



Parenting and eldercare: positive and normative analyses

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Abstract

Global trends in delayed childbearing and population aging have intertwined parenting and eldercare, necessitating concurrent attention to young children and elderly parents. This paper develops an overlapping generations model where young adults, exhibiting two-sided altruism, educate their children to promote human capital accumulation and provide caregiving for their aging parents. Education can be attained through financial investments and the implementation of harsh discipline, which demands minimal parental resources but can strain parent–child relations. Eldercare is labor intensive, with its quality decreasing with the frequency of childhood discipline. Our positive analysis suggests that increased longevity may reduce the prevalence of harsh parenting, while enhanced altruism towards the elderly benefits them but can undermine children’s human capital development. We then examine the first-best optimality and the second-best public policies in the steady state. When intergenerational altruism is limited, we advocate for the idea of taxing labor and subsidizing education from a novel perspective of adjusting parenting styles and promoting eldercare.

Keywords Parenting style · Human capital · Longevity · Long-term care · Two-sided altruism

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1 Introduction

Economic literature typically treats child-rearing and elderly care as separate issues, presuming households face only one caregiving burden at a time. Evolving demographic trends—specifically delayed childbearing and senior boom—have led to a potential overlap between these two concerns within families (Zhong & Peng, 2024). In a three-generation family unit, the working parent, who harbors altruism towards both the young child and dependent grandparent, simultaneously attends to their distinct needs. The time demands of eldercare obligations compel the parent to scale back her work commitment (Pestieau & Sato, 2008; De Donder & Leroux, 2017), and the subsequent loss of income curtails household investment in education, thereby undermining the child’s human capital formation (Bogan, 2015). Conversely, prioritizing the child’s education may inadvertently overlook the assistance required by the elderly. The dual responsibilities place financial strain on the parent, forcing cuts to her employment and consumption (Burke, 2017). This paper investigates the intergenerational allocation of family resources and explores the role of government actions—with a focus on educational policies and welfare systems—in shaping household planning.

Family dynamics encompasses how parents and children support each other, balance work and personal life, navigate conflicts, and express affection throughout their shared lifetime. A foremost channel for intergenerational transfers is education. It is commonly argued that young parents act altruistically, dedicating substantial time and money to cultivate their children’s potential (Becker & Tomes, 1986; Heckman et al., 2018). Beyond financial commitments to formal education, the parent may alternatively practice structured disciplinary techniques at home. A prominent example is “tiger parenting,” a demanding child-rearing style prevalent in East Asian communities that uses stern methods to ensure educational attainment.¹ Unlike pedagogical effort, harsh discipline (e.g., silent treatment, corporal punishment, and verbal aggression) requires little investment of time or money, making it a low-cost strategy for parents with competing priorities (Weinberg, 2001). However, the authoritarian nature of harsh parenting is under increasing scrutiny for hindering family communication. In response to this controlling environment, young children—feeling frustrated and powerless—often develop entrenched resentment and hostility. The profound relational damage incurred in earlier years leads to lasting intergenerational estrangement that resists later repair (Hwang et al., 2023).

In aging societies worldwide, where the need for filial caregiving intensifies, persistent familial tensions can compromise the quality of informal eldercare provided by adult children (Vangen & Herlofson, 2024). While a young adult may care for her parent out of altruism, the power reversal inherent in caregiving—where she over-

¹ The term “tiger mother” is popularized by Yale Law professor Amy Chua in her memoir *Battle Hymn of the Tiger Mother*. Chua coins this concept to portray a parent who adopts hard-line methods with their children, aiming for top grades, emphasizing extracurricular, and prohibiting social activities. By recounting her own journey of relentlessly pushing her two daughters to excel academically and professionally, Chua argues that pressure always transforms into motivation. Extreme actions—such as threatening to burn toys and not to offer food, using derogatory language, calling children names, and setting high standards without dialogue—are also part of her harsh discipline.

sees her dependent parent's life—can reignite old wounds (Kong & Moorman, 2015; Kong & Moorman, 2016). Apart from this psychological cost, the labor-intensive nature of eldercare crowds out the child's work time, reducing her total earnings and well-being and further discouraging her from providing support. From the perspective of the elderly, these dynamics are equally fraught. For instance, a parent who was stern earlier in life struggles to trust his child's compassion, which diminishes his own emotional comfort. He may also be burdened by remorse for his past harshness, a feeling of unworthiness that poisons the very support he receives.

This paper introduces a positive and normative analysis that connects parenting to eldercare. We establish an overlapping-generations model, wherein individuals progress through childhood, young adulthood, and old adulthood. Children do not make decisions but accumulate human capital, treating parental inputs as given. Young adults exhibit two-sided altruism: they are concerned with promoting their children's human capital (known as “impure parental altruism”) and find joy in assisting their elderly parents. They are endowed with one unit of time, which they choose to allocate between work and filial care. Meanwhile, young adults invest in their children's education through financial means and implementation of harsh discipline, both facilitating their children's human capital development. The elderly consume their savings and value their children's support and attention; however, engaging in harsh parenting early on can undermine the perceived quality of future eldercare offered by their children.

Our analysis starts with a *laissez-faire* economy, in which young adults choose parenting style, work time, educational spending, and saving without policy interventions. In Sect. 3, we suggest that as life expectancy rises, parents tend to employ harsh tactics less frequently with their children. Driven by longer lifespans, parents are becoming increasingly aware of their future reliance on their children for informal care. This foresight motivates them to maintain stronger familial relationships by mitigating harsh parenting practices. A deeper sense of fulfillment derived from caring for old parents strengthens dedication to labor-intensive eldercare, but it can result in a decrease in both financial investments in education and harsh disciplinary measures. In contrast, a heightened focus on child development diverts attention and effort away from eldercare obligations. Both altruistic drives tend to generate undesirable consequences on personal saving and consumption.

In Sect. 4, we examine the social optimum where a social planner maximizes the individual utility in the steady state. By defining the steady state as the constant level of human capital in each generation, our examination reveals a difference between the *laissez-faire* choices and the first-best optimality conditions. This disparity results largely from impure parental altruism, where parents prioritize their children's human capital but do not directly take into account their children's utility (Becker, 1993). In particular, if parental concern about children's human capital is low, educational investments fall short of the first-best level; likewise, when children's altruism towards parents is weak, eldercare provision dips below the first-best level. Instruments to restore the social optimum are lacking, which prompts us to explore the second-best policies.

Section 4 also investigates the optimal policies that the government can implement to improve the steady-state social welfare. Policy instruments at the govern-

ment's disposal include taxation or subsidization on labor earnings and educational funds, which guides individuals to balance work commitment, consumption, child development, and eldercare provision. Although the government cannot intervene in unobserved parenting styles directly, public policies can still alter the broader constraints—such as the steady-state human capital stock—that indirectly affect parental behavior. Our analysis indicates that, under certain circumstances (e.g., when intergenerational altruism is low), the optimal intervention entails taxing labor and subsidizing formal education. Moreover, we conduct a simulation deriving numerical solutions to the optimal income taxation and educational subsidy and the individual responses in equilibrium. We then provide numerical comparative static analyses under various parameter configurations.

This study complements the theoretical literature on population aging and informal eldercare. Existing theories focus on the effect of rising longevity on human capital formation and economic development (Cervellati & Sunde, 2005; Hazan & Zoabi, 2006; de la Croix & Ponthière, 2010). Fanti and Gori (2014) investigate how grandparental child care moderates the relationship between longevity and long-term growth. Our model addresses the effect of longevity on informal eldercare provision from the perspective of parenting styles. There are three alternative motives of long-term care within families, namely exchange contract, social norm, and pure altruism (Cremer et al., 2012; Klimaviciute & Pestieau, 2023). Canta and Cremer (2019) hold that parents can commit to a rule specifying a bequest conditional on the level of eldercare. Barigozzi et al. (2020) suggest that social expectation plays a crucial role in determining the dynamics of family eldercare, which contributes to the emergence of gender gap in care provision. Ponthière (2013) introduces heterogenous altruism into an OLG model, analyzing the optimal design of long-term care policy when the state intervention can crowd out informal care. Pestieau and Ponthière (2016) examine endogenous birth timing in an OLG model, showing that encouraging early births can effectively reduce the social burden of informal care provision. Our model adopts a life cycle perspective, emphasizing that the quality of eldercare provided by adult children is rooted in the history of the parent–child relationship.

Our paper contributes to the burgeoning research on parenting styles. Weinberg (2001) shows that parenting styles tend to be affected by parental income. Burton et al. (2002) present a model of parent–child interactions in which parenting is determined by both the parent's stress and the child's (mis)behavior. Doepke and Zilibotti (2017) explore the effects of socioeconomic environment (e.g., income inequality, division of labor, and technological advancement) on child-rearing practices, attributing the long-term decline of authoritarian parenting to rising returns to independence. This paper is most closely related to Fan et al. (*forthcoming*), who analyze the relationship between parenting and eldercare. We extend their OLG framework in three important directions. First, while they conceptualize family eldercare as driven by societal norms, we model the motivation for eldercare as stemming from children's pure altruism (i.e., the joy of giving). Second, while they include parental time investment but mostly abstract from educational expenditure, our model explicitly incorporates such expenditures. The most substantial extension lies in our normative analysis. We derive the first-best social optimum and characterize the second-best policy mix—educational subsidies and earnings taxes—a central question that the

earlier study does not address. Therefore, this paper not only complements but also advances the research agenda initiated in Fan et al. (*forthcoming*).

