# Discrete and asymptotically normal estimators for both parameters of the Harmonizable fractional stable motion

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**Abstract**

Statistical estimation of the Hurst parameter and the stability parameter of harmonizable fractional stable motion (HFSM) is a difficult issue mainly due to the lack of ergodicity of this process and heaviness of tails of its marginal distributions. Very few estimators have been introduced in the literature so far. Those proposed in [4] offer the advantages to be strongly consistent and asymptotically normal. Yet, they have the drawback to require the knowledge of a whole continuous trajectory of HFSM on the real line, whereas, in practice, only a discretized one is available. The goal of the present article is to overcome this drawback by introducing discretized versions of them, obtained through values of HFSM on dyadic numbers.

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## 1 Introduction

Self-similar stochastic processes have been widely studied since a few decades and provide powerful stochastic models to mimic persistent phenomena in various domains. For instance, self-similarity is observed in network traffic [22, 28] and hydrology [26]. The most famous self-similar process is the fractional Brownian motion (FBM). Introduced in [17] and later formalized in [25], the FBM is a Gaussian centered self-similar stochastic process with stationary increments. However, many natural phenomena exhibit some heavy tails, which prevents to model them by a FBM. As an alternative, self-similar $\alpha$-stable random processes have been proposed to model heavy-tailed persistent phenomena, , see e.g. [22, 28].

Especially, harmonizable fractional stable motions (HFSM) and linear fractional stable motions (LFSM), introduced in [11, 33], are two very classical stable extensions of FBM. They

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are defined generalizing the harmonizable and moving average representations of a FBM to the stable framework. On the one hand, the LFSM $\{L(t)\}_{t\in\mathbb{R}}$ is defined through a stable stochastic integral in the time domain:

$$L(t) := \int_{\mathbb{R}} \left( (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right) dM_\alpha(s), \ t \in \mathbb{R}$$

where $M_\alpha$ is a real-valued $\alpha$-stable random measure. On the other hand, the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ is defined as a stable stochastic integral in the frequency domain:

$$X(t) := \Re\left( \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{H+1/\alpha}} \, d\widetilde{M}_\alpha(\xi) \right), \quad t \in \mathbb{R}$$

where $\widetilde{M}_\alpha$ is a complex-valued isotropic $\alpha$-stable random measure with Lebesgue control measure. For details on stable random measures and stochastic integrals, we refer the reader to [32]. Both HFSM and LFSM are then parametrized by the Hurst parameter $H \in (0,1)$ and the stability parameter $\alpha \in (0,2]$. While the parameter $H$ governs their self-similarity property, the parameter $\alpha$ controls the heaviness of their tails. When $\alpha = 2$, both HFSM and LFSM coincide with FBM. However, when $\alpha \in (0,2)$, HFSM and LFSM are very different. For instance, while LFSM is ergodic, HFSM is not (see Theorem 4 in [10]); LFSM has also multifractal sample paths (see [6]), which are unbounded on any interval when $H < 1/\alpha$ (see e.g. [32, 15]) whereas the trajectories of HFSM are locally Hölder continuous of any order $\gamma \in (0,H)$ (see [18, 19, 32, 15]) and then monofractal by Corollary 4.4 in [5].

Estimating the model parameters is a key challenge when modeling natural phenomena. Assuming that $\alpha = 2$, i.e. for a FBM, many articles deal with the estimation of the Hurst parameter $H$, see e.g. [7, 12, 21, 8, 3, 30] and references therein. In the framework of LFSM and related moving average stable processes, several estimators of both self-similarity and stability parameters have also been proposed, see e.g. [14, 31, 29, 2, 1, 13, 27, 24, 23]. The situation is different in the case of HFSM; actually, as emphasized in the rather recent article [9], statistical inference in the framework of the latter process is far from being a trivial issue due to lack ergodicity. Indeed, as far we know, only the recently published paper [4] and the bit more recent work [20] deal with statistical estimation of parameters of HFSM. The strategy of [20] is significantly different from that in [4]. Indeed, the consistent estimators for the two parameters $H \in (0,1)$ and $\alpha \in (0,2)$ of a non-Gaussian HFSM, proposed in [20], rely on an estimation of the periodogram frequency and a LePage series representation. While, in [4] the non-ergodicity is bypassed thanks to the introduction of some new integral transforms of HFSM which are to a certain extent inspired by discrete wavelet transforms. The strongly consistent estimators for these two parameters proposed in [4] cover the non-Gaussian case $\alpha \in (0,2)$ as well as the Gaussian case $\alpha = 2$ in which HFSM reduces to FBM. Moreover, they are asymptotic normal, which is crucial to build some confidence intervals and perform hypothesis testing. Nevertheless, they are defined assuming that a whole continuous trajectory of HFSM on the real line is observed, whereas, in practice, only a discretized one is available.

The aim of the present paper is to extend the main results of [4] to the important setting where only one single discretized trajectory of HFSM is observed. More precisely, the estimators introduced in [4] rely on some independent real-valued stable variables

$$Y_{j,k} := \frac{2}{2\pi} \int_{\mathbb{R}} \Re\Big(\widehat{\psi}_{j,k}(t)\Big) X(t)\, dt,$$

where

$$\psi_{j,k}(t) := \psi(2^j t - k), \quad \text{for all } t \in \mathbb{R},$$

with $\psi$ being a "nice" real-valued function. For each parameter, [4] proposed one estimator based on empirical means of logarithms of some $|Y_{j,k}|$'s, and another one relying on sums of some $|Y_{j,k}|$'s raised to a certain power $\gamma$, respectively named logarithmic and power estimators in the present article. Our main strategy is to replace $Y_{j,k}$ by discretizing the previous integral. We then prove that all the discretized estimators obtained by this way are still strongly consistent and asymptotic normal.

The paper is organized as follows. In Section 2, after some background on stochastic integration with respect to complex-valued isotropic stable random measures and on harmonizable fractional stable motions, our main assumptions are introduced. In this same section, some results of [4] are also recalled, since they are the starting point for defining and studying estimators of both parameters of a HFSM based on a single discretized trajectory. Then Section 3 is devoted to the study of discretized logarithmic estimators and Section 4 focuses on discretized power estimators.

## 2 Preliminaries

Harmonizable random fields are defined through stochastic integrals in the frequency domain with respect to complex-valued isotropic stable random measures. We refer the reader to [32] for a detailed presentation of such random measures and the associated stochastic integrals. Thus we only recall now the essential properties needed for our study.

Throughout this paper, for $\alpha \in (0, 2]$, $\widetilde{M}_\alpha$ denotes a complex-valued isotropic $\alpha$-stable random measure with Lebesgue control measure. The stochastic integral

$$\int_{\mathbb{R}} g(\xi)\, \widetilde{M}_\alpha(d\xi)$$

is then well-defined for any deterministic complex-valued function $g$ in the Lebesgue space $L^\alpha(\mathbb{R})$ and is a complex-valued isotropic $\alpha$-stable random variable. This stable stochastic integral $\int_{\mathbb{R}} (\,\cdot\,)\, d\widetilde{M}_\alpha$ is a linear map on $L^\alpha(\mathbb{R})$ such that for any $g \in L^\alpha(\mathbb{R})$

$$\forall z \in \mathbb{C},\ \mathbb{E}\left(\exp\left(\Re\left\{\overline{z} \int_{\mathbb{R}} g(\xi)\, \widetilde{M}_\alpha(d\xi)\right)\right\}\right) = \exp\left(-|z|^\alpha \int_{\mathbb{R}} |g(\xi)|^\alpha\, d\xi\right).$$

Especially, for any deterministic function $g \in L^\alpha(\mathbb{R})$, the real part $\Re\big\{ \int_{\mathbb{R}} g(\xi)\, d\widetilde{M}_\alpha(\xi)\big\}$ is a real-valued symmetric $\alpha$-stable (S$\alpha$S) random variable with scale parameter

$$\sigma\left(\Re\left\{\int_{\mathbb{R}} g(\xi)\, d\widetilde{M}_\alpha(\xi)\right\}\right)^\alpha = \int_{\mathbb{R}} \big|g(\xi)\big|^\alpha\, d\xi. \tag{2.1}$$

In the particular case where the stability parameter $\alpha = 2$, $\widetilde{M_2}$ is an isotropic complex-valued Gaussian measure and the stochastic integral $\int_{\mathbb{R}} (\,\cdot\,)\, d\widetilde{M_2}$ is an isometry from $L^2(\mathbb{R})$ onto the space of square integrable random variables.

Before we introduce harmonizable fractional stable motion, we recall an essential property of real-valued S$\alpha$S random variables.

**Remark 2.1.** For any arbitrary $\alpha \in (0, 2]$, let $Z_\alpha$ be a real-valued S$\alpha$S random variable with scale parameter equals to $\sigma(Z_\alpha)$ and $W_\alpha$ be a real-valued S$\alpha$S random variable with scale parameter $\sigma(W_\alpha) = 1$. Then

$$Z_\alpha \overset{d}{=} \sigma(Z_\alpha) W_\alpha$$

where the equality is in distribution.

The harmonizable fractional stable motion (HFSM) $\{X(t)\}_{t\in\mathbb{R}}$ is then defined by

$$X(t) := \Re\Big( \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/\alpha}} \, d\widetilde{M_\alpha}(\xi) \Big), \quad t \in \mathbb{R}, \tag{2.2}$$

where $H \in (0, 1)$ is its Hurst parameter and $\alpha \in (0, 2]$ its stability parameter. In the special case where $\alpha = 2$, $X$ reduces to the well-known fractional Brownian motion (FBM) with Hurst parameter $H$, usually denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$. Recently, [4] has introduced strongly consistent and asymptotic normal estimators for both parameters $H$ and $\alpha$ assuming that a single continuous trajectory on the real-line is observed. This work provides a first step towards the estimation of a HFSM parameters. However, the drawback is that, in practice, only a discretized trajectory is available. The aim of this paper is to address this issue by adapting the estimators proposed in [4] to the discrete-time observation setting.

Before we recall the estimators proposed in [4] and present our strategy to discretized them, let us give our main assumptions. All along this paper, we consider a real-valued function $\psi$ which satisfies the two following assumptions:

($\mathcal{A}_1$) $\psi$ is an even (i.e. $\psi(-\xi) = \psi(\xi)$ for all $\xi \in \mathbb{R}$), real-valued, continuous function on $\mathbb{R}$, with a compact non-empty support, denoted by $I$, such that

$$I := \operatorname{supp} \psi := \overline{\big\{\xi \in \mathbb{R},\, \psi(\xi) \neq 0\big\}} \subseteq \big[-4^{-1}, 4^{-1}\big]\,. \tag{2.3}$$

($\mathcal{A}_2$) There exist an integer $L \geq 2$ and a constant $c_L$ (depending on $L$) such that $\widehat{\psi}$ the Fourier of $\psi$, and its derivative $\widehat{\psi}'$ satisfy

$$\big|\widehat{\psi}(t)\big| + \big|\widehat{\psi}'(t)\big| \leq c_L \big(1 + |t|\big)^{-L}, \quad \text{for every } t \in \mathbb{R}. \tag{2.4}$$

We mention in passing that throughout this article we use the very common convention that the Fourier transform $\widehat{f}$ of an arbitrary function of $L^1(\mathbb{R})$ is defined, for all $t \in \mathbb{R}$, as $\widehat{f}(t) := \int_{\mathbb{R}} e^{-it\xi} f(\xi)\, d\xi$. Also, we mention that the only difference between our two assumptions on $\psi$ and that made in [4] comes from (2.4), according to which, in contrast to (1.13) in [4], not only of vanishing at infinity of the function $\widehat{\psi}$ itself is governed by the exponent $L$, but also that of its derivative $\widehat{\psi}'$. Actually, this additional assumption on $\widehat{\psi}'$ allows to control the difference between the estimators proposed in [4], which rely on a continuous trajectory, and their discretized versions.

**Example 2.2.** Let us consider, as in [4], the piecewise linear triangle function:

$$\psi(\xi) := \big(\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-1,1]}\big)(8\xi) = \big(2 - |8\xi|\big)\mathbb{1}_{[-4^{-1},4^{-1}]}(\xi), \quad \text{for all } \xi \in \mathbb{R}, \tag{2.5}$$

where "$*$" denotes the usual convolution product. Thus, using some basic properties of Fourier transform, it turns out that $\widehat{\psi}(0) = 1/2$ and

$$\widehat{\psi}(t) = \frac{\sin^2(t/8)}{2(t/8)^2}, \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{2.6}$$

Then, standard calculations allow to show that the function $\psi$ in (2.5) satisfies Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ with $L = 2$. One mentions that, for any chosen arbitrarily large integer $L \geq 2$, by taking a conveniently shrinked version of the compactly supported function and $L-2$ differentiable function on $\mathbb{R}$ provided by the convolution product of the indicator function $\mathbb{1}_{[-1,1]}$, $L$ times with itself, one can obtain an explicit function $\psi$ which fulfills Assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ for this chosen arbitrarily large $L$. Also one mentions that one can derive from Lemma 3.8 in Section 3 and a careful inspection of the proofs of our main theorems in Sections 3 and 4, that increasing the value of the exponent $L$ has, to certain extent, a positive impact on the rates of convergence of the discretized logarithmic and power estimators. Yet, this advantage is counterbalanced by the drawback that the lager $L$ is taken the more complicated is the form of a corresponding function $\psi$.

Let us now present the main results in [4]. To this aim, for $(j,k) \in \mathbb{N}^2$, we set

$$\psi_{j,k}(t) := \psi(2^j t - k), \quad \text{for all } t \in \mathbb{R},$$

and define $Y_{j,k}$ as the pathwise Lebesgue integral

$$Y_{j,k} := \frac{2}{2\pi}\int_{\mathbb{R}} \Re\Big(\widehat{\psi}_{j,k}(t)\Big) X(t)\,dt = \frac{2^{1-j}}{2\pi}\int_{\mathbb{R}} \cos(2^{-j}k\,t)\widehat{\psi}(2^{-j}t)X(t)\,dt, \tag{2.7}$$

where $\widehat{\psi}_{j,k}$ is the Fourier transform of $\psi_{j,k}$. We recall in the next remark several useful representations of the random variables $Y_{j,k}$ and properties of the Fourier transforms $\widehat{\psi}_{j,k}$.

