1)_p$, $\Delta(\mathbf{A}) \hookrightarrow K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$.

Now, let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathcal{B}_n$ and let $T : \mathbf{A} \rightarrow \mathbf{B}$; we can write

$$K^{\mathbf{B}}(1, t_1, \dots, t_n, Ta) \leq \|T\|_{A_0, B_0} K^{\mathbf{A}}(1, t_1 \frac{\|T\|_{A_1, B_1}}{\|T\|_{A_0, B_0}}, \dots, t_n \frac{\|T\|_{A_n, B_n}}{\|T\|_{A_0, B_0}}, a),$$

so that, for $\alpha_j = \frac{\|T\|_{A_j, B_j}}{\|T\|_{A_0, B_0}}$, we have

$$\begin{aligned} \|Ta\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{B})} &\leq \|T\|_{A_0, B_0} \left(\int_{(0, \infty)^n} \left(\frac{K^{\mathbf{A}}(1, u_1, \dots, u_n, a)^p \frac{du_1}{u_1} \dots \frac{du_n}{u_n}}{\phi_1(u_1/\alpha_1) \dots \phi_n(u_n/\alpha_n)} \right)^{1/p} \right. \\ &\quad \left. \leq \|T\|_{A_0, B_0} \overline{\phi_1}(\alpha_1) \dots \overline{\phi_n}(\alpha_n) \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})}, \right. \end{aligned}$$

thus completing the demonstration. $\square$

Remark 3.14. If $(\mathcal{H}_1)_p$ is not satisfied, then $K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = \{0\}$ and thus $K_p^{\phi_1, \dots, \phi_n}$ is not an interpolation functor, since $\Delta(\mathbf{A}) \hookrightarrow K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ is not true when $\Delta(\mathbf{A}) \neq \{0\}$.

Remark 3.15. The fact that we employ assumption $(\mathcal{H}_1)_p$ rather than a more restrictive type of $(\mathcal{H}_2)$ in Proposition 3.13 indicates that we are addressing “limiting cases”, following [24]. This provides an additional justification for using Boyd functions to generalize real interpolation spaces, since the conditions guaranteeing, for instance, that the equivalence theorem remains valid (see [24] for the case $n = 1$, and Remark 4.6) can be stated straightforwardly in terms of Boyd functions.

Definition 3.16. Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $\overline{(\mathcal{H}_2)}$; $J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ is the set of all $a \in \Sigma(\mathbf{A})$ such that there exists a function $u : \mathbb{P}_+^n \rightarrow \Delta(\mathbf{A})$ for which we have

$$a = \int u(t) d\omega(t) = \int_{(0, \infty)^n} u(1, t_1, \dots, t_n) \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n} \quad \text{in } \Sigma(\mathbf{A})$$

and

$$\Phi_p^{\phi_1, \dots, \phi_n}(J(\cdot, u(\cdot))) < \infty.$$

This space is equipped with the norm

$$\|a\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} = \inf_u \Phi_p^{\phi_1, \dots, \phi_n}(J(\cdot, u(\cdot))),$$

the infimum being taken on all $u : \mathbb{P}_+^n \rightarrow \Delta(\mathbf{A})$ such that $a = \int u(t) d\omega(t)$.

Proposition 3.17. *Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$; the functor $J_p^{\phi_1, \dots, \phi_n}$ is an exact interpolation functor of type f where*

$$f(t_0, \dots, t_n) = t_0 \overline{\phi_1}(t_1/t_0) \cdots \overline{\phi_n}(t_n/t_0).$$

Moreover, there exists $C > 0$ depending on $p, \phi_1, \dots, \phi_n$ such that for $a \in \Delta(\mathbf{A})$, $t_1, \dots, t_n > 0$,

$$\|a\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} \leq C \frac{1}{\phi_1(t_1) \cdots \phi_n(t_n)} J(1, t_1, \dots, t_n, a).$$

Proof. For $a \in \Delta(\mathbf{A})$, we have

$$a = \frac{1}{(\log 2)^n} \int_{(0, \infty)^n} a \chi_{(1, 2)^n}(t_1, \dots, t_n) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n},$$

and

$$\begin{aligned} & \|a\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} \\ & \leq \left(\int_{(0, \infty)^n} \left(\frac{J(1, t_1, \dots, t_n, a \frac{\chi_{(1, 2)^n}(t_1, \dots, t_n)}{(\log 2)^n})}{\phi_1(t_1) \cdots \phi_n(t_n)} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p} \\ & \leq \left(\int_{(0, \infty)^n} \left(\frac{\max\{1, \frac{t_1}{s_1}, \dots, \frac{t_n}{s_n}\} J(1, s_1, \dots, s_n, a)}{\phi_1(t_1) \cdots \phi_n(t_n) (\log 2)^n} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p} \\ & \leq C \frac{1}{\phi_1(s_1) \cdots \phi_n(s_n)} J(1, s_1, \dots, s_n, a). \end{aligned}$$

In particular, by taking $s_1 = \dots = s_n = 1$ in the previous inequality, one gets

$$\|a\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} \leq C \|a\|_{\Delta(\mathbf{A})},$$

so that $\Delta(\mathbf{A}) \hookrightarrow J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$. Moreover, with the natural inclusion of the equivalence theorem (see Proposition 4.1), one has

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A}).$$

Now, let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathcal{B}_n$ and let $T : \mathbf{A} \rightarrow \mathbf{B}$. If $a = \int u(s) d\omega(s)$ belongs to $J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$, then $Ta = \int Tu(s) d\omega(s) \in J_p^{\phi_1, \dots, \phi_n}(\mathbf{B})$ and

$$\begin{aligned} & J^{\mathbf{B}}(1, s_1, \dots, s_n, Tu(1, s_1, \dots, s_n)) \\ & \leq \|T\|_{A_0, B_0} J^{\mathbf{A}}(1, s_1 \frac{\|T\|_{A_1, B_1}}{\|T\|_{A_0, B_0}}, \dots, s_n \frac{\|T\|_{A_n, B_n}}{\|T\|_{A_0, B_0}}, u(1, s_1, \dots, s_n)), \end{aligned}$$

so that we have, for $\alpha_j = \frac{\|T\|_{A_j, B_j}}{\|T\|_{A_0, B_0}}$,

$$\begin{aligned} & \|Ta\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{B})} \\ & \leq \|T\|_{A_0, B_0} \left(\int_{(0, \infty)^n} \left(\frac{J^{\mathbf{A}}(1, t_1 \alpha_1, \dots, t_n \alpha_n, u(t))}{\phi_1(t_1) \cdots \phi_n(t_n)} \right)^p \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \right)^{1/p} \\ & = \|T\|_{A_0, B_0} \left(\int_{(0, \infty)^n} \left(\frac{J^{\mathbf{A}}(1, s_1, \dots, s_n, u(t))}{\phi_1(s_1/\alpha_1) \cdots \phi_n(s_n/\alpha_n)} \right)^p \frac{ds_1}{s_1} \cdots \frac{ds_n}{s_n} \right)^{1/p} \\ & \leq \|T\|_{A_0, B_0} \overline{\phi_1}(\alpha_1) \cdots \overline{\phi_n}(\alpha_n) \|a\|_{J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})}, \end{aligned}$$

which ends the proof. $\square$

Remark 3.18. Similarly to Proposition 3.13, a milder assumption can be adopted in the last proposition. Given $\phi_1, \dots, \phi_n \in \mathcal{B}$, $p \in [1, \infty]$, we say that condition $(\mathcal{H}_3)$ is satisfied when

$$\int_{(0, \infty)^n} \left(\frac{1}{\phi_1(t_1) \cdots \phi_n(t_n)} \min\{1, 1/t_1, \dots, 1/t_n\} \right)^{p'} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} < \infty,$$

where p' denotes the conjugate exponent of p . The last proposition remains valid upon replacing $(\mathcal{H}_2)$ by $(\mathcal{H}_3)$. Moreover, note that $(\mathcal{H}_2)$ implies $(\mathcal{H}_3)$ and if $p = 1$, $(\mathcal{H}_2)$ implies $(\mathcal{H}_3)$.