2 The model setup

Consider an overlapping-generations small open economy where a unit mass of identical individuals live for childhood, young adulthood, and old adulthood. Children do not make any decision and accumulate human capital by treating parental inputs as given. Young adults are endowed with one unit of time, which they allocate between work and care of their parents. Alongside this time-allocation decision, they each raise a child, providing education through both harsh discipline and financial investments in schooling.² We assume that the enforcement of harsh discipline does not consume parental time or resources. In addition, young adults allocate their income between consumption, saving, and educational expenditure. Old adults retire and live for a fraction $\pi \in (0, 1)$ of period, during which they depend on their children's caregiving support and consume savings without leaving any bequest. A larger value of the parameter π signifies aging.

Denote individuals who spend their young adulthood in period t as generation t . In period t , a representative member of generation t has a human capital stock of H_t , which is governed by her parental human capital (H_{t-1}), the frequency of harsh discipline implemented by her parent (d_{t-1}), and educational expenditure of her parent (e_{t-1}), as expressed by

$$H_t = \Phi(H_{t-1}, d_{t-1}, e_{t-1}). \quad (1)$$

For generation $t - 1$, H_{t-1} is a state variable, while d_{t-1} and e_{t-1} are choice variables. In particular, a high value of d_{t-1} corresponds to harsh parenting, and a low value reflects loving parenting. The function Φ satisfies the following properties:

$$\frac{\partial \Phi}{\partial H_t} > 0 \quad \frac{\partial \Phi}{\partial d_t} > 0 \quad \frac{\partial \Phi}{\partial e_t} > 0, \quad (2)$$

$$\frac{\partial^2 \Phi}{\partial H_t^2} < 0 \quad \frac{\partial^2 \Phi}{\partial d_t^2} < 0 \quad \frac{\partial^2 \Phi}{\partial e_t^2} < 0, \quad (3)$$

$$\frac{\partial^2 \Phi}{\partial e_t \partial H_t} > 0 \quad \frac{\partial^2 \Phi}{\partial d_t \partial H_t} \geq 0 \quad 0 \leq \frac{\partial^2 \Phi}{\partial d_t \partial e_t} < \sqrt{\frac{\partial^2 \Phi}{\partial d_t^2} \frac{\partial^2 \Phi}{\partial e_t^2}}. \quad (4)$$

Conditions (2) and (3) indicate that each parental input exhibits diminishing marginal returns. Condition (4) states that the cross-partial derivatives in the human capi-

²Our model treats educational investment as a direct monetary cost, which is functionally equivalent to the common approach of modeling it as a time cost (e.g., Fan et al. *forthcoming*), because both take away a part of parental disposable income. We apply this monetary framework to formal education, for which it is a direct fit.

tal production function are non-negative. Specifically, parental human capital and educational expenditure are complementary in building children's human capital, while parenting does not exert a countervailing impact on the effectiveness of parental human capital or educational spending. In particular, the last expression suggests that d_t and e_t are perfect substitutes (i.e., $\frac{\partial^2 \Phi}{\partial d_t \partial e_t} = 0$) or weak complements (i.e., a small positive value of $\frac{\partial^2 \Phi}{\partial d_t \partial e_t}$).

In period t , the young adult allocates a fraction $a_t \in (0, 1)$ of time to caring for her parent when her parent is alive. She experiences a pure joy of giving, as depicted by the function $g(\pi a_t)$, where $g' > 0$ and $g'' < 0$. The aggregate production function of the economy is given by $F(K_t, L_t)$, where F represents a constant-returns-to-scale technology, K_t denotes the physical capital stock, and $L_t = (1 - \pi a_t)H_t$ denotes the labor supply in efficiency units. We assume that in this small open economy, the prices of production factors are exogenously given. Specifically, the wage rate per efficiency unit of labor is fixed at w , while the intertemporal interest rate is fixed at r . A young adult, who earns a total income $w(1 - \pi a_t)H_t$, faces a budget constraint as expressed by

$$c_t^y + s_t + e_t = w(1 - \pi a_t)H_t, \quad (5)$$

where c_t^y and s_t are young-age consumption and saving, respectively. She obtains utility of $u(c_t^y)$ from consumption, where the law of diminishing marginal utility applies (i.e., $u' > 0$ and $u'' < 0$).

In period $t + 1$, each member of generation t becomes old and spends all her savings on consumption. With c_t^o representing the old-age consumption level per unit of time, the old-age budget constraint implies that

$$\pi c_t^o = (1 + r)s_t. \quad (6)$$

For each unit of time, the elderly derives utility from consumption, denoted as $u(c_t^o)$, and from her child's attention, denoted as $v(a_{t+1}, d_t)$. Formally, the following derivatives of v are satisfied:

$$\frac{\partial v}{\partial a_{t+1}} > 0 \quad \frac{\partial^2 v}{\partial a_{t+1}^2} \leq 0 \quad \frac{\partial v}{\partial d_t} < 0 \quad \frac{\partial^2 v}{\partial d_t^2} < 0. \quad (7)$$

In essence, the child's physical companion exhibits weakly diminishing returns, while harsh parenting causes severe disutility consequences. For analytical tractability, we further assume

$$\frac{\partial^2 v}{\partial d_t \partial a_{t+1}} = 0. \quad (8)$$

This assumption indicates that v is additively separable in d_t and a_{t+1} , which implicates no interaction effect between parental harshness and the child's eldercare time

in determining the child's attention to their parent. In sum, the lifetime utility of a member of generation t is given by

$$U_t = u(c_t^y) + \gamma g(\pi a_t) + \alpha H_{t+1} + \beta \pi [u(c_t^o) + v(a_{t+1}, d_t)], \quad (9)$$

where $\gamma > 0$ reflects the intensity of altruism towards her parent, and $\alpha > 0$ measures the degree of parental altruism linked to the child's human capital formation. The final term captures the old-age utility, where $\beta > 0$ is the time preference factor.

3 The Laissez-faire equilibrium

Our analysis starts with the *laissez-faire* economy. Substituting (1), (5), and (6) into (9) rewrites the individual lifetime utility as

$$U_t = u[(1 - \pi a_t)wH_t - s_t - e_t] + \gamma g(\pi a_t) + \alpha \Phi(H_t, d_t, e_t) + \beta \pi \left[u\left(\frac{(1+r)s_t}{\pi}\right) + v(a_{t+1}, d_t) \right].$$

The individual maximizes U_t by choosing (d_t, e_t, a_t, s_t) in young adulthood. The optimal choices are in accordance with the following four first order conditions:

$$\left(\frac{\partial v}{\partial d_t}\right)^{-1} \frac{\partial \Phi}{\partial d_t} = -\frac{\beta \pi}{\alpha}, \quad (10)$$

$$u'(c_t^y) = \alpha \frac{\partial \Phi}{\partial e_t} = \frac{\gamma}{wH_t} g'(\pi a_t) = \beta(1+r)u'(c_t^o). \quad (11)$$

Equation (10) defines the equilibrium parental harshness as a trade-off: it boosts the child's human capital development at the cost of future care from the child. The first equality of (11) characterizes the educational expenditure in equilibrium, where the marginal gain of the child's human capital equals the marginal cost associated with parental consumption loss. Likewise, the second equality of (11) reveals that the optimal time allocated to eldercare is achieved when the utility derived from providing care just offsets the utility cost of the forgone income.

We develop the next proposition demonstrating how parental behaviors change in response to rising longevity (measured by π).

Proposition 1 *An increase in longevity leads to both lower disciplinary harshness and lower education spending.*

Proof See Appendix. □

Proposition 1 states that a greater value of π results in a decrease in d_t : with greater longevity, parents adopt less authoritarian strategies towards their children. Driven by the prospect of longer lifespans, parents grow increasingly mindful of their future caregiving needs (Pestieau, 2022). This foresight prompts them to consider the vital

role their adult children will assume in providing care and assistance as they age. The quality of early-life intergenerational relationship often establishes a persistent life-long trajectory (Hwang et al., 2023). Recognizing that stronger bonds will translate into more emotional support from their children in later years, parents are inclined to reduce stern discipline in early life to prevent familial discord.

Empirical research increasingly confirms that childhood trauma caused by harsh parenting undermines the quality of care that adult children offer to their aging parents. Paris (2001) finds that individuals who report less controlling mothering indicate greater willingness to provide eldercare to mothers in the US. Kim et al. (2013) discover that tough parenting in Chinese American families is linked to a greater sense of alienation. (Kong & Moorman, 2015) present evidence from Wisconsin indicating that adults who care for their perpetrating parents had more frequent depressive symptoms during the caregiving period. They later reveal that a history of maternal childhood abuse is linked to adult children providing less emotional support to their mothers in the US (Kong & Moorman, 2016). Similarly, Parker et al. (2018) show that adults with histories of childhood abuse and neglect report a reduced sense of obligation towards their families. Hence, rational and forward-looking parents, who anticipate a high chance of needing informal eldercare, will strive to maintain closeness with their children from an early age.

Proposition 1 also predicts a potential negative correlation between π and e_t , underpinned by several rationales. First, individuals expecting a longer retirement horizon increase their life-cycle savings during their working years. This heightened imperative for self-insurance may crowd out financial resources they would otherwise dedicate to their children's education. Second, increased longevity means that old parents require more extensive care, which compels working-age children to resume caregiving responsibilities and thus divert time away from income-generating activities. With their earning capacity restricted, young adults experience lower total income, which in turn curtails investments in their children's schooling through an income effect. These dynamics may be compounded if parenting and financial investments exhibit complementarity in promoting children's human capital. In this case, the decline in harsh disciplinary practices—a consequence of extended lifespans—lowers the marginal return on educational expenditure, thereby discouraging those very investments. This result is consistent with Poterba's (1997) finding that an increasing fraction of the old population results in a decrease in education spending across the US states.

In short, Proposition 1 predicts that rising life expectancy can result in a decrease in discipline and educational investments. We can further infer from equation (1) that, all else being equal, this generates a detrimental effect on the human capital stock of the younger generation. Empirical research consistently supports this inference, highlighting the adverse economic consequences of an aging population. Acemoglu & Johnson, 2007 establish that increased longevity can depress human capital development and per capita income. This finding is reinforced by the cross-country analysis of Alper et al. (2016), who identify a negative correlation between average age and per capita income. Fraumeni et al. (2019) show that population aging curtails the growth rate of human capital in China.