**Remark 2.3.** Let $(j,k) \in \mathbb{N}^2$. First, one knows from [4, Lemma 2.1] that, the real-valued S$\alpha$S random variable $Y_{j,k}$ can be rewritten almost surely as

$$Y_{j,k} = \Re\Big(\int_{\mathbb{R}} \frac{\widetilde{\psi}_{j,k}(\xi)}{|\xi|^{H+1/\alpha}}\,d\widetilde{M}_\alpha(\xi)\Big), \tag{2.8}$$

where

$$\widetilde{\psi}_{j,k}(\xi) := \psi(2^j\xi + k) + \psi(2^j\xi - k), \quad \text{for every } \xi \in \mathbb{R}. \tag{2.9}$$

Moreover, as noticed in Remark 2.6 of [4], one also has

$$Y_{j,k} = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\widetilde{\psi}}_{j,k}(t)X(t)\,dt\,. \tag{2.10}$$

since

$$\widehat{\widetilde{\psi}}_{j,k}(t) = 2^{1-j}\cos(2^{-j}kt)\widehat{\psi}(2^{-j}t) \quad \text{for all } t \in \mathbb{R}. \tag{2.11}$$

Finally, (2.11) and (2.4) imply that there exists a finite deterministic constant $c_{j,L} > 0$, only depending on $j$ and $L$, such that, for any $k \in \mathbb{N}$, one has

$$\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big| \le c_{j,L}\big(1+|t|\big)^{-L}, \quad \text{for every } t \in \mathbb{R}, \tag{2.12}$$

and

$$\big|\widehat{\widetilde{\psi}}'_{j,k}(t)\big| \le c_{j,L}\, k\big(1+|t|\big)^{-L}, \quad \text{for every } t \in \mathbb{R}. \tag{2.13}$$

It follows that $\widehat{\widetilde{\psi}}_{j,k}$ and its derivative belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

We have now all the ingredients needed to recall the main results of [4], which are our starting point. The following theorem corresponds to Theorems 1.5 and 1.7 in [4]. It provides strongly consistent and asymptotically normal estimators for the unknown parameters $\alpha$ and $H$ of HFSM. These estimators are built through empirical means of logarithms of some $|Y_{j,k}|$'s. Thus, in the present article, they are called the logarithmic estimators.

**Theorem 2.4** ([4, Theorems 1.5 and 1.7])**.** *For every $m \in \mathbb{N}$, one sets*

$$\widehat{\alpha}^{-1}_{m,\log_2} := \frac{1}{m}\left(\sum_{p=1}^{m}\Big(\log_2|Y_{1,2p-1}| - \log_2|Y_{2,4p-1}|\Big)\right) \tag{2.14}$$

*and*

$$\widehat{H}_{m,\log_2} := \frac{1}{m}\left(\sum_{p=1}^{m}\Big(\log_2|Y_{2,2p-1}| - \log_2|Y_{1,2p-1}|\Big)\right), \tag{2.15}$$

*where $Y_{j,k}$ is defined through (2.7). Then, the following results hold.*

*(i) $\widehat{\alpha}^{-1}_{m,\log_2}$ and $\widehat{H}_{m,\log_2}$ are strongly consistent (almost surely convergent) estimators respectively of the inverse $\alpha^{-1}$ of the stability parameter $\alpha$ and of the Hurst parameter $H$.*

*(ii) For all $m \in \mathbb{N}$, the random variable $D_{1,m,\log_2}$ is defined as*

$$D_{1,m,\log_2} := G\Big(\max\{\widehat{\alpha}^{-1}_{m,\log_2}, 2^{-1}\}\Big)m^{1/2}\big(\widehat{H}_{m,\log_2} - H\big), \tag{2.16}$$

*where the function $G$ is defined on the interval $[2^{-1}, +\infty)$ by*

$$G(\beta) := \Big(2\,\mathrm{Var}\big(\log_2|W_{\beta^{-1}}|\big)\Big)^{-1/2} = \frac{\sqrt{6}\ln(2)}{\pi\sqrt{1+2\beta^2}} \tag{2.17}$$

*with $W_{\beta^{-1}}$ a symmetric stable random variable with stability parameter $\beta^{-1}$ and scale parameter $\sigma = 1$.*

*Then, when $m$ goes to $+\infty$, the random variable $D_{1,m,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

*(iii) For all $m \in \mathbb{N}$, the random variable $D_{2,m,\log_2}$ is defined as*

$$D_{2,m,\log_2} := G\Big(\max\{\widehat{\alpha}^{-1}_{m,\log_2}, 2^{-1}\}\Big)m^{1/2}\big(\widehat{\alpha}^{-1}_{m,\log_2} - \alpha^{-1}\big). \tag{2.18}$$

*Then, when $m$ goes to $+\infty$, the random variable $D_{2,m,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

**Remark 2.5.** Basically the quantity $G\Big(\max\{\widehat{\alpha}_{m,\log_2}^{-1}, 2^{-1}\}\Big)$, in (2.16) and (2.18), plays the role of an asymptotic normalization factor, since it has been shown in [4] that, when $m$ goes to $+\infty$, the two random variables $m^{1/2}\big(\widehat{H}_{m,\log_2} - H\big)$ and $m^{1/2}\big(\widehat{\alpha}_{m,\log_2}^{-1} - \alpha^{-1}\big)$ converge in distribution to a same centered Gaussian random variable with variance equals $2\,\mathrm{Var}\big(\log_2|W_\alpha|\big)$. It is noteworthy that, despite the fact that these two random variables play different roles, they have a same asymptotic variance which depends only on the unknown parameter $\alpha$ and not on the unknown parameter $H$.

The following theorem, which is in fact Theorem 1.8 in [4], shows that, when a bit knowledge on the positive lower bound of the range of the unknown stability parameter $\alpha$ of HFSM is available, then other strongly consistent estimators for it and the Hurst parameter $H$ of HFSM can be constructed through sums of random variables $|Y_{j,k}|$ raised to certain power $\gamma$. Thus, in the present article, the latter estimators are called the power estimators.

**Theorem 2.6** ([4, Theorem 1.8])**.** *One assumes that $\alpha \in [\underline{\alpha}, 2]$, where the lower bound $\underline{\alpha} \in (0, 2]$ is known. Also, one assumes that $\gamma \in (0, 4^{-1}\underline{\alpha})$ is arbitrary and fixed. Let $(m_j)_{j\in\mathbb{N}}$ be an arbitrary non-decreasing sequence (that is $m_j \le m_{j+1}$, for all $j \in \mathbb{N}$) of integers larger than 2 which satisfy the condition*

$$m_j \ge j\,, \quad \text{for all } j \in \mathbb{N}. \tag{2.19}$$

*For all $j \in \mathbb{N}$, one sets*

$$\widehat{H}_{j,\gamma} := \gamma^{-1}\log_2\left(\frac{\sum_{p=1}^{m_j}\big|Y_{2,2p-1}\big|^\gamma}{\sum_{p=1}^{m_j}\big|Y_{1,2p-1}\big|^\gamma}\right) \tag{2.20}$$

*and*

$$\widehat{\alpha}_{j,\gamma}^{-1} := \gamma^{-1}\left(1 - \log_2\left(\frac{\sum_{p=1}^{m_{j+1}}\big|Y_{2,2p-1}\big|^\gamma}{\sum_{p=1}^{m_j}\big|Y_{1,2p-1}\big|^\gamma}\right)\right). \tag{2.21}$$

*(i) $\widehat{H}_{j,\gamma}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter $H$ of HFSM.*

*(ii) Under the condition*

$$\lim_{j\to+\infty}\log_2\left(\frac{m_{j+1}}{m_j}\right) = 1, \tag{2.22}$$

*$\widehat{\alpha}_{j,\gamma}^{-1}$ is a strongly consistent (almost surely convergent) estimator of the inverse $\alpha^{-1}$ of the stability parameter $\alpha$ of HFSM.*

In order to state a Central-Limit Theorem for the estimators in (2.20) and (2.21), we use the following remark.

**Remark 2.7.** Let $\underline{\alpha} \in (0, 2]$ be as in Theorem 2.6 and $W_\alpha$ be an arbitrary real-valued S$\alpha$S random variable with scale parameter equals to 1. Let also $\gamma$ be arbitrary and such that

$$0 < \gamma < \frac{\underline{\alpha}}{2\underline{\alpha} + 2}. \tag{2.23}$$

Then, for every $(H, \alpha^{-1}) \in [0,1] \times [2^{-1}, \underline{\alpha}^{-1}]$, one can set

$$F_\gamma(H, \alpha^{-1}) := \frac{\mathbb{E}\big(|W_\alpha|^\gamma\big)\big(1 - 2\gamma(H + \alpha^{-1})\big)^{1/2}}{\big(\mathrm{Var}\big(|W_\alpha|^\gamma\big)\big)^{1/2}\big(1 - \gamma(H + \alpha^{-1})\big)}. \tag{2.24}$$

According to Remark 1.9 in [4], the positive function $F_\gamma$ can be explicitely expressed in terms of the Gamma function and is continuous on the compact rectangle $[0,1]\times[2^{-1},\underline{\alpha}^{-1}]$.

Strengthening condition (2.22) when dealing with the estimation of the stability index, [4] obtains the asymptotic normality of the power estimators introduced in Theorem 2.19.

**Theorem 2.8.** *Let $\underline{\alpha}$ be as in Theorem 2.8 and let $\gamma\in(0,4^{-1}\underline{\alpha})$ be arbitrary and such that* (2.23) *holds. Let $(m_{1,j})_{j\in\mathbb{N}}$ and $(m_{2,j})_{j\in\mathbb{N}}$ be two arbitrary non-decreasing sequences of integers larger than* 2 *which satisfy the condition (2.19). Also, one assumes that $(m_{2,j})_{j\in\mathbb{N}}$ satisfies the following strengthened version of the condition (2.22):*

$$\lim_{j\to+\infty}(m_{2,j})^{1/2}\left|\log_2\left(\frac{m_{2,j+1}}{m_{2,j}}\right)-1\right|=0\,. \tag{2.25}$$

*For each $j\in\mathbb{N}$, one denotes by $\widehat{H}_{1,j,\gamma}$ the estimator of $H$ of HFSM defined through (2.20) with $m_j=m_{1,j}$, and one denotes by $\widehat{\alpha}_{2,j,\gamma}^{-1}$ the estimator of $\alpha^{-1}$ defined through (2.21) with $m_j=m_{2,j}$ and $m_{j+1}=m_{2,j+1}$. For all $j\in\mathbb{N}$, one sets*

$$D_{1,j,\gamma}:=2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}),\tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1})\big)(m_{1,j})^{1/2}\Big(\widehat{H}_{1,j,\gamma}-H\Big) \tag{2.26}$$

*and*

$$D_{2,j,\gamma}:=(2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}),\tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1})\big)(m_{2,j})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1}-\alpha^{-1}\Big), \tag{2.27}$$

*where $F_\gamma$ is the positive continuous function on $[0,1]\times[2^{-1},\underline{\alpha}^{-1}]$ defined in* (2.24), *and $\tau_1$ and $\tau_2$ are the two continuous "truncation" functions defined, for all $x\in\mathbb{R}$, as*

$$\tau_1(x):=\max\Big\{0,\min\{x,1\}\Big\}\quad\text{and}\quad\tau_2(x):=\max\Big\{2^{-1},\min\{x,\underline{\alpha}^{-1}\}\Big\}\,. \tag{2.28}$$

*When $j$ goes to $+\infty$, the two random variables $D_{1,j,\gamma}$ and $D_{2,j,\gamma}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

In order to define discretized versions of the logarithmic and the power estimators introduced in Theorems 2.4 and 2.6 respectively, we assume that we observe a single trajectory of an HFSM on dyadic numbers. Our strategy in the current paper strongly relies then on the following approximation of the random variable $Y_{j,k}$.

**Definition 2.9.** For every $n\in\mathbb{N}$, the random variable $Y_{j,k}^{n,T_n}$ is defined as the finite sum:

$$\begin{aligned} Y_{j,k}^{n,T_n} &:= \frac{1}{2^{n+1}\pi}\sum_{|m|\le 2^nT_n}X(d_{n,m})\widehat{\overline{\psi}}_{j,k}(d_{n,m}) \qquad (2.29)\\ &= \Re\left(\int_{\mathbb{R}}|\xi|^{-H-1/\alpha}\left(\frac{1}{2^{n+1}\pi}\sum_{|m|\le 2^nT_n}\big(e^{id_{n,m}\xi}-1\big)\widehat{\overline{\psi}}_{j,k}(d_{n,m})\right)d\widetilde{M}_\alpha(\xi)\right), \end{aligned}$$

where $(T_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of positive integers which satisfy, for some $\zeta\in(0,1]$,

$$T_n\ge 2^{n\zeta},\quad\text{for all } n\in\mathbb{N}, \tag{2.30}$$

and where, for each $(n,m)\in\mathbb{N}\times\mathbb{Z}$, the dyadic number $d_{n,m}:=2^{-n}\,m$.

One mentions that for the discretized logarithmic estimators, which will be precisely defined in Section 3, the size $m$ of the empirical means in (2.14) and (2.15) will depend on $n$. Also, one mentions that for the discretized power estimators, which will be precisely defined in Section 4, the sizes $m_j$ and $m_{j+1}$ of the sums in (2.20) and (2.21) will depend on $n$ as well. Then, the asymptotic properties of the discretized logarithmic and power estimators will be studied as $n$ tends to $\infty$, and so when the number [1] $v_n = 2^{n+1}T_n$ of the observations $\{X(d_{n,m}); d_{n,m} \in [-T_n, T_n]\}$ tends to $+\infty$. This leads us to introduce some subsequence $(\mu_n)_{n\in\mathbb{N}}$.