By following the methodology outlined in [36, Remark 4.6], one can readily verify the following equivalent definitions.

Proposition 3.19. *Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$; an equivalent norm on $K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ is given by*

$$\inf \left(\int \|a_0(t)\|_{A_0}^p d\omega(t) \right)^{1/p} + \sum_{j=1}^n \left(\int \left(\frac{t_j \phi_1(t_0) \cdots \phi_n(t_0)}{t_0 \phi_1(t_1) \cdots \phi_n(t_n)} \|a_j(t)\|_{A_j}^p d\omega(t) \right)^{1/p} \right),$$

where the infimum is taken over all decompositions $a = a_0(t) + \cdots + a_n(t)$, $a_j(t) \in A_j$.

Proposition 3.20. *Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$; an equivalent norm on $J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ is given by*

$$\inf_u \max \left\{ \left(\int \|u(t)\|_{A_0}^p d\omega(t) \right)^{1/p}, \max_{1 \leq j \leq n} \left(\int \left(\frac{t_j \phi_1(t_0) \cdots \phi_n(t_0)}{t_0 \phi_1(t_1) \cdots \phi_n(t_n)} \|u(t)\|_{A_j}^p d\omega(t) \right)^{1/p} \right) \right\},$$

where the infimum is taken over all $u : \mathbb{P}_+^n \rightarrow \Delta(\mathbf{A})$ such that $a = \int u(t) d\omega(t)$ in $\Sigma(\mathbf{A})$.

4 On the equivalence theorem

The equivalence between the K - and J -methods is one of the fundamental structural properties of real interpolation. In the classical setting of Banach couples, this equivalence provides two complementary realizations of the same interpolation functor. For finite families of Banach spaces, however, the situation is more subtle: the J -method is no longer equivalent to the K -method in general, and additional structural assumptions are required.

The results presented in this section place this phenomenon within the framework developed above. We first recall the general inclusion between the two methods, provide a counterexample showing that the reverse inclusion fails in general, and then introduce a sufficient condition ensuring equivalence. These results will also be used later to investigate further structural properties of the interpolation functors.

For $n > 1$, the K - and the J - methods are not equivalent, with the K -method being more general.

Proposition 4.1. *Given $p \in [1, \infty]$, if $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfy $(\mathcal{H}_2)$, then*

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}).$$

Proof. The arguments provided in the proof of Theorem 1 in [1] are sufficient to conclude. $\square$

Now, we provide a counterexample for the reverse inclusion, inspired by the one used in the classical case ([36, 39]):

Example 4.2. Let A_0 and A_1 be two Banach spaces and $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi) \leq \bar{b}(\phi) < 1$, such that

$$A_1 \subsetneq K_\infty^\phi(A_0, A_1) \subset A_0.$$

Let $a \in K_\infty^\phi(A_0, A_1) \setminus A_1$ and let A_2 denote the (one dimensional) vector space generated by a with norm inherited from A_0 . We then consider the triplet $\mathbf{A} = (A_0, A_1, A_2)$. It is clear that $\Delta(\mathbf{A}) = 0$, so that $J_p^{\phi_1, \phi_2}(\mathbf{A}) = 0$ for all p , ϕ_1 , and ϕ_2 .

We can construct $\phi_1, \phi_2 \in \mathcal{B}$ such that $a \in K_p^{\phi_1, \phi_2}(\mathbf{A})$. Without loss of generality, one can assume that

$$\max\{\|a\|_{A_0}, \|a\|_{K_\infty^\phi(A_0, A_1)}\} \leq 1.$$

By the given assumptions, for $t_0, t_1, t_2 > 0$, we obtain

$$\begin{aligned} K(t_0, t_1, t_2, a) &\leq \min\{t_0\|a\|_{A_0}, K^{(A_0, A_1)}((t_0, t_1), a), t_2\|a\|_{A_2}\} \\ &\leq \min\{t_0, \frac{t_0\phi(t_1)}{\phi(t_0)}, t_2\}. \end{aligned}$$

Therefore, using $(s_1, s_2) = (\phi(t_1), t_2)$, one has

$$\begin{aligned} \|a\|_{K_p^{\phi_1, \phi_2}(\mathbf{A})} &\leq \left(\int_{\mathbb{P}_+^2} \left(\frac{\phi_1(t_0)\phi_2(t_0)}{t_0\phi_1(t_1)\phi_2(t_2)} \min\{t_0, \frac{t_0\phi(t_1)}{\phi(t_0)}, t_2\} \right)^p dw(t) \right)^{1/p} \\ &= \left(\int_{(0, \infty)^2} \left(\frac{1}{\phi_1(t_1)\phi_2(t_2)} \min\{1, \phi(t_1), t_2\} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/p} \\ &\leq C \left(\int_{(0, \infty)^2} \left(\frac{1}{\phi_1(\xi(s_1))\phi_2(s_2)} \min\{1, s_1, s_2\} \right)^p \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right)^{1/p}, \end{aligned}$$

where $\xi = \phi^{-1}$. Thus, choosing $\phi_1, \phi_2 \in \mathcal{B}$ such that

$$\bar{b}(\phi_1 \circ \xi) + \bar{b}(\phi_2) = \bar{b}(\phi_1)/\underline{b}(\phi) + \bar{b}(\phi_2) < 1,$$

we have $a \in K_p^{\phi_1, \phi_2}(\mathbf{A})$.

We introduce a sufficient condition for the equivalence theorem to hold.

Definition 4.3. We denote by $\sigma(\mathbf{A})$ the set of all $a \in \Sigma(\mathbf{A})$ for which

$$\int \frac{K(t, a)}{\max t} d\omega(t) < \infty.$$

Definition 4.4. The condition $\mathcal{F}(\mathbf{A})$ is satisfied if, for every $a \in \sigma(\mathbf{A})$, there exists a function $u : \mathbb{P}_+^n \rightarrow \Delta(\mathbf{A})$ such that

$$a = \int u(t) d\omega(t) \quad \text{in } \Sigma(\mathbf{A})$$

and

$$J(t, u(t)) \leq C(\mathbf{A})K(t, a).$$

In the case $n = 1$, $\mathcal{F}(\mathbf{A})$ is satisfied for any Banach couple $\mathbf{A}$ [36, Remark 5.3]; for $n > 1$ however, it is not necessarily the case.

Theorem 4.5. *Let $p \in [1, \infty]$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$; if $\mathcal{F}(\mathbf{A})$ is satisfied, then*

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}). \quad (3)$$

Proof. This follows from Theorem 1 in [1] upon taking the particular case (1). $\square$

If the condition is satisfied, we denote the space (3) by $\mathbf{A}_p^{\phi_1, \dots, \phi_n}$.

Remark 4.6. If one wishes to impose a weaker assumption than $(\mathcal{H}_2)$ (given that $\mathcal{F}(\mathbf{A})$ is known to be optimal), one must consider conditions involving the Boyd functions and their derivatives in order to establish a modified fundamental lemma. We do not provide the details here, but the adaptation from [24] is straightforward.

The condition $\mathcal{F}(\mathbf{A})$ is a structural property of the underlying family of spaces. The following result establishes the stability of the construction with respect to retracts. We recall their definition. Let A and B be two objects in a given category, we say that B is a retract of A if there exist morphisms P and I such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\text{id}} & B \\ & \searrow I \quad \nearrow P & \\ & A & \end{array}$$

is commutative, i.e. $P \circ I = \text{id}$.