We develop the following proposition to analyze the effect of intergenerational altruism on the distribution of care and attention within families.

Proposition 2

- (i) *An increase in parental altruism leads to an increase in disciplinary harshness and education spending and a decrease in eldercare time and saving;*
- (ii) *an increase in filial altruism leads to an increase in eldercare time and a decrease in educational spending and saving;*
- (iii) *when parental discipline and educational spending are complementary in fostering a child's human capital development, greater filial altruism leads to reduced disciplinary harshness. When they are substitutes, filial altruism has no effect on harshness.*

Proof See Appendix. □

Proposition 2 explores how shifting priorities, shaped by concerns over future potential in children versus eldercare needs in parents, affect parenting styles and household resources allocation. When young adults are increasingly concerned about their children (larger α), prioritizing human capital development becomes paramount, which manifests through financial investments in education and the implementation of structured discipline (larger e_t and d_t). Yet this heightened concern for children necessitates young parents to increase their labor supply to boost earnings. As a result, they direct less attention towards elderly parents who also need care and assistance (smaller a_t).

Conversely, as the concerns of young adults turn to the well-being of their aging parents, family eldercare takes precedence. Part (ii) of Proposition 2 demonstrates that the escalating concern for the elderly (larger γ) leads to a reallocation of resources to ensure the caregiving support for aging parents under some conditions (larger a_t), which in turn causes adverse labor market consequences. Van Houtven et al. (2013) find that female caregivers (i.e., daughters) decrease work hours and attain a lower wage in the US. Furthermore, when the imperative to prioritize eldercare amplifies, another potential consequence emerges—a lack of attention to children's developmental needs. The diversion of time away from income-generating endeavors crowds out financial investments for children's education (smaller e_t) through the income effect. This implication also aligns with Bogan (2015) empirical finding that having an elderly dependent in the household significantly mitigates the likelihood of investing funds in offspring education in the US.

Part (iii) of Proposition 2 highlights that the influence of the child's altruism parameter γ on the intensity of parental discipline d_t is not straightforward—it depends on the interaction between discipline and educational investment. According to Condition (4), parenting and formal education can be either complements or substitutes in fostering children's human capital development. In the case of complements, a decrease in educational spending induced by stronger altruism tends to impair the marginal productivity of parental discipline. As a result, parents optimally choose to reduce the adoption of harsh discipline, which implies that such altruism indirectly

leads to a more loving parenting style. When parenting and formal education are perfect substitutes, a change in educational spending does not affect the marginal benefit of parenting. Even if the child's stronger altruistic preference results in a lower level of educational investment, it does not necessarily alter the optimal choice of discipline.

By examining the intergenerational distribution of care and resources, Proposition 2 also sheds light on its consequence and implication for the working-age population. Young adults find themselves sandwiched between the dual responsibilities of supporting aging parents and raising their own children. Greater altruism towards parents diverts time from work to eldercare, thereby reducing their incomes and hence savings. Conversely, heightened altruism towards children results in a greater commitment to educational spending and hence crowds out savings. In short, an increase in intergenerational altruism—in terms of either α or γ —can impose a significant financial strain on the young, leaving them less funds for future retirement (smaller s_t) and taking a toll on their own well-being. Bogan (2015) presents evidence that the American households with both children and elderly dependents have much less stockholding in order to smooth consumption.

Finally, we develop the following proposition to analyze the comparative statics of (e_t, d_t) with respect to H_t , where H_t is a state variable for generation t :

Proposition 3

- (i) *An increase in parental human capital leads to an increase in educational spending;*
- (ii) *an increase in parental human capital leads to an increase in disciplinary harshness if parental human capital and discipline are complements.*

Proof See Appendix. □

Proposition 3 explores how parental human capital (H_t) influences parenting and formal education (d_t and e_t). Part (i) suggests that parents with a higher stock of human capital invest more financial resources in their children's education. This positive relationship between H_t and e_t is largely attributed to their complementarity in promoting children's human capital (see the first inequality of Condition (4)) and has received extensive support from the theoretical and empirical literature (e.g., Becker, 1993; Brown, 2006; Kornrich and Furstenberg, 2013).

Part (ii) of Proposition 3 suggests that parental human capital—whether in the form of education, income, or social status—may lead to more intensive disciplinary strategies. This relationship becomes particularly salient when parental human capital and harsh discipline function as complements in the production of children's human capital. In such cases, an increase in a parent's human capital enhances the marginal productivity of discipline, thereby incentivizing a more intense use of harsh discipline as part of the child-rearing process. Some empirical studies demonstrate the positive association between parental socioeconomic status and the adoption of harsh discipline in the household. For example, Bronfenbrenner (1958) finds that higher socioeconomic status parents exert greater pressure on their children and are

more likely to use psychological techniques of discipline in the US between 1930 and 1955. Wang and Liu (2014) show that Chinese parents from higher socioeconomic status backgrounds are more likely to implement corporal punishment on their children, perceiving strict discipline as an investment in their children's future success.

4 Social optimum and optimal policies

This section aims to identify the socially optimal outcome for the economy and investigate the welfare-maximizing public policies. We define social welfare in period t as the lifetime utility of generation t (U_t). For simplification, we focus on maximizing social welfare in the steady state—a condition where the human capital stock is constant across generations (i.e., $H_t = H$ for all t). As shown in Eq. (1), the evolution of human capital is path-dependent. Upon converging towards the steady state, individual decisions also become time-invariant (i.e., $d_t = d$, $e_t = e$, $a_t = a$, and $s_t = s$ for all t). Consequently, the steady-state condition can be characterized by the fixed-point equation $H = \Phi(H, d, e)$, which implicates that each generation enjoys the same level of utility U .

Our subsequent analysis unfolds in three parts. In Section 4.1, we examine the problem facing a utilitarian social planner who maximizes the steady-state social welfare. In Section 4.2, we explore how policy instruments (e.g., subsidies, taxes) can be designed to maximize the steady-state social welfare, addressing potential inefficiencies that arise from decentralized decision-making. In Section 4.3, we employ specific functional forms and simulations to illustrate the steady-state outcomes under optimal government policies.

4.1 First-best optimality

We start with the utilitarian social planner's optimization problem in the first-best world. As usual this kind of problem (Hammond, 1987; Harsanyi, 1995), we eliminate the influences of altruism by laundering out both altruistic terms (i.e., $\alpha = \gamma = 0$). The steady-state social welfare can thus be written as

$$U = u(c^y) + \beta\pi[u(c^o) + v(a, d)]. \quad (12)$$

The social planner's problem is to optimally choose (d, e, a, s) subject to the resource constraints (5) and (6) and the human capital production (1). By (12), we set up the Lagrangian as follows:

$$\mathcal{L} = u[(1 - \pi a)wH - e - s] + \beta\pi u\left(\frac{(1 + r)s}{\pi}\right) + \beta\pi v(a, d) - \lambda[H - \Phi(H, d, e)],$$

where λ denotes the Lagrange multiplier associated with human capital formation. The first order conditions with respect to (d, e, a, s) can be rearranged as

$$\left(\frac{\partial v}{\partial d}\right)^{-1} \frac{\partial \Phi}{\partial d} = -\frac{\beta \pi}{\lambda}, \quad (13)$$

$$u'(c^y) = \lambda \frac{\partial \Phi}{\partial e} = \frac{\beta}{wH} \frac{\partial v}{\partial a} = \beta(1+r)u'(c^o), \quad (14)$$

which determines the social planner's choices of (d, e, a, s) .

By comparing (13) and (14) with (10) and (11), we establish the next proposition to characterize the conditions under which the decentralized equilibrium outcome aligns with the social optimum:

Proposition 4 *The social optimum coincides with the laissez-faire equilibrium if and only if*

$$\alpha = \lambda, \quad \gamma = \frac{\beta}{g'(\pi a)} \frac{\partial v}{\partial a}. \quad (15)$$

Proposition 4 demonstrates the sufficient and necessary condition for a decentralized economy to achieve the social optimum. Specifically, condition (15) outlines two equalities, each imposing a separate requirement on the altruistic parameter. These two conditions are generally independent and may not hold simultaneously in practice. The violation of either equality leads to a misalignment between private incentives and social welfare. Consequently, achieving the first-best outcome in a decentralized setting seems highly restrictive.

When parental altruism is limited ($\alpha < \lambda$), the steady-state parental choices of d and e are lower than desired (see Proposition 2(i)). To ensure greater educational investments for young children, it seems necessary to introduce a subsidy on formal education. Besides, a more frequent application of discipline is necessary to balance the marginal returns from human capital accumulation against its marginal costs associated with eldercare. Note that the choice variable d , which is not readily observable, can only be controlled indirectly. For instance, subsidizing formal education influences the practice of parental discipline through the function Φ .

When adult children demonstrate a low degree of altruism towards their old parents (small γ), the *laissez-faire* economy will see an eldercare deficit in the steady state (i.e., an overly small a). To provide adequate caregiving support for the elderly, the social planner should reallocate young adults' time from employment towards eldercare provision. From a policy-making perspective, taxing labor earnings can be an effective tool to incentivize greater effort in eldercare. As a result, each young adult should reduce their savings and allocate fewer resources to their children's education. In the next subsection, we resort to a second-best approach to demonstrate how government intervention can enhance welfare by improving on the decentralized market outcome.

