An Envelope-Based Operational Scheme for Distribution Networks: From Close-to-Real-Time Envelope Adjustment to Market-Based Reallocation

***Abstract*—In this paper, we propose a static security management scheme for distribution networks that aims to improve the use of the flexibility available in the network and increase users' access to power. Prior to operation, all users have a bilateral agreement with the distribution system operator (DSO) defining their conditions of access to the network. This agreement defines a maximum range of exchanged power with the network, which we call a contractual envelope (CE). It comprises a firm part that is guaranteed and a flexible part that may be reduced close to real time. An operating envelope (OE) is the power range within which a user is required to operate over a given period. It is computed by the DSO close to real time and always contains the firm part of the CE. The OE allows the user to operate freely within it without compromising the static security of the network. The scheme differentiates users whose power can be reliably predicted by the DSO, the so-called predictable users, from those whose power cannot, referred to as unpredictable users. This distinction yields OEs that allocate network capacity more efficiently than worst-case approaches, i.e., approaches that secure the network against every possible power consumption/injection of every user. The OEs' computation problem is formulated as a finite sequence of MILPs. Because the resulting OEs may nevertheless not match users' needs, the operational scheme comes with a secondary market that allows for the exchange of OE bounds among users. This secondary market is cleared close to real time to maximize social welfare while preserving static security for every power consumption/injection within the OEs. The market-clearing problem is formulated as a single MILP.**

I. Introduction

The growing penetration of distributed energy resources (DERs), such as photovoltaic (PV) panels and batteries, together with the electrification of transport, heating, and industrial processes, is pushing distribution networks close to their physical limits. Distribution system operators (DSOs) are consequently unable to accommodate all the requests for new or increased grid connections they receive. They are stepping up investment in network reinforcement, which includes the installation or upgrade of substations [1], but such work remains capital-intensive and slow, constrained by lengthy permitting and construction lead times [2]. Network hosting capacity therefore cannot grow at the required pace. Network access is thus becoming a scarce resource, and users increasingly face connection delays [3]. In response, DSOs have introduced flexible connection contracts. Under these contracts, the consumption or injection of a user may be limited close-to-real-time whenever network constraints would otherwise be violated. Operating such contracts increases the complexity of distribution network operation, as it compels DSOs to take close-to-real-time control actions to maintain network assets within their acceptable operating ranges.

Operating envelopes (OEs) offer a practical way to allocate network capacity while preserving operational security. An OE is a power range, delimited by an upper and a lower bound, that the DSO assigns to a user and within which the latter may freely withdraw or inject power without compromising the network's static security [4]. Rather than imposing strict set points, the DSO thus constrains only this range, which it can adapt dynamically over time. It also preserves confidentiality, as the DSO grants the range without accessing the internal costs or dispatch decisions of the user, which the latter optimizes privately within that range [5].

OEs have been applied in different contexts. Among others, they have been leveraged for guaranteeing fair network access within local energy markets [6], and for demand response [5, 7, 8]. OEs have also been proposed to enhance coordination between transmission and distribution system operators, as detailed in [9–11]. These works, however, treat the envelope as a single quantity, allocated by the DSO and used by the grid user as granted over time. Because DER behavior is uncertain, the DSO must size envelopes so that security holds for every operating point a user might adopt within its OE, which leads to conservative allocations, i.e., tight OEs. Market-based reallocation has been proposed to make better use of these OEs, by letting users exchange them according to their needs while preserving the same security guarantee. The secondary congestion option platform exchange lets a curtailed producer draw on local flexibility to relieve injection congestion [12], and SecuLEx formalizes the same principle using OEs, turning envelope bounds into assets traded on a continuously cleared market [13]. In [13], the envelope granted to a user remains a single quantity, sized to hold for every operating point of every user and allocated according to an egalitarian criterion.

In this paper, the OEs are computed differently. Prior to operation, a *contractual envelope* (CE) is agreed upon between the DSO and the user, combining two complementary parts: (i) a *firm envelope*, i.e., a range that guarantees the user a baseline access to grid power, and (ii) a *flexible extension*, which the DSO may reduce prior to each period according to production, consumption, and network conditions. The OE of each user is computed by the DSO through the adjustment of the flexible extensions. Moreover, security is not assessed for every operating point of every user, but accounts for the predictability of users. Users whose power can be reliably predicted by the DSO are represented by a predicted power consumption/injection value within their OE, and security is ensured for this value only. Security assessment is thus made conditional for predictable users. By contrast, for users whose

power cannot be predicted, network capacity must be reserved over their whole OE.

The operational scheme proceeds in two sequential stages. In close-to-real-time, the DSO adjusts the flexible extensions to grant users as much operating range as possible while preserving the aforementioned conditional security guarantee and following regulatory priority rules. The mismatch between resulting envelopes and users' needs motivates a second stage. This second stage consists in a secondary market on which users trade envelope bounds against financial compensation. A market operator validates these trades only if the resulting OEs satisfy the same security requirements as those of the initial flexible extension adjustment. The market is secondary in that users exchange among themselves the range already granted to them, rather than the DSO procuring it from them as in a local primary flexibility market, where it purchases flexibility and bears the associated cost [14].

The contribution of this paper is therefore threefold. First, the OE granted to a user is delimited by a contractual object, the CE, decomposed into a firm envelope, matching the firm capacity of a flexible connection agreement, and a flexible extension. Second, the security of the OEs is made conditional on the information related to the predictability of grid users. Predictable users are represented in the adjustment problem by a predicted power consumption/injection, whereas the security constraints are enforced over the entire envelope only for unpredictable users. Third, the flexible extension adjustment problem, i.e., the OE computational problem, is combined with a market-based reallocation of the kind proposed in [13]. The market reallocation is here organized as a single pool auction per period and settled pay-as-clear. This market ensures that the security requirements of the initial flexible extension allocation are still respected.

The remainder of this paper is organized as follows. In Section II, an overview of the operational scheme through a running example is provided, from the initial close-to-real-time flexible extension adjustment to the market-based reallocation mechanism of OE bounds. Section III formalizes the envelope adjustment problem, while Section IV formalizes the market-based envelope reallocation mechanism. Section V discusses the limitations of the proposed operational scheme and Section VI concludes this work.

II. An overview of the envelope-based operational scheme

This section presents the envelope-based network operational scheme. In Section II-A, contractual and operating envelopes are formally defined and applied on an illustrative example, while Section II-B formalizes the close-to-real-time flexible envelope adjustment problem. Once OEs have been allocated, a market opens in which users bid to release part of their OE, or to extend it, against financial compensation. The market-clearing problem is formalized in Section II-C.

A. An envelope-based distribution network operation

Network model

Let us consider a balanced three-phase distribution network serving a set $\mathcal{U} \subseteq \mathbb{N}$ of N grid users. The time horizon is discretised into T periods, indexed by $t \in \{0, \dots, T-1\} = \mathcal{T}$. For each user $u \in \mathcal{U}$, we denote by $P_u \in \mathbb{R}$ the effective net power exchanged by user u with the network during a given time period. This power is assumed to be constant over every period t. By convention, positive values correspond to power withdrawal (consumption) and negative values to power injection (production). Only active power is considered.

Contractual and operating envelope definitions

A *contractual envelope* (CE) is the power range agreed upon between the DSO and a grid user, defining the maximum level of power access granted to that user. It is a contractual object consisting of two components, a firm part and a flexible part, hereafter referred to as *firm envelope* and *maximum flexible extension*, respectively.

The *firm envelope* imposed on user u is a power range $[P_u^{f,\min},\ P_u^{f,\max}] \subset \mathbb{R}$, guaranteed to the user at every period.[1]

The *maximum flexible extension* of user u is a pair of non-negative quantities $(\bar{\delta}_u^{\min},\ \bar{\delta}_u^{\max}) \in \mathbb{R}^+ \times \mathbb{R}^+$, stating the largest downward and upward extensions of the firm envelope the DSO may ever grant. As with the firm envelope, the maximum flexible extension is stipulated in the connection contract. The CE of user u therefore reads

$$\bar{\Delta}_u = \left[P_u^{f,\min} - \bar{\delta}_u^{\min},\ P_u^{f,\max} + \bar{\delta}_u^{\max}\right]. \tag{1}$$

An *operating envelope* (OE) of user u at a given period is a power range $\Delta_u \subset \mathbb{R}$ within which the user can operate, alongside the other users within theirs, without compromising the static security of the network. The user is required to operate within its OE during the given period, i.e., $P_u \in \Delta_u$. Unlike the CE, which only bounds the range the user may ever be granted, the OE thus carries a security guarantee.