Proposition 4.7. *If $\mathcal{F}(\mathbf{A})$ is satisfied and $\mathbf{B}$ is a retract of $\mathbf{A}$, then $\mathcal{F}(\mathbf{B})$ is also satisfied.*

Proof. Let $b \in \sigma(\mathbf{B})$, so that $Ib \in \sigma(\mathbf{A})$. Therefore, there exists $u : \mathbb{P}_+^n \rightarrow \Delta(\mathbf{A})$ such that

$$Ib = \int u(t) d\omega(t) \quad \text{in } \Sigma(\mathbf{A})$$

and

$$J(t, u(t)) \leq C(\mathbf{A})K(t, Ib).$$

We have

$$b = P Ib = \int Pu(t) d\omega(t) \quad \text{in } \Sigma(\mathbf{B})$$

and

$$J^{\mathbf{B}}(t, Pu(t)) \leq C J^{\mathbf{A}}(t, u(t)) \leq C K^{\mathbf{A}}(t, Ib) \leq C K^{\mathbf{B}}(t, b),$$

bringing the proof to a close. $\square$

By interpolation, we have the following result:

Proposition 4.8. *If $\mathbf{B}$ is a retract of $\mathbf{A}$, then $K_p^{\phi_1, \dots, \phi_n}(\mathbf{B})$ is a retract of $K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ and $J_p^{\phi_1, \dots, \phi_n}(\mathbf{B})$ is a retract of $J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$, with the same P and I .*

We conclude this section by showing that the equivalence theorem is compatible with reiteration procedures, thereby recovering the expected stability of the interpolation scale.

Theorem 4.9. *Let $1 \leq m \leq n-1$, $0 < \lambda_1 + \dots + \lambda_m < 1$, $\eta_0, \eta_{m+1}, \dots, \eta_n \in \mathcal{B}$ be such that $0 < \underline{b}(\eta_0) + \underline{b}(\eta_{m+1}) + \dots + \underline{b}(\eta_n)$ and $\bar{b}(\eta_0) + \bar{b}(\eta_{m+1}) + \dots + \bar{b}(\eta_n) < 1$. Set*

$$(\phi_1, \dots, \phi_m, \phi_{m+1}, \dots, \phi_n) = (\eta_0^{\lambda_1}, \dots, \eta_0^{\lambda_m}, \eta_{m+1}, \dots, \eta_n)$$

and let $\mathbf{B}^{(j)} = (A_j, A_{m+1}, \dots, A_n)$ for $j \in \{0, \dots, m\}$. If

$$K_p^{\eta_0, \eta_{m+1}, \dots, \eta_n}(\mathbf{B}^{(j)}) = J_p^{\eta_0, \eta_{m+1}, \dots, \eta_n}(\mathbf{B}^{(j)})$$

(this is the case if $\mathcal{F}(\mathbf{B}^{(j)})$ is satisfied) for $j \in \{0, \dots, m\}$ and if

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$((\mathbf{B}^{(0)})_p^{\eta_0, \eta_{m+1}, \dots, \eta_n}, \dots, (\mathbf{B}^{(m)})_p^{\eta_0, \eta_{m+1}, \dots, \eta_n})^{\lambda_1, \dots, \lambda_m} = \mathbf{A}_p^{\phi_1, \dots, \phi_n}.$$

Proof. This follows as a direct consequence of the reiteration theorem in [1], with Φ defined by (1). $\square$

5 Some properties

We now establish several structural properties of the interpolation functors introduced above. These properties show that the proposed framework preserves the natural features of real interpolation, such as monotonicity with respect to the interpolation parameter, invariance under permutations of the underlying family, and compatibility with reductions to lower-order interpolation schemes.

From now on, we assume that condition $(\mathcal{H}_2)$ is always satisfied. The first property expresses the monotonicity of the interpolation scale with respect to the exponent p .

Proposition 5.1. *For $1 \leq p \leq q \leq \infty$, we have*

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow K_q^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

and

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow J_q^{\phi_1, \dots, \phi_n}(\mathbf{A}).$$

The next result shows that the construction is intrinsic to the family of spaces and does not depend on the particular ordering chosen for its realization. Given $\phi \in \mathcal{B}$, let us set $\phi_q(t) = \phi(t^q)^{1/q}$, with the usual modification in the case $q = \infty$. We directly get the following result:

Proposition 5.2. *Let π be a permutation of $\{0, 1, \dots, n\}$ and set*

$$\mathbf{A}^{(\pi)} = (A_{\pi(0)}, A_{\pi(1)}, \dots, A_{\pi(n)}),$$

$\phi_0(t) = t/(\phi_1(t) \cdots \phi_n(t))$ and

$$(\psi_0, \psi_1, \dots, \psi_n) = ((\phi_{\pi(0)})_p, (\phi_{\pi(1)})_p, \dots, (\phi_{\pi(n)})_p).$$

We have

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = K_p^{\psi_1, \dots, \psi_n}(\mathbf{A}^{(\pi)})$$

and

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\psi_1, \dots, \psi_n}(\mathbf{A}^{(\pi)}).$$

A natural consistency requirement for an interpolation theory of finite families is that repeated spaces can be merged without changing the resulting interpolation space. The following proposition verifies this property.

Proposition 5.3. *If $A_{n-1} = A_n$ ($n > 1$), then*

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = K_p^{\phi_1, \dots, \phi_{n-1}\phi_n}(A_0, \dots, A_{n-1})$$

and

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\phi_1, \dots, \phi_{n-1}\phi_n}(A_0, \dots, A_{n-1}).$$

Proof. Let us show the result for the case K . For $\mathbf{A}' = (A_0, \dots, A_{n-1})$, we have

$$K^{\mathbf{A}}(1, t_1, \dots, t_n, a) = K^{\mathbf{A}'}(1, t_1, \dots, \min\{t_{n-1}, t_n\}, a).$$

This identity allows to conclude, since

$$\begin{aligned} & \int_{(0, \infty)^2} \left(\frac{1}{\phi_{n-1}(t_{n-1})\phi_n(t_n)} K^{\mathbf{A}}(1, t_1, \dots, t_n, a) \right)^p \frac{dt_{n-1}}{t_{n-1}} \frac{dt_n}{t_n} \\ &= C \int_0^\infty \left(\frac{1}{\phi_{n-1}(s)\phi_n(s)} K^{\mathbf{A}'}(1, t_1, \dots, s, a) \right)^p \frac{ds}{s}. \end{aligned}$$

□

Let us highlight the following outcome from [36].

Proposition 5.4. *If $A_{n-1} = A_n$ and $\mathcal{F}(A_0, \dots, A_{n-1})$ is satisfied, then $\mathcal{F}(\mathbf{A})$ is also satisfied.*

6 Power theorem

The stability of interpolation spaces under power transformations is one of the fundamental structural properties of real interpolation. In particular, it shows that the interpolation construction is compatible with the natural extension of the theory from Banach spaces to quasi-normed spaces. We recall the required notation and establish the corresponding power theorem in the present setting of finite families and Boyd-function parameters.

The real interpolation method does not rely on all the properties of a norm; in particular, the triangle inequality can be replaced by a quasi-triangle inequality of the form

$$|a + b| \leq C(|a| + |b|),$$

which naturally leads to an extension of the theory to quasi-normed spaces [3]. Let us remind ourselves that if $\|\cdot\|$ is a quasi-norm on A , then $\|\cdot\|^\alpha$ is also a quasi-norm for $\alpha > 0$. We denote the space equipped with this functional by $A^{(\alpha)}$. If $\mathbf{A} = (A_0, \dots, A_n)$ is in $\mathcal{B}_n$, we naturally define

$$\mathbf{A}^{(p)} = (A_0^{(p)}, \dots, A_n^{(p)}).$$

We also set $\phi_p(t) = \phi(t^p)^{1/p}$ for all $t > 0$.