**Remark 2.10.** One denotes by $(\mu_n)_{n\in\mathbb{N}}$ an arbitrary non-decreasing sequence of positive integers such that

$$\frac{n}{10} \leq \mu_n \leq 2^{n-2}, \quad \text{for all } n \geq 4, \tag{2.31}$$

and

$$\lim_{n\to+\infty} \frac{\log_2(\mu_n)}{n} = 0. \tag{2.32}$$

Consequently, for any fixed real number $a > 0$, one has that

$$\lim_{n\to+\infty} 2^{-an}\mu_n = 0. \tag{2.33}$$

A typical example of such a sequence $(\mu_n)_{n\in\mathbb{N}}$ is given by

$$\mu_n := \left\lfloor 2^{\frac{n}{\log_2(n+1)}} \right\rfloor, \quad \text{for all } n \in \mathbb{N}, \tag{2.34}$$

where $\lfloor\cdot\rfloor$ denotes the integer part function.

# 3 Logarithmic estimators

The goal of this section is to derive the following two theorems which provide discrete versions of the logarithmic estimators introduced in Theorem 2.4.

**Theorem 3.1.** *The fixed integer $n_0 \geq 4$ is the same as in Lemma 3.4 below. We define, for all integer $n \geq n_0$,*

$$\widehat{\alpha}^{-1,dis}_{n,\log_2} := \frac{1}{\mu_n}\left(\sum_{p=1}^{\mu_n}\Big(\log_2|Y^{n,T_n}_{1,2p-1}| - \log_2|Y^{n,T_n}_{2,4p-1}|\Big)\right) \tag{3.1}$$

*The following two results hold.*

*(i) $\widehat{\alpha}^{-1,dis}_{n,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the inverse $\alpha^{-1}$ of the stability parameter $\alpha$ of HFSM.*

*(ii) For all $n \in \mathbb{N}$, the random variable $D^{dis}_{2,n,\log_2}$ is defined as*

$$D^{dis}_{2,n,\log_2} := G\Big(\max\{\widehat{\alpha}^{-1,dis}_{n,\log_2}, 2^{-1}\}\Big)\mu_n^{1/2}\big(\widehat{\alpha}^{-1,dis}_{n,\log_2} - \alpha^{-1}\big),$$

*where the positive bounded $C^\infty$ function $G$ is as in (2.17). When $n$ goes to $+\infty$, the random variable $D^{dis}_{2,n,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

[1] Observe that one does not take into account the observation $X(0)$ since it always equal 0.

One can introduce, in the same vein, a discrete estimator for the Hurst parameter $H$ of the HFSM.

**Theorem 3.2.** *The fixed integer $n_0 \geq 4$ is the same as in Lemma 3.4 below. We define, for all integer $n \geq n_0$,*

$$\widehat{H}^{dis}_{n,\log_2} := \frac{1}{\mu_n}\left(\sum_{p=1}^{\mu_n}\Big(\log_2|Y^{n,T_n}_{2,2p-1}| - \log_2|Y^{n,T_n}_{1,2p-1}|\Big)\right). \tag{3.2}$$

*The following two results hold.*

*(i) $\widehat{H}^{dis}_{n,\log_2}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter $H$ of HFSM.*

*(ii) For all $n \in \mathbb{N}$, the random variable $D^{dis}_{1,n,\log_2}$ is defined as*

$$D^{dis}_{1,n,\log_2} := G\Big(\max\{\widehat{\alpha}^{-1,dis}_{n,\log_2}, 2^{-1}\}\Big)\mu_n^{1/2}\big(\widehat{H}^{dis}_{n,\log_2} - H\big),$$

*where $G$ and $\widehat{\alpha}^{-1,dis}_{n,\log_2}$ are as in (2.17) and (3.1). When $n$ goes to $+\infty$, the random variable $D^{dis}_{1,n,\log_2}$ converges in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

**Remark 3.3.** The rate of the central limit theorems in Parts $(ii)$ of Theorems 3.1 and 3.2 is given in terms $\mu_n$. It is worth mentioning that, when one takes, for all $n \in \mathbb{N}$, $T_n := 2^{n-1}$ and $\mu_n$ as in (2.34), this rate can also be expressed in terms of the number of observations $v_n := 2T_n 2^n = 2^{2n}$. Indeed, for such a choice of the sequences $(T_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$, one clearly has, for all $n \in \mathbb{N}$, that $n = 2^{-1}\log_2(v_n)$ which entails that

$$\mu_n = \left\lfloor v_n^{\left(2\log_2(2^{-1}\log_2(v_n)+1)\right)^{-1}} \right\rfloor.$$

Before digging into the proofs of Theorem 3.1 and 3.2, we first notice that in order that the estimators $\widehat{\alpha}^{-1,\mathrm{dis}}_{n,\log_2}$ and $\widehat{H}^{\mathrm{dis}}_{n,\log_2}$ in them to be almost surely well-defined and finite, each random variable $Y^{n,T_n}_{j,k}$ in (3.1) and (3.2) should be almost surely non-vanishing, which is the case if and only if its scale parameter $\sigma(Y^{n,T_n}_{j,k})$ does not vanish. The following lemma provides a weak sufficient condition for the latter fact to be simultaneously true for all those random variables $Y^{n,T_n}_{j,k}$, provided that $n$ is large enough.

**Lemma 3.4.** *Assume that the continuous function $\widehat{\psi}$ in (2.7) satisfies $\widehat{\psi}(0) > 0$ and let a fixed integer $n_0 \geq 4$ be such that*

$$\min_{t\in[0,2^{-n_0}]} \widehat{\psi}(t) > 0. \tag{3.3}$$

*Then, for all integers $j \in \mathbb{N}$ and $n \geq n_0$, one has*

$$\min_{1\leq k\leq 4\mu_n} \sigma(Y^{n,T_n}_{j,k}) > 0. \tag{3.4}$$

*Therefore, with probability 1, none of the values of any one of the $S\alpha S$ random variables $Y^{n,T_n}_{j,k}$, with $j \in \mathbb{N}$, $n \geq n_0$ and $1 \leq k \leq 4\mu_n$, is equal to 0.*

**Remark 3.5.** Observe that in the case where the function $\psi$ is given by (2.5), according to (2.6), one can simply take $n_0 = 4$.

*Proof.* of Lemma 3.4 Suppose ad absurdum that there are some integers $j_1 \geq 1$, $n_1 \geq n_0 \geq 4$ and $k_1 \in \{1, \ldots, 4\mu_{n_1}\}$ for which $\sigma(Y^{n_1,T_{n_1}}_{j_1,k_1}) = 0$. Then, it results from (2.29) and (2.1) that

$$\int_{\mathbb{R}} |\xi|^{-\alpha H-1} \Big| \sum_{|m|\leq 2^{n_1} T_{n_1}} \big(e^{id_{n_1,m}\xi} - 1\big) \widehat{\widehat{\psi}}_{j_1,k_1}(d_{n_1,m}) \Big|^{\alpha} d\xi = 0,$$

which implies that

$$\sum_{|m|\leq 2^{n_1} T_{n_1}} \big(e^{id_{n_1,m}\xi} - 1\big) \widehat{\widehat{\psi}}_{j_1,k_1}(d_{n_1,m}) = 0, \quad \text{for all } \xi \in \mathbb{R}.$$

Since the continuous functions $\big(e^{id_{n_1,m}(\cdot)} - 1\big)$, $m \in \{-2^{n_1}T_{n_1}, \ldots, 2^{n_1}T_{n_1}\} \setminus \{0\}$, are linearly independent, one gets that

$$\widehat{\widehat{\psi}}_{j_1,k_1}(d_{n_1,m}) = 0, \quad \text{for every } m \in \{-2^{n_1}T_{n_1}, \ldots, 2^{n_1}T_{n_1}\} \setminus \{0\}.$$

In the particular case $m = 1$, using (2.11) and $d_{n_1,1} = 2^{-n_1}$, the latter equality can be expressed as

$$\cos(2^{-j_1-n_1}k_1)\widehat{\psi}(2^{-j_1-n_1}) = 0. \tag{3.5}$$

The equality (3.5) cannot hold. Indeed, since $j_1 \geq 1$, $n_1 \geq 4$ and $1 \leq k_1 \leq 4\mu_{n_1}$, one knows from (2.31) that $0 \leq 2^{-j_1-n_1}k_1 \leq 2^{2-n_1}\mu_{n_1} \leq 1$ and consequently that $\cos(2^{-j_1-n_1}k_1) \geq \cos(1) > 0$; combining the latter inequalities with the inequalities $j_1 \geq 1$, $n_1 \geq n_0$ and (3.3), one gets that

$$\cos(2^{-j_1-n_1}k_1)\widehat{\psi}(2^{-j_1-n_1}) \geq \cos(1) \min_{t\in[0,2^{-n_0}]} \widehat{\psi}(t) > 0.$$

□

**Remark 3.6.** Proving Theorems 3.1 and 3.2 reduces to show that Theorem 2.4 remains valid when $\widehat{\alpha}^{-1}_{\mu_n,\log_2}$ and $\widehat{H}_{\mu_n,\log_2}$ are replaced by $\widehat{\alpha}^{-1,\mathrm{dis}}_{n,\log_2}$ and $\widehat{H}^{\mathrm{dis}}_{n,\log_2}$. Thus, in view of (2.33) and of the fact that $G$ is a bounded $C^\infty$ positive function, it is enough to show that, for some positive deterministic real number $a$, one has almost surely

$$\lim_{n\to+\infty} 2^{na}\big|\widehat{\alpha}^{-1}_{\mu_n,\log_2} - \widehat{\alpha}^{-1,\mathrm{dis}}_{n,\log_2}\big| = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^{na}\big|\widehat{H}_{\mu_n,\log_2} - \widehat{H}^{\mathrm{dis}}_{n,\log_2}\big| = 0. \tag{3.6}$$

Notice that, in view of (3.1), (2.14), (3.2) and (2.15), one can obtain (3.6) by proving, almost surely, that

$$\lim_{n\to+\infty} 2^{na}\big|A_n - \widetilde{A}_n\big| = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^{na}\big|A'_n - \widetilde{A}'_n\big| = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^{na}\big|B_n - \widetilde{B}_n\big| = 0 \tag{3.7}$$

where

$$A_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y_{1,2p-1}| \quad \text{and} \quad \widetilde{A}_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y^{n,T_n}_{1,2p-1}|, \tag{3.8}$$

$$A'_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y_{2,2p-1}| \quad \text{and} \quad \widetilde{A}'_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y^{n,T_n}_{2,2p-1}|, \tag{3.9}$$

$$B_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y_{2,4p-1}| \quad \text{and} \quad \widetilde{B}_n := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \log_2 |Y^{n,T_n}_{2,4p-1}|. \tag{3.10}$$

The global goal of the rest of this section is to establish that the first equality in (3.7) holds; the two other equalities in it can be obtained in the same way, this is why we skip their proofs. In order to reach this goal, we need several preliminary results.

Our next aim is to show, when $n$ goes to $+\infty$, that $Y^{n,T_n}_{j,k}$ is almost surely convergent to $Y_{j,k}$ uniformly in $k \in \{1, \ldots, 4\mu_n\}$, and to estimate the rate of convergence. To this end, a very fundamental ingredient is the following lemma, which improves and generalizes Lemma 2.8 in [4].

**Lemma 3.7.** *Let $(j,k) \in \mathbb{N}^2$ be arbitrary and fixed, for each $n \in \mathbb{N}$, the S$\alpha$S random variable $R^{n,T_n}_{j,k}$ is defined as*

$$R^{n,T_n}_{j,k} := Y^{n,T_n}_{j,k} - Y_{j,k}. \tag{3.11}$$

*Then, there is a finite constant $c > 0$, which does not depend on $n$ and $k$, such that the scale parameter $\sigma(R^{n,T_n}_{j,k})$ satisfies*

$$\sigma(R^{n,T_n}_{j,k}) \leq c\, k \max\big\{2^{-nH}, T_n^{-(L-H-1)}\big\}, \tag{3.12}$$

*where the integer $L \geq 2$ is the same as in (2.4).*

The following proof adapts that of Lemma 3.3 from [4]. Definition (2.29) introduces an additional term to control and the role of $T_n$, equal to $2^n$ in [4], must also be taken into account.

*Proof.* Let $(j,k) \in \mathbb{N}^2$ and $n \in \mathbb{N}$ be arbitrary and fixed. Combining (3.11), the second equality in (2.29) and (2.8), the S$\alpha$S random variable $R^{n,T_n}_{j,k}$ rewrites as

$$\begin{aligned} R^{n,T_n}_{j,k} &= \Re\Bigg( \int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Bigg( \frac{1}{2\pi} \sum_{|m|\leq 2^n T_n} 2^{-n}\big(e^{i d_{n,m}\xi} - 1\big)\widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widetilde{\psi}_{j,k}(\xi)\Bigg)\, d\widetilde{M}_\alpha(\xi)\Bigg) \\ &= \Re\Bigg( \int_{\mathbb{R}} |\xi|^{-H-1/\alpha} \Bigg( \frac{1}{2\pi} \sum_{|m|\leq 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{i d_{n,m}\xi} - 1\big)\widehat{\widetilde{\psi}}_{j,k}(d_{n,m})\, dt - \widetilde{\psi}_{j,k}(\xi)\Bigg)\, d\widetilde{M}_\alpha(\xi)\Bigg). \end{aligned}$$

Note that (2.9), the fact that $\psi$ is an even function, and (2.3) imply that

$$\widetilde{\psi}_{j,k}(0) = 2\psi(k) = 0.$$

Therefore, applying the inverse Fourier transform, $R^{n,T_n}_{j,k}$ can be expressed as

$$
\begin{aligned}
R^{n,T_n}_{j,k} &= \Re\Bigg( \int_{\mathbb{R}} |\xi|^{-H-1/\alpha}\Bigg( \frac{1}{2\pi} \sum_{|m|\le 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi}-1\big)\widehat{\widehat{\psi}}_{j,k}(d_{n,m})\,dt \\
&\qquad\qquad -\frac{1}{2\pi}\int_{\mathbb{R}} \big(e^{it\xi}-1\big)\widehat{\widehat{\psi}}_{j,k}(t)\,dt\Bigg)\,d\widetilde{M}_\alpha(\xi)\Bigg) \\
&= \frac{1}{2\pi}\Re\Bigg( \int_{\mathbb{R}} |\xi|^{-H-1/\alpha}\Bigg( \sum_{|m|\le 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{itd_{n,m}\xi}-1\big)\Big(\widehat{\widehat{\psi}}_{j,k}(d_{n,m})-\widehat{\widehat{\psi}}_{j,k}(t)\Big)\,dt \\
&\qquad\qquad + \sum_{|m|\le 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi}-e^{it\xi}\big)\widehat{\widehat{\psi}}_{j,k}(t)\,dt \\
&\qquad\qquad - \int_{\{t\notin[-T_n,T_n+2^{-n})\}} \big(e^{it\xi}-1\big)\widehat{\widehat{\psi}}_{j,k}(t)\,dt\Bigg)\,d\widetilde{M}_\alpha(\xi)\Bigg).
\end{aligned}
$$