The DSO computes the OEs by granting, at each period, a *flexible extension* $(\delta_u^{\min},\ \delta_u^{\max}) \in \mathbb{R}^+ \times \mathbb{R}^+$ of the firm envelope of each user, within the maximum values stated in the contract,

$$0 \leq \delta_u^{\min} \leq \bar{\delta}_u^{\min}, \qquad 0 \leq \delta_u^{\max} \leq \bar{\delta}_u^{\max}. \tag{2}$$

The flexible extensions are chosen such that, for every user, the range

$$\Delta_u = \left[\Delta_u^{\min},\ \Delta_u^{\max}\right] = \left[P_u^{f,\min} - \delta_u^{\min},\ P_u^{f,\max} + \delta_u^{\max}\right] \tag{3}$$

is an OE. By construction, $[P_u^{f,\min}, P_u^{f,\max}] \subseteq \Delta_u \subseteq \bar{\Delta}_u$. The flexible extensions and the OEs of all users are gathered in the matrices $\delta = (\delta^{\min}, \delta^{\max})$ and $\Delta = (\Delta^{\min}, \Delta^{\max})$ of $\mathbb{R}^{N \times 2}$, whose row u contains $(\delta_u^{\min}, \delta_u^{\max})$ and $(\Delta_u^{\min}, \Delta_u^{\max})$, respectively. Similarly, the maximum contractual flexible extensions on all users are gathered in $\bar{\delta}^{\min}$ and $\bar{\delta}^{\max}$.

Illustrative example

To ground the concepts introduced above, we introduce a running example that will be used throughout the paper. We consider three grid users, 1, 2 and 3, connected to a distribution network:

[1]The firm envelope is guaranteed by default, but it is not inalienable. A user may temporarily release part of it on a secondary market in exchange for financial compensation, as further detailed in Section IV.

- 1: a wind farm of 10 MW installed capacity, which can only inject power,
- 2: an aggregated residential load of approximately 500 houses, with a peak withdrawal of 3 MW,
- 3: a battery rated at 6 MW, which can both withdraw and inject power.

Their respective CEs are provided in Table I. The transformer connecting the distribution network to the rest of the grid has a nominal capacity of 10 MW, so the net power flowing through it must remain within the secure range $[-10, 10]$ MW at all periods. The distribution network of the illustrative example is provided in Figure 1.

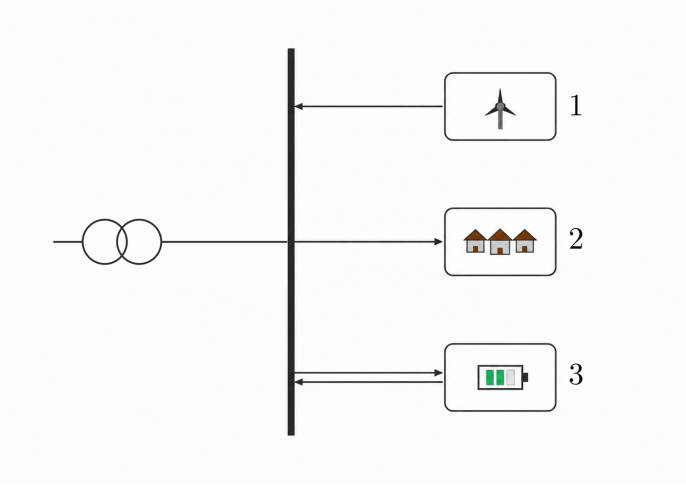

Fig. 1: Distribution network of the illustrative example: a wind farm (user 1), an aggregated residential load (user 2) and a battery (user 3) share a common bus, connected to the upstream grid through the substation transformer. Arrows indicate whether the CE of each user covers withdrawal, injection, or both.

	$[P_u^{f,\min},\ P_u^{f,\max}]$	$\bar{\delta}_u^{\min}$	$\bar{\delta}_u^{\max}$
1	$[-4,\ 0]$	6	0
2	$[\ 0,\ 3]$	0	0
3	$[-2,\ 2]$	4	4

TABLE I: Firm envelopes and maximum flexible extensions defining the CE of each user of the illustrative example (in MW).

Prior to a given period, the DSO adjusts the flexible extension of every grid user. The envelope adjustment process is further detailed in Section II-B. The resulting flexible extensions must keep the transformer within its secure operating range, i.e., $[-10, 10]$ MW. The secure operating range of a network asset is hereafter called the security envelope (SE). Table II gives the flexible extension values $\delta_u^{\min}, \delta_u^{\max}$ for the given period, and the resulting OE Δ_u.

During this period t, the effective power consumption/injection $P_1 = -8$ MW, $P_2 = 2$ MW, and $P_3 = -4$ MW. Neglecting losses, the net power seen by the transformer is $(-8)+2+(-4) = (-10)$ MW, which lies within the range $[-10, 10]$ MW. The transformer thus operates securely.

	$\delta_u^{\min}$	$\delta_u^{\max}$	Δ_u	P_u
1	6	0	$[-10,\ 0]$	-8
2	0	0	$[\ 0,\ 3]$	2
3	2	3	$[-4,\ 5]$	-4

TABLE II: Flexible extensions $\delta_u^{\min}, \delta_u^{\max}$, resulting OEs Δ_u, and power consumption/injection P_u of each user at the given period t (MW).

B. Close-to-real-time envelope adjustment

Prior to each period, the DSO adjusts the flexible extensions of grid users, such that network assets are operated securely. Users are not all represented in the same way when the DSO computes the adjustment, and two categories are distinguished accordingly. *Predictable users* are those whose power can be predicted with a sufficiently high degree of confidence given knowledge of their OE, whereas *unpredictable users* are those whose power cannot.

The adjustment proceeds in two steps. Firstly, the DSO establishes how each user is to be represented in the adjustment problem. A predictable user is represented by its power consumption/injection as a function of its OE. An unpredictable user is represented by its whole envelope, as it is assumed that this type of users might reach any point within their OEs. Secondly, the DSO tightens the flexible extensions until the resulting OEs are consistent with security requirements. This tightening is performed while keeping the OEs granted to users as wide as possible, and is subject to the security constraints and to regulatory requirements such as a priority ordering over users. The adjustment is formalized as an optimization problem in Section III.

Illustrative example

We revisit the running example to illustrate how the OEs Δ_1, Δ_2, and Δ_3 are computed prior to the upcoming period, such that the power flowing through the transformer stays within $[-10, 10]$ MW.

In the first step, the DSO predicts the power consumption or injection of each predictable user as a function of its OE. The wind farm (user 1) is predicted to inject 10 MW during the upcoming period if its OE accommodates it, while the load (user 2) is predicted to withdraw 1 MW. The battery (user 3) is unpredictable, so no forecast can be made for it. Its power is instead taken over the whole envelope under consideration. With the extensions at their contractual maxima, this envelope is the contractual envelope $\bar{\Delta}_3 = [-6, 6]$ MW, and security is verified against the bounds of that range. The net power flowing through the transformer can therefore take any value in $[(-10)+1+(-6),\ (-10)+1+6] = [-15, -3]$ MW, whose lower end falls outside the transformer SE $[-10, 10]$ MW.

In the second step, the DSO tightens the flexible extensions until network security is ensured. The DSO minimizes the total tightening of the flexible extensions required to restore network security. This objective determines the overall amount of flexibility that must be withdrawn, but it does not identify the users that should bear the corresponding reductions. That choice does not fall to the DSO. It follows from a regula-

tory priority ordering, which specifies how the adjustment is distributed among users. In this paper, we adopt a last-in-first-out (LIFO) rule, under which users are ranked according to their connection date. The most recently connected users have their envelopes modulated first, while earlier users are modulated only once the flexibility of all more recent users has been exhausted. In the illustrative example, the battery (user 3) is assumed to be the most recently connected user and is therefore modulated before the wind farm (user 1). The latter is tightened only once the battery's downward flexible extension has been fully exhausted.

Accordingly, the lower flexible bound of the battery $\delta_3^{\min}$ is first reduced from its maximum value of 4 to $0\,\text{MW}$, removing $4\,\text{MW}$ of injection and yielding the OE $[-2, 6]\,\text{MW}$. This alone is insufficient, since the possible exchange power range at the transformer is $[(-10) + 1 + (-2), (-10) + 1 + 6] = [-11, -3]\,\text{MW}$. This range still has a lower end that falls outside the transformer SE. The wind farm is therefore modulated next. Its lower flexible bound $\delta_1^{\min}$ is reduced from 6 to $5\,\text{MW}$, yielding the OE $[-9, 0]\,\text{MW}$ for user 1. As a predictable user, user 1 is represented by a single power value in the adjustment problem. Its forecast injection of $10\,\text{MW}$ lies outside this OE, so that the prediction returns the lower bound and user 1 is expected to inject $9\,\text{MW}$ in the upcoming period. The OE of u_2 remains at its firm value $[0, 3]\,\text{MW}$.