4.2 Optimal government policies

In this section, we consider that a benevolent government designs public policies to help families balance eldercare, child development, and work commitment. Policy tools at disposal include an income tax (at a rate of τ), an educational subsidy (a portion θ of a child's received educational fund), and a demogrant (A). In period t , the interactions between the government and generation t proceed in two stages. The government moves first to maximize the steady-state social welfare by optimally choosing (τ, θ, A) subject to the balanced budget, as follows:

$$\tau(1 - \pi a_t)wH_t = A + \theta e_t, \quad (16)$$

which indicates that tax revenues are directed towards educational subsidies and cash transfers. By observing the three policy instruments, generation t maximizes their utility by choosing (d_t, e_t, a_t, s_t) during young adulthood.

In this subsection, we present a theoretical analysis of the optimal policies and the corresponding equilibrium. By backward induction, we first examine the individual optimal choices given the policies. Each member of generation t receives a disposable income of $(1 - \tau)(1 - \pi a_t)wH_t + A$ and spends an amount of $(1 - \theta)e_t$ on their child's education. Accordingly, their lifetime utility of generation t can be written as

$$U_t = u[(1 - \tau)(1 - \pi a_t)wH_t + A - s_t - (1 - \theta)e_t] + \gamma g(\pi a_t) + \alpha H_{t+1} \\ + \beta \pi \left[u\left(\frac{(1 + r)s_t}{\pi}\right) + v(a_{t+1}, d_t) \right].$$

Given $(a_{t+1}, H_t; \tau, \theta, A)$, each member of generation t maximizes U_t by choosing (d_t, e_t, a_t, s_t) . In the steady state, their optimal choices satisfy

$$\left(\frac{\partial v}{\partial d}\right)^{-1} \frac{\partial H}{\partial d} = -\frac{\beta \pi}{\alpha}, \quad (17)$$

$$u'(c^y) = \frac{\alpha}{1 - \theta} \frac{\partial H}{\partial e} = \frac{\gamma g'(\pi a)}{(1 - \tau)wH} = \beta(1 + r)u'(c^o). \quad (18)$$

which indicates that the optimal solutions to (d, e, a, s) are a function of (τ, θ, A) .

Next, we probe into the optimal design of policies by the benevolent government. The government chooses policy tools (τ, θ, A) to maximize the steady-state social welfare, taking into account the individual optimal responses. Set up the Lagrangian in the steady state, where the two altruistic components have been removed (i.e., $\alpha = \gamma = 0$):

$$\begin{aligned}\mathfrak{L} = & u[(1 - \pi a)(1 - \tau)wH - s - (1 - \theta)e + A] + \beta\pi \left[u \left(\frac{(1 + r)s}{\pi} \right) + v(a, d) \right] \\ & + \mu [\tau (1 - \pi a) wH - \theta e - A],\end{aligned}$$

where μ is the multiplier associated with the government's budget constraint (16). Taking the first order conditions of $\mathfrak{L}$ with respect to (τ, θ, A) and then using (17) and (18), we derive the interior solutions as follows (see full derivation in Appendix):

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial \tau} = & [(1 - \pi a)(1 - \tau)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \tau} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a}{\partial \tau} \\ & - (1 - \pi a)wH[u'(c^y) - \mu] + \mu \Delta_\tau = 0,\end{aligned}\quad (19)$$

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial \theta} = & [(1 - \pi a)(1 - \tau)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \theta} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a}{\partial \theta} \\ & + e[u'(c^y) - \mu] + \mu \Delta_\theta = 0,\end{aligned}\quad (20)$$

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial A} = & [(1 - \pi a)(1 - \tau)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial A} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial A} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a}{\partial A} \\ & + [u'(c^y) - \mu] + \mu \Delta_A = 0,\end{aligned}\quad (21)$$

$$\text{where} \quad \Delta_z = \tau(1 - \pi a)w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial z} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial z} \right) - \tau\pi wH \frac{\partial a}{\partial z} - \theta \frac{\partial e}{\partial z} \quad \text{for}$$

$$z \in \{\tau, \theta, A\}.$$

We can now use these conditions to rewrite them in compensated terms with the notation $\frac{\partial x^C}{\partial z} = \frac{\partial x}{\partial z} + \frac{dA}{dz} \frac{\partial x}{\partial A}$, where $x = \{\mathfrak{L}, d, e, a\}$, $z \in \{\tau, \theta\}$, and the superscript C denotes compensation. By compensation, we look at the impact of the income tax rate (or the educational subsidy) as well as the impact of demogrant implied by the government's budget constraint (16), as follows:

$$\begin{aligned}\frac{\partial \mathfrak{L}^C}{\partial \tau} = & [(1 - \tau)(1 - \pi a)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial e} \frac{\partial e^C}{\partial \tau} + \frac{\partial H}{\partial d} \frac{\partial d^C}{\partial \tau} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a^C}{\partial \tau} + \mu \Delta_\tau^C, \\ \frac{\partial \mathfrak{L}^C}{\partial \theta} = & [(1 - \tau)(1 - \pi a)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial e} \frac{\partial e^C}{\partial \theta} + \frac{\partial H}{\partial d} \frac{\partial d^C}{\partial \theta} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a^C}{\partial \theta} + \mu \Delta_\theta^C,\end{aligned}$$

where $\Delta_z^C = \Delta_z + \frac{dA}{dz} \Delta_A$ for $z \in \{\tau, \theta\}$. At the point $\tau = \theta = A = 0$, we have $\Delta_\tau = \Delta_\theta = \Delta_A = 0$ and hence $\Delta_z^C = 0$. It follows that

$$\left. \frac{\partial \mathfrak{L}^C}{\partial \tau} \right|_{\tau=\theta=0} = [(1 - \pi a)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial e} \frac{\partial e^C}{\partial \tau} + \frac{\partial H}{\partial d} \frac{\partial d^C}{\partial \tau} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a^C}{\partial \tau}. \quad (22)$$

$$\left. \frac{\partial \mathfrak{L}^C}{\partial \theta} \right|_{\tau=\theta=0} = [(1 - \pi a)wu'(c^y) - \alpha] \left(\frac{\partial H}{\partial e} \frac{\partial e^C}{\partial \theta} + \frac{\partial H}{\partial d} \frac{\partial d^C}{\partial \theta} \right) + \pi \left[\beta \frac{\partial v}{\partial a} - \gamma g'(\pi a) \right] \frac{\partial a^C}{\partial \theta}. \quad (23)$$

It is straightforward to see that in equilibrium, $\tau > 0$ when the right-hand side (RHS) of Eq. (22) is positive, and $\theta > 0$ when the RHS of Eq. (23) is positive. Suppose that

the direct effect of τ on a outweighs its indirect effects on (e, d) , and likewise, the direct effect of θ on e dominates its indirect effects on (a, d) . Based on (22) and (23), we develop the next proposition:

Proposition 5 *The government taxes earnings ($\tau > 0$) and subsidizes education ($\theta > 0$) when*

$$\alpha < (1 - \pi a)wu'(c^y), \quad \gamma < \frac{\beta}{g'(\pi a)} \frac{\partial v}{\partial a}. \quad (24)$$

Proposition 5 demonstrates the signs of the tax/subsidy rely on the degree of intergenerational altruism. A benevolent government implements a policy of taxing labor and subsidizing formal education under certain circumstances. Condition (24) specifies that such a policy is particularly pertinent provided that the parameters of intergenerational altruism remain at a low value. When parents have limited concern about their children's human capital (small α), the government should promote and subsidize education to address the deficit in parental attention. With insufficient filial altruism (small γ), the optimal public policy involves discouraging labor supply and encouraging eldercare, achievable through imposing earnings taxes on the workforce. Cremer et al. (2016) make a similar argument regarding the optimal design of social long-term care insurance in the presence of two-sided altruism.

While labor income taxation and education subsidies are ubiquitous fiscal tools, their optimal design remains incomplete without considering the dual function of young adults as caregivers to the elderly and investors in their children's human capital. Current policies that compartmentalize eldercare and child-rearing overlook the critical intergenerational trade-offs faced by the young adults. Proposition 5 argues for a more cohesive social policy that explicitly integrates these interconnected responsibilities.

4.3 An example

Building upon the analytical framework established in Section 4.2, this subsection transitions to a more concrete examination by employing explicit functional forms and numerical simulations to derive the steady-state solutions under optimal policies. Our analysis is structured around two core objectives. First, we establish the existence, uniqueness, and stability of this long-run equilibrium, thereby verifying that, under the given assumptions and policy configurations, the model converges to a well-defined steady state. Second, we undertake a comparative static analysis that illuminates how the steady-state outcome responds to key demographic and behavioral changes—specifically, varying degrees of population aging and different levels of two-sided altruism. To ensure analytical tractability and intuitive clarity, we consider simple forms for u , g , Φ , and v , as follows:

$$u(x) = g(x) = \ln x, \quad (25)$$

$$H_{t+1} = \sqrt{H_t}(d_t^\rho + e_t^\delta) \quad \text{where } \rho, \delta \in (0, 1), \quad (26)$$

$$v(a_{t+1}, d_t) = a_{t+1} - \frac{\eta}{2} d_t^2 \quad \text{where } \eta > 0. \quad (27)$$

With these functions and the given policy tools (τ, θ, A) , we establish the following proposition to determine the steady-state solutions:

Proposition 6 *The steady-state solutions to (H, d, e) satisfy*

$$d = \left(\frac{\alpha\rho}{\beta\pi\eta} \sqrt{H} \right)^{\frac{1}{2-\rho}}, \quad (28)$$

$$\frac{1 + \gamma + \beta\pi}{\alpha\delta\sqrt{H}} e^{1-\delta} + e = \frac{(1-\tau)wH + A}{1-\theta}, \quad (29)$$

$$\sqrt{H} = d^\rho + e^\delta. \quad (30)$$

The steady state always exists, may or may not be unique, and is locally stable if and only if

$$\frac{e^\delta}{2\sqrt{H}} \left[\delta(2-\rho) \frac{3(1-\tau)wH + A - (1-\theta)e}{(1-\delta)[(1-\tau)wH + A] + (1-\theta)\delta e} - \rho \right] + \rho < 1. \quad (31)$$