Thus, by (2.1), one obtains that

$$
\begin{aligned}
\sigma(R^{n,T_n}_{j,k})^\alpha = \frac{1}{(2\pi)^\alpha}\int_{\mathbb{R}} |\xi|^{-\alpha H-1}\Bigg| & \sum_{|m|\le 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{itd_{n,m}\xi}-1\big)\Big(\widehat{\widehat{\psi}}_{j,k}(d_{n,m})-\widehat{\widehat{\psi}}_{j,k}(t)\Big)\,dt \\
&+ \sum_{|m|\le 2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \big(e^{id_{n,m}\xi}-e^{it\xi}\big)\widehat{\widehat{\psi}}_{j,k}(t)\,dt \\
&- \int_{\{t\notin[-T_n,T_n+2^{-n})\}} \big(e^{it\xi}-1\big)\widehat{\widehat{\psi}}_{j,k}(t)\,dt\Bigg|^\alpha d\xi.
\end{aligned}
\tag{3.13}
$$

Then, combining (3.13) with the triangle inequality and the inequality $|a+b|^\alpha \le 2^\alpha\big(|a|^\alpha+|b|^\alpha\big)$, for all complex numbers $a$ and $b$, and using that $\widehat{\widehat{\psi}}_{j,k}$ is even, one gets that

$$
\sigma(R^{n,T_n}_{j,k})^\alpha \le \frac{4^\alpha}{\pi^\alpha}\big(\mathcal{T}^n_{j,k}+\mathcal{U}^n_{j,k}+\mathcal{V}^n_{j,k}\big) \tag{3.14}
$$

where

$$
\mathcal{T}^n_{j,k} := \int_{\mathbb{R}} |\xi|^{-\alpha H-1}\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} \big|e^{id_{n,m}\xi}-1\big|\left|\widehat{\widehat{\psi}}_{j,k}(d_{n,m})-\widehat{\widehat{\psi}}_{j,k}(t)\right|\,dt\right)^\alpha d\xi, \tag{3.15}
$$

$$
\mathcal{U}^n_{j,k} := \int_{\mathbb{R}} |\xi|^{-\alpha H-1}\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} \big|e^{i(d_{n,m}-t)\xi}-1\big|\big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\,dt\right)^\alpha d\xi \tag{3.16}
$$

and

$$
\mathcal{V}^n_{j,k} := \int_{\mathbb{R}} |\xi|^{-\alpha H-1}\Big(\int_{T_n}^{\infty} \big|e^{it\xi}-1\big|\big|\widehat{\widehat{\psi}}_{j,k}(t)\big|\,dt\Big)^\alpha d\xi. \tag{3.17}
$$

Let us first bound $\mathcal{U}^n_{j,k}$. In the following, we will use the classical inequality:

$$
\big|e^{i\theta}-1\big| \le \min\big\{|\theta|,2\big\}, \quad \text{for all } \theta\in\mathbb{R}. \tag{3.18}
$$

We split $\mathcal{U}_{j,k}^n$ as

$$\mathcal{U}_{j,k}^n = \mathcal{U}_{j,k}^{n,0} + \mathcal{U}_{j,k}^{n,1},$$

where

$$\mathcal{U}_{j,k}^{n,0} := \int_{\{|\xi|\le 2^n\}} |\xi|^{-\alpha H-1} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| e^{i(d_{n,m}-t)\xi} - 1 \right| \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} d\xi \tag{3.19}$$

and

$$\mathcal{U}_{j,k}^{n,1} := \int_{\{|\xi|> 2^n\}} |\xi|^{-\alpha H-1} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| e^{i(d_{n,m}-t)\xi} - 1 \right| \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} d\xi. \tag{3.20}$$

On one hand, combining (3.19) and (3.18), one gets that

$$\begin{aligned} \mathcal{U}_{j,k}^{n,0} &\le \left( \int_{\{|\xi|\le 2^n\}} |\xi|^{\alpha(1-H)-1}\, d\xi \right) \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| t - d_{n,m} \right| \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} \\ &\le c_1\, 2^{(1-H)n\alpha} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| t - d_{n,m} \right| \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} \end{aligned}$$

where $c_1 = \frac{2}{\alpha(1-H)}$. Then, since

$$|t - d_{n,m}| \le 2^{-n}, \quad \text{for all } (n,m) \in \mathbb{N}\times\mathbb{N} \text{ and } t \in [d_{n,m}, d_{n,m+1}],$$

applying inequality (2.12) one gets that

$$\mathcal{U}_{j,k}^{n,0} \le c_1\, 2^{-n\alpha H} \left( \int_{\mathbb{R}_+} \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} \le c_2\, 2^{-n\alpha H}, \tag{3.21}$$

where the finite positive constant $c_2$ does not depend on $n$ and $k$. On another hand, combining (3.20) and (3.18), one obtains that

$$\begin{aligned} \mathcal{U}_{j,k}^{n,1} &\le 2^{\alpha} \left( \int_{\{|\xi|> 2^n\}} |\xi|^{-\alpha H-1}\, d\xi \right) \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} \\ &\le 2^{\alpha} \left( \int_{\{|\xi|> 2^n\}} |\xi|^{-\alpha H-1}\, d\xi \right) \left( \int_{\mathbb{R}_+} \left| \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha}. \end{aligned}$$

Then inequality (2.12) implies that

$$\mathcal{U}_{j,k}^{n,1} \le c_3\, 2^{-n\alpha H}, \tag{3.22}$$

where the finite positive constant $c_3$ does not depend on $n$ and $k$. Finally, combining (3.21) and (3.22), one derives that

$$\mathcal{U}_{j,k}^n = \mathcal{U}_{j,k}^{n,0} + \mathcal{U}_{j,k}^{n,1} \le c_4\, 2^{-n\alpha H}, \tag{3.23}$$

where the finite positive constant $c_4$ does not depend on $n$ and $k$.

Let us now bound $\mathcal{V}_{j,k}^n$. We split it as

$$\mathcal{V}_{j,k}^n = \mathcal{V}_{j,k}^{n,0} + \mathcal{V}_{j,k}^{n,1},$$

where

$$\mathcal{V}_{j,k}^{n,0} := \int_{\{|\xi|\le 1/T_n\}} |\xi|^{-\alpha H-1}\Big(\int_{T_n}^{\infty} \big|e^{it\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\Big)^{\alpha} d\xi \tag{3.24}$$

and

$$\mathcal{V}_{j,k}^{n,1} := \int_{\{|\xi|> 1/T_n\}} |\xi|^{-\alpha H-1}\Big(\int_{T_n}^{\infty} \big|e^{it\xi}-1\big|\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\Big)^{\alpha} d\xi. \tag{3.25}$$

Next, notice that using (3.18), for every $\varepsilon \in (0,1)$ and for every $(t,\xi)\in\mathbb{R}^2$, one has

$$\big|e^{it\xi}-1\big| = \big|e^{it\xi}-1\big|^{\varepsilon}\big|e^{it\xi}-1\big|^{1-\varepsilon} \le 2^{\varepsilon}|t\xi|^{1-\varepsilon}. \tag{3.26}$$

Let $\widetilde{\tau}\in(0,1-H)\subset(0,1)$. Combining (3.26) with $\varepsilon=\widetilde{\tau}$ and (3.24), one gets that

$$\begin{aligned}
\mathcal{V}_{j,k}^{n,0} &\le 2^{\widetilde{\tau}\alpha}\left(\int_{\{|\xi|\le 1/T_n\}} |\xi|^{\alpha(1-\widetilde{\tau}-H)-1}\,d\xi\right)\left(\int_{T_n}^{\infty} t^{1-\widetilde{\tau}}\,\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\right)^{\alpha}\\
&\le c_5\,T_n^{-(1-\widetilde{\tau}-H)\alpha}\left(\int_{T_n}^{\infty} t^{1-\widetilde{\tau}}\,\big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\right)^{\alpha},
\end{aligned}$$

where $c_5=\frac{2^{1+\widetilde{\tau}\alpha}}{\alpha(1-\widetilde{\tau}-H)}$. Then, since $L+\widetilde{\tau}-1\ge 1+\widetilde{\tau}>1$, the inequality (2.12) implies that

$$\begin{aligned}
\mathcal{V}_{j,k}^{n,0} &\le c_6\,T_n^{-(1-\widetilde{\tau}-H)\alpha}\left(\int_{T_n}^{\infty} (1+t)^{-(L+\widetilde{\tau}-1)}dt\right)^{\alpha}\\
&\le c_7\,T_n^{-(L-1-H)\alpha},
\end{aligned} \tag{3.27}$$

where $c_6$ and $c_7$ are two finite positive constants which do not depend on $n$ and $k$. Moreover, putting together (3.18), (3.25) and (2.12), one obtains that

$$\begin{aligned}
\mathcal{V}_{j,k}^{n,1} &\le 2^{\alpha}\left(\int_{\{|\xi|> 1/T_n\}} |\xi|^{-\alpha H-1}\,d\xi\right)\left(\int_{T_n}^{\infty} \big|\widehat{\widetilde{\psi}}_{j,k}(t)\big|\,dt\right)^{\alpha}\\
&\le c_8\,T_n^{H\alpha}\left(\int_{T_n}^{\infty}(1+t)^{-L}\,dt\right)^{\alpha}\\
&\le c_9\,T_n^{-(L-1-H)\alpha},
\end{aligned} \tag{3.28}$$

where the finite positive constants $c_8$ and $c_9$ do not depend on $n$ and $k$. Finally, combining (3.27) and (3.28), one derives that

$$\mathcal{V}_{j,k}^n = \mathcal{V}_{j,k}^{n,0} + \mathcal{V}_{j,k}^{n,1} \le c_{10}\,T_n^{-(L-1-H)\alpha}, \tag{3.29}$$

where the finite positive constant $c_{10}$ does not depend on $n$ and $k$.

Let us now bound $\mathcal{T}_{j,k}^n$. We split it as

$$\mathcal{T}_{j,k}^n = \mathcal{T}_{j,k}^{n,0} + \mathcal{T}_{j,k}^{n,1},$$

where

$$\mathcal{T}_{j,k}^{n,0} := \int_{\{|\xi|\le 1\}} |\xi|^{-\alpha H-1} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| e^{i d_{n,m}\xi} - 1 \right| \left| \widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} d\xi \tag{3.30}$$

and

$$\mathcal{T}_{j,k}^{n,1} := \int_{\{|\xi|> 1\}} |\xi|^{-\alpha H-1} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left| e^{i d_{n,m}\xi} - 1 \right| \left| \widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widehat{\widetilde{\psi}}_{j,k}(t) \right| dt \right)^{\alpha} d\xi. \tag{3.31}$$

Applying (3.26) with $t = d_{n,m}$ and $\varepsilon = \widetilde{\tau}$, Taylor formula and (2.13), one gets, for all $\xi \in \mathbb{R}$, for every $(n,m) \in \mathbb{N}\times\mathbb{N}$ and for every $t \in [d_{n,m}, d_{n,m+1}]$,

$$\begin{aligned}
\left| e^{i d_{n,m}\xi} - 1 \right| \left| \widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widehat{\widetilde{\psi}}_{j,k}(t) \right| &\le 2^{\widetilde{\tau}} |\xi|^{1-\widetilde{\tau}} d_{n,m}^{1-\widetilde{\tau}} \left| \widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widehat{\widetilde{\psi}}_{j,k}(t) \right| \\
&\le 2^{\widetilde{\tau}} |\xi|^{1-\widetilde{\tau}} d_{n,m}^{1-\widetilde{\tau}} \int_{d_{n,m}}^{t} \left| \widehat{\widetilde{\psi}}'_{j,k}(u) \right| du \\
&\le 2^{\widetilde{\tau}} c_{j,L}\, k |\xi|^{1-\widetilde{\tau}} \int_{d_{n,m}}^{t} d_{n,m}^{1-\widetilde{\tau}} (1+u)^{-L}\, du \\
&\le 2^{\widetilde{\tau}} c_{j,L}\, k |\xi|^{1-\widetilde{\tau}} \int_{d_{n,m}}^{t} (1+u)^{1-\widetilde{\tau}-L}\, du,
\end{aligned}$$

where the last inequality follows from the fact that, for all $u \in [d_{n,m}, t]$, one clearly has that $d_{n,m}^{1-\widetilde{\tau}} \le u^{1-\widetilde{\tau}} \le (1+u)^{1-\widetilde{\tau}}$. Then combining the previous inequalities and (3.30) and using Fubini-Tonelli Theorem, one derives that

$$\begin{aligned}
\mathcal{T}_{j,k}^{n,0} &\le c_{11}\, k^{\alpha} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} \left( \int_{d_{n,m}}^{t} (1+u)^{1-\widetilde{\tau}-L}\, du \right) dt \right)^{\alpha} \\
&\le c_{11}\, k^{\alpha} \left( \sum_{m=0}^{2^n T_n} \int_{d_{n,m}}^{d_{n,m+1}} (d_{n,m+1} - u)(1+u)^{1-\widetilde{\tau}-L}\, du \right)^{\alpha} \\
&\le c_{12}\, 2^{-n\alpha} k^{\alpha},
\end{aligned} \tag{3.32}$$

where

$$c_{11} = 2^{\widetilde{\tau}\alpha} c_{j,L}^{\alpha} \left( \int_{\{|\xi|\le 1\}} |\xi|^{\alpha(1-\widetilde{\tau}-H)-1}\, d\xi \right) \quad \text{and} \quad c_{12} = c_{11} \left( \int_{\mathbb{R}_+} (1+u)^{1-\widetilde{\tau}-L}\, du \right)^{\alpha}$$

are two positive finite constants not depending on $n$ and $k$.