Once the flexible extensions are modulated, the net power through the transformer is guaranteed to remain within $[(-9) + 1 + (-2), (-9) + 1 + 6] = [-10, -2]\,\text{MW}$, which is compatible with the secure operating interval of the transformer $[-10, 10]\,\text{MW}$. The resulting envelopes are summarized in Table III. Note that the upward flexible bounds are left at their maximum values. Since only the injection side is reaching the SE bounds, tightening them would bring no security benefit while needlessly shrinking the users' operating range.

	$\delta_u^{\min}$	$\delta_u^{\max}$	Δ_u	P_u
1	5	0	$[-9,\ 0]$	-9
2	0	0	$[\ 0,\ 3]$	1
3	0	4	$[-2,\ 6]$	6

TABLE III: Flexible extensions $\delta_u^{\min}, \delta_u^{\max}$, resulting operating envelopes Δ_u, and power consumption/injection P_u of each user for the upcoming period (MW).

C. Market-based envelope reallocation

The flexible extension adjustment of Section II-B distributes the available capacity according to the regulatory priority ordering, and not according to what each user needs for the coming period. Some users may not exploit the range granted to them, while others would need a wider one. To address this, we propose a secondary market on which users trade envelope bounds against financial compensation. The market is organized as a pool, cleared once per period. After the DSO has notified the envelopes computed for a given period, users submit their orders to the market operator. At gate closure, the operator clears the whole order book in a single auction, maximizing social welfare while preserving network security, and notifies the resulting envelopes before the period begins. This mechanism is illustrated hereafter and formalized in Section IV.

Illustrative example

We revisit the running example to illustrate the market mechanism. After the envelope adjustment, the downward flexible extension of the battery (user 3) has been reduced to zero. Its injection capability is thus limited by the negative bound of the firm part of its CE. The wind farm (user 1) has retained part of its downward extension and is limited to 9 rather than $10\,\text{MW}$. During the upcoming period, the battery does not expect to use its injection capacity. It is therefore willing to raise its lower bound by up to $2\,\text{MW}$, provided it receives at least 20 €/MW. The wind farm, in contrast, wants to lower its own bound by $1\,\text{MW}$ so as to inject its full forecast production, and is willing to pay up to 50 €/MW for it. The resulting order book is given in Table IV.

User	Type	Side	Quantity (MW)	Price (€/MW)
1	buy	min	1	50
3	sell	min	2	20

TABLE IV: Order book submitted to the market for the upcoming period.

The two orders are matched, since the price offered by the wind farm exceeds the price required by the battery. User 1 bids for $1\,\text{MW}$ whereas user 3 offers $2\,\text{MW}$. The cleared quantity is therefore limited to $1\,\text{MW}$ by the buy order, and only half of the sell order is accepted. The battery is thus the marginal participant of the auction, being the only one whose order is partially accepted, and under a pay-as-clear settlement it is its limit price that sets the clearing price, namely 20 €/MW. Clearing $1\,\text{MW}$ at that price lowers the bound of user 1 to $-10\,\text{MW}$ and raises that of user 3 to $-1\,\text{MW}$, against a payment of $1\,\text{MW} \times 20$ €/MW $= 20$ € from user 1 to user 3.

The megawatt gained by user 1 is exactly the one released by user 3, so the sum of the lower bounds is unchanged and the largest injection the transformer may see remains $-10\,\text{MW}$, which sits within its SE. The range released by user 3 belongs to its firm envelope, its downward extension having already been reduced to zero. The firm envelope is thus guaranteed but not inalienable. A user may give it up voluntarily, for a single period and against compensation. Conversely, the range acquired by user 1 brings its envelope back to the widest one its connection contract allows. Table V summarizes the envelopes before and after clearing and the operational scheme is illustrated in Figure 2.

III. Computation of flexible envelope extensions

This section formalizes the close-to-real-time envelope adjustment as an optimization problem, hereafter called the *flexible extension adjustment problem*, which the DSO solves prior to each period to determine the flexible extensions

	$[P_u'^{f,\min}, P_u'^{f,\max}]$	$\delta_u'^{\min}$	$\delta_u'^{\max}$	Δ_u'	P_u'
1	$[-4,\ 0]$	6	0	$[-10,\ 0]$	-10
2	$[\ 0,\ 3]$	0	0	$[\ 0,\ 3]$	1
3	$[-1,\ 2]$	0	4	$[-1,\ 6]$	6

TABLE V: Updated firm envelopes, flexible extensions, operating envelopes, and operating points for the upcoming period after market clearing (MW).

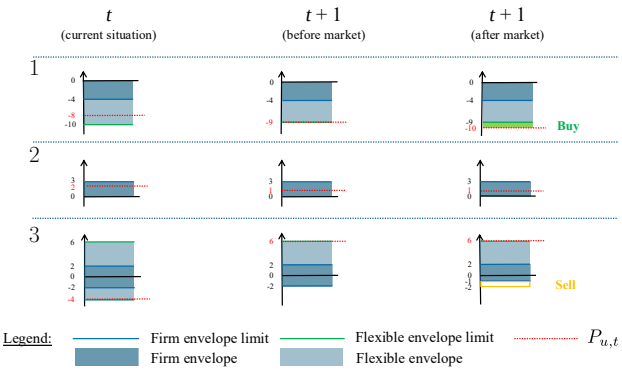

Fig. 2: Evolution of the operating envelopes of the three users across the operational scheme.

to be applied to each user. Its solution yields OEs carrying the security guarantee of Section II-A. Consistent with the running example, the formulation is restricted to active power, and the security constraints considered are the thermal capacities of the network edges. Each edge corresponds to a physical asset, namely the substation transformer, a line, or a cable. Section III-A presents the model of the radial network and the resulting security conditions. Section III-B states the flexible extension adjustment problem in its generic form, Section III-C instantiates this optimization problem, Section III-D derives a tractable counterpart and Section III-E applies it to the running example.

A. Radial network and security envelopes

The distribution network is modeled as a directed tree $\mathfrak{G} = (\mathcal{N}, \mathcal{E})$ rooted at the substation node 0, which connects to the upstream grid. Edges are directed away from node 0. Each user is connected to a single node of the network, and $\mathcal{C}(n) \subseteq \mathcal{U}$ denotes the set of users connected to node n. In this section, we consider a fixed time period for which the DSO must derive OEs. All physical quantities correspond to that period. For each node n, the net active power exchanged with the grid is $P_n = \sum_{u \in \mathcal{C}(n)} P_u$, following the sign convention of Section II, i.e., positive for withdrawal, and negative for injection. The active power flowing from node n to node m along edge $(n, m) \in \mathcal{E}$ is denoted $F_{n,m} \in \mathbb{R}$.

The network is assumed lossless, so that the active power entering a line equals the active power leaving it and a single flow variable can be associated with each line. The power

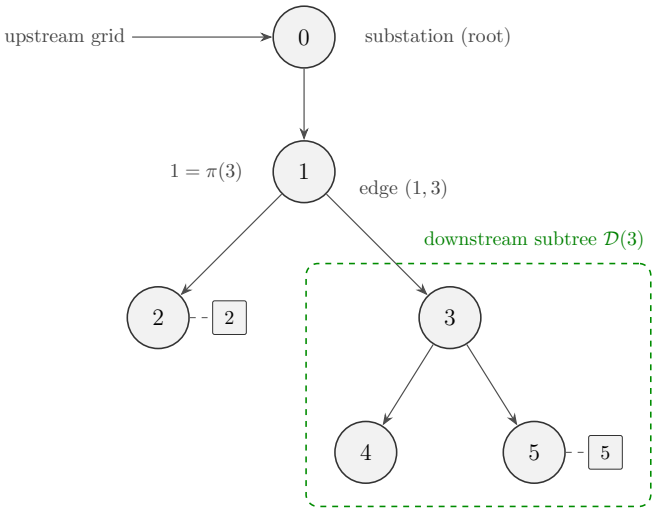

Fig. 3: Radial distribution network as a directed tree rooted at the substation node 0. Edges point away from the root, giving each node $n \neq 0$ a unique parent $\pi(n)$. The dashed region is the downstream subtree $\mathcal{D}(n)$, and users 2 and 5 are attached to nodes 2 and 5, respectively.