Theorem 6.1. *Let $p \in [1, \infty]$, $q > 0$ and $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$; we have*

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}^{(q)}) = K_{pq}^{(\phi_1)_q, \dots, (\phi_n)_q}(\mathbf{A})^{(q)}.$$

Proof. Let us define

$$K_\infty^{\mathbf{A}}(t, a) = K_\infty(t, a) = \inf_a \max\{t_0\|a_0\|_{A_0}, t_1\|a_1\|_{A_1}, \dots, t_n\|a_n\|_{A_n}\},$$

where the infimum is taken over all the decompositions $a = a_0 + \dots + a_n$, with $a_j \in A_j$. It can be observed that

$$K_\infty(t, a) \leq K(t, a) \leq nK_\infty(t, a).$$

If $p = \infty$, we have

$$\begin{aligned} \|a\|_{K_\infty^{\phi_1, \dots, \phi_n}(\mathbf{A}^{(q)})} &\asymp \sup_s \frac{1}{\phi_1(s_1)} \cdots \frac{1}{\phi_n(s_n)} K_\infty^{\mathbf{A}^{(q)}}(1, s_1, \dots, s_n, a) \\ &= \sup_t \frac{1}{(\phi_1)_q(t_1)^q} \cdots \frac{1}{(\phi_n)_q(t_n)^q} K_\infty^{\mathbf{A}}(1, t_1, \dots, t_n, a)^q. \end{aligned}$$

If $p < \infty$, we get

$$\begin{aligned} \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}^{(q)})}^p &\asymp \int_{(0, \infty)^n} \left(\frac{1}{\phi_1(s_1)} \cdots \frac{1}{\phi_n(s_n)} K_\infty(1, s_1, \dots, s_n) \right)^p \frac{ds_1}{s_1} \cdots \frac{ds_n}{s_n} \\ &= \int_{(0, \infty)^n} \left(\frac{1}{(\phi_1)_q(t_1)} \cdots \frac{1}{(\phi_n)_q(t_n)} K_\infty(1, t_1, \dots, t_n) \right)^{pq} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \end{aligned}$$

thereby concluding the proof. $\square$

7 Stability

In this section, we establish several stability properties of the interpolation spaces introduced above. These results show that the generalized K - and J -methods behave consistently with respect to changes of parameters, reiteration procedures, and interpolation of intermediate spaces. In particular, we prove stability results extending classical properties of real interpolation to the present multi-parameter framework.

In this section, we use X to denote an intermediate space with respect to $\mathbf{A}$.

Definition 7.1. Let $\phi_1, \dots, \phi_n \in \mathcal{B}$ satisfying $\overline{(\mathcal{H}_2)}$,

1. X is of class $\mathcal{C}_K(\phi_1, \dots, \phi_n; \mathbf{A})$ if there exists $C > 0$ depending on $\phi_1, \dots, \phi_n$ such that

$$K(1, t_1, \dots, t_n, a) \leq C \phi_1(t_1) \cdots \phi_n(t_n) \|a\|_X,$$

for all $a \in X$ and $t_1, \dots, t_n > 0$;

2. X is of class $\mathcal{C}_J(\phi_1, \dots, \phi_n; \mathbf{A})$ if there exists $C > 0$ depending on $\phi_1, \dots, \phi_n$ such that

$$\|a\|_X \leq C \frac{1}{\phi_1(t_1) \cdots \phi_n(t_n)} J(1, t_1, \dots, t_n, a),$$

for all $a \in \Delta(\mathbf{A})$ and $t_1, \dots, t_n > 0$.

We say that X is of class $\mathcal{C}(\phi_1, \dots, \phi_n; \mathbf{A})$ if X is of class $\mathcal{C}_K(\phi_1, \dots, \phi_n; \mathbf{A})$ and of class $\mathcal{C}_J(\phi_1, \dots, \phi_n; \mathbf{A})$.

The following two propositions are immediate consequences of these definitions.

Proposition 7.2. *X is of class $\mathcal{C}_K(\phi_1, \dots, \phi_n; \mathbf{A})$ if and only if*

$$X \hookrightarrow K_{\infty}^{\phi_1, \dots, \phi_n}(\mathbf{A}).$$

Proposition 7.3. *X is of class $\mathcal{C}_J(\phi_1, \dots, \phi_n; \mathbf{A})$ if and only if*

$$J_1^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow X.$$

We have the following result.

Proposition 7.4. *Let $p \in [1, \infty]$, $m \geq n$, $0 < \lambda_1 + \dots + \lambda_m < 1$ and $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$ for all $j \in \{1, \dots, m\}$. Set*

$$(\phi_1, \dots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \cdots (\phi_1^{(m)})^{\lambda_m}, \dots, (\phi_n^{(1)})^{\lambda_1} \cdots (\phi_n^{(m)})^{\lambda_m})$$

and suppose that for all $t_1, \dots, t_n > 0$, the rank of the matrix

$$\left(\frac{t_k D \phi_k^{(j)}(t_k)}{\phi_k^{(j)}(t_k)} \right)_{1 \leq k \leq n, 1 \leq j \leq m}$$

is n . Finally, let $\mathbf{X} = (X_0, \dots, X_m)$.

- (1) *If X_j is of class $\mathcal{C}_K(\phi_1^{(j)}, \dots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \dots, m\}$, then, with obvious notations,*

$$K_p^{\lambda_1, \dots, \lambda_m}(\mathbf{X}) \hookrightarrow K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}).$$

- (2) *If X_j is of class $\mathcal{C}_J(\phi_1^{(j)}, \dots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \dots, m\}$, then*

$$J_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) \hookrightarrow J_p^{\lambda_1, \dots, \lambda_m}(\mathbf{X}).$$

Proof. Remark that since $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ satisfy $(\mathcal{H}_2)$, they are differentiable. First suppose that $m = n$ and remark that

$$\begin{cases} s_1 = \phi_1^{(1)}(t_1) \cdots \phi_n^{(1)}(t_n) \\ \vdots \\ s_n = \phi_1^{(n)}(t_1) \cdots \phi_n^{(n)}(t_n) \end{cases}$$

is an appropriate change of variable if and only if the vectors

$$\begin{pmatrix} \frac{t_1 D\phi_1^{(j)}(t_1)}{\phi_1^{(j)}(t_1)} \\ \vdots \\ \frac{t_n D\phi_n^{(j)}(t_n)}{\phi_n^{(j)}(t_n)} \end{pmatrix} \quad (j \in \{1, \dots, n\})$$

are linearly independent.

For $a = a_0 + \dots + a_n \in \Sigma(\mathbf{X})$, we get

$$\begin{aligned} K^{\mathbf{A}}(1, t_1, \dots, t_n, a) &\leq \|a_0\|_{A_0} + \sum_{k=1}^n K^{\mathbf{A}}(1, t_1, \dots, t_n, a_k) \\ &\leq C(\|a_0\|_{X_0} + \phi_1^{(1)}(t_1) \cdots \phi_n^{(1)}(t_n) \|a_1\|_{X_1} \\ &\quad + \dots + \phi_1^{(n)}(t_1) \cdots \phi_n^{(n)}(t_n) \|a_n\|_{X_n}). \end{aligned}$$

Therefore, using

$$\begin{cases} s_1 = \phi_1^{(1)}(t_1) \cdots \phi_n^{(1)}(t_n) \\ \vdots \\ s_n = \phi_1^{(n)}(t_1) \cdots \phi_n^{(n)}(t_n) \end{cases},$$

one gets (1), since

$$\begin{aligned} \|a\|_{K_p^{\phi_1, \dots, \phi_n}(\mathbf{A})} &= \Phi_p^{\phi_1, \dots, \phi_n}(K^{\mathbf{A}}(t, a)) \\ &\leq C\Phi_p^{\lambda_1, \dots, \lambda_m}(K^{\mathbf{X}}(s, a)) \\ &= C\|a\|_{K_p^{\lambda_1, \dots, \lambda_m}(\mathbf{X})}. \end{aligned}$$

Let us prove (2). Set $a = \int u(t)d\omega(t) \in J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$ and define v (see (3.21) of [36] for the construction) so that

$$C(s)v(s(t)) = u(t).$$

We have

$$\begin{aligned} \|a\|_{J_p^{\lambda_1, \dots, \lambda_m}(\mathbf{X})} &= \Phi_p^{\lambda_1, \dots, \lambda_m}(J^{\mathbf{X}}(s, v(s))) \\ &= C\Phi_p^{\phi_1, \dots, \phi_n}(J^{\mathbf{X}}(s(t), v(s(t)))) \\ &\leq C\Phi_p^{\phi_1, \dots, \phi_n}(J^{\mathbf{A}}(t, u(t))). \end{aligned}$$

Now suppose $m > n$ and consider the $(m+1)$ -tuple

$$\mathbf{A}' = (A_0, \dots, A_n, A_n, \dots, A_n).$$

Using Proposition 5.3, we get

$$K_p^{\psi_1, \dots, \psi_m}(\mathbf{A}') = K_p^{\psi_1, \dots, \psi_{n-1}, \psi_n, \dots, \psi_m}(\mathbf{A}),$$

bringing the proof to a close. $\square$

By adjusting the reasoning from the preceding demonstration, we derive the following results.