Let us now fix $\tau \in (0,H) \subset (0,1)$. Applying (3.26) with $t = d_{n,m}$ and $\varepsilon = 1-\tau$ and using arguments similar to that which have allowed to obtain (3.32), one derives from (3.31) that

$$\begin{aligned}
\mathcal{T}^{n,1}_{j,k} &\le 2^{(1-\tau)\alpha}\left(\int_{\{|\xi|>1\}} |\xi|^{-\alpha(H-\tau)-1} d\xi\right)\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} d_{n,m}^{\tau}\left|\widehat{\widetilde{\psi}}_{j,k}(d_{n,m}) - \widehat{\widetilde{\psi}}_{j,k}(t)\right| dt\right)^{\alpha}\\
&\le c_{14}k^{\alpha}\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} d_{n,m}^{\tau}(d_{n,m+1}-u)(1+u)^{-L}\, du\right)^{\alpha}\\
&\le c_{14}2^{-n\alpha}k^{\alpha}\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} d_{n,m}^{\tau}(1+u)^{-L}\, du\right)^{\alpha}\\
&\le c_{14}2^{-n\alpha}k^{\alpha}\left(\sum_{m=0}^{2^nT_n}\int_{d_{n,m}}^{d_{n,m+1}} (1+u)^{\tau-L}\, du\right)^{\alpha}\\
&\le c_{14}2^{-n\alpha}k^{\alpha},
\end{aligned}\tag{3.33}$$

where the two constants

$$c_{14} := 2^{(1-\tau)\alpha}c_{j,L}^{\alpha}\left(\int_{\{|\xi|>1\}} |\xi|^{-\alpha(H-\tau)-1} d\xi\right) \quad \text{and} \quad c_{15} := c_{14}\left(\int_{\mathbb{R}_+} (1+u)^{\tau-L}du\right)^{\alpha}$$

are finite since $L-\tau > 1$ and $H-\tau>0$. Then, combining (3.32) and (3.33), one gets that

$$\mathcal{T}^{n}_{j,k} = \mathcal{T}^{n,0}_{j,k} + \mathcal{T}^{n,1}_{j,k} \le c_{16}2^{-n\alpha}k^{\alpha}. \tag{3.34}$$

Finally, putting together (3.14), (3.23), (3.29) and (3.34), it follows that (3.12) is satisfied. □

The following lemma provides, for all $k \in \{1,\dots,4\mu_n\}$, the almost sure uniform convergence of $Y^{n,T_n}_{j,k}$ to $Y_{j,k}$, when $n$ goes to $+\infty$, as well as an estimate of the uniform of convergence.

**Lemma 3.8.** *Let $L \ge 2$ be the same integer as in (2.4). Let $j$ be an arbitrary fixed positive integer and let $\tau$ and $\widetilde{\tau}$ be arbitrary fixed real numbers satisfying*

$$0 < \tau < H \quad \text{and} \quad 0 < \widetilde{\tau} < 1-H. \tag{3.35}$$

*Then, there exists a positive finite random variable $C^*_{L,j}(\tau,\widetilde{\tau})$ such that one has, almost surely for all $n \in \mathbb{N}$,*

$$\sup_{1\le k\le 4\mu_n}\left|Y_{j,k} - Y^{n,T_n}_{j,k}\right| \le C^*_{L,j}(\tau,\widetilde{\tau})\,\max\left\{2^{-n\tau}, T_n^{-(L+\widetilde{\tau}-2)}\right\}, \tag{3.36}$$

*where $(\mu_n)_{n\in\mathbb{N}}$ belongs to the class of admissible sequences introduced in Remark 2.10.*

*Proof.* One knows from Lemma 3.7 that there is a finite constant $c_1 > 0$, which does not depend on $n$ and $k$, such that

$$\sigma\big(Y_{j,k} - Y^{n,T_n}_{j,k}\big) \le c_1\, k \max\left\{2^{-nH}, T_n^{-(L-H-1)}\right\}. \tag{3.37}$$

Let $\gamma \in (0,\alpha)$ be arbitrary and fixed. One denotes by $W$ an arbitrary real-valued S$\alpha$S random variable with scale parameter equals to 1. It follows from standard calculations, Markov inequality and (3.37) that, for all $(j,n)\in\mathbb{N}^2$,

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{1\le k\le 4\mu_n}\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big| > \max\big\{2^{-n\tau}, T_n^{-(L+\widetilde{\tau}-2)}\big\}\Big)\\
&=\mathbb{P}\Big(\sup_{1\le k\le 4\mu_n}\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big|^{\gamma} > \max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\}\Big)\\
&\le \Big(\max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\}\Big)^{-1}\mathbb{E}\Big(\sup_{1\le k\le 4\mu_n}\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big|^{\gamma}\Big)\\
&\le \Big(\max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\}\Big)^{-1}\sum_{k=1}^{4\mu_n}\mathbb{E}\Big(\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big|^{\gamma}\Big)\\
&=\mathbb{E}(|W|^{\gamma})\Big(\max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\}\Big)^{-1}\sum_{k=1}^{4\mu_n}\sigma\big(Y_{j,k}-Y_{j,k}^{n,T_n}\big)^{\gamma}\\
&\le c_2\mu_n^{1+\gamma}\Big(\max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\}\Big)^{-1}\max\big\{2^{-n\gamma H}, T_n^{-\gamma(L-H-1)}\big\},
\end{aligned}
\tag{3.38}
$$

where $c_2 := 4^{1+\gamma}c_1^{\gamma}\,\mathbb{E}(|W|^{\gamma})$ is a positive finite constant. In the sequel, one sets

$$
b_n := \max\big\{2^{-n\gamma(H-\tau)}, T_n^{-\gamma(1-H-\widetilde{\tau})}\big\},\quad \text{for all } n\in\mathbb{N}.
$$

Observe that

$$
\max\big\{2^{-n\gamma H}, T_n^{-\gamma(L-H-1)}\big\} \le b_n\max\big\{2^{-n\gamma\tau}, T_n^{-\gamma(L+\widetilde{\tau}-2)}\big\},\quad \text{for every } n\in\mathbb{N}. \tag{3.39}
$$

Moreover, one knows from (3.35) and (2.30) that there exists $\varepsilon>0$ such that

$$
b_n \le 2^{-\varepsilon n},\quad \text{for all } n\in\mathbb{N}. \tag{3.40}
$$

Putting together (3.38), (3.39) and (3.40), it follows that, for every $(j,n)\in\mathbb{N}^2$,

$$
\mathbb{P}\Big(\sup_{1\le k\le 4\mu_n}\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big| > \max\big\{2^{-n\tau}, T_n^{-(L+\widetilde{\tau}-2)}\big\}\Big) \le c_2\mu_n^{1+\gamma}\,2^{-\varepsilon n}.
$$

Then, one can derive from (2.33) that

$$
\sum_{n=1}^{+\infty}\mathbb{P}\Big(\sup_{1\le k\le 4\mu_n}\big|Y_{j,k}-Y_{j,k}^{n,T_n}\big| > \max\big\{2^{-n\tau}, T_n^{-(L+\widetilde{\tau}-2)}\big\}\Big) < +\infty.
$$

Thus, applying Borel-Cantelli Lemma one gets (3.36). □

Let us now recall Lemma 3.2 from [4] which will be useful in the sequel.

**Lemma 3.9.** *[4, Lemma 3.2] For all $(j,k)\in\mathbb{N}^2$, the scale parameter of the real-valued S$\alpha$S random variable $Y_{j,k}$ (see (2.7) and (2.8)) satisfies*

$$
\sigma(Y_{j,k}) = 2^{jH+1/\alpha}\left(\int_{-4^{-1}}^{4^{-1}}\frac{\big|\psi(\eta)\big|^{\alpha}}{(\eta+k)^{\alpha H+1}}\,d\eta\right)^{1/\alpha},
$$

*which clearly implies (see (2.3)) that*

$$\|\psi\|_{L^\alpha(\mathbb{R})} 2^{jH+1/\alpha}\big(k+4^{-1}\big)^{-(H+1/\alpha)} \\ \leq \sigma(Y_{j,k}) \leq \|\psi\|_{L^\alpha(\mathbb{R})} 2^{jH+1/\alpha}\big(k-4^{-1}\big)^{-(H+1/\alpha)}, \tag{3.41}$$

*where*

$$\|\psi\|_{L^\alpha(\mathbb{R})} := \Big(\int_{\mathbb{R}} \big|\psi(\eta)\big|^\alpha\,d\eta\Big)^{1/\alpha} = \Big(\int_{-4^{-1}}^{4^{-1}} \big|\psi(\eta)\big|^\alpha\,d\eta\Big)^{1/\alpha}.$$

Now, we need to introduce some additional notations.

**Definition 3.10.** For any arbitrary fixed $\delta \in (0,+\infty)$ and for all $n \in \mathbb{N}$, the four random variables $A_{0,n}(\delta)$, $A_{1,n}(\delta)$, $\widetilde{A}_{0,n}(\delta)$ and $\widetilde{A}_{1,n}(\delta)$ are defined as

$$A_{0,n}(\delta) := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \big(\log_2 |Y_{1,2p-1}|\big)\mathbb{1}_{\big\{|Y_{1,2p-1}|\leq 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}, \tag{3.42}$$

$$A_{1,n}(\delta) := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \big(\log_2 |Y_{1,2p-1}|\big)\mathbb{1}_{\big\{|Y_{1,2p-1}|> 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}, \tag{3.43}$$

$$\widetilde{A}_{0,n}(\delta) := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \big(\log_2 |Y^{n,T_n}_{1,2p-1}|\big)\mathbb{1}_{\big\{|Y_{1,2p-1}|\leq 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}, \tag{3.44}$$

and

$$\widetilde{A}_{1,n}(\delta) := \frac{1}{\mu_n}\sum_{p=1}^{\mu_n} \big(\log_2 |Y^{n,T_n}_{1,2p-1}|\big)\mathbb{1}_{\big\{|Y_{1,2p-1}|> 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}. \tag{3.45}$$

Observe that, it easily results from (3.8) that, for all $\delta \in (0,+\infty)$, one has

$$A_n = A_{0,n}(\delta) + A_{1,n}(\delta) \quad \text{and} \quad \widetilde{A}_n = \widetilde{A}_{0,n}(\delta) + \widetilde{A}_{1,n}(\delta). \tag{3.46}$$

The following two lemmas show that, when $n$ goes to $+\infty$, the two positive random variables $\big|A_{0,n}(\delta)\big|$ and $\big|\widetilde{A}_{0,n}(\delta)\big|$ converge almost surely to zero at a geometric rate which depends on $\delta$.

**Lemma 3.11.** *For each fixed $\delta \in (0,+\infty)$, one has almost surely*

$$\lim_{n\to+\infty} 2^{n\theta}\big|A_{0,n}(\delta)\big| = 0, \quad \textit{for all } \theta \in (0,\delta).$$

**Lemma 3.12.** *For each fixed $\delta \in (0,+\infty)$, one has almost surely*

$$\lim_{n\to+\infty} 2^{n\theta}\big|\widetilde{A}_{0,n}(\delta)\big| = 0, \quad \textit{for all } \theta \in (0,\delta).$$

*Proof of Lemma 3.11.* First observe that, one knows from Borel-Cantelli Lemma that for proving Lemma 3.11 it is enough to show that, for any arbitrary fixed $\varepsilon > 0$, the following is true:

$$\sum_{n=1}^{+\infty} \mathbb{P}\big(2^{n\theta}\big|A_{0,n}(\delta)\big| > \varepsilon\big) < +\infty. \tag{3.47}$$

Moreover, since, for all $n \in \mathbb{N}$, one has

$$\mathbb{P}\big(2^{n\theta}\big|A_{0,n}(\delta)\big| > \varepsilon\big) = \mathbb{P}\Big(2^{n\theta/2}\big|A_{0,n}(\delta)\big|^{1/2} > \varepsilon^{1/2}\Big),$$

thanks to Markov inequality, it turns out that (3.47) can be obtained by showing that

$$\sum_{n=1}^{+\infty} 2^{n\theta/2}\,\mathbb{E}\Big(\big|A_{0,n}(\delta)\big|^{1/2}\Big) < +\infty. \tag{3.48}$$

From now on, our goal is to provide, for all $n \in \mathbb{N}$, an appropriate upper bound for the expectation $\mathbb{E}\Big(\big|A_{0,n}(\delta)\big|^{1/2}\Big)$, which allows to get (3.48). Using (3.42), the triangle inequality, the inequality

$$\Big(\sum_{p=1}^{m}|x_p|\Big)^{1/2} \le \sum_{p=1}^{m}|x_p|^{1/2}, \quad \text{for all } m \in \mathbb{N} \text{ and } x_1,\ldots,x_m \in \mathbb{R}, \tag{3.49}$$

and Cauchy-Schwarz inequality it follows that

$$\begin{aligned}
&\mathbb{E}\Big(\big|A_{0,n}(\delta)\big|^{1/2}\Big) = \mu_n^{-1/2}\,\mathbb{E}\Big(\Big|\sum_{p=1}^{\mu_n}\big(\log_2|Y_{1,2p-1}|\big)\mathbb{1}_{\big\{|Y_{1,2p-1}|\le 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}\Big|^{1/2}\Big)\\
&\le \mu_n^{-1/2}\,\mathbb{E}\Big(\Big(\sum_{p=1}^{\mu_n}\big|\log_2|Y_{1,2p-1}|\big|\mathbb{1}_{\big\{|Y_{1,2p-1}|\le 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}\Big)^{1/2}\Big)\\
&\le \mu_n^{-1/2}\sum_{p=1}^{\mu_n}\mathbb{E}\Big(\big|\log_2|Y_{1,2p-1}|\big|^{1/2}\,\mathbb{1}_{\big\{|Y_{1,2p-1}|\le 2^{-n\delta}\sigma(Y_{1,2p-1})\big\}}\Big)\\
&\le \mu_n^{-1/2}\sum_{p=1}^{\mu_n}\Big(\mathbb{E}\big|\log_2|Y_{1,2p-1}|\big|\Big)^{1/2}\Big(\mathbb{P}\big(|Y_{1,2p-1}|\le 2^{-n\delta}\sigma(Y_{1,2p-1})\big)\Big)^{1/2}.
\end{aligned} \tag{3.50}$$