Proof See Appendix. □

In Proposition 6, we first present a system of three equations, which characterizes solutions to three key steady-state variables (H, d, e) . Equation (28) expresses d as an explicit function of H , while Eq. (29) defines e implicitly in terms of H . By substituting them into Eq. (30), we can analytically determine the equilibrium level of H . Clearly, (H, d, e) are a function of three policy tools (τ, θ, A) . Accordingly, the optimal saving and eldercare time are solved as

$$s(H, e) = \frac{\beta\pi(1-\theta)}{\alpha\delta\sqrt{H}} e^{1-\delta}, \quad (32)$$

$$a(H, s) = \frac{\gamma s}{\beta(1-\tau)\pi^2 w H}. \quad (33)$$

We then establish the properties of the steady-state outcome by examining its existence, uniqueness, and stability. It is revealed that the dynamic system is guaranteed to have a steady state, and under certain conditions, can exhibit multiple steady states. In addition, the system is locally stable around $H_t = H$ if and only if condition (31) is met. In other words, existence is clearly established, but uniqueness and stability

Table 1 The optimal policies and equilibrium results under various configurations

	Variables	[1] Benchmark	[2] Reduced longevity	[3] Greater parental altruism	[4] Greater filial altruism
			$\pi = 0.4$	$\alpha = 0.7$	$\gamma = 0.7$
Note: Benchmark parameter values are $\pi = 0.5$, $\alpha = \gamma = 0.5$, $\beta = 0.9$, $\rho = \delta = 0.15$, $\eta = 5$, $w = 2$, $r = 1$	τ	0.313	0.315	0.296	0.289
	θ	0.780	0.764	0.741	0.782
	A	0.317	0.353	0.081	0.184
	H	3.548	3.606	3.805	3.471
	d	0.224	0.254	0.274	0.223
	e	1.720	1.720	2.223	1.520
	a	0.506	0.665	0.465	0.632
	s	1.111	0.946	1.122	1.002

remain elusive even for the simple functions. To further examine these two properties of the steady state, we resort to a numerical exercise.

Our simulation focuses on examining the steady-state outcome under the optimal public policies. In light of the individual responses, the government's objective is

$$\max_{\tau, \theta, A} (1 + \beta\pi) \ln s + \gamma \ln a + \alpha H + \beta\pi \left(a - \frac{\eta}{2} d^2 \right).$$

To perform the simulation, we rely on the following set of parameters: $\pi = 0.5$, $\alpha = \gamma = 0.5$, $\beta = 0.9$, $\rho = \delta = 0.15$, $\eta = 5$, $w = 2$, $r = 1$. The simulation outcomes are summarized in the first column of Table 1. The government sets an optimal income tax at the rate of 31.3% while offering a transfer payment of 0.317 to each individual. Each individual receives the educational funding totaling 1.720, with 22% being covered by her parent and the remaining 78% being subsidized by the government. Moreover, each individual allocates 50.6% of her time to family eldercare during her parent's lifetime. Out of her total earnings of $(1 - \pi a)H = 2.650$, she saves an amount of 1.111, representing a saving rate of 41.9%. The next three columns record our comparative static analyses. As presented in all four configurations, the steady state is unique and locally stable.

Column [2] differs from column [1] only by a fall in π from 0.5 to 0.4. With a decline in life expectancy, the government opts for a higher rate of the optimal income tax (e.g., 31.5% > 31.3%), which in turn discourages individuals from working (i.e., $1 - \pi a$ falls from 0.747 to 0.734). While the steady-state human capital stock increases slightly from 3.548 to 3.606, the educational investment per child remains relatively stable. The government covers a smaller proportion (76.4%), with parents contributing the remaining 23.6%. Due to the shortened life expectancy, parents tend to save less (i.e., $0.946 < 1.111$) and impose harsh discipline more frequently (i.e., $0.254 > 0.224$).

Column [3] differs from column [1] only in that α increases from 0.5 to 0.7, which reflects a growing emphasis among parents on promoting their children's human capital accumulation. Both the frequency of harsh discipline and the amount of educational investments experience an uptick, which helps to achieve a higher level of the steady-state human capital (i.e., $3.805 > 3.548$). Also, the government supports

education by allocating a budget of $\theta e = 1.647$ and by reducing the income tax (i.e., $29.6\% < 31.3\%$). The shift towards prioritizing children's educational advancement results in a decreased attention towards elderly parents, which is manifested by the reduction of a from 0.506 to 0.465.

In column [4], the sole change from column [1] is that γ rises from 0.5 to 0.7. As adult children develop a heightened concern about the well-being of their aging parents, they choose to dedicate a greater portion of their time in eldercare when their parents are alive (i.e., $0.632 > 0.506$). As their commitments to work decrease, the government responds by lowering the rate of income tax (i.e., $28.8\% < 31.3\%$) to prevent further discouragement of labor supply. However, the increased joy of giving towards elderly parents comes at a cost for young children: parents impose less discipline and cut back on educational investments. While parents' educational expenditure, as measured by $(1 - \theta)e$, drops from 0.378 to 0.331, the government reduces its subsidy on formal education (θe) from 1.342 to 1.189. With a societal reorientation away from nurturing the younger generation, the human capital stock ends with a lower level (i.e., $3.471 < 3.548$).

5 Conclusion

Historically, many parents, particularly in East Asian societies, have favored a strict and authoritarian approach to child-rearing in pursuit of academic excellence. This demanding style, although requiring minimal time and monetary investments, can strain the parent-child relationship, leading to an enduring negative influence on the care that children provide to their parents in later years. As life expectancies continue to rise, along with an increased likelihood of encountering physical and cognitive challenges in old age, there will be a marked trend towards adopting more nurturing parenting practices.

This paper establishes an OLG model analyzing the linkages between parenting styles, human capital accumulation, and informal eldercare. The framework captures a life course where parents educate their children in young adulthood and demand caregiving support from children in old adulthood. While educational funding and disciplinary measures are both effective in facilitating children's human capital development, the latter exerts a detrimental effect on the perceived quality of eldercare that children provide later in life. Young adults exhibit two-sided altruism, characterized by a concern for their children's educational attainment and a joy of giving from supporting their elderly parents.

We begin by analyzing individual decisions in a decentralized economy without policy intervention. Population aging, as measured by the lengthening of lifetime, discourages the use of harsh parenting. Balancing the imperative of nurturing the potential of the upcoming generation with the duty of caring for aging parents poses a challenge for young adults. A heightened focus on eldercare fulfillment can diminish their educational expenditures for their children and engagement in disciplinary measures. Conversely, a greater emphasis on child development can detract from the care provided to the elderly. Both altruistic priorities—directed upwards to parents and downwards to children—simultaneously depress personal savings and consumption.

We then delve into the normative aspect of what the optimal public policy should be concerning individual choices. Our evaluation reveals that decentralized equilibrium coincidentally aligns with first-best optimality, grounded in parents' impure altruism towards their children (i.e., parent care about their children's human capital but not lifetime utility). We outline the conditions under which the government should tax labor and subsidize formal education, proposing that such policies can enhance social welfare especially when intergenerational altruism remains low. This result is not novel, as the idea of taxing labor and subsidizing education represent standard policy instruments established well in the economic literature. What distinguishes our paper is its distinctive rationale behind this policy recommendation: aimed at adjusting parenting styles and promoting eldercare.

To further develop the idea presented in this paper, future research may extend our analysis in several important directions. With a focus on altruistic preferences in eldercare, this paper abstracts from intergenerational strategic exchanges (i.e., cash and in kind transfers from elderly parents to adult children). A compelling avenue for subsequent work would be to investigate the interactions between our core mechanism and other *quid pro quo* arrangements—such as grandparents providing child-care as a down payment on future eldercare (Lefebvre et al., 2025) or parents using bequests as a bargaining chip in the “time for money” exchange (Haberkern, 2023). Furthermore, endogenizing fertility represents another promising next step. Parents may strategically choose both the number of children—which directly impacts population aging—and their parenting styles to secure future eldercare by one or multiple children. Finally, while introducing public long-term care insurance would be an interesting extension, it would not fundamentally change our results. To the extent that formal and informal care are complementary (Perelman & Pestieau, 2025), the elderly cannot avoid relying on informal care—the central mechanism of our model.