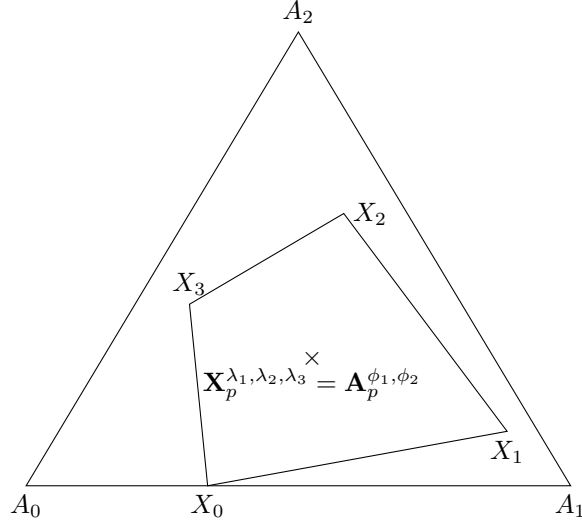


Figure 1: Illustration inspired of [36, Figure 9.1] of Theorem 7.5 with $m = 3$ and $n = 2$.

Theorem 7.5. Let $p \in [1, \infty]$, $m \geq n$, $0 < \lambda_1 + \dots + \lambda_m < 1$ and $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$ for all $j \in \{1, \dots, m\}$. Set

$$(\phi_1, \dots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \dots (\phi_1^{(m)})^{\lambda_m}, \dots, (\phi_n^{(1)})^{\lambda_1} \dots (\phi_n^{(m)})^{\lambda_m})$$

and suppose that for all $t_1, \dots, t_n > 0$, the rank of the matrix

$$\left(\frac{t_k D\phi_k^{(j)}(t_k)}{\phi_k^{(j)}(t_k)} \right)_{1 \leq k \leq n, 1 \leq j \leq m}$$

is n . Finally, let $\mathbf{X} = (X_0, \dots, X_m)$. If X_j is of class $\mathcal{C}(\phi_1^{(j)}, \dots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \dots, m\}$ and

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$\mathbf{X}_p^{\lambda_1, \dots, \lambda_m} = \mathbf{A}_p^{\phi_1, \dots, \phi_n}.$$

Corollary 7.6. Let $m \geq n$, $p, p_0, \dots, p_m \in [1, \infty]$, $0 < \lambda_1 + \dots + \lambda_m < 1$ and $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$ for all $j \in \{1, \dots, m\}$. Set

$$(\phi_1, \dots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \dots (\phi_1^{(m)})^{\lambda_m}), \dots, (\phi_n^{(1)})^{\lambda_1} \dots (\phi_n^{(m)})^{\lambda_m})$$

and suppose that for all $t_1, \dots, t_n > 0$, the rank of the matrix

$$\left(\frac{t_k D\phi_k^{(j)}(t_k)}{\phi_k^{(j)}(t_k)} \right)_{1 \leq k \leq n, 1 \leq j \leq m}$$

is n . If $K_{p_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A}) = J_{p_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A})$ for $j \in \{0, \dots, m\}$ and if

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$(\mathbf{A}_{p_0}^{\phi_1^{(0)}, \dots, \phi_n^{(0)}}, \dots, \mathbf{A}_{p_m}^{\phi_1^{(m)}, \dots, \phi_n^{(m)}})^{\lambda_1, \dots, \lambda_m} = \mathbf{A}_p^{\phi_1, \dots, \phi_n}.$$

Proposition 7.7. Let $m \geq n$, $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ be such that $(\mathcal{H}_2)$ is satisfied for all $j \in \{1, \dots, m\}$. Suppose that for all $t_1, \dots, t_n > 0$, the rank of the matrix

$$\left(\frac{t_k D\phi_k^{(j)}(t_k)}{\phi_k^{(j)}(t_k)} \right)_{1 \leq k \leq n, 1 \leq j \leq m}$$

is n . Finally, let $\mathbf{X} = (X_0, \dots, X_m)$ such that X_j is of class $\mathcal{C}(\phi_1^{(j)}, \dots, \phi_n^{(j)}; \mathbf{A})$ for all $j \in \{0, \dots, m\}$. If $\mathcal{F}(\mathbf{A})$ is satisfied, then $\mathcal{F}(\mathbf{X})$ is also satisfied.

Additional stability results can be achieved by imposing restrictions on $\lambda_1, \dots, \lambda_m$, p , $p_0, \dots, p_m$ instead of $\phi_1^{(j)}, \dots, \phi_n^{(j)}$. The ensuing result can be inferred from [36]:

Theorem 7.8. Let $p, p_0, \dots, p_m \in [1, \infty]$, $0 < \lambda_1 + \dots + \lambda_m < 1$ and $\phi_1^{(j)}, \dots, \phi_n^{(j)} \in \mathcal{B}$ satisfying $(\mathcal{H}_2)$ for all $j \in \{1, \dots, m\}$. Set

$$(\phi_1, \dots, \phi_n) = ((\phi_1^{(1)})^{\lambda_1} \dots (\phi_1^{(m)})^{\lambda_m}), \dots, (\phi_n^{(1)})^{\lambda_1} \dots (\phi_n^{(m)})^{\lambda_m})$$

and suppose that

$$\frac{1}{p} = \sum_{j=0}^m \frac{\lambda_j}{p_j}.$$

If $K_{p_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A}) = J_{p_j}^{\phi_1^{(j)}, \dots, \phi_n^{(j)}}(\mathbf{A})$ for $j \in \{0, \dots, m\}$ and if

$$K_p^{\phi_1, \dots, \phi_n}(\mathbf{A}) = J_p^{\phi_1, \dots, \phi_n}(\mathbf{A})$$

(this is the case if $\mathcal{F}(\mathbf{A})$ is satisfied), then

$$(\mathbf{A}_{p_0}^{\phi_1^{(0)}, \dots, \phi_n^{(0)}}, \dots, \mathbf{A}_{p_m}^{\phi_1^{(m)}, \dots, \phi_n^{(m)}})^{\lambda_1, \dots, \lambda_m} = \mathbf{A}_p^{\phi_1, \dots, \phi_n}.$$

8 Interpolation of generalized Sobolev spaces

We now apply the abstract interpolation framework developed above to a concrete and important class of function spaces. The aim of this section is to show that the multi-parameter interpolation method naturally extends classical results on Sobolev spaces. More precisely, we show that interpolation of generalized Sobolev spaces associated with Fourier multipliers yields generalized Besov spaces. This provides a direct extension of the classical reiteration results for fractional Sobolev spaces to a broader scale of smoothness parameters.

First, let us provide a brief introduction to generalized Sobolev and Besov spaces ([31, 9]).

Definition 8.1. Let $q \in [1, \infty]$ and $\phi \in \mathcal{B}$; if X is a Banach space, the space $l_\phi^q(X)$ consists of all sequences $(a_j)_j$ of X such that

$$(\phi(2^j)\|a_j\|_X)_j \in l^q.$$

This space is equipped with the norm

$$\|(a_j)_j\|_{l_\phi^q(X)} = \|(\phi(2^j)\|a_j\|_X)_j\|_{l^q}.$$

- If $\phi(t) = t^s$ ($s \in \mathbb{R}$), we set $l_s^q(X) = l_\phi^q(X)$;
- If $X = \mathbb{C}$, then we set $l_\phi^q = l_\phi^q(X)$.