balance at a node $n \neq 0$ then reads

$$F_{\pi(n),n} = P_n + \sum_{k\,:\,(n,k)\in\mathcal{E}} F_{n,k}, \tag{4}$$

where $\pi(n)$ is the parent of node n in the tree. Let $\mathcal{D}(n) \subseteq \mathcal{N}$ denote the set of nodes of the subtree rooted at node n, node n included. The set of users connected downstream of node n is then

$$\mathcal{U}(n) = \bigcup_{n' \in \mathcal{D}(n)} \mathcal{C}(n'). \tag{5}$$

Removing an edge (m, n) disconnects the subtree $\mathcal{D}(n)$ from the rest of the network. Summing the nodal balance (4) over the nodes of $\mathcal{D}(n)$, the flows of the edges internal to the subtree cancel, so that the flow through (m, n) is the sum of the power consumption/injection of the users connected to that subtree:

$$F_{m,n} = \sum_{u \in \mathcal{U}(n)} P_u, \quad \forall (m, n) \in \mathcal{E}. \tag{6}$$

An example of distribution network is illustrated in Figure 3. Each edge $(m, n) \in \mathcal{E}$ is secure whenever the flow does not exceed its thermal rating $\overline{F}_{m,n} \in \mathbb{R}^+$. Each edge is therefore associated with a SE, written as $\Sigma_{m,n}$, such that:

$$\Sigma_{m,n} \triangleq \left[-\overline{F}_{m,n},\ \overline{F}_{m,n}\right], \qquad \forall (m, n) \in \mathcal{E}. \tag{7}$$

The network is secure if

$$F_{m,n} \in \Sigma_{m,n}, \qquad \forall (m, n) \in \mathcal{E}. \tag{8}$$

B. A generic flexible extension adjustment problem

The flexible extension adjustment illustrated in Section II-B is formalized as an optimization problem. Prior to each period,

the DSO computes the flexible extensions granted to the users. These extensions are used to define the OE of each user.

Objective function

The DSO is assumed to maximise an objective function $\mathrm{Flex}(\Delta)$ that quantifies the flexibility provided to grid users. This objective function measures the operating range granted to the users through their OEs. As a reminder, we denote $\Delta = (\Delta^{\min}, \Delta^{\max})$ as the resulting OEs, with $\Delta^{\min} = (\Delta_1^{\min}, \dots, \Delta_N^{\min})$ and $\Delta^{\max} = (\Delta_1^{\max}, \dots, \Delta_N^{\max})$, $\Delta_u^{\min}$ and $\Delta_u^{\max}$ being the lower and upper bounds of the OE Δ_u of user u. The exact form of the objective function is left generic at this stage. It may, for instance, be the total width of the OEs, or weight the users differently.

Security and power flow constraints

Let $P = (P_1, \dots, P_N)$ denote the vector of the power consumption/injection of the users, and F the vector of the flows $F_{m,n}$ on the edges $(m,n) \in \mathcal{E}$ of the network. The *power flow constraint*

$$\mathrm{PowerFlow}\big(P, F\big) = 0 \tag{9}$$

is verified if vectors P and F respect the physical laws of the network. The *security constraint*

$$\mathrm{Security}\big(P, F\big) = 0 \tag{10}$$

holds if the network is secure, i.e., when every edge is operated within its SE.

Predictable and unpredictable users

The power values that a user may realise within its OE are described by a *prediction constraint*

$$\mathrm{Pred}\big(P,\, \Delta\big) = 0. \tag{11}$$

This constraint is respected when P is a possible behavior of users when they are given the OEs Δ. For a *predictable user*, we assume there is a single power value for which the constraint is zero, for any given OE. For an *unpredictable user*, we assume all values of power within the OE respect the constraint.

Regulatory requirements

When the network cannot accommodate every flexible extension at its contractual maximum, several combinations of reduced extensions may satisfy the constraints. However, extensions cannot be chosen freely as they must respect regulatory conditions. These conditions define what possible adjustment schemes are allowed. It will later be based on a preferred order settled by regulation. To express this requirement, we define the *regulatory rule compliance constraint*

$$\mathrm{Reg}(\Delta) = 0. \tag{12}$$

An adjustment complies with the regulatory framework if $\mathrm{Reg}(\Delta) = 0$.

Optimization problem

Prior to a given period, the DSO solves the following optimization problem, whose solution gives the flexible extensions to be applied during this period, subject to the modifications the market may subsequently bring:

$$\begin{aligned}
&\max_{\delta} \ \mathrm{Flex}\big(\Delta\big) && (\mathcal{P}_1)\\
&\text{s.t. } (a)\ 0 \le \delta^{\min} \le \bar{\delta}^{\min},\\
&\qquad\quad 0 \le \delta^{\max} \le \bar{\delta}^{\max},\\
&\qquad (b)\ \mathrm{Security}\big(P, F\big) = 0,\\
&\qquad\qquad \forall\, (P, F) :\\
&\qquad\qquad\quad (c)\ \mathrm{PowerFlow}\big(P, F\big) = 0,\\
&\qquad\qquad\quad (d)\ \mathrm{Pred}\big(P, \Delta\big) = 0,\\
&\qquad (e)\ \mathrm{Reg}\big(\Delta\big) = 0.
\end{aligned}$$

The objective function Flex measures the operating range granted to the users through their OEs. Condition (a) keeps each flexible extension within its contractual bounds. Condition (b) requires the network to be secure for every power consumption/injection P and flows F that respect the physics of the network (c) and correspond to a possible future behaviour of the users given their OEs (d). Condition (e) imposes the regulatory rules to be respected.

C. A tractable flexible extension adjustment problem

Problem $\mathcal{P}_1$ cannot be passed to a solver as its functions are left generic. The following specifications instantiate every function of $\mathcal{P}_1$.

1) *Objective.* The objective function is the total width of the OEs. The width of the OE of user u is denoted $\omega(\Delta_u) = \Delta_u^{\max} - \Delta_u^{\min}$, so that $\mathrm{Flex}(\Delta) = \sum_{u \in \mathcal{U}} \omega(\Delta_u)$.
2) *Security.* $\mathrm{Security}\big(P, F\big) = 0$ if and only if (8) holds on every edge. The firm envelopes are assumed to be jointly secure, so that zero flexible extensions satisfy condition (b).
3) *Power flow.* The network is the radial and lossless network of Section III-A, and PowerFlow enforces the nodal balance (4).
4) *Prediction.* The prediction constraint is respected independently by individual users. Individual users are either classified as predictable or unpredictable. Let $\mathcal{U}^{\mathrm{pred}} \subseteq \mathcal{U}$ and $\mathcal{U}^{\mathrm{unpred}} \subseteq \mathcal{U}$ denote the sets of predictable and unpredictable users, respectively, with $\mathcal{U}^{\mathrm{pred}} \cup \mathcal{U}^{\mathrm{unpred}} = \mathcal{U}$ and $\mathcal{U}^{\mathrm{pred}} \cap \mathcal{U}^{\mathrm{unpred}} = \emptyset$. For unpredictable users, the power consumption/injection may take any value within its OE, i.e., $P_u \in \Delta_u, \forall u \in \mathcal{U}^{\mathrm{unpred}}$. For predictable users, we assume to have access to a forecast $\hat{P}_u \in \mathbb{R}$, which is clipped within the OE, i.e.,
$$P_u = \begin{cases} \Delta_u^{\min}, & \forall u \in \mathcal{U}^{\mathrm{pred}} :\ \hat{P}_u < \Delta_u^{\min},\\ \Delta_u^{\max}, & \forall u \in \mathcal{U}^{\mathrm{pred}} :\ \hat{P}_u > \Delta_u^{\max},\\ \hat{P}_u, & \forall u \in \mathcal{U}^{\mathrm{pred}} :\ \text{otherwise}, \end{cases} \tag{13}$$
5) *Regulatory rule.* The regulatory rule is the last-in-first-out (LIFO) rule announced in Section II-B. Users are assumed to be sorted by rank, such that $u < u'$ means that user u has a higher priority than user u'. This rule is formalised as a lexicographic condition. For two OEs'

matrices Δ and $\tilde{\Delta}$, $\tilde{\Delta}$ is lexicographically greater than or equal to Δ, written $\Delta \preceq_{\text{lex}} \tilde{\Delta}$, if both are equal for each user, or if $\tilde{\Delta}$ provides a larger envelope to the first user for which the envelopes differ, i.e.,

$$\begin{aligned} &\Delta \preceq_{\text{lex}} \tilde{\Delta} \text{ if and only if} \\ &\omega(\Delta_u) = \omega(\tilde{\Delta}_u) \ \ \forall u, \quad \text{or} \\ &\exists\, u: \ \omega(\Delta_{u'}) = \omega(\tilde{\Delta}_{u'}) \ \ \forall u' < u \\ &\qquad \text{and} \ \ \omega(\Delta_u) < \omega(\tilde{\Delta}_u). \end{aligned} \tag{14}$$

The OEs comply with the LIFO rule if no other feasible OEs are lexicographically greater, i.e., if they belong to

$$\mathcal{L} = \big\{\Delta \mid \forall \tilde{\Delta} \text{ that respect Eq (b) in } \mathcal{P}_1 : \ \tilde{\Delta} \preceq_{\text{lex}} \Delta\big\}. \tag{15}$$

The regulatory constraint $\text{Reg}(\Delta) = 0$ is respected if $\Delta \in \mathcal{L}$.