Appendix: Mathematical proofs

Proof of Proposition 1

Totally differentiating (10) with respect to π , d_t , e_t , and a_{t+1} and then applying (8) obtains

$$\left[\left(\frac{\partial v}{\partial d_t} \right)^{-1} \frac{\partial^2 \Phi}{\partial d_t^2} - \left(\frac{\partial v}{\partial d_t} \right)^{-2} \frac{\partial^2 v}{\partial d_t^2} \frac{\partial \Phi}{\partial d_t} \right] dd_t + \left(\frac{\partial v}{\partial d_t} \right)^{-1} \frac{\partial^2 \Phi}{\partial d_t \partial e_t} de_t = -\frac{\beta}{\alpha} d\pi$$

$$\Leftrightarrow \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{de_t}{d\pi} + \Omega_t \frac{dd_t}{d\pi} = -\frac{\beta}{\alpha} \frac{\partial v}{\partial d_t}, \quad (\text{A1})$$

where $\Omega_t := \frac{\partial^2 \Phi}{\partial d_t^2} + \left(-\frac{\partial v}{\partial d_t}\right)^{-1} \frac{\partial^2 v}{\partial d_t^2} \frac{\partial \Phi}{\partial d_t} < \frac{\partial^2 \Phi}{\partial d_t^2} < 0$. Totally differentiating each equality of equation (11) with respect to (d_t, e_t, a_t, s_t) and π and then rearranging obtains

$$\frac{\beta(1+r)^2}{\pi} \frac{ds_t}{d\pi} - \frac{\gamma g''(\pi a_t) \pi}{w H_t u''(c_t^o)} \frac{da_t}{d\pi} = \frac{\beta(1+r)^2 s_t}{\pi^2} + \frac{\gamma g''(\pi a_t) a_t}{w H_t u''(c_t^o)}, \quad (\text{A2})$$

$$\frac{\beta(1+r)^2}{\pi} \frac{ds_t}{d\pi} - \frac{\alpha}{u''(c_t^o)} \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{dd_t}{d\pi} - \frac{\alpha}{u''(c_t^o)} \frac{\partial^2 \Phi}{\partial e_t^2} \frac{de_t}{d\pi} = \frac{\beta(1+r)^2 s_t}{\pi^2}, \quad (\text{A3})$$

$$\left[w H_t + \frac{\gamma g''(\pi a_t)}{w H_t u''(c_t^y)} \right] \pi \frac{da_t}{d\pi} + \frac{ds_t}{d\pi} + \frac{de_t}{d\pi} = - \left[w H_t + \frac{\gamma g''(\pi a_t)}{w H_t u''(c_t^y)} \right] a_t. \quad (\text{A4})$$

We introduce the notation $\Psi_t := \alpha \left[\frac{\pi}{\beta(1+r)^2 u''(c_t^o)} + \frac{1}{u''(c_t^y)} + \frac{(w H_t)^2}{\gamma g''(\pi a_t)} \right] < 0$. Combining the above four equations and rearranging yields:

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{de_t}{d\pi} = - \underbrace{\frac{s \Omega_t}{\pi \Psi_t}}_{+} + \underbrace{\frac{\beta}{\alpha} \frac{\partial v}{\partial d_t}}_{-} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} < 0, \quad (\text{A5})$$

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{dd_t}{d\pi} = - \underbrace{\frac{\beta}{\alpha} \frac{\partial v}{\partial d_t} \frac{\partial^2 \Phi}{\partial e_t^2}}_{+} - \underbrace{\frac{\beta}{\alpha \Psi_t} \frac{\partial v}{\partial d_t}}_{+} + \underbrace{\frac{s}{\pi \Psi_t} \frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{- \text{ or } 0} < 0. \quad (\text{A6})$$

By (4), the bracketed term on the left-hand side (LHS) in expressions (A5) and (A6) satisfies

$$\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} > \Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 > \frac{\partial^2 \Phi}{\partial d_t^2} \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 > 0, \quad (\text{A7})$$

which implies that $\frac{dd_t}{d\pi} < 0$ and $\frac{de_t}{d\pi} < 0$.

Proof of Proposition 2

Totally differentiating (10) with respect to α , d_t , e_t , and a_{t+1} and then applying (8) obtains

$$\begin{aligned} & \left(\frac{\partial v}{\partial d_t} \right)^{-1} \left(\frac{\partial^2 \Phi}{\partial d_t^2} dd_t + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} de_t \right) - \left(\frac{\partial v}{\partial d_t} \right)^{-2} \frac{\partial \Phi}{\partial d_t} \frac{\partial^2 v}{\partial d_t^2} dd_t = \frac{\beta \pi}{\alpha^2} d\alpha \\ \Leftrightarrow & \quad \Omega_t \frac{dd_t}{d\alpha} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{de_t}{d\alpha} = \frac{\beta \pi}{\alpha^2} \frac{\partial v}{\partial d_t}. \end{aligned} \quad (\text{A8})$$

Totally differentiating each equality of (11) with respect to (d_t, e_t, a_t, s_t) and α and then rearranging obtains

$$\frac{\beta(1+r)^2 u''(c_t^e)}{\pi} \frac{ds_t}{d\alpha} - \frac{\gamma \pi g''(\pi a_t)}{w H_t} \frac{da_t}{d\alpha} = 0, \quad (\text{A9})$$

$$\left[\frac{\beta(1+r)^2 u''(c_t^e)}{\pi u''(c_t^y)} + 1 \right] \frac{ds_t}{d\alpha} + \pi w H_t \frac{da_t}{d\alpha} + \frac{de_t}{d\alpha} = 0, \quad (\text{A10})$$

$$\frac{\partial^2 \Phi}{\partial e_t^2} \frac{de_t}{d\alpha} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{dd_t}{d\alpha} - \frac{\gamma \pi g''(\pi a_t)}{\alpha w H_t} \frac{da_t}{d\alpha} = -\frac{1}{\alpha} \frac{\partial \Phi}{\partial e_t}. \quad (\text{A11})$$

Combining the above four equations and rearranging yields:

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{da_t}{d\alpha} = \underbrace{\frac{w H_t}{\gamma \pi g''(\pi a_t) \Psi_t}}_{+} \underbrace{\left(\frac{\partial \Phi}{\partial e_t} \Omega_t \right)}_{-} + \underbrace{\frac{\beta \pi}{\alpha} \frac{\partial v}{\partial d_t}}_{-} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} < 0, \quad (\text{A12})$$

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{ds_t}{d\alpha} = \underbrace{\frac{\pi}{\beta(1+r)^2 u''(c_t^e) \Psi_t}}_{+} \underbrace{\left(\frac{\partial \Phi}{\partial e_t} \Omega_t \right)}_{-} + \underbrace{\frac{\beta \pi}{\alpha} \frac{\partial v}{\partial d_t}}_{-} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} < 0, \quad (\text{A13})$$

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{de_t}{d\alpha} = \underbrace{-\frac{1}{\alpha}}_{-} \underbrace{\left(\frac{\partial \Phi}{\partial e_t} \Omega_t \right)}_{-} + \underbrace{\frac{\beta \pi}{\alpha} \frac{\partial v}{\partial d_t}}_{-} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} > 0, \quad (\text{A14})$$

$$\frac{dd_t}{d\alpha} = \underbrace{\frac{\beta \pi}{\alpha^2 \Omega_t} \frac{\partial v}{\partial d_t}}_{+} - \underbrace{\frac{1}{\Omega_t} \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{de_t}{d\alpha}}_{+ \text{ or } 0}. \quad (\text{A15})$$

By using (A7), we can infer from equations (A12–A14) that $\frac{da_t}{d\alpha} < 0$, $\frac{ds_t}{d\alpha} < 0$, and $\frac{de_t}{d\alpha} > 0$. Given that $\frac{de_t}{d\alpha} > 0$, it is straightforward to show that $\frac{dd_t}{d\alpha} > 0$ in equation (A15).

Totally differentiating (10) with respect to γ , d_t , e_t , and a_{t+1} and then applying (8) obtains

$$\begin{aligned} & \left(\frac{\partial v}{\partial d_t} \right)^{-1} \left(\frac{\partial^2 \Phi}{\partial d_t^2} dd_t + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} de_t \right) - \left(\frac{\partial v}{\partial d_t} \right)^{-2} \frac{\partial \Phi}{\partial d_t} \frac{\partial^2 v}{\partial d_t^2} dd_t = 0 \\ \Leftrightarrow & \quad \Omega_t \frac{dd_t}{d\gamma} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{de_t}{d\gamma} = 0. \end{aligned} \quad (\text{A16})$$

Totally differentiating each equality of (11) with respect to (d_t, e_t, a_t, s_t) and γ and then rearranging obtains

$$\frac{\beta(1+r)^2 u''(c_t^o)}{\pi} \frac{ds_t}{d\gamma} - \frac{\gamma \pi g''(\pi a_t)}{w H_t} \frac{da_t}{d\gamma} = \frac{g'(\pi a_t)}{w H_t}, \quad (\text{A17})$$

$$\left[\frac{\beta(1+r)^2 u''(c_t^o)}{\pi u''(c_t^y)} + 1 \right] \frac{ds_t}{d\gamma} + \pi w H_t \frac{da_t}{d\gamma} + \frac{de_t}{d\gamma} = 0, \quad (\text{A18})$$

$$\frac{\partial^2 \Phi}{\partial e_t^2} \frac{de_t}{d\gamma} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{dd_t}{d\gamma} - \frac{\gamma \pi g''(\pi a_t)}{\alpha w H_t} \frac{da_t}{d\gamma} = \frac{g'(\pi a_t)}{\alpha w H_t}. \quad (\text{A19})$$

Combining the above four equations and rearranging yields:

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{de_t}{d\gamma} = \frac{\Omega_t}{\Psi_t} \frac{w H_t g'(\pi a_t)}{\gamma g''(\pi a_t)} < 0, \quad (\text{A20})$$

$$\begin{aligned} & \left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{da_t}{d\gamma} = \underbrace{\frac{\Omega_t}{\Psi_t} \frac{-g'(\pi a_t)}{\gamma \pi g''(\pi a_t)}}_{+} \\ & - \underbrace{\frac{\alpha g'(\pi a_t)}{\gamma \pi g''(\pi a_t) \Psi_t}}_{+} \underbrace{\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 \right]}_{+} \underbrace{\left[\frac{1}{u''(c_t^y)} + \frac{\pi}{\beta(1+r)^2 u''(c_t^o)} \right]}_{-} > 0, \end{aligned} \quad (\text{A21})$$

$$\begin{aligned} & \left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{ds_t}{d\gamma} \\ & = \underbrace{\frac{\alpha \pi w H_t g'(\pi a_t)}{\beta(1+r)^2 \gamma g''(\pi a_t) u''(c_t^o) \Psi_t}}_{-} \underbrace{\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 \right]}_{+} < 0, \end{aligned} \quad (\text{A22})$$

$$\left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{dd_t}{d\gamma} = - \underbrace{\frac{w H_t g'(\pi a_t)}{\gamma g''(\pi a_t) \Psi_t}}_{+} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} \leq 0. \quad (\text{A23})$$

By using (A7), we can infer from equations (A20–A22) that $\frac{de_t}{d\gamma} < 0$, $\frac{da_t}{d\gamma} > 0$, and $\frac{ds_t}{d\gamma} < 0$. Moreover, we have $\frac{dd_t}{d\gamma} < 0$ when $\frac{\partial^2 \Phi}{\partial d_t \partial e_t} > 0$ and $\frac{dd_t}{d\gamma} = 0$ when $\frac{\partial^2 \Phi}{\partial d_t \partial e_t} = 0$.