Let $L^p := L^p(\mathbb{R}^d)$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the space of test functions, $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions and $\mathcal{F}$ the Fourier transform on $\mathcal{S}'$.

Definition 8.2. Let $\mathcal{J}$ be the Bessel operator of order s , that is

$$\mathcal{J}^s f = \mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2} \mathcal{F}f),$$

for $f \in \mathcal{S}'$ and $s \in \mathbb{R}$. Given $s \in \mathbb{R}$ and $p \in [1, \infty]$, the fractional Sobolev space H_p^s is defined by

$$H_p^s = \{f \in \mathcal{S}' : \|\mathcal{J}^s f\|_{L^p} < \infty\}.$$

We denote by $\mathcal{B}''$ the set of functions $\phi \in \mathcal{B}$ which are C^∞ on $[1, \infty)$ and such that

$$t^m |D^m \phi(t)| \leq C_m \phi(t),$$

for all $t \in [1, \infty)$ and all $m \in \mathbb{N}$.

Example 8.3. Given $\theta, \gamma \in \mathbb{R}$, let us set

$$\phi(t) = t^\theta (1 + |\log t|)^\gamma,$$

for $t > 0$; it can be readily verified that $\phi \in \mathcal{B}''$.

Definition 8.4. Given $\phi \in \mathcal{B}''$, the generalized Bessel operator $\mathcal{J}^\phi$ is defined by

$$\mathcal{J}^\phi f = \mathcal{F}^{-1}(\phi(\sqrt{1 + |\cdot|^2}) \mathcal{F}f),$$

for $f \in \mathcal{S}'$.

Remark 8.5. The previous definition makes sense, since $\phi \in \mathcal{B}''$ and $f \in \mathcal{S}'$ imply $\phi(\sqrt{1 + |\cdot|^2}) \mathcal{F}f \in \mathcal{S}'$. Moreover, it is clear that $\mathcal{J}^\phi$ is a linear bijective operator on $\mathcal{S}'$ such that $(\mathcal{J}^\phi)^{-1} = \mathcal{J}^{1/\phi}$ and $\mathcal{J}^\phi(\mathcal{S}) = \mathcal{S}$.

Definition 8.6. Given $\phi \in \mathcal{B}''$ and $p \in [1, \infty]$, the generalized (fractional) Sobolev space H_p^ϕ is defined by

$$H_p^\phi = \{f \in \mathcal{S}' : \|\mathcal{J}^\phi f\|_{L^p} < \infty\}.$$

Remark 8.7. For $\phi(t) = t^s$, it is self-evident that $H_p^\phi = H_p^s$.

Definition 8.8. For $p \in [1, \infty]$, the space of *Fourier multipliers* on L^p is defined by

$$M^p = \{f \in \mathcal{S}' : \forall g \in \mathcal{S}, \|(\mathcal{F}^{-1}f) * g\|_{L^p} \leq C\|g\|_{L^p}\}.$$

This space is equipped with the norm

$$\|f\|_{M^p} = \sup\{\|(\mathcal{F}^{-1}f) * g\|_{L^p} : g \in \mathcal{S} \text{ and } \|g\|_{L^p} \leq 1\}.$$

Let's recall the following classical result (Lemma 6.1.5 in [3]):

Lemma 8.9. Let $N \in \mathbb{N}$, if $f \in \mathcal{S}'$ is such that $f \in L^2$ and $D^\alpha f \in L^2$ for $|\alpha| = N > d/2$, then $f \in M^p$ and

$$\|f\|_{M^p} \leq C\|f\|_{L^2}^{1-\frac{d}{2N}} \left(\sup_{|\alpha|=N} \|D^\alpha f\|_{L^2} \right)^{\frac{d}{2N}},$$

for some constant $C > 0$.

Definition 8.10. Let φ be a function in $\mathcal{S}$ with support included in the set $\{x \in \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 2\}$, such that $\varphi(x) > 0$ if $\frac{1}{2} \leq |x| \leq 2$ and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}x) = 1,$$

for $x \neq 0$. Define functions ψ_0 and Φ_j ($j \in \mathbb{Z}$) in $\mathcal{S}$ such that

$$\mathcal{F}\Phi_j(x) = \varphi(2^{-j}x) \quad \text{and} \quad \mathcal{F}\psi_0(x) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}x)$$

and set $\varphi_0 = \psi_0$, $\varphi_j = \Phi_j$ for all $j \in \mathbb{N}_0$. We call $(\varphi_j)_j$ a system of test functions.

Proposition 8.11. Let $(\varphi_j)_j$ be a system of test functions, $f \in \mathcal{S}'$, $\phi \in \mathcal{B}''$ and $p \in [1, \infty]$. If $\varphi_j * f$ belongs to L^p for all $j \in \mathbb{N}$, then, for all $j \in \mathbb{N}$,

$$\|\mathcal{J}^\phi \varphi_j * f\|_{L^p} \leq C\phi(2^j)\|\varphi_j * f\|_{L^p},$$

for some positive constant C .

Proof. Since $\varphi_j * f = \sum_{l=-1}^1 \varphi_{j+l} * \varphi_j * f$, we have

$$\|\mathcal{J}^\phi \varphi_j * f\|_{L^p} \leq \sum_{l=-1}^1 \|\mathcal{F}\mathcal{J}^\phi \varphi_{j+l}\|_{M^p} \|\varphi_j * f\|_{L^p}.$$

Hence, it suffices to demonstrate that

$$\|\mathcal{F}\mathcal{J}^\phi \varphi_{j+l}\|_{M^p} \leq C\phi(2^j). \tag{4}$$

From

$$\mathcal{F}\mathcal{J}^\phi \varphi_{j+l} = \phi(\sqrt{1 + |\cdot|^2})\mathcal{F}\varphi_{j+l},$$

we deduce that (4) is equivalent to

$$\|\phi(2^{j+l}\sqrt{2^{-2(j+l)} + |\cdot|^2})\varphi(\cdot)\|_{M^p} \leq C\phi(2^j). \tag{5}$$

Now for $\alpha \in \mathbb{N}^d$ and $g = \phi(2^{j+l}\sqrt{2^{-2(j+l)} + |\cdot|^2})\varphi(\cdot)$, the term $|D^\alpha g(x)|$ is bounded by expressions of the form

$$C2^{(j+l)\alpha_1}|\phi^{(\alpha_1)}(2^{j+l}\sqrt{2^{-2(j+l)} + |x|^2})|$$

in finite number, with $0 \leq \alpha_1 \leq |\alpha|$. Since $\phi \in \mathcal{B}''$, $|D^\alpha g(x)|$ is bounded by a finite sum of terms of the form

$$C\phi(2^j)\bar{\phi}(\sqrt{2^{-2(j+l)} + |x|^2})(2^{-2(j+l)} + |x|^2)^{-\alpha_1/2}.$$

From Lemma 8.9 with g , the properties of $\bar{\phi}$ and the fact that the support of φ is in $\{x \in \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 2\}$, we obtain

$$\|g\|_{M^p} \leq C\|g\|_{L^2}^{1-\frac{d}{2N}} \left(\sup_{|\alpha|=N} \|D^\alpha f\|_{L^2} \right)^{\frac{d}{2N}} \leq C\phi(2^j),$$

which means that inequality (5) is satisfied. $\square$

Lemma 8.12. *If $\phi \in \mathcal{B}''$ is such that $\underline{b}(\phi) > 0$, then $\mathcal{J}^{1/\phi}$ maps L^p into L^p continuously.*

Proof. Since $\underline{b}(\phi) > 0$, $1/\phi \in \mathcal{B}''$ and from Proposition 8.11, we get

$$\begin{aligned} \|\mathcal{J}^{1/\phi} f\|_{L^p} &\leq \sum_j \|\mathcal{J}^{1/\phi} \varphi_j * f\|_{L^p} \leq C \sum_j \frac{1}{\phi}(2^j) \|\varphi_j * f\|_{L^p} \\ &\leq C \left(\sum_j \frac{1}{\phi}(2^j) \right) \|f\|_{L^p} \\ &\leq C \left(\int_0^1 \bar{\phi}(t) \frac{dt}{t} \right) \|f\|_{L^p} \\ &\leq C \|f\|_{L^p}, \end{aligned}$$

which ends the proof. $\square$

Remark that if $\phi_0, \phi_1 \in \mathcal{B}''$, then $\phi_0 \phi_1 \in \mathcal{B}''$ and $\mathcal{J}^{\phi_0 \phi_1}(f) = \mathcal{J}^{\phi_0}(\mathcal{J}^{\phi_1}(f))$. We deduce the following result.