Recall that, one knows from Lemma 3.9 that, for every $p \in \mathbb{N}$, one has $\sigma(Y_{1,2p-1}) > 0$. Also, one knows from Remark 2.1 that, for each $p \in \mathbb{N}$, the real-valued random variable $Y_{1,2p-1}/\sigma(Y_{1,2p-1})$ is S$\alpha$S with scale parameter equals to 1. In the sequel, one denotes by $W$ an arbitrary real-valued S$\alpha$S random variable with scale parameter equals to 1. On one hand, since $W$ has a continuous bounded probability density function over the real line, one has, for some positive finite constant $c_1$, not depending on $p$ and $n$,

$$\mathbb{P}\big(|Y_{1,2p-1}| \le 2^{-n\delta}\sigma(Y_{1,2p-1})\big) = \mathbb{P}\big(|W| \le 2^{-n\delta}\big) \le c_1 2^{-n\delta}, \text{ for all integers } 1 \le p \le \mu_n. \tag{3.51}$$

On the other hand, one can derive from the triangle inequality and (3.49) that, for all integers $1 \le p \le \mu_n$,

$$\begin{aligned}
\Big(\mathbb{E}\big|\log_2|Y_{1,2p-1}|\big|\Big)^{1/2} &= \Big(\mathbb{E}\Big|\log_2|W| + \log_2\big(\sigma(Y_{1,2p-1})\big)\Big|\Big)^{1/2}\\
&\le \Big(\mathbb{E}\big|\log_2|W|\big| + \big|\log_2\big(\sigma(Y_{1,2p-1})\big)\big|\Big)^{1/2}\\
&\le \Big(\mathbb{E}\big|\log_2|W|\big|\Big)^{1/2} + \big|\log_2\big(\sigma(Y_{1,2p-1})\big)\big|^{1/2}.
\end{aligned} \tag{3.52}$$

Next, setting $c_2 := c_1^{1/2}\Big(\mathbb{E}\big|\log_2|W|\big|\Big)^{1/2}$ and putting together (3.50), (3.51) and (3.52), one obtains, for all $n\in\mathbb{N}$, that

$$\mathbb{E}\Big(\big|A_{0,n}(\delta)\big|^{1/2}\Big) \le c_2\,\mu_n^{1/2}\,2^{-n\delta/2} + c_1^{1/2}\,2^{-n\delta/2}\mu_n^{-1/2}\sum_{p=1}^{n}\big|\log_2\left(\sigma(Y_{1,2p-1})\right)\big|^{1/2}. \tag{3.53}$$

Next notice that (3.41) implies that there exists two finite constants $0<c_3<c_4$ such that, for all $p\in\mathbb{N}$, one has

$$c_3(p+1)^{-H-1/\alpha} \le \sigma(Y_{1,2p-1}) \le c_4(p+1)^{-H-1/\alpha}. \tag{3.54}$$

Thus, setting $c_5 := |\log_2(c_3)|+|\log_2(c_4)|+H+1/\alpha$, one can derive from the second inequality in (3.54) that

$$\log_2\big(\sigma(Y_{1,2p-1})\big) \le \log_2(c_4)-(H+1/\alpha)\log_2(p+1) \le c_5\log_2(p+1), \tag{3.55}$$

and one can derive from the first inequality in (3.54) that

$$-\log_2\big(\sigma(Y_{1,2p-1})\big) \le -\log_2(c_3)+(H+1/\alpha)\log_2(p+1) \le c_5\log_2(p+1). \tag{3.56}$$

Then, combining (3.55) and (3.56), one gets, for every $p\in\mathbb{N}$, that

$$\big|\log_2\big(\sigma(Y_{1,2p-1})\big)\big| \le c_5\log_2(p+1),$$

and consequently that, for all $n\in\mathbb{N}$,

$$\mu_n^{-1/2}\sum_{p=1}^{\mu_n}\big|\log_2\big(\sigma(Y_{1,2p-1})\big)\big|^{1/2} \le c_5^{1/2}\mu_n^{1/2}\big(\log_2(\mu_n+1)\big)^{1/2}. \tag{3.57}$$

Finally, since $\theta<\delta$, it results from (3.53), (3.57) and (2.33) that (3.48) holds. $\square$

For proving Lemma 3.12, one needs the following two lemmas. The first one of them is basically a consequence of Lemma 3.7.

**Lemma 3.13.** *Let $L\ge 2$ be the same integer as in* (2.4).

*(i) When $\alpha\in[1,2]$, there exists a finite constant $c'>0$ such that, for all $n\in\mathbb{N}$,*

$$\sup_{1\le p\le\mu_n}\big|\sigma(Y_{1,2p-1})-\sigma(Y_{1,2p-1}^{n,T_n})\big| \le c'\mu_n\max\big\{2^{-nH},T_n^{-(L-H-1)}\big\}. \tag{3.58}$$

*(ii) When $\alpha\in(0,1)$, there exists a finite constant $c''>0$ such that, for all $n\in\mathbb{N}$,*

$$\sup_{1\le p\le\mu_n}\big|\sigma(Y_{1,2p-1})^{\alpha}-\sigma(Y_{1,2p-1}^{n,T_n})^{\alpha}\big| \le c''\mu_n^{\alpha}\max\big\{2^{-n\alpha H},T_n^{-\alpha(L-H-1)}\big\}. \tag{3.59}$$

*Proof.* One knows from (2.8) and (2.29) that, for all positive integers $n$ and $p$, one has

$$Y_{1,2p-1}=\Re\Big(\int_{\mathbb{R}}F_p(\xi)d\widetilde{M}_\alpha(\xi)\Big) \quad\text{and}\quad Y_{1,2p-1}^{n,T_n}=\Re\Big(\int_{\mathbb{R}}F_p^n(\xi)d\widetilde{M}_\alpha(\xi)\Big),$$

where the functions $F_p$ and $F_p^n$ belong to $L^\alpha(\mathbb{R})$. Thus, it results from (2.1) that

$$\sigma(Y_{1,2p-1}) = \|F_p\|_{L^\alpha(\mathbb{R})}, \quad \sigma(Y^n_{1,2p-1}) = \|F^n_p\|_{L^\alpha(\mathbb{R})} \text{ and } \sigma(Y_{1,2p-1} - Y^n_{1,2p-1}) = \|F_p - F^n_p\|_{L^\alpha(\mathbb{R})}.$$

When $\alpha \in [1,2]$, using the inequality $\big|\|F_p\|_{L^\alpha(\mathbb{R})} - \|F^n_p\|_{L^\alpha(\mathbb{R})}\big| \le \|F_p - F^n_p\|_{L^\alpha(\mathbb{R})}$ and (3.12), one obtains (3.58).

When $\alpha \in (0,1)$, using the inequality $\big|\|F_p\|^\alpha_{L^\alpha(\mathbb{R})} - \|F^n_p\|^\alpha_{L^\alpha(\mathbb{R})}\big| \le \|F_p - F^n_p\|^\alpha_{L^\alpha(\mathbb{R})}$ and (3.12), one gets (3.59). □

**Lemma 3.14.** *There are a positive integer $n_1$ and two finite constants $0 < c_1 < c_2$ such that, for all integers $n \ge n_1$ and $1 \le p \le \mu_n$, one has*

$$c_1(p+1)^{-H-1/\alpha} \le \sigma(Y^{n,T_n}_{1,2p-1}) \le c_2(p+1)^{-H-1/\alpha}. \tag{3.60}$$

*Proof.* First we suppose that $\alpha \in [1,2]$. Let $c_3$ and $c_4$ be the same positive constants as in (3.54); thus one has, for all integers $n \ge 1$ and $1 \le p \le \mu_n$,

$$c_3(p+1)^{-H-1/\alpha} \le \sigma(Y^{n,T_n}_{1,2p-1}) - \big(\sigma(Y^{n,T_n}_{1,2p-1}) - \sigma(Y_{1,2p-1})\big) \le c_4(p+1)^{-H-1/\alpha}$$

and consequently that

$$\begin{aligned} &-\sup_{1\le p\le\mu_n} \big|\sigma(Y_{1,2p-1}) - \sigma(Y^{n,T_n}_{1,2p-1})\big| + c_3(p+1)^{-H-1/\alpha} \\ &\le \sigma(Y^{n,T_n}_{1,2p-1}) \le c_4(p+1)^{-H-1/\alpha} + \sup_{1\le p\le\mu_n} \big|\sigma(Y_{1,2p-1}) - \sigma(Y^n_{1,2p-1})\big|. \end{aligned}$$

Then, one can derive from (3.58) that one has for some finite constant $c_5 > 0$ and for all integers $n \ge 1$ and $1 \le p \le \mu_n$,

$$\begin{aligned} &-c_5\,\mu_n \max\big\{2^{-nH}, T_n^{-(L-H-1)}\big\} + c_3(p+1)^{-H-1/\alpha} \\ &\le \sigma(Y^{n,T_n}_{1,2p-1}) \le c_4(p+1)^{-H-1/\alpha} + c_5\,\mu_n \max\big\{2^{-nH}, T_n^{-(L-H-1)}\big\}. \end{aligned} \tag{3.61}$$

Next, observe that, since $L \ge 2 > H+1$, one knows from (2.30) and (2.33) that

$$\lim_{n\to+\infty} (\mu_n+1)^{H+1/\alpha}\,\mu_n \max\big\{2^{-nH}, T_n^{-(L-H-1)}\big\} = 0.$$

Thus, there exists a positive integer $n_1'$ such that, for all integer $n \ge n_1$,

$$c_5\,\mu_n \max\big\{2^{-nH}, T_n^{-(L-H-1)}\big\} \le 2^{-1}\min\{c_3, c_4\}(\mu_n+1)^{-H-1/\alpha}. \tag{3.62}$$

Next, combining (3.61) and (3.62), it follows that, for every integers $n \ge n_1'$ and $1 \le p \le \mu_n$, one has

$$(1/2)c_3(p+1)^{-H-1/\alpha} \le \sigma(Y^{n,T_n}_{1,2p-1}) \le (3/2)c_4(p+1)^{-H-1/\alpha},$$

which shows that (3.60) holds.

Let us now suppose that $\alpha \in (0,1)$. Using (3.54), one has, for all integers $n \ge 1$ and $1 \le p \le \mu_n$,

$$c_3^\alpha(p+1)^{-\alpha H-1} \le \sigma(Y^{n,T_n}_{1,2p-1})^\alpha - \big(\sigma(Y^{n,T_n}_{1,2p-1})^\alpha - \sigma(Y_{1,2p-1})^\alpha\big) \le c_4^\alpha(p+1)^{-\alpha H-1}$$

and consequently that

$$
\begin{aligned}
&- \sup_{1\le p\le \mu_n} \big|\sigma(Y_{1,2p-1})^{\alpha} - \sigma(Y_{1,2p-1}^{n,T_n})^{\alpha}\big| + c_3^{\alpha}(p+1)^{-\alpha H-1} \\
&\le \sigma(Y_{1,2p-1}^{n,T_n})^{\alpha} \le c_4^{\alpha}(p+1)^{-\alpha H-1} + \sup_{1\le p\le \mu_n} \big|\sigma(Y_{1,2p-1})^{\alpha} - \sigma(Y_{1,2p-1}^{n,T_n})^{\alpha}\big|.
\end{aligned}
$$

Then, one can derive from (3.59) that one has for some constant $c_6 > 0$ and for all integers $n \ge 1$ and $1 \le p \le \mu_n$,

$$
\begin{aligned}
&- c_6\, \mu_n^{\alpha} \max\big\{2^{-n\alpha H}, T_n^{-\alpha(L-H-1)}\big\} + c_3^{\alpha}(p+1)^{-\alpha H-1} \\
&\le \sigma(Y_{1,2p-1}^{n,T_n})^{\alpha} \le c_4^{\alpha}(p+1)^{-\alpha H-1} + c_6\, \mu_n^{\alpha} \max\big\{2^{-n\alpha H}, T_n^{-\alpha(L-H-1)}\big\}.
\end{aligned}
\tag{3.63}
$$

Similarly to (3.62), it can be shown that there exists a positive integer $n_1''$ such that, for all integer $n \ge n_1''$,

$$
c_6\, \mu_n^{\alpha} \max\big\{2^{-n\alpha H}, T_n^{-\alpha(L-H-1)}\big\} \le 2^{-1} \min\{c_3^{\alpha}, c_4^{\alpha}\}(\mu_n+1)^{-\alpha H-1}. \tag{3.64}
$$

Next, combining (3.63) and (3.64), it follows that, for every integers $n \ge n_1''$ and $1 \le p \le \mu_n$, one has

$$
(1/2)c_3^{\alpha}(p+1)^{-\alpha H-1} \le \sigma(Y_{1,2p-1}^{n,T_n})^{\alpha} \le (3/2)c_4^{\alpha}(p+1)^{-\alpha H-1},
$$

which shows that (3.60) holds. □

Let us now prove Lemma 3.12.