Proof of Proposition 3

Totally differentiating (10) with respect to H_t , d_t , e_t , and a_{t+1} and then applying (8) obtains

$$\begin{aligned} \left(\frac{\partial v}{\partial d_t} \right)^{-1} \left(\frac{\partial^2 \Phi}{\partial d_t^2} dd_t + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} de_t + \frac{\partial^2 \Phi}{\partial d_t \partial H_t} dH_t \right) &= \left(\frac{\partial v}{\partial d_t} \right)^{-2} \frac{\partial \Phi}{\partial d_t} \frac{\partial^2 v}{\partial d_t^2} dd_t \\ \Leftrightarrow \Omega_t \frac{dd_t}{dH_t} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{de_t}{dH_t} &= - \frac{\partial^2 \Phi}{\partial d_t \partial H_t}. \end{aligned} \quad (\text{A24})$$

Totally differentiating each equality of (11) with respect to (d_t, e_t, a_t, s_t) and H_t and then rearranging obtains

$$\frac{\beta(1+r)^2}{\pi} \frac{ds_t}{dH_t} - \frac{\gamma \pi g''(\pi a_t)}{w H_t u''(c_t^o)} \frac{da_t}{dH_t} = - \frac{\gamma g'(\pi a_t)}{w H_t^2 u''(c_t^o)}, \quad (\text{A25})$$

$$\left[\frac{\beta(1+r)^2 u''(c_t^o)}{\pi u''(c_t^y)} + 1 \right] \frac{ds_t}{dH_t} + \pi w H_t \frac{da_t}{dH_t} + \frac{de_t}{dH_t} = (1 - \pi a_t) w, \quad (\text{A26})$$

$$\frac{\partial^2 \Phi}{\partial e_t^2} \frac{de_t}{dH_t} + \frac{\partial^2 \Phi}{\partial d_t \partial e_t} \frac{dd_t}{dH_t} - \frac{\gamma \pi g''(\pi a_t)}{\alpha w H_t} \frac{da_t}{dH_t} = - \left[\frac{\gamma g'(\pi a_t)}{\alpha w H_t^2} + \frac{\partial^2 \Phi}{\partial e_t \partial H_t} \right]. \quad (\text{A27})$$

Combining the above four equations and rearranging yields

$$\begin{aligned} \left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{de_t}{dH_t} &= \underbrace{-\Omega_t}_{+} \underbrace{\frac{\partial^2 \Phi}{\partial e_t \partial H_t}}_{+} \\ &+ \underbrace{\left\{ \frac{\partial^2 \Phi}{\partial d_t \partial H_t} + w \frac{\Omega_t}{\Psi_t} \left[(1 - \pi a_t) - \frac{g'(\pi a_t)}{g''(\pi a_t)} \right] \right\}}_{+ \text{ or } 0} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} > 0, \end{aligned} \quad (\text{A28})$$

$$\begin{aligned}
& \left[\Omega_t \frac{\partial^2 \Phi}{\partial e_t^2} - \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t} \right)^2 + \frac{\Omega_t}{\Psi_t} \right] \frac{dd_t}{dH_t} = - \underbrace{\left(\frac{1}{\Psi_t} + \frac{\partial^2 \Phi}{\partial e_t^2} \right)}_{-} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial H_t}}_{+ \text{ or } 0} \\
& + \left\{ \underbrace{\frac{\partial^2 \Phi}{\partial e_t \partial H_t}}_{+} - \underbrace{\frac{w}{\Psi_t} \left[(1 - \pi a_t) - \frac{g'(\pi a_t)}{g''(\pi a_t)} \right]}_{-} \right\} \underbrace{\frac{\partial^2 \Phi}{\partial d_t \partial e_t}}_{+ \text{ or } 0} \geq 0.
\end{aligned} \tag{A29}$$

Because the bracketed term of the LHS of Eq. (A28) is positive by (A7), we can derive that $\frac{\partial e_t}{\partial H_t} > 0$. Likewise, we have $\frac{\partial d_t}{\partial H_t} \geq 0$, where the strict inequality holds when $\frac{\partial^2 \Phi}{\partial d_t \partial H_t} > 0$.

Deriving equations (19–21)

Taking the first order conditions of $\mathcal{L}$ with respect to (τ, θ, A) obtains

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \tau} &= u'(c^y) \left[-(1 - \pi a)wH - (1 - \tau)\pi wH \frac{\partial a}{\partial \tau} + (1 - \pi a)(1 - \tau)w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \tau} \right) \right. \\
&\quad \left. - \frac{\partial s}{\partial \tau} - (1 - \theta) \frac{\partial e}{\partial \tau} \right] + \beta \left[u'(c^o)(1 + r) \frac{\partial s}{\partial \tau} + \pi \frac{\partial v}{\partial a} \frac{\partial a}{\partial \tau} + \pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial \tau} \right] \\
&\quad + \mu \left[(1 - \pi a)wH + \tau(1 - \pi a)w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \tau} \right) - \tau\pi wH \frac{\partial a}{\partial \tau} - \theta \frac{\partial e}{\partial \tau} \right] = 0 \\
\Rightarrow \frac{\partial \mathcal{L}}{\partial \tau} &= (1 - \tau)(1 - \pi a)wu'(c^y) \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \tau} \right) + \beta\pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial \tau} - (1 - \theta)u'(c^y) \frac{\partial e}{\partial \tau} \\
&\quad + \pi \left[\beta \frac{\partial v}{\partial a} - (1 - \tau)wHu'(c^y) \right] \frac{\partial a}{\partial \tau} + [\beta(1 + r)u'(c^o) - u'(c_t^y)] \frac{\partial s}{\partial \tau} \\
&\quad - (1 - \pi a)wHu'(c^y) + \mu [(1 - \pi a)wH + \Delta_\tau] = 0,
\end{aligned} \tag{A30}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= u'(c^y) \left[-\pi wH(1 - \tau) \frac{\partial a}{\partial \theta} + (1 - \pi a)(1 - \tau)w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \theta} \right) - \frac{\partial s}{\partial \theta} \right. \\
&\quad \left. + e - (1 - \theta) \frac{\partial e}{\partial \theta} \right] + \beta \left[u'(c^o)(1 + r) \frac{\partial s}{\partial \theta} + \pi \frac{\partial v}{\partial a} \frac{\partial a}{\partial \theta} + \pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial \theta} \right] \\
&\quad + \mu \left[-\tau\pi wH \frac{\partial a}{\partial \theta} + \tau(1 - \pi a)w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \theta} \right) - e - \theta \frac{\partial e}{\partial \theta} \right] = 0 \\
\Rightarrow \frac{\partial \mathcal{L}}{\partial \theta} &= (1 - \tau)(1 - \pi a)wu'(c^y) \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial \theta} \right) + \beta\pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial \theta} - (1 - \theta)u'(c^y) \frac{\partial e}{\partial \theta} \\
&\quad + \pi \left[\beta \frac{\partial v}{\partial a} - (1 - \tau)wHu'(c^y) \right] \frac{\partial a}{\partial \theta} + [\beta(1 + r)u'(c^o) - u'(c_t^y)] \frac{\partial s}{\partial \theta} + eu'(c^y) \\
&\quad + \mu (-e + \Delta_\theta) = 0,
\end{aligned} \tag{A31}$$

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial A} &= u'(c^y) \left[-\pi w H (1-\tau) \frac{\partial a}{\partial A} + (1-\pi a)(1-\tau) w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial A} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial A} \right) \right. \\
 &\quad \left. - \frac{\partial s}{\partial A} - (1-\theta) \frac{\partial e}{\partial A} + 1 \right] + \beta \left[u'(c^o)(1+r) \frac{\partial s}{\partial A} + \pi \frac{\partial v}{\partial a} \frac{\partial a}{\partial A} + \pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial A} \right] \\
 &\quad + \mu \left[-\tau \pi w H \frac{\partial a}{\partial A} + \tau (1-\pi a) w \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial A} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial A} \right) - \theta \frac{\partial e}{\partial A} - 1 \right] = 0 \\
 \Leftrightarrow \frac{\partial \mathcal{L}}{\partial A} &= (1-\tau)(1-\pi a) w u'(c^y) \left(\frac{\partial H}{\partial d} \frac{\partial d}{\partial A} + \frac{\partial H}{\partial e} \frac{\partial e}{\partial A} \right) + \beta \pi \frac{\partial v}{\partial d} \frac{\partial d}{\partial A} - (1-\theta) u'(c^y) \frac{\partial e}{\partial A} \\
 &\quad + \pi \left[\beta \frac{\partial v}{\partial a} - (1-\tau) w H u'(c^y) \right] \frac{\partial a}{\partial A} + [\beta(1+r) u'(c^o) - u'(c_t^y)] \frac{\partial s}{\partial A} + u'(c^y) \\
 &\quad + \mu(-1 + \Delta_\theta) = 0.
 \end{aligned} \tag{A32}$$

Substituting (17) and (18) into (A30–A32) and then rearranging obtains (19–21).