Proposition 8.13. *Let $\phi_0, \phi_1 \in \mathcal{B}''$, $\phi = \phi_0/\phi_1$, $p \in [1, \infty]$; if $\underline{b}(\phi) > 0$, then we have $H_p^{\phi_0} \hookrightarrow H_p^{\phi_1}$.*

Proof. From Lemma 8.12, we directly get

$$\|\mathcal{J}^{\phi_1} f\|_{L^p} = \|\mathcal{J}^{1/\phi} \mathcal{J}^{\phi_0} f\|_{L^p} \leq C \|\mathcal{J}^{\phi_0} f\|_{L^p},$$

hence the conclusion. $\square$

Definition 8.14. Let $\phi \in \mathcal{B}$ and $(\varphi_k)_k$ be a system of test functions. For $p, q \in [1, \infty]$, the generalized Besov Space is the space

$$B_{p,q}^\phi = \{f \in \mathcal{S}' : (\varphi_k * f)_k \in l_\phi^q(L^p)\}.$$

This space is equipped with the norm

$$\|f\|_{B_{p,q}^\phi} = \|(\varphi_k * f)_k\|_{l_\phi^q(L^p)} = \|(\|\varphi_k * f\|_{L^p})_k\|_{l_\phi^q}.$$

For $\phi(t) = t^s$ ($s \in \mathbb{R}$), we set $B_{p,q}^s = B_{p,q}^\phi$, i.e.,

$$B_{p,q}^s = \{f \in \mathcal{S}' : (\varphi_k * f)_k \in l_s^q(L^p)\}.$$

Let $\gamma \in \mathcal{B}$, $\phi_0, \phi_1 \in \mathcal{B}''$ and set $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$. For $p, q \in [1, \infty]$, it is well known (see Theorem 12 in [31]) that if $(\underline{b}(f) > 0 \text{ or } \bar{b}(f) < 0)$ and $0 < \underline{b}(\gamma) \leq \bar{b}(\gamma) < 1$, then we have

$$K_q^\gamma(H_p^{\phi_0}, H_p^{\phi_1}) = B_{p,q}^\psi.$$

Let us generalize this result to several spaces.

Theorem 8.15. *Let $p, q \in [1, \infty]$, $\gamma_1, \dots, \gamma_n \in \mathcal{B}$, $\phi_0, \phi_1, \dots, \phi_n \in \mathcal{B}''$ and for $l_1, l_2 \in \{1, \dots, n\}$, set $f_{l_1, l_2} = \phi_{l_1}/\phi_{l_2}$. If $(\underline{b}(f_{l_1, l_2}) > 0 \text{ or } \bar{b}(f_{l_1, l_2}) < 0)$ for $l_1 < l_2$ and if $0 < \underline{b}(\gamma_k) \leq \bar{b}(\gamma_k) < 1$ for $k \in \{1, \dots, n\}$, then*

$$K_q^{\gamma_1, \dots, \gamma_n}(H_p^{\phi_0}, \dots, H_p^{\phi_n}) = B_{p,q}^\psi,$$

with $\psi = \phi_0/(\gamma_1 \circ f_{0,1} \cdots \gamma_n \circ f_{0,n})$.

Proof. Let us assume that $\underline{b}(f_{l_1, l_2}) > 0$ for $l_1 < l_2$; it is evident that $\psi \in \mathcal{B}$. Moreover, we can suppose that $f_{l_1, l_2} \in \mathcal{B}_+^*$.

Let us first show that

$$K_q^{\gamma_1, \dots, \gamma_n}(H_p^{\phi_0}, \dots, H_p^{\phi_n}) = K_q^{\eta_2, \dots, \eta_n}(H_p^{\phi_1}, \dots, H_p^{\phi_n}), \quad (6)$$

with

$$\eta_2(t) = \frac{\gamma_1(f_{0,1}(f_{1,2}^{-1}(t)))\gamma_2(f_{0,2}(f_{1,2}^{-1}(t)))}{f_{0,1}(f_{1,2}^{-1}(t))}$$

and

$$\eta_l(t) = \gamma_l(f_{0,l}(f_{1,l}^{-1}(t))),$$

for $l \in \{3, \dots, n\}$. Since $H_p^{\phi_0} \hookrightarrow H_p^{\phi_1}$, we already know that

$$\Sigma(H_p^{\phi_0}, \dots, H_p^{\phi_n}) = \Sigma(H_p^{\phi_1}, \dots, H_p^{\phi_n}).$$

From there, we can get (6) using $s_l = f_{1,l}^{-1}(t_l)$ ($l = 2, \dots, n$), Proposition 5.3 and

$$(u_1, \dots, u_n) = (f_{0,1}(s_1), \dots, f_{0,n}(s_n)).$$

Applying the same procedure, we obtain

$$K_q^{\gamma_1, \dots, \gamma_n}(H_p^{\phi_0}, \dots, H_p^{\phi_n}) = K_q^\eta(H_p^{\phi_{n-1}}, H_p^{\phi_n}), \quad (7)$$

with η such that $\psi = \phi_{n-1}/(\eta \circ f_{(n-1),n})$. Given that the right-hand side of (7) is equal to $B_{p,q}^\psi$, we can draw the conclusion. $\square$

In the case where $\phi_j(t) = t^{s_j}$ ($s_j \in \mathbb{R}$) for $j \in \{0, \dots, n\}$, we recover a result of [36]:

$$K_q^{\theta_1, \dots, \theta_n}(H_p^{s_0}, \dots, H_p^{s_n}) = B_{p,q}^s,$$

with $s = (1 - \sum_{j=1}^n \theta_j)s_0 + \sum_{j=1}^n \theta_j s_j$, where $\theta_1, \dots, \theta_n > 0$ are such that $\sum_{j=1}^n \theta_j < 1$.

9 Extension of the Stein-Weiss Theorem

To illustrate the scope of the framework developed in this paper, we consider a weighted interpolation problem for generalized Lorentz spaces. This setting is particularly relevant since, although the interpolation of two weighted Lorentz spaces does not generally preserve the same scale of spaces, the use of an intermediate three-space interpolation formula allows us to recover a natural block-Lorentz structure. In this way, we extend the Stein-Weiss theorem to the generalized setting introduced here.

We begin by recalling the definition of the generalized Lorentz spaces [27]:

Definition 9.1. Let (Ω, μ) be a σ -finite measure space and let ω be a positive measurable function on Ω . We denote by $\omega d\mu$ the measure defined by

$$d\nu = \omega d\mu.$$

Given $\phi \in \mathcal{B}$ and $p \in [1, \infty]$, the space $\Lambda_p^\phi(\omega d\mu)$ consists of all $\omega d\mu$ -measurable functions a on Ω such that

$$\|a\|_{\Lambda_p^\phi(\omega d\mu)} = \|\phi a_{\omega d\mu}^*\|_{L^p((0, \infty), dt/t)}$$

is finite, where $a_{\omega d\mu}^*$ denotes the non-increasing rearrangement of $|a|$ with respect to the measure $\omega d\mu$. Equivalently, if

$$S_{a, \omega}(\alpha) = \int_{\{x \in \Omega : |a(x)| > \alpha\}} \omega d\mu,$$

then

$$a_{\omega d\mu}^*(t) = \inf\{\alpha \geq 0 : S_{a, \omega}(\alpha) \leq t\}.$$

The space $\Lambda_p^\phi(\omega d\mu)$ is quasi-normed with respect to $\|\cdot\|_{\Lambda_p^\phi}$.