*Proof of Lemma 3.12.* One sets $n_2 := \max\{n_0, n_1\}$, where the integers $n_0$ and $n_1$ are the same as in Lemmas 3.4 and 3.14. By using similar arguments as those in the beginning of the proof of Lemma 3.11, it turns out that for proving Lemma 3.12 it is enough to show that

$$
\sum_{n=n_2}^{+\infty} 2^{n\theta/2}\, \mathbb{E}\Big(\big|\widetilde{A}_{0,n}(\delta)\big|^{1/2}\Big) < +\infty. \tag{3.65}
$$

Moreover, by using (3.44) and arguments similar to those which have allowed to obtain (3.50), it can be shown, for all integer $n \ge n_2$, that

$$
\mathbb{E}\Big(\big|\widetilde{A}_{0,n}(\delta)\big|^{1/2}\Big) \le \mu_n^{-1/2} \sum_{p=1}^{\mu_n} \Big(\mathbb{E}\big|\log_2 |Y_{1,2p-1}^{n,T_n}|\big|\Big)^{1/2} \Big(\mathbb{P}\big(|Y_{1,2p-1}| \le 2^{-n\delta}\sigma(Y_{1,2p-1})\big)\Big)^{1/2}.
$$

Therefore, (3.51) implies that

$$
\mathbb{E}\Big(\big|\widetilde{A}_{0,n}(\delta)\big|^{1/2}\Big) \le c_1^{1/2}\, 2^{-n\delta/2} \mu_n^{-1/2} \sum_{p=1}^{\mu_n} \Big(\mathbb{E}\big|\log_2 |Y_{1,2p-1}^{n,T_n}|\big|\Big)^{1/2}, \tag{3.66}
$$

where the finite constant $c_1 > 0$, not depending on $n$, is the same as in (3.51). Next, thanks to (3.66) and (3.4), similarly to (3.53) it can be shown that

$$
\mathbb{E}\Big(\big|\widetilde{A}_{0,n}(\delta)\big|^{1/2}\Big) \le c_2\, \mu_n^{1/2}\, 2^{-n\delta/2} + c_1^{1/2}\, 2^{-n\delta/2} \mu_n^{-1/2} \sum_{p=1}^{\mu_n} \big|\log_2\big(\sigma(Y_{1,2p-1}^{n,T_n})\big)\big|^{1/2}, \tag{3.67}
$$

where the finite constant $c_2 > 0$, not depending on $n$, is the same as in (3.53). Moreover, by using Lemma 3.14, similarly to (3.57), it can be shown, for all integer $n \geq n_2$, that

$$\mu_n^{-1/2} \sum_{p=1}^{\mu_n} \big| \log_2 \big(\sigma(Y^{n,T_n}_{1,2p-1})\big)\big|^{1/2} \leq c_3\, \mu_n^{1/2}\big(\log_2(\mu_n+1)\big)^{1/2}, \tag{3.68}$$

for some finite constant $c_3 > 0$ not depending on $n$. Finally, since $\theta < \delta$, combining (3.67) and (3.68) with (2.33), one obtains (3.65). $\square$

We are now ready to prove Theorems 3.1 and 3.2.

*Proof of Theorems 3.1 and 3.2 .*

As already mentioned in Remark 3.6, in order to show that the two equalities in (3.6) hold almost surely it is enough to prove that the three equalities in (3.7) are valid almost surely; in the sequel, we only give the proof of the first one of them since those of the two others are similar.

Let $n_0$ be the same positive integer as in Lemma 3.4 and let $a$ be an arbitrary fixed positive real number satisfying

$$a < 2^{-1} \min\{H, \zeta(1-H)\}, \tag{3.69}$$

where $\zeta \in (0,1]$ is the same as in (2.30). Then, the positive real number $\varepsilon$ is defined as

$$\varepsilon := 2^{-1}\big(2^{-1}\min\{H,\zeta(1-H)\} - a\big). \tag{3.70}$$

Using (3.46) with $\delta = a + \varepsilon$ and the triangle inequality, one obtains, for all integer $n \geq n_0$,

$$2^{na}\big|A_n - \widetilde{A}_n\big| \leq 2^{na}\big|A_{0,n}(a+\varepsilon)\big| + 2^{na}\big|\widetilde{A}_{0,n}(a+\varepsilon)\big| + 2^{na}\big|A_{1,n}(a+\varepsilon) - \widetilde{A}_{1,n}(a+\varepsilon)\big|. \tag{3.71}$$

Observe that one knows from Lemmas 3.11 and 3.12 that one has, almost surely,

$$\lim_{n\to+\infty} 2^{na}\big|A_{0,n}(a+\varepsilon)\big| = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^{na}\big|\widetilde{A}_{0,n}(a+\varepsilon)\big| = 0. \tag{3.72}$$

Thus, in view of (3.71) and (3.72), in order to show that the first equality in (3.7) holds almost surely, it remains to prove that the following equality is valid almost surely:

$$\lim_{n\to+\infty} 2^{na}\big|A_{1,n}(a+\varepsilon) - \widetilde{A}_{1,n}(a+\varepsilon)\big| = 0. \tag{3.73}$$

Using (3.43), (3.45) and the triangle inequality, one gets, for all integer $n \geq n_0$,

$$\begin{aligned} &\big|A_{1,n}(a+\varepsilon) - \widetilde{A}_{1,n}(a+\varepsilon)\big| \\ &\leq \mu_n^{-1} \sum_{p=1}^{\mu_n} \big| \log_2|Y_{1,2p-1}| - \log_2|Y^{n,T_n}_{1,2p-1}|\big| \mathbb{1}_{\left\{|Y_{1,2p-1}| > 2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}. \end{aligned} \tag{3.74}$$

Next, for each $p \in \{1, \ldots, \mu_n\}$, let $\Lambda^{n,T_n}_{1,2p-1}$ be the almost surely positive random variable defined as

$$\Lambda^{n,T_n}_{1,2p-1} := \min\big\{|Y_{1,2p-1}|, |Y^{n,T_n}_{1,2p-1}|\big\}. \tag{3.75}$$

It follows from (3.74), the mean-value theorem applied to the function $\log_2$, (3.75) and the triangle inequality that

$$
\begin{aligned}
&\big|A_{1,n}(a+\varepsilon)-\widetilde{A}_{1,n}(a+\varepsilon)\big|\\
&\le (\log 2)^{-1}\mu_n^{-1}\sum_{p=1}^{\mu_n}\frac{\big||Y_{1,2p-1}|-|Y_{1,2p-1}^{n,T_n}|\big|}{\Lambda_{1,2p-1}^{n,T_n}}\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\\
&\le (\log 2)^{-1}\mu_n^{-1}\sum_{p=1}^{\mu_n}\frac{|Y_{1,2p-1}-Y_{1,2p-1}^{n,T_n}|}{\Lambda_{1,2p-1}^{n,T_n}}\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}.
\end{aligned}\tag{3.76}
$$

Next, notice that (3.69) and (3.70) imply that

$$
2a+2\varepsilon<\min\{H,\zeta(1-H)\}.
$$

Thus, one can derive from (3.36) with $\tau=2a+2\varepsilon$ and $\widetilde{\tau}=(2a+2\varepsilon)/\zeta$ that there exists a positive finite random variable $C_1$ such that one has, almost surely, for all integer $n\ge n_0$,

$$
\sup_{1\le p\le\mu_n}|Y_{1,2p-1}-Y_{1,2p-1}^{n,T_n}|\le C_1\max\left\{2^{-2n(a+\varepsilon)},T_n^{-(L-2)-(2a+2\varepsilon)/\zeta}\right\}\le C_1 2^{-2n(a+\varepsilon)},\tag{3.77}
$$

where the last inequality follows from (2.30). On another hand, observe that (3.75) implies, for all $p\in\{1,\dots,\mu_n\}$, that

$$
\Lambda_{1,2p-1}^{n,T_n}\ge|Y_{1,2p-1}|-|Y_{1,2p-1}-Y_{1,2p-1}^{n,T_n}|.
$$

Combining the latter inequality with (3.77), one obtains almost surely, for all $n\ge n_0$ and $p\in\{1,\dots,\mu_n\}$, that

$$
\Lambda_{1,2p-1}^{n,T_n}\ge|Y_{1,2p-1}|-C_1 2^{-2n(a+\varepsilon)}.\tag{3.78}
$$

Moreover, (3.78) and the first inequality in (3.54) entail almost surely, for all $n\ge n_0$ and $p\in\{1,\dots,\mu_n\}$, that

$$
\begin{aligned}
&\Lambda_{1,2p-1}^{n,T_n}\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\ge\Big(|Y_{1,2p-1}|-C_1 2^{-2n(a+\varepsilon)}\Big)\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\\
&\ge 2^{-n(a+\varepsilon)}\Big(\sigma(Y_{1,2p-1})-C_1 2^{-n(a+\varepsilon)}\Big)\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\\
&\ge 2^{-n(a+\varepsilon)}\Big(c_2(p+1)^{-H-1/\alpha}-C_1 2^{-n(a+\varepsilon)}\Big)\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\\
&\ge 2^{-n(a+\varepsilon)}\Big(c_2(\mu_n+1)^{-H-1/\alpha}-C_1 2^{-n(a+\varepsilon)}\Big)\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}\\
&\ge 2^{-n(a+\varepsilon)}(\mu_n+1)^{-H-1/\alpha}\Big(c_2-C_1 2^{-n(a+\varepsilon)}(\mu_n+1)^{H+1/\alpha}\Big)\mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}},
\end{aligned}\tag{3.79}
$$

where $c_2$ denotes the positive deterministic constant $c_3$ in (3.54). Next, observe that one knows from (2.32) that $\lim_{n\to+\infty}2^{-n(a+\varepsilon)}(\mu_n+1)^{H+1/\alpha}=0$. Thus, there exists a random integer $\nu\ge n_0$ such that, for all integer $n\ge\nu$, one has

$$
c_2-C_1 2^{-n(a+\varepsilon)}(\mu_n+1)^{H+1/\alpha}\ge 2^{-1}c_2.\tag{3.80}
$$

Then setting $c_3 := 2^{-1}c_2$ and combining (3.79) and (3.80), it follows that one has almost surely, for every integer $n \geq \nu$ and $p \in \{1, \dots, \mu_n\}$,

$$\Lambda_{1,2p-1}^{n,T_n} \mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}} \geq c_3 2^{-n(a+\varepsilon)}(\mu_n+1)^{-H-1/\alpha} \mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}.$$

This amounts to say that one has almost surely, for every integer $n \geq \nu$ and $p \in \{1, \dots, \mu_n\}$,

$$\left(\Lambda_{1,2p-1}^{n,T_n}\right)^{-1} \mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}} \leq c_3^{-1}\, 2^{n(a+\varepsilon)}(\mu_n+1)^{H+1/\alpha} \mathbb{1}_{\left\{|Y_{1,2p-1}|>2^{-n(a+\varepsilon)}\sigma(Y_{1,2p-1})\right\}}. \tag{3.81}$$

Next, let $C_4$ be the positive finite random variable defined as $C_4 := c_3^{-1}(\log 2)^{-1}C_1$. Putting together (3.76), (3.77) and (3.81) one obtains almost surely, for all integer $n \geq \nu$,

$$\left|A_{1,n}(a+\varepsilon) - \widetilde{A}_{1,n}(a+\varepsilon)\right| \leq C_4 2^{-n(a+\varepsilon)}(\mu_n+1)^{H+1/\alpha}. \tag{3.82}$$

Finally, a straightforward consequence of (2.32) and (3.82) is that the equality (3.73) holds almost surely. □

# 4 Power estimators

In this Section, we construct discrete versions of the power estimators introduced in Theorems 2.6 and 2.8, which are nothing else than Theorems 1.7 and 1.10 in [4]. The first main result of the section concerns consistency of these discrete estimators; it holds up to considering a subsequence $\left(\mu_{n_j}\right)_{j\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$, rather similarly to what has been done in [4] for the continuous power estimators. More precisely, the first main result of the section is the following theorem; one mentions in passing that the estimators $\widehat{H}_{j,\gamma}^{\mathrm{dis}}$ and $\widehat{\alpha}_{j,\gamma}^{-1,\mathrm{dis}}$ in it are almost surely well-defined thanks to Lemma 3.4.

**Theorem 4.1.** *One assumes that $\alpha \in [\underline{\alpha}, 2]$, where the lower bound $\underline{\alpha} \in (0,2]$ is known. Also, one assumes that $\gamma \in (0, 4^{-1}\underline{\alpha})$ is arbitrary and fixed. Let $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers greater than $n_0$ (see Lemma 3.4) and such that*

$$\lim_{j\to+\infty} n_j = +\infty.$$

*One defines, for all $j \in \mathbb{N}$,*

$$\widehat{H}_{j,\gamma}^{dis} := \gamma^{-1} \log_2 \left( \frac{\sum_{p=1}^{\mu_{n_j}} \left|Y_{2,2p-1}^{n_j,T_{n_j}}\right|^\gamma}{\sum_{p=1}^{\mu_{n_j}} \left|Y_{1,2p-1}^{n_j,T_{n_j}}\right|^\gamma} \right) \tag{4.1}$$

*and*

$$\widehat{\alpha}_{j,\gamma}^{-1,dis} := \gamma^{-1} \left( 1 - \log_2 \left( \frac{\sum_{p=1}^{\mu_{n_j+1}} \left|Y_{2,2p-1}^{n_j,T_{n_j}}\right|^\gamma}{\sum_{p=1}^{\mu_{n_j}} \left|Y_{1,2p-1}^{n_j,T_{n_j}}\right|^\gamma} \right) \right). \tag{4.2}$$

*The following two results hold.*

*(i) $\widehat{H}_{j,\gamma}^{dis}$ is a strongly consistent (almost surely convergent) estimator of the Hurst parameter $H$ of HFSM.*

*(ii) Under the condition*

$$\lim_{j\to+\infty} \log_2\left(\frac{\mu_{n_{j+1}}}{\mu_{n_j}}\right) = 1, \tag{4.3}$$

*$\widehat{\alpha}_{j,\gamma}^{-1,dis}$ is a strongly consistent (almost surely convergent) estimator of the inverse $\alpha^{-1}$ of the stability parameter $\alpha$ of HFSM.*

The second main result of the section is following theorem which shows that the estimators $\widehat{H}_{j,\gamma}^{\mathrm{dis}}$ and $\widehat{\alpha}_{j,\gamma}^{-1,\mathrm{dis}}$ are asymptotically normal under some additional conditions.