Proof of Proposition 6

With the explicit functions, the lifetime utility of each member of generation t can be rewritten as

$$U_t = \ln[(1 - \pi a_t) w H_t - s_t - e_t] + \gamma \ln(\pi a_t) + \alpha \sqrt{H_t} (d_t^\rho + e_t^\delta) + \beta \pi \left[\ln \left(\frac{(1+r)s_t}{\pi} \right) + a_{t+1} - \frac{\eta}{2} d_t^2 \right].$$

The optimal choices of (d_t, e_t, a_t, s_t) satisfy the following first order conditions:

$$\begin{aligned}
 \frac{\partial U_t}{\partial d_t} &= \alpha \sqrt{H_t} \rho d_t^{\rho-1} - \beta \pi \eta d_t = 0, \\
 \frac{\partial U_t}{\partial e_t} &= \frac{-1}{(1 - \pi a_t) w H_t - s_t - e_t} + \alpha \sqrt{H_t} \delta e_t^{\delta-1} = 0, \\
 \frac{\partial U_t}{\partial a_t} &= \frac{-\pi w H_t}{(1 - \pi a_t) w H_t - s_t - e_t} + \frac{\gamma}{a_t} = 0, \\
 \frac{\partial U_t}{\partial s_t} &= \frac{-1}{(1 - \pi a_t) w H_t - s_t - e_t} + \frac{\beta \pi}{s_t} = 0,
 \end{aligned}$$

which can be rearranged as below:

$$d_t = \left(\frac{\alpha \rho}{\beta \pi \eta} \sqrt{H_t} \right)^{\frac{1}{2-\rho}}, \tag{A33}$$

$$\frac{1 + \gamma + \beta \pi}{\alpha \delta \sqrt{H_t}} e_t^{1-\delta} + e_t = \frac{(1-\tau) w H_t + A}{1-\theta}, \tag{A34}$$

$$s_t = \frac{\beta \pi (1-\theta)}{\alpha \delta \sqrt{H_t}} e_t^{1-\delta}, \tag{A35}$$

$$a_t = \frac{\gamma s_t}{\beta(1-\tau)\pi^2 w H_t}. \quad (\text{A36})$$

In the steady state, Eq. (30) follows directly from the explicit function for H_{t+1} , Eq. (A33) can be written as (28), and Eq. (A34) can be written as (29). Substituting (30) into (29) and then rearranging obtains

$$1 + \frac{1 + \gamma + \beta\pi}{\alpha\delta\sqrt{H}(\sqrt{H} - d\rho)} = \frac{(1-\tau)wH + A}{(1-\theta)(\sqrt{H} - d\rho)^{\frac{1}{\delta}}}. \quad (\text{A37})$$

In the limit as $H \rightarrow 0$, Eq. (28) implies that $d \rightarrow 0$. Consequently, the LHS of (A37) goes to infinity, while the RHS of (A37) approaches zero. In the limit as $H \rightarrow \infty$, then $d \rightarrow \infty$ by (28). It follows that the LHS of (A37) goes to zero, contrasting with its RHS, which goes to infinity. By continuity, the solution to H must exist.

By (29), we introduce the following definition:

$$Z \equiv \frac{1 + \gamma + \beta\pi}{\alpha\delta} H^{-\frac{1}{2}} e^{1-\delta} + e - \frac{(1-\tau)w}{1-\theta} H - \frac{A}{1-\theta} = 0.$$

Differentiating Z with respect to H yields

$$\begin{aligned} \frac{dZ}{dH} &= -\frac{1 + \gamma + \beta\pi}{2\alpha\delta} H^{-\frac{3}{2}} e^{1-\delta} + \frac{1 + \gamma + \beta\pi}{\alpha\delta} H^{\frac{1}{2}} e^{-\delta} \frac{de}{dH} + \frac{de}{dH} - \frac{(1-\tau)w}{1-\theta} \\ &= \left[\underbrace{1 + \frac{(1 + \gamma + \beta\pi)(1-\delta)}{\alpha\delta\sqrt{H}e^\delta}}_{+} \right] \frac{\partial e}{\partial H} - \frac{1}{H} \left[\underbrace{\frac{(1 + \gamma + \beta\pi)e^{1-\delta}}{2\alpha\delta\sqrt{H}} + \frac{(1-\tau)wH}{1-\theta}}_{+} \right]. \end{aligned} \quad (\text{A38})$$

Inserting Eq. (28) into Eq. (30) derives e as a function of H , as below:

$$e = \left[\sqrt{H} - \left(\frac{\alpha\rho}{\beta\pi\eta} \sqrt{H} \right)^{\frac{\rho}{2-\rho}} \right]^{\frac{1}{\delta}}. \quad (\text{A39})$$

Differentiating e in (A39) with respect to H yields and then using (A39) obtains

$$\begin{aligned} \frac{\partial e}{\partial H} &= \frac{1}{\delta} \left[\sqrt{H} - \left(\frac{\alpha\rho}{\beta\pi\eta} \sqrt{H} \right)^{\frac{\rho}{2-\rho}} \right]^{\frac{1}{\delta}-1} \left[\frac{1}{2\sqrt{H}} - \left(\frac{\alpha\rho}{\beta\pi\eta} \right)^{\frac{\rho}{2-\rho}} \frac{\rho}{2-\rho} (\sqrt{H})^{\frac{\rho}{2-\rho}-1} \frac{1}{2\sqrt{H}} \right] \\ &= \frac{e^{1-\delta}}{2\delta\sqrt{H}} \left[1 - \frac{\rho}{2-\rho} \left(\frac{\alpha\rho\sqrt{H}}{\beta\pi\eta} \right)^{\frac{\rho}{2-\rho}} \frac{1}{\sqrt{H}} \right] \\ &= \frac{e}{\delta(2-\rho)\sqrt{H}} \left(\frac{1-\rho}{e^\delta} + \frac{\rho}{2\sqrt{H}} \right) > 0. \end{aligned} \quad (\text{A40})$$

Combining (A38) with (A40) suggests that $\frac{dZ}{dH}$ has an ambiguous sign, which means that there may exist multiple solutions to H .

Finally, we investigate the local stability around the steady state equilibrium H . It follows from Eq. (A33) that

$$\frac{dd_t}{dH_t} = \frac{1}{2-\rho} \left(\frac{\alpha\rho}{\beta\pi\eta} \sqrt{H_t} \right)^{\frac{1}{2-\rho}-1} \frac{\alpha\rho}{\beta\pi\eta} \frac{1}{2\sqrt{H_t}} = \frac{d_t}{2(2-\rho)H_t}. \quad (\text{A41})$$

The total differentiation of Eq. (A34) with respect to H_t and e_t yields

$$\begin{aligned} & \frac{1+\gamma+\beta\pi}{\alpha\delta} \left[\frac{(1-\delta)e_t^{-\delta}}{\sqrt{H_t}} de_t - \frac{e_t^{1-\delta}}{2H_t^{\frac{3}{2}}} dH_t \right] + de_t = \frac{(1-\tau)w}{1-\theta} dH_t \\ \Leftrightarrow & \left[\frac{1+\gamma+\beta\pi}{\alpha\delta\sqrt{H_t}e_t^\delta} (1-\delta) + 1 \right] de_t = \frac{1}{H_t} \left[\frac{1+\gamma+\beta\pi}{2\alpha\delta\sqrt{H_t}} e_t^{1-\delta} + \frac{(1-\tau)wH_t}{1-\theta} \right] dH_t \\ \Leftrightarrow & \left[\delta + (1-\delta) \frac{(1-\tau)wH_t + A}{(1-\theta)e_t} \right] \frac{de_t}{dH_t} = \frac{3(1-\tau)wH_t + A - (1-\theta)e_t}{2(1-\theta)H_t} \quad (\text{A42}) \\ \Leftrightarrow & \frac{de_t}{dH_t} = \frac{e_t}{2H_t} \frac{3(1-\tau)wH_t + A - (1-\theta)e_t}{(1-\delta)[(1-\tau)wH_t + A] + (1-\theta)\delta e_t}. \end{aligned}$$

Differentiating H_{t+1} with respect to H_t and using the above two expressions obtains

$$\begin{aligned} \frac{dH_{t+1}}{dH_t} &= \frac{d_t^p + e_t^\delta}{2\sqrt{H_t}} + \sqrt{H_t} \left(\rho d_t^{p-1} \frac{dd_t}{dH_t} + \delta e_t^{\delta-1} \frac{de_t}{dH_t} \right) \\ &= \frac{H_{t+1}}{2H_t} + \frac{1}{2\sqrt{H_t}} \left[\frac{\rho d_t^p}{2-\rho} + \delta e_t^\delta \frac{3(1-\tau)wH_t + A - (1-\theta)e_t}{(1-\delta)[(1-\tau)wH_t + A] + (1-\theta)\delta e_t} \right] \\ &= \frac{H_{t+1}}{(2-\rho)H_t} + \frac{e_t^\delta}{2\sqrt{H_t}} \left[\delta \frac{3(1-\tau)wH_t + A - (1-\theta)e_t}{(1-\delta)[(1-\tau)wH_t + A] + (1-\theta)\delta e_t} - \frac{\rho}{2-\rho} \right]. \quad (\text{A43}) \end{aligned}$$

The dynamic system is locally stable around the steady state if and only if $\left. \frac{dH_{t+1}}{dH_t} \right|_H < 1$, which can be rearranged as

$$\begin{aligned} & \frac{H}{(2-\rho)H} + \frac{e^\delta}{2\sqrt{H}} \left[\delta \frac{3(1-\tau)wH + A - (1-\theta)e}{(1-\delta)[(1-\tau)wH + A] + (1-\theta)\delta e} - \frac{\rho}{2-\rho} \right] < 1, \\ \Leftrightarrow & 1 + \frac{e^\delta}{2\sqrt{H}} \left[\delta(2-\rho) \frac{3(1-\tau)wH + A - (1-\theta)e}{(1-\delta)[(1-\tau)wH + A] + (1-\theta)\delta e} - \rho \right] < 2-\rho, \quad (\text{A44}) \end{aligned}$$

which can be rearranged as condition (31).

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Declarations

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