Under Specifications 1 to 5, the problem $\mathcal{P}_1$ writes as

$$\max_{\delta} \quad \sum_{u \in \mathcal{U}} \omega(\Delta_u) \tag{$\mathcal{P}_2$}$$

s.t. $\quad \forall u \in \mathcal{U}^{\text{unpred}}, \ \ \forall P_u \in \Delta_u :$

$$(a) \qquad \begin{aligned} &0 \le \delta^{\min} \le \bar{\delta}^{\min} \\ &0 \le \delta^{\max} \le \bar{\delta}^{\max} \end{aligned}$$

$$(b) \qquad F_{m,n} \in \Sigma_{m,n}, \qquad \forall (m,n) \in \mathcal{E},$$

$$(c) \qquad F_{m,n} = \sum_{u \in \mathcal{U}(n)} P_u, \quad \forall (m,n) \in \mathcal{E},$$

$$(d) \qquad \begin{aligned} &P_u = \Delta_u^{\min}, && \forall u \in \mathcal{U}^{\text{pred}} : \hat{P}_u < \Delta_u^{\min}, \\ &P_u = \Delta_u^{\max}, && \forall u \in \mathcal{U}^{\text{pred}} : \hat{P}_u > \Delta_u^{\max}, \\ &P_u = \hat{P}_u, && \forall u \in \mathcal{U}^{\text{pred}} : \hat{P}_u \in \Delta_u, \end{aligned}$$

$$(e) \qquad \Delta \in \mathcal{L}.$$

Condition (b) must hold for every power consumption/injection value an unpredictable user might inject/consume within its OE, which amounts to infinitely many constraints. Problem $\mathcal{P}_2$ thus cannot be passed to a solver as it stands. A tractable reformulation of this problem nevertheless exists and is presented in the rest of this section.

D. Tractable reformulation

The reformulation of $\mathcal{P}_2$ leads to a sequence of N MILPs. It rests on three observations, which address the robust security condition (b), the clipping in condition (d) and the lexicographic condition (e), respectively.

1) Robust security condition: Under Specification 3, condition (c) defines the flow of each edge as the sum of the powers of the downstream users. For given OEs, the powers of the predictable users are fixed by condition (d). Each flow is thus affine and increasing in the powers of the unpredictable users downstream. Since these users vary independently within their OEs, the flow of edge (m,n) ranges over the interval $[F_{m,n}^{\min}, F_{m,n}^{\max}]$, whose bounds are obtained when every unpredictable user downstream operates at the lower, respectively the upper, bound of its OE:

$$\begin{aligned} F_{m,n}^{\min} &= \sum_{u \in \mathcal{U}^{\text{pred}}(n)} P_u + \sum_{u \in \mathcal{U}^{\text{unpred}}(n)} \Delta_u^{\min}, \\ F_{m,n}^{\max} &= \sum_{u \in \mathcal{U}^{\text{pred}}(n)} P_u + \sum_{u \in \mathcal{U}^{\text{unpred}}(n)} \Delta_u^{\max}, \end{aligned} \tag{16}$$

where $\mathcal{U}^{\text{pred}}(n) = \mathcal{U}^{\text{pred}} \cap \mathcal{U}(n)$ and $\mathcal{U}^{\text{unpred}}(n) = \mathcal{U}^{\text{unpred}} \cap \mathcal{U}(n)$ denote the predictable and unpredictable users downstream of node n. If the SE is respected for the highest and for the lowest consumption/injection of unpredictable users, it is respected for any intermediate value if all other variables are fixed. The robust condition (b) of $\mathcal{P}_2$ is thus equivalent to requiring $F_{m,n}^{\min}, F_{m,n}^{\max} \in \Sigma_{m,n}$ for every edge (m,n). The infinitely many constraints indexed by the power consumption/injection of the unpredictable users are thus replaced by two constraints per edge.

2) Clipping of the predictable users: Condition (d) is implemented using two binary variables $b_u, c_u \in \{0,1\}$ per predictable user, indicating whether the lower or the upper bound of its OE is active, and a constant $M > 0$ that exceeds the width of each CE and the distance between each forecast and the bounds of the corresponding CE.

3) LIFO rule: The LIFO rule is implemented through solving a lexicographic optimization problem where the flexible extensions of each user are fixed iteratively.

Based on these three observations, the flexible envelope adjustment problem $\mathcal{P}_2$ is solved through a sequence of N tractable optimisation problems, one per user, taken in order of priority. At step $K = 1, \ldots, N$, the largest width that can be granted to user K without narrowing the OEs of the users of higher priority is obtained by solving

$$\max_{\delta} \quad \omega(\Delta_K) \tag{$\mathcal{P}_3^{(K)}$}$$

s.t.

$$(a) \qquad \begin{aligned} &0 \le \delta^{\min} \le \bar{\delta}^{\min} \\ &0 \le \delta^{\max} \le \bar{\delta}^{\max} \end{aligned}$$

$$(b) \qquad F_{m,n}^{\min}, F_{m,n}^{\max} \in \Sigma_{m,n}, \qquad \forall (m,n) \in \mathcal{E},$$

$$(c) \qquad \begin{aligned} F_{m,n}^{\min} &= \sum_{u \in \mathcal{U}^{\text{pred}}(n)} P_u + \sum_{u \in \mathcal{U}^{\text{unpred}}(n)} \Delta_u^{\min}, \\ F_{m,n}^{\max} &= \sum_{u \in \mathcal{U}^{\text{pred}}(n)} P_u + \sum_{u \in \mathcal{U}^{\text{unpred}}(n)} \Delta_u^{\max}, \\ &\qquad\qquad \forall (m,n) \in \mathcal{E}, \end{aligned}$$

$$(d) \qquad \begin{aligned} &\Delta_u^{\min} \le P_u \le \Delta_u^{\max}, && \forall u \in \mathcal{U}^{\text{pred}}, \\ &\hat{P}_u - M c_u \le P_u \le \hat{P}_u + M b_u, && \forall u \in \mathcal{U}^{\text{pred}}, \\ &P_u \le \Delta_u^{\min} + M(1 - b_u), && \forall u \in \mathcal{U}^{\text{pred}}, \\ &P_u \ge \Delta_u^{\max} - M(1 - c_u), && \forall u \in \mathcal{U}^{\text{pred}}, \\ &b_u,\, c_u \in \{0,1\}, && \forall u \in \mathcal{U}^{\text{pred}}, \end{aligned}$$

$$(e) \qquad \omega(\Delta_u) \ge \omega_u^{\star}, \qquad \forall u < K,$$

where $\omega_u^\star$ denotes the optimal value of $\mathcal{P}_3^{(u)}$. At the first step, $u = 1$, no user has a higher priority, and $\mathcal{P}_3^{(1)}$ grants user 1 the widest OE the network can accommodate. Since $\omega_{u'}^\star$ is the largest width attainable by user u' once the OEs of the users of higher priority are computed, each inequality holds with equality at every feasible point of the subsequent steps. After the N steps, the OEs of any optimal solution of $\mathcal{P}_3^{(N)}$ therefore belong to $\mathcal{L}$, and, since the objective of $\mathcal{P}_2$ takes the same value on every element of $\mathcal{L}$, they solve $\mathcal{P}_2$. Each problem of the sequence is a MILP with $2|\mathcal{U}^{\text{pred}}|$ binary variables.

E. Back to the illustrative example

The sequence of problems $\mathcal{P}_3^{(K)}$ is now applied to the example introduced in Section II. All users share a single bus, so that the tree reduces to the single edge $(0, 1)$ representing the substation transformer, with SE $\Sigma_{0,1} = [-10, 10]$ MW. Users 1 and 2 are predictable, with forecasts $\hat{P}_1 = -10$ MW and $\hat{P}_2 = 1$ MW, whereas user 3 is unpredictable. Users 1, 2 and 3 are assumed to have been connected in this order, so that user 1 has the highest priority and user 3 the lowest.