Example 9.2. When $\phi(t) = t^{1/q}$, one recovers the usual Lorentz-Zygmund space (see, e.g., [3]):

$$\Lambda_p^\phi(d\mu) = L^{q, p}(d\mu).$$

There are thus two kind of weights here: ω that is a positive measurable function on Ω and $\phi \in \mathcal{B}$ that generalizes the function $t \mapsto t^{1/q}$ appearing in the usual Lorentz-Zygmund space.

We now consider the generalized interpolation of two such spaces. On the one hand, the classical Stein-Weiss theorem can be stated as follows (see, for example, Theorem 5.5.1 in [3]):

Theorem 9.3. Let $p_0, p_1 \in [1, \infty]$, $\theta \in (0, 1)$, ω_0, ω_1 measurable functions on Ω and suppose $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Define

$$\omega = \omega_0^{(1-\theta)p/p_0} \omega_1^{\theta p/p_1}.$$

Then,

$$K_p^\theta(L^{p_0}(\omega_0 d\mu), L^{p_1}(\omega_1 d\mu)) = L^p(\omega d\mu). \quad (8)$$

On the other hand, in the unweighted setting, we have the following result for the generalized case (see Theorem 3 in [31]):

Theorem 9.4. Let $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ and define $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$. Let $p, p_0, p_1, q \in [1, \infty]$. If $(\underline{b}(f) > 0$ or $\bar{b}(f) < 0)$ and $0 < \underline{b}(\gamma), \bar{b}(\gamma) < 1$, then

$$K_q^\gamma(\Lambda_{p_0}^{\phi_0}(d\mu), \Lambda_{p_1}^{\phi_1}(d\mu)) = \Lambda_q^\psi(d\mu). \quad (9)$$

Outline of the proof. We apply the reiteration theorem for the real method with a function parameter (Theorem 2 in [31]) to the couple

$$(X_0, X_1) = (L^{r,\infty}(d\mu), L^\infty(d\mu)),$$

for a suitable $r > 0$. The generalized Lorentz spaces can be represented as intermediate spaces of this couple, namely

$$\Lambda_{p_i}^{\phi_i}(d\mu) = K_{p_i}^{\eta_i}(X_0, X_1), \quad i = 0, 1,$$

for suitable parameters η_i . Applying the reiteration theorem with parameter γ gives again an interpolation space of the same couple, with parameter corresponding to

$$\psi = \frac{\phi_0}{\gamma \circ f}, \quad f = \frac{\phi_0}{\phi_1}.$$

Hence

$$K_q^\gamma(\Lambda_{p_0}^{\phi_0}(d\mu), \Lambda_{p_1}^{\phi_1}(d\mu)) = \Lambda_q^\psi(d\mu),$$

with equivalence of quasi-norms. $\square$

Combining (9) and (8), one might naturally expect a relation of the form

$$K_q^\gamma(\Lambda_p^{\phi_0}(\omega_0 d\mu), \Lambda_p^{\phi_1}(\omega_1 d\mu)) = \Lambda_q^\psi(\omega d\mu), \quad (10)$$

for a suitably chosen ω . However, this identity (10) is generally false (see Counterexample 6.3 in [2]). In fact, even in the classical setting, interpolation of two Lorentz spaces typically yields a so-called block-Lorentz space [2]. Strikingly, proving such a result requires the use of reiteration formulas involving three spaces.

We now show how this can be extended within the framework developed here.

Definition 9.5. Let $\phi \in \mathcal{B}$, $p \in [1, \infty]$ and ω be a weight on Ω ; we consider the family of quasi-Banach function lattices $L_p^\phi(\omega d\mu)$ defined by the quasi-norm

$$\|a\|_{L_p^\phi(\omega d\mu)} = \left(\int_\Omega |\phi(a(x)\omega(x))|^p d\mu(x) \right)^{1/p},$$

with the usual modification if $p = \infty$.

We will need the following “intermediate” spaces.

Definition 9.6. Let $\phi \in \mathcal{B}$, $p, q \in [1, \infty]$ and let ω be a weight on Ω ; the block-Lorentz space $L_{p,q}^\phi(\omega d\mu)$ is defined as the space of measurable functions a for which the quasi-norm

$$\left(\sum_{k \in \mathbb{Z}} (\|a \chi_{\Omega_k^\omega}\|_{\Lambda_p^\phi(\omega d\mu)})^q \right)^{1/q}$$

is finite, with the usual modification if $q = \infty$. Here, the level sets Ω_k^ω are defined by

$$\Omega_k^\omega = \{x \in \Omega : 2^k \leq \omega(x) < 2^{k+1}\}.$$

In general, the interpolation of two spaces of the form $L_p^\phi(\omega d\mu)$ does not yield a block-Lorentz space [2]. However, a more favorable situation arises when interpolating three such spaces.

Lemma 9.7. *Let $p_0, p_1, p_2, q \in [1, \infty]$, $\phi_0, \phi_1, \phi_2 \in \mathcal{B}$, ω a weight on Ω and $\theta_1, \theta_2 \in (0, 1)$. Suppose that, for all $t > 0$, the points $(\frac{tD\phi_j(t)}{\phi_j(t)}, \frac{1}{p_j})$, $j = 0, 1, 2$, are not collinear. Then*

$$K_q^{\theta_1, \theta_2}(L_{p_0}^{\phi_0}(\omega d\mu), L_{p_1}^{\phi_1}(\omega d\mu), L_{p_2}^{\phi_2}(\omega d\mu)) = L_{p, q}^\psi(\omega d\mu),$$

where $1/p = \sum_{j=0}^2 \theta_j/p_j$ with $\theta_0 = 1 - \theta_1 - \theta_2$, and $\psi = \phi_0^{\theta_0} \phi_1^{\theta_1} \phi_2^{\theta_2}$.

Proof. A direct and routine adaptation of Lemmata 4.4 and 4.5 from [2], combined with similar arguments, yields the result. $\square$

We now state a result that generalizes Corollary 6.2 in [2] within our framework:

Theorem 9.8. *Let $\gamma = \cdot^\theta$, $\omega, \omega_0, \omega_1$ be weights on Ω and $\phi_0, \phi_1 \in \mathcal{B}$. Define $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$, and*

$$\psi \circ \omega = \frac{(\phi_0 \circ \omega_0) \cdot (\gamma \circ \phi_1 \circ \omega_1)}{\gamma \circ \phi_0 \circ \omega_0}.$$

Let $p, p_0, p_1, q \in [1, \infty]$ satisfy $1/p = (1 - \theta)/p_0 + \theta/p_1$. If $(\underline{b}(f) > 0$ or $\bar{b}(f) < 0)$ and $0 < \underline{b}(\gamma), \bar{b}(\gamma) < 1$, then

$$K_q^\gamma(\Lambda_{p_0}^{\phi_0}(\omega_0 d\mu), \Lambda_{p_1}^{\phi_1}(\omega_1 d\mu))$$

coincides with the block-Lorentz-type space whose quasi-norm is given by

$$\|a\| = \left(\sum_{k \in \mathbb{Z}} (\|a \chi_{\Omega_k^{\omega_0/\omega_1}}\|_{\Lambda_p^\phi(\omega d\mu)})^q \right)^{1/q}.$$

Proof. This result is obtained by extending the proof of Theorem 6.2 in [2], using Lemma 9.7 to accommodate the generalized framework. $\square$

Data Availability Statement

No data were generated or analysed during the current study.

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