With the values of Table I, the lower bound of the OE of user 1 is $\Delta_1^{\min} = -4 - \delta_1^{\min} \geq -10$, since $\delta_1^{\min} \leq 6$. Its forecast $\hat{P}_1 = -10$ therefore never lies above this bound, and the clipped forecast (13) gives $P_1 = \Delta_1^{\min} = -4 - \delta_1^{\min}$. The forecast $\hat{P}_2 = 1$ of user 2 lies within its OE $\Delta_2 = [0, 3]$, so that $P_2 = 1$. The minimum and maximum flows through the transformer are therefore

$$\begin{aligned} F_{0,1}^{\min} &= -5 - \delta_1^{\min} - \delta_3^{\min}, \\ F_{0,1}^{\max} &= -1 - \delta_1^{\min} + \delta_3^{\max}. \end{aligned} \tag{17}$$

The upper flow never exceeds 3 MW and thus remains within the SE, whereas the lower flow remains within it if and only if

$$\delta_1^{\min} + \delta_3^{\min} \leq 5. \tag{18}$$

Problem $\mathcal{P}_3^{(1)}$ maximises the width $\omega(\Delta_1) = 4 + \delta_1^{\min}$ of the OE of user 1, which gives $\delta_1^{\min} = 5$ and $\omega_1^\star = 9$ MW. The OE of user 2 has no flexible extension, so that $\mathcal{P}_3^{(2)}$ returns $\omega_2^\star = 3$ MW. Problem $\mathcal{P}_3^{(3)}$ finally maximises the width $\omega(\Delta_3) = 4 + \delta_3^{\min} + \delta_3^{\max}$ of the OE of user 3 with $\omega(\Delta_1) \geq 9$, which imposes $\delta_3^{\min} = 0$ by (18) and gives $\delta_3^{\max} = 4$ and $\omega_3^\star = 8$ MW. The most recently connected user, i.e., user 3, is thus fully reduced on its injection side before the earlier user 1 is reduced by 1 MW. This recovers the OEs $\Delta_1 = [-9, 0]$, $\Delta_2 = [0, 3]$ and $\Delta_3 = [-2, 6]$ MW of Table III.

IV. Formalization of the Market-Based Reallocation Mechanism

The flexible extension adjustment problem of Section III guarantees that the OEs allocated by the DSO are compatible with the SEs of the network, but it does not ensure that these envelopes match the users' needs. Some users may not require the full range allocated to them, while others would value an extension of their OE. To address this, we formalize a market-based reallocation mechanism, first introduced in Section II-C. Organized after the DSO has solved the flexible extension adjustment problem, this market reallocates OEs' bounds between users while preserving security.

After the market clearing, the OE of user u, originally denoted $\Delta_u = [\Delta_u^{\min}, \Delta_u^{\max}]$, evolves into

$$\Delta'_u = [\Delta_u^{'\min}, \Delta_u^{'\max}]. \tag{19}$$

Note that what is exchanged on the market are downward and upward parts of OEs, irrespective of the decomposition into a firm and a flexible part. A user may therefore release range that belongs to its firm envelope, and not only the flexible extension granted for the period. The firm envelope is therefore guaranteed but not inalienable. It may be given up voluntarily, for a single period and against compensation.

A. Order format

An order submitted by a user on this market is defined as the tuple

$$o = (id,\, u,\, \text{type},\, s,\, \bar{q},\, \pi)\,, \tag{20}$$

where $id \in \mathbb{N}$ is a unique identifier, $u \in \mathcal{U}$ is the user submitting the order, $\text{type} \in \{\text{buy}, \text{sell}\}$ indicates whether the user wants to acquire or release operating range, $s \in \{\min, \max\}$ specifies whether the lower or the upper bound is concerned, $\bar{q} \in \mathbb{R}_+$ is the maximum quantity offered, and $\pi \in \mathbb{R}_+$ is the limit price.

A buy order on side $\min$ means that the user wants to decrease its lower bound, thereby obtaining more room to inject. A sell order on side $\min$ means that the user accepts to increase its lower bound, thereby releasing part of this room. Conversely, a buy order on side $\max$ increases the upper bound, whereas a sell order on side $\max$ decreases it.

Let $\mathcal{O}$ denote the set of submitted orders. Clearing an order consists in accepting all or part of the offered quantity. Therefore, we associate with each order $o \in \mathcal{O}$ a *cleared quantity* $q_o \in [0, \bar{q}_o]$, the decision variable of the market-clearing problem. To lighten the notation, we introduce the sign

$$\sigma_o = \begin{cases} +1, & \text{type} = \text{buy}, \\ -1, & \text{type} = \text{sell}, \end{cases} \tag{21}$$

so that a cleared order widens the envelope when $\sigma_o = +1$ and tightens it when $\sigma_o = -1$.

B. Market-clearing problem

The market-clearing problem illustrated in Section II-B is now formalized as an optimization problem. Following the flexible envelope adjustment performed by the DSO, a market operator gathers buy and sell bids from grid users who require additional grid power or want to valorize unused capacity. The market operator then solves an optimization problem to clear the market.

Objective function

The market operator clears the orders by maximising the social welfare, i.e., the sum of the surpluses of the buyers

and the sellers resulting from the cleared quantities, as in the double-sided auctions of wholesale electricity markets. The social welfare is denoted $\mathrm{Wel}(q)$, where $q \in \mathbb{R}^{|\mathcal{O}|}$ is the vector of all cleared quantities.

Network security

Network security requirements for the market clearing are enforced exactly as in Section III-B, with the post-market OEs in place of the adjusted ones.

Predictable/unpredictable users

Predictable users are represented by their predicted power consumption/injection, obtained by evaluating the prediction function, i.e., (11) on the post-market OEs Δ' in place of the adjusted ones.

A generic market-clearing optimization problem

The market operator solves the following optimization problem:

$$
\begin{aligned}
\max_{q} \ & \mathrm{Wel}(q) && (\mathcal{P}_4)\\
\text{s.t. } & (a)\ 0 \le q_o \le \bar{q}_o, \\
& \qquad \forall o \in \mathcal{O}, \\
& (b)\ \sum_{o\,:\,s_o = s} \sigma_o\, q_o = 0, \\
& \qquad \forall s \in \{\min, \max\}, \\
& (c)\ \Delta_u'^{\min} = \Delta_u^{\min} - \sum_{\substack{o \in \mathcal{O}\,:\\ u_o = u,\ s_o = \min}} \sigma_o\, q_o, \\
& \qquad \Delta_u'^{\max} = \Delta_u^{\max} + \sum_{\substack{o \in \mathcal{O}\,:\\ u_o = u,\ s_o = \max}} \sigma_o\, q_o, \\
& \qquad \forall u \in \mathcal{U}, \\
& (d)\ \Delta_u' \subseteq \bar{\Delta}_u, \\
& \qquad \Delta_u'^{\min} \le \Delta_u'^{\max}, \\
& \qquad \forall u \in \mathcal{U}, \\
& (e)\ \mathrm{Security}\big(P', F'\big) = 0, \\
& \qquad \forall\, (P', F') : \\
& \qquad\quad (f)\ \mathrm{PowerFlow}\big(P', F'\big) = 0, \\
& \qquad\quad (g)\ \mathrm{Pred}\big(P', \Delta'\big) = 0,
\end{aligned}
$$

where $\Delta' = (\Delta'^{\min}, \Delta'^{\max})$ is the matrix of the post-market OEs. Vectors P' and F' denote the feasible power consumption/injection of the users and the flows on the edges of the network after the market clearing. Condition (a) limits each cleared quantity to the quantity offered. Condition (b) states that, on each side, the signed cleared quantities sum to zero, so that the market only reallocates operating range already granted. Condition (c) gives the post-market bounds of each user, obtained by modifying the bounds of its OE by the quantities cleared on its orders. Condition (d) keeps each post-market OE within the CE of the user and non-empty. Condition (e) requires the network to be secure for every power consumption/injection P' and flows F' that respect the physics of the network (f) and correspond to a possible future behaviour of the users given their post-market OEs (g).

C. A tractable formulation for the market-clearing optimization problem

Problem $\mathcal{P}_4$ has the same security, power flow and prediction conditions as $\mathcal{P}_2$, imposed on the post-market OEs Δ' instead of the OEs Δ. Specifications 2 to 4 of Section III-C thus apply unchanged, and so do the first two observations of Section III-D, i.e., the robust security condition reduces to two constraints per edge, on both the minimum and maximum flows computed with the post-market OEs, and the clipping of each predictable user is expressed with two binary variables. The market-clearing optimization problem is thus reformulated as a single MILP, which is:

$$
\begin{aligned}
\max_{q_o} \ & \sum_{o \in \mathcal{O}} \sigma_o\, \pi_o\, q_o && (\mathcal{P}_5)\\
\text{s.t. } & (a)\ 0 \le q_o \le \bar{q}_o, \\
& \qquad \forall o \in \mathcal{O}, \\
& (b)\ \sum_{o\,:\,s_o = s} \sigma_o\, q_o = 0, \\
& \qquad \forall s \in \{\min, \max\}, \\
& (c)\ \Delta_u'^{\min} = \Delta_u^{\min} - \sum_{\substack{o \in \mathcal{O}\,:\\ u_o = u,\ s_o = \min}} \sigma_o\, q_o, \\
& \qquad \Delta_u'^{\max} = \Delta_u^{\max} + \sum_{\substack{o \in \mathcal{O}\,:\\ u_o = u,\ s_o = \max}} \sigma_o\, q_o, \\
& \qquad \forall u \in \mathcal{U}, \\
& (d)\ \Delta_u' \subseteq \bar{\Delta}_u, \\
& \qquad \Delta_u'^{\min} \le \Delta_u'^{\max}, \\
& \qquad \forall u \in \mathcal{U}, \\
& (e)\ P_{m,n}'^{\min},\ P_{m,n}'^{\max} \in \Sigma_{m,n}, \\
& \qquad \forall (m,n) \in \mathcal{E}, \\
& (f)\ P_{m,n}'^{\min} = \sum_{u \in \mathcal{U}^{\mathrm{pred}}(n)} P_u' + \sum_{u \in \mathcal{U}^{\mathrm{unpred}}(n)} \Delta_u'^{\min}, \\
& \qquad P_{m,n}'^{\max} = \sum_{u \in \mathcal{U}^{\mathrm{pred}}(n)} P_u' + \sum_{u \in \mathcal{U}^{\mathrm{unpred}}(n)} \Delta_u'^{\max}, \\
& \qquad \forall (m,n) \in \mathcal{E}, \\
& (g)\ \Delta_u'^{\min} \le P_u' \le \Delta_u'^{\max}, \\
& \qquad P_u' \le \hat{P}_u + M\, b_u, \\
& \qquad P_u' \ge \hat{P}_u - M\, c_u, \\
& \qquad P_u' \le \Delta_u'^{\min} + M\,(1 - b_u), \\
& \qquad P_u' \ge \Delta_u'^{\max} - M\,(1 - c_u), \\
& \qquad \forall u \in \mathcal{U}^{\mathrm{pred}}, \\
& (h)\ b_u,\ c_u \in \{0,1\}, \\
& \qquad \forall u \in \mathcal{U}^{\mathrm{pred}}.
\end{aligned}
$$

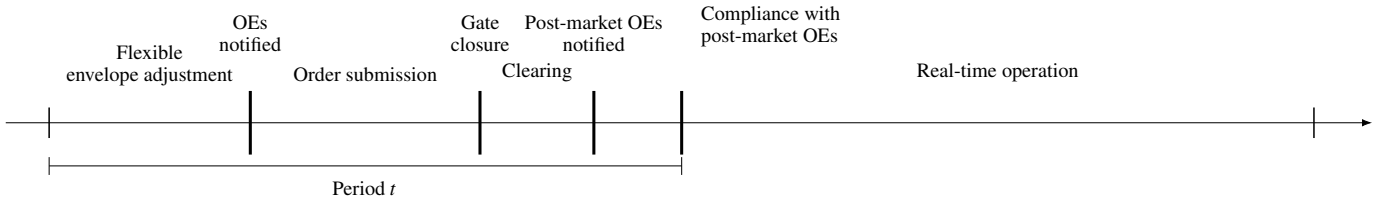

Fig. 4: Timeline of the envelope-based operational scheme. During a given period t, the DSO adjusts the flexible extensions and notifies the resulting OEs, after which users submit their orders until gate closure. The market operator then clears the order book and notifies the post-market OEs. Users then have some time left for compliance with the notified post-market OEs before the next period begins. Each user then operates within its post-market OE over the whole of that period.

D. Clearing price

Problem $\mathcal{P}_5$ is a MILP with $2|\mathcal{U}^{\mathrm{pred}}|$ binary variables. The market is settled pay-as-clear, with a single clearing price per side. Every cleared order on a side is therefore priced identically, and the payments of the buyers exactly cover the remuneration of the sellers. In a linear program, the clearing price of a side would be given by the dual variable of its balance constraint (b). Dual variables are, however, not defined for a MILP. Following [15], the clearing prices are therefore obtained in three steps:

1) Problem $\mathcal{P}_5$ is solved.
2) The binary variables b_u and c_u are fixed at their optimal values, which turns $\mathcal{P}_5$ into a linear program (LP).
3) This LP is solved, and the clearing price of side s is given by the optimal dual variable of constraint (b) for that side.

Fixing the binary variables leaves the optimal cleared quantities unchanged, since the optimal solution of $\mathcal{P}_5$ remains feasible and optimal for the LP.

E. Operational scheme timeline

The market is organized as a double-sided pool, cleared once per period by a market operator. All orders are collected over a submission window and cleared simultaneously in a single auction. The following timeline summarizes the different steps of the proposed operational scheme:

1) the DSO solves the flexible extension adjustment problem $\mathcal{P}_2$ and notifies each user of its OE for the next period;
2) the submission window opens, during which users send their orders to the market operator;
3) at gate closure, the order book $\mathcal{O}$ is fixed and the market operator clears it by solving $\mathcal{P}_6$;
4) the resulting post-market OEs are notified to the users, who have some time left for compliance before the next period begins.

This timeline is depicted in Figure 4. Each user then operates within its post-market OE over the whole of the period, and the procedure is repeated for the following period. The envelopes traded are thus valid for a single period, after which the firm envelope is restored and the flexible extensions are recomputed by the DSO.

V. Discussions

Different aspects of the operational scheme deserve to be further detailed. The regulatory framework under which the scheme would be deployed and the representation of predictable users are two of them. The network model, for its part, has been chosen for its tractability rather than for its realism.

Moreover, a commercial actor may operate assets at several locations of the grid, in which case it cannot be represented as a single user of the grid. Instead, it should be modelled as a collection of users, one per connection point, each holding its own OE.

To discuss these aspects, among others, this section is organized as follows. Section V-A confronts the scheme with the regulatory framework, at both the European and regional levels. The Walloon region, in Belgium, is taken as a candidate for deployment. Its regulation is briefly examined to assess how far it already accommodates the scheme.

Section V-B discusses the prediction function and the binary classification of users it rests on. Section V-C examines how commercial actors might operate their assets at several network locations and how they may participate in the secondary market. Section V-D positions the scheme with respect to existing electricity markets, balancing services, and the energy-management systems of users. Section V-E sets out settlement, non-delivery and liability principles. Section V-F further discusses practical market rules. Finally, Section V-G reviews the limitations of the simplified network model.

A. Regulatory framework

The practical implementation of the scheme presupposes a regulation that (i) recognizes the envelope as a contractual object, (ii) entitles the DSO to adjust the flexible part of the envelope without owing compensation to the user, and (iii) allows users to exchange envelope bounds among themselves. These conditions are only partially met today.

The envelope proposed in this paper can be considered as an implementation of the flexible connection agreements introduced in European law. Directive (EU) 2019/944, as amended by Directive (EU) 2024/1711 [16, 17], requires a framework under which system operators offer flexible connection agreements in areas where network capacity is limited. Article 6a

provides that such agreements may specify maximum firm injection and withdrawal capacities, together with additional flexible capacities. This distinction maps directly onto the two components of the contractual envelope of Section II, namely the firm envelope $[P_u^{f,\min}, P_u^{f,\max}]$ and the maximum flexible extensions $(\bar{\delta}_u^{\min}, \bar{\delta}_u^{\max})$.

In Wallonia, taken here as a candidate region for deployment, the decree of 19 December 2025 transposes Article 6a and extends to withdrawal a flexibility regime that previously concerned injection only [18–21]. Connection contracts concluded in congested areas must clearly define a firm capacity together with the additional flexible capacity. Determining how the grid capacity requested by a user should be divided into a firm and a flexible part is a complex problem whose mathematical formalization is left for future work.

Two gaps nevertheless remain. Firstly, the market of Section IV has no legal basis, as grid power access capacity remains attached to a connection point and cannot be transferred. Their tradability should be recognized and a market operator designated. Secondly, the flexibility requirements apply only to new connections above a power threshold in congested areas, whereas the proposed scheme assigns an envelope to every user of the network.

B. The notion of predictability

The formalization of Section III rests on two modeling choices: (i) a binary classification of users into predictable and unpredictable ones, and (ii) a deterministic prediction function. The instantiation of the prediction function proposed in the illustrative example, (13), is not unique and other instantiations of this function may be proposed. For example, this function may encode a behavioral model of the user, such as the anticipated response of a price-driven asset to a market signal.

The binary classification itself may prove too restrictive. It amounts to assuming that the DSO either holds perfect information on the future behavior of a user, who is then called predictable, or no information at all, in which case the user is unpredictable. Many real cases sit between these two extremes. An aggregated load may be predictable fifteen minutes ahead but not two hours ahead, a battery may be predictable only when its operation is governed by a contractual agreement, and renewable generation typically comes with a quantifiable forecast error. In all these cases, the information available is partial but not absent. The predictable and unpredictable user categories can be generalized by defining an uncertainty set around the predicted power consumption/injection of each user. Building this set from probabilistic forecasts allows a small risk of SE violation to be accepted, as first proposed in [22].

C. Multi-site portfolios and aggregated participation

Users may be attached to any node of the network. A commercial actor operating assets at several network locations is therefore represented as a set of users, one per connection point, each holding its own contractual envelope. The internal constraints and load shapes of these assets remain private. The commercial actor dispatches its portfolio freely within the OEs granted, which is precisely the confidentiality property that motivates the envelope-based approach.

Commercial actors may also use the market of Section IV to reallocate range within their own portfolio, by submitting matching sell and buy orders for two of their sites. Whether the trade clears, however, is not guaranteed due to the security requirement integrated within the market-clearing problem.

Participation as a single market party, rather than as a collection of users, raises different questions. The first question is whether orders at several sites may be linked. The order format of Section IV-A may be extended with a group identifier, such that all orders of a group are cleared in the same proportion, i.e., $q_o/\bar{q}_o = q_{o'}/\bar{q}_{o'}$ for every pair of orders (o, o') of the same group. A sell order at one site and a buy order at another then clear together or not at all, which allows an aggregator to shift range across connection points in a single transaction. The linking constraint being linear, the clearing problem remains a MILP.

The second question is the level at which predictability is assessed. A portfolio may be predictable in aggregate while its individual sites are not, as aggregation smooths out individual variability. The aggregated residential load of the illustrative example already relies on this effect. Portfolio predictability, however, only informs the sum of the powers of the sites, and not their distribution across the network. It may therefore be exploited on the edges downstream of which the whole portfolio is connected, whereas an edge separating two sites of the portfolio must still be secured against the individual OEs. The prediction function of Section III would then be defined on a set of users, and would be used for assessing security on an edge only where the set is entirely downstream of that edge.

D. Interaction with existing markets and energy-management systems

The scheme does not replace the existing markets in which grid users participate, and its position with respect to them must be clarified. An OE is a grid capacity constraint at a connection point. It is neither an energy position nor a flexibility product, and the market of Section IV reallocates capacity access, not energy. The energy position of a user therefore continues to be settled under its balance responsible party (BRP), through the day-ahead and intraday markets and the imbalance settlement. The two are nevertheless coupled. An OE narrower than the position a user has nominated forces a deviation from that position, and hence an imbalance charged to its BRP. The secondary market gives the user an alternative, namely to acquire the missing range from another user, at a price it may compare with the expected imbalance cost. Conversely, a user whose position leaves part of its OE unused may release it. The timeline of Section IV-E should place the notification of the post-market OEs after the day-ahead clearing and within the intraday trading horizon, so that the

user retains the possibility of adjusting its energy position once its OE is known.

Balancing and other flexibility services raise a further coordination question. Reserve capacity offered to the transmission system operator, such as mFRR, is deliverable only if the corresponding power lies within the OE of the period, so that the OE bounds the capacity a distribution-connected asset may commit. The firm envelope provides a floor on which prequalification may rely, whereas the flexible extension may be committed only once notified.

From the point of view of the user, the OE enters its energy-management system (EMS) as a time-varying box constraint on the power at the connection point. The EMS may also generate the orders submitted to the market, the limit price of a buy order being the marginal value of one megawatt of range to the user, and that of a sell order the opportunity cost of releasing it.

E. Settlement and liability

The financial flows of the scheme should follow two identified principles. First, no compensation is due for an adjustment of the flexible extension by the DSO, the flexible part of the CE being contractually reducible, consistently with the Walloon framework discussed in Section V-A. This holds as long as the number and depth of the reductions of the flexible extension remain within the limits stipulated in the connection contract.

Second, the liability of a predictable user for a deviation from its predicted power consumption/injection depends on who produces the prediction. The conditional security guarantee of Section III holds only if the user operates at the value returned by the prediction function. In case that value derives from an announcement made by the user, any deviation from the announcement, even within the OE, then breaches the contract and the user is liable for it. Where the DSO produces the forecast itself, from historical measurements or exogenous data, as discussed in Section V-B, the user has made no commitment and cannot be held liable for a deviation that remains within its OE. The forecast error is then borne by the DSO, which must cover it by other means, such as an uncertainty set around the predicted value. The choice between the two instantiations therefore allocates the liability for the forecast error.

Enforcing these principles requires that any excursion outside the OE be observable, hence metering at the connection point at the resolution of the period, and a penalty proportional to the excursion or deviation. The penalty must exceed the value of range on the secondary market, so that a user preferring more range acquires it rather than exceeding its OE. The residual risk borne by the DSO, namely the security violation that an excursion or a deviation may cause, is then covered by the corrective actions of the operator, financed through the network tariff and the penalties collected.

F. Market rules and investment decisions

The market of Section IV has been formalised with a minimal set of rules, namely a single pool auction per period, divisible orders and a pay-as-clear settlement. A practical deployment would require complementary rules. The first rule concerns the minimum quantity of an order. Orders of a few kilowatts bring little welfare, while increasing the number of transactions to settle and the metering effort required to verify the post-market OEs.

The second rule concerns price limits. The penalty discussed in Section V-E bounds the price a rational user is willing to pay, since exceeding the OE becomes cheaper than acquiring range above that price. A price cap set at the level of the penalty therefore does not exclude any order a rational user would submit, while protecting buyers against extreme prices.

The third rule concerns market power. A constraint on an edge can only be relieved by the users connected downstream of it, which, on a distribution feeder, are typically few. A single user may thus hold most of the range offered on a side, and withhold part of it to raise the clearing price. The regulator may therefore monitor the orders, as on wholesale markets, or open the market only in network areas where a minimum number of users participate.

The clearing prices also provide the DSO with an indication of investment priorities. A high clearing price on a side reveals that users value additional range on that side beyond what the network accommodates, i.e., that network access is scarce. Conversely, clearing prices persistently close to zero indicate that the flexible extensions granted by the DSO are sufficient, and that a reinforcement would bring little benefit to the users of the area.

G. Network model limitations

The network model of Section III-A assumes a balanced three-phase, lossless network, and enforces security through the thermal capacities of the edges only. Only active power is considered. While these assumptions preserve the tractability of the flexible envelope adjustment and market-clearing problems, they provide only a rough representation of real-world three-phase distribution networks, where phase unbalance is common.

A more accurate representation of the network might be explored. Voltage constraints should be considered, particularly in networks with significant photovoltaic generation. These constraints cannot be captured by a per-edge power range, and would require SEs of a different nature. Reactive power and losses should be taken into account as well. These extensions affect both stages of the scheme. The flexible envelope adjustment and market-clearing problems keep their structure, but their security conditions become more complex, so that their tractable formulations would need to be revisited.

VI. Conclusions

This paper proposes a static security management scheme for distribution networks built on both contractual and operat-

ing envelopes, addressing the lack of grid power access users might face. Each contractual envelope combines a firm part, which is guaranteed, and a flexible part that the DSO adjusts according to network conditions. The scheme is organized in two successive stages. First, close-to-real-time, the DSO adjusts the flexible extensions to grant users an OE as wide as possible, under a LIFO rule. We formalize this flexible extension adjustment problem as a sequence of N MILPs. Resulting OEs ensure that every network asset remains within its SE, provided each predictable user operates at its predicted power consumption/injection.

Because the resulting OEs may not satisfy the users' needs, a secondary market allows users to trade OE bounds against financial compensation, reallocating operating range to those who value it most while preserving security. The market-clearing problem of this secondary market is formalized as a single MILP, which integrates the same security requirements as the flexible envelope adjustment problem. A running example on a radial network illustrates both stages. The work could be extended along several lines. First, the formulation could be extended to complex power envelopes, so as to capture voltage-related constraints. Second, the scheme could be assessed on realistic, unbalanced three-phase networks. Third, the binary classification of users could be extended, by defining probabilistic forecasts and uncertainty sets around the predicted power consumption/injection. Finally, the strategic behavior of users bidding into the secondary market should be studied.

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