**Theorem 4.2.** *Let $\underline{\alpha}$ be as in Theorems 2.6 and 4.1, and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and such that (2.23) holds. One denotes by $(\mu_{1,n})_{n\in\mathbb{N}}$ and $(\mu_{2,n})_{n\in\mathbb{N}}$ two non-decreasing sequences of integers larger than 2, which satisfy the conditions (2.31) and (2.32). Moreover, one denotes $(n_{1,j})_{j\in\mathbb{N}}$ and $(n_{2,j})_{j\in\mathbb{N}}$ two non-decreasing sequence of positive integers larger $n_0$ such that*

$$\lim_{j\to+\infty} n_{1,j} = \lim_{j\to+\infty} n_{2,j} = +\infty,$$

*and one assumes that $(\mu_{2,n_{2,j}})_{j\in\mathbb{N}}$ satisfies the following strengthened version of the condition (4.3):*

$$\lim_{j\to+\infty} (\mu_{2,n_{2,j}})^{1/2} \left|\log_2\left(\frac{\mu_{2,n_{2,j+1}}}{\mu_{2,n_{2,j}}}\right) - 1\right| = 0\,. \tag{4.4}$$

*For all $j\in\mathbb{N}$, let $\widehat{H}_{1,j,\gamma}^{dis}$ be the estimator of $H$ obtained by replacing in (4.1) $n_j$ and $\mu_{n_j}$ by $n_{1,j}$ and $\mu_{1,n_{1,j}}$ respectively, and let $\widehat{\alpha}_{2,j,\gamma}^{-1,dis}$ be the estimator of $\alpha^{-1}$ obtained by replacing in (4.2) $n_j$, $\mu_{n_j}$ and $\mu_{n_{j+1}}$ by $n_{2,j}$, $\mu_{2,n_{2,j}}$ and $\mu_{2,n_{2,j+1}}$ respectively. The two random variables $D_{1,j,\gamma}^{dis}$ and $D_{2,j,\gamma}^{dis}$ are then defined as*

$$D_{1,j,\gamma}^{dis} := 2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}^{dis}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1,dis})\big)(\mu_{1,n_{1,j}})^{1/2}\Big(\widehat{H}_{1,j,\gamma}^{dis} - H\Big)$$

*and*

$$D_{2,j,\gamma}^{dis} := (2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}^{dis}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1,dis})\big)(\mu_{2,n_{2,j}})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1,dis} - \alpha^{-1}\Big)\,,$$

*where $F_\gamma$ is the positive continuous function on $[0,1]\times[2^{-1},\underline{\alpha}^{-1}]$ introduced in (2.24), and $\tau_1$ and $\tau_2$ are the two continuous "truncation" functions introduced in (2.28). When $j$ goes to $+\infty$, the two random variables $D_{1,j,\gamma}^{dis}$ and $D_{2,j,\gamma}^{dis}$ converge in distribution to a random variable having a $\mathcal{N}(0,1)$ Gaussian distribution.*

**Remark 4.3.** Assumption (4.4) is not required for the asymptotic normality of $D_{1,j,\gamma}^{\mathrm{dis}}$. Moreover, one can replace in $D_{1,j,\gamma}^{\mathrm{dis}}$ the estimator $\widehat{\alpha}_{2,j,\gamma}^{-1,\mathrm{dis}}$ by any strongly consistent estimator of the inverse $1/\alpha$.

Let us start by introducing some notations. For all $n\in\mathbb{N}$ and $\ell\in\{1,2\}$, we set

$$V_{\ell,\gamma}^{\mu_n} := \sum_{p=1}^{\mu_n} \left|Y_{\ell,2p-1}\right|^\gamma \tag{4.5}$$

and

$$\mathcal{V}_{\ell,\gamma}^{\mu_n,n} := \sum_{p=1}^{\mu_n} |Y_{\ell,2p-1}^{n,T_n}|^{\gamma}. \tag{4.6}$$

Let us mention that, with this notation, one has, for all $j \in \mathbb{N}$

$$\widehat{H}_{j,\gamma}^{\text{dis}} := \gamma^{-1} \log_2 \left( \frac{\mathcal{V}_{2,\gamma}^{\mu_{n_j},n_j}}{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}} \right)$$

and

$$\widehat{\alpha}_{j,\gamma}^{-1,\text{dis}} := \gamma^{-1} \left( 1 - \log_2 \left( \frac{\mathcal{V}_{2,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}} \right) \right)$$

as well as

$$\widehat{H}_{j,\gamma} := \gamma^{-1} \log_2 \left( \frac{V_{2,\gamma}^{\mu_{n_j}}}{V_{1,\gamma}^{\mu_{n_j}}} \right)$$

and

$$\widehat{\alpha}_{j,\gamma}^{-1} := \gamma^{-1} \left( 1 - \log_2 \left( \frac{V_{2,\gamma}^{\mu_{n_{j+1}}}}{V_{1,\gamma}^{\mu_{n_j}}} \right) \right).$$

Roughly speaking, we aim at showing that Theorems 2.6 and 2.8 remain valid when $\widehat{H}_{j,\gamma}$ and $\widehat{\alpha}_{j,\gamma}^{-1}$ are respectively replaced by $\widehat{H}_{j,\gamma}^{\text{dis}}$ and $\widehat{\alpha}_{j,\gamma}^{-1,\text{dis}}$. To this end, we first state the following result which is a straightforward consequence of Lemma 4.1 in [4].

**Lemma 4.4.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1. Under the sole condition $j \leq \mu_{n_j}$ for all $j \in \mathbb{N}$, one has, for all fixed $\gamma \in (0, 4^{-1}\underline{\alpha})$ and any $\ell \in \{1,2\}$,*

$$\lim_{j\to+\infty} \frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]} = 1, \tag{4.7}$$

*where the convergence holds almost surely.*

Now, we control the distance between the variation $V_{\ell,\gamma}^{\mu_{n_j}}$ and its discretized version $\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}$ in $L^1$. Note that applying (2.30) and (2.33), next Lemma implies that the discretized version converges in $L^1$ to $V_{\ell,\gamma}^{\mu_{n_j}}$.

**Lemma 4.5.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of integers satisfying the conditions* (2.31) *and* (2.32) *and $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers. Then, there is a finite constant $c > 0$ such that, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,*

$$\mathbb{E}[|\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j} - V_{\ell,\gamma}^{\mu_{n_{j+k}}}|] \leq c\mu_{n_{j+k}} \max\left\{2^{-n_j H\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\}.$$

*Proof.* In the sequel, one denotes by $W$ an arbitrary real-valued S$\alpha$S random variable with scale parameter equals to 1. From the expressions in equations (4.5) and (4.6), the triangle inequality, the inequality

$$||x|^{\gamma} - |y|^{\gamma}| \leq |x-y|^{\gamma}, \quad \text{for all } x, y \in \mathbb{R},$$

since $\gamma \in (0,1)$, and the Remark 2.1, we get

$$\begin{aligned}
\mathbb{E}[|\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j} - V_{\ell,\gamma}^{\mu_{n_{j+k}}}|] &\le \mathbb{E}\left[\left|\sum_{p=1}^{\mu_{n_{j+k}}} \left(|Y_{\ell,2p-1}^{n_j,T_{n_j}}|^\gamma - |Y_{\ell,2p-1}|^\gamma\right)\right|\right] \\
&\le \sum_{p=1}^{\mu_{n_{j+k}}} \mathbb{E}\left[\left|Y_{\ell,2p-1}^{n_j,T_{n_j}} - Y_{\ell,2p-1}\right|^\gamma\right] \\
&= \sum_{p=1}^{\mu_{n_{j+k}}} \mathbb{E}\left[|W|^\gamma \sigma\left(Y_{\ell,2p-1}^{n_j,T_{n_j}} - Y_{\ell,2p-1}\right)^\gamma\right].
\end{aligned}$$

Then, we get, from inequality (3.12) in Lemma 3.7, the existence of a finite constant $c>0$ such that, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,

$$\mathbb{E}[|\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j} - V_{\ell,\gamma}^{\mu_{n_{j+k}}}|] \le c \sum_{p=1}^{\mu_{n_{j+k}}} \max\left\{2^{-n_j H}, T_{n_j}^{-(L-1-H)}\right\}^\gamma = c\mu_{n_{j+k}} \max\left\{2^{-n_j H\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\}.$$

□

In the sequel, we will need the following Lemma, which is obtained in the course of the proof of [4, Lemma 4.1] (see inequality (4.24) in this last paper).

**Lemma 4.6.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions (2.31) and (2.32) and $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges and such that (4.3) holds. Then, there exists a positive finite constant c such that, for all $j \ge 1$ and $\ell \in \{1,2\}$,*

$$\left(\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]\right)^{-1} \le c\, \mu_{n_j}^{\gamma(H+1/\alpha)-1}.$$

The two following corollaries will be important to revisit Theorems 2.6 and 2.8 with the estimators (4.1) and (4.2).

**Corollary 4.7.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions (2.31) and (2.32) and $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges. For any $\ell \in \{1,2\}$, we have*

$$\lim_{j\to+\infty} \frac{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]} = 1. \tag{4.8}$$

*Moreover, if $(\mu_n)_{n\in\mathbb{N}}$ also satisfies assumption (4.3), then, for any $\ell \in \{1,2\}$, we have*

$$\lim_{j\to+\infty} \frac{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]} = 1. \tag{4.9}$$

*Proof.* From Lemmata 4.5 and 4.6, there exists a finite constant $c > 0$ such that, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,

$$\begin{aligned}\left|\frac{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]} - 1\right| &= \frac{\left|\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}] - \mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]\right|}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]} \\ &\le \left(\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]\right)^{-1} \mathbb{E}\left[\left|\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j} - V_{\ell,\gamma}^{\mu_{n_{j+k}}}\right|\right] \\ &\le c\mu_{n_{j+k}}^{\gamma(H+1/\alpha)} \max\left\{2^{-n_j H\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\}. \end{aligned} \tag{4.10}$$

Then, taking $k = 0$, (4.8) follows from (2.30) and (2.33). Taking $k = 1$ and assuming that $(\mu_n)_{n\in\mathbb{N}}$ also satisfies (4.3), we have

$$\lim_{j\to+\infty} \frac{\mu_{n_{j+1}}}{\mu_{n_j}} = 2 \tag{4.11}$$

and then (4.9) also follows from (2.30) and (2.33). □

**Corollary 4.8.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 2.6 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions (2.31) and (2.32) and let $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges. Then, there is a finite constant $c > 0$ such that, for all $j \in \mathbb{N}$ and $\ell \in \{1,2\}$,*

$$\left|\log_2\left(\frac{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]}\right)\right| \le c\mu_{n_j}^{\gamma(H+1/\alpha)} \max\left\{2^{-n_j H\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\}. \tag{4.12}$$

*Moreover, if $(\mu_n)_{n\in\mathbb{N}}$ also satisfies assumption (4.3), then, for any $\ell \in \{1,2\}$, we have*

$$\left|\log_2\left(\frac{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]}\right)\right| \le c\mu_{n_{j+1}}^{\gamma(H+1/\alpha)} \max\left\{2^{-n_j H\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\}. \tag{4.13}$$

*Proof.* Using

$$|\log_2(1+x)| \le |x| \quad \text{for every } x \in [-2^{-1}, +\infty),$$

inequalities (4.12) and (4.13) then follow directly from Lemma 4.5 and Corollary 4.7 and (**??**). □

**Lemma 4.9.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions (2.31) and (2.32) and let $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges. Let also $\tau'$ be an arbitrary and fixed real number satisfying $0 < \tau' < \min\{H, 1-H\}$. There exists a positive finite random variable $C^*(\tau')$ such that one has, almost surely, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,*

$$\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]}\right| \le C^*(\tau') \ \min\{2^{-n_j\tau'\gamma}, T_{n_j}^{-\tau'\gamma}\}. \tag{4.14}$$

*Moreover, if $(\mu_n)_{n\in\mathbb{N}}$ also satisfies assumption (4.3), then, almost surely, for any $j \in \mathbb{N}$ and $\ell \in \{1,2\}$, one also has*

$$\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+1}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]}\right| \le C^*(\tau') \min\{2^{-n_j\tau'\gamma}, T_{n_j}^{-\tau'\gamma}\}. \tag{4.15}$$

*Proof.* For each $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$, using Markov inequality and the triangle inequality, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right| \geq \min\{2^{n_j}, T_{n_j}\}^{-\tau'\gamma}\right) \\
&\leq \min\{2^{n_j}, T_{n_j}\}^{\tau'\gamma}\,\mathbb{E}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right|\right) \\
&\leq \min\{2^{n_j}, T_{n_j}\}^{\tau'\gamma}\,\mathbb{E}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}\left(\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}] - \mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]\right)}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}} - \mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right|\right) \\
&\leq 2\min\{2^{n_j}, T_{n_j}\}^{\tau'\gamma}\,\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]^{-1}\mathbb{E}[|\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j} - V_{\ell,\gamma}^{\mu_{n_{j+k}}}|].
\end{aligned}$$

Then from (4.10), there exists a finite constant $c > 0$ such that, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right| \geq \min\{2^{n_j}, T_{n_j}\}^{-\tau'\gamma}\right) \\
&\leq 2c\min\left\{2^{n_j}, T_{n_j}\right\}^{\tau'\gamma}\mu_{n_{j+k}}^{\gamma(H+1/\alpha)}\max\left\{2^{-n_jH\gamma}, T_{n_j}^{-(L-1-H)\gamma}\right\} \\
&\leq 2c\mu_{n_{j+k}}^{\gamma(H+1/\alpha)}\max\left\{2^{-n_j(H-\tau')\gamma}, T_{n_j}^{-(L-1-H-\tau')\gamma}\right\}.
\end{aligned}$$

Moreover, since $\gamma > 0$, $H - \tau' > 0$ and $L - 1 - H - \tau' \geq 1 - H - \tau' > 0$, one knows from (2.30) that there exists $\varepsilon > 0$ such that

$$\max\left\{2^{-n_j(H-\tau')\gamma}, T_{n_j}^{-(L-1-H-\tau')\gamma}\right\} \leq 2^{-\varepsilon n_j}, \quad \text{for all } j \in \mathbb{N}.$$

and then

$$\mathbb{P}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right| \geq \min\{2^{n_j}, T_{n_j}\}^{-\tau'\gamma}\right) \leq 2c\mu_{n_{j+k}}^{\gamma(H+1/\alpha)}2^{-\varepsilon n_j}.$$

Thus, when $k = 0$, one can derive from (2.33) that

$$\sum_{j\in\mathbb{N}}\mathbb{P}\left(\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]}\right| \geq \min\{2^{n_j}, T_{n_j}\}^{-\tau'\gamma}\right) < \infty.$$

If $(\mu_n)_{n\in\mathbb{N}}$ also satisfies assumption (4.3), we use again (4.11) and then derive (4.15) in the same way as previously done for obtaining (4.14). $\square$

As an immediate consequence of Lemma 4.9, we obtain the following corollary, which can be viewed as an extension of Lemma 4.4 to the random variables $\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}$ defined through (4.6).

**Corollary 4.10.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 4.1. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions* (2.31) *and* (2.32) *and let $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges. For all fixed $\gamma \in (0, 4^{-1}\underline{\alpha})$ and any $\ell \in \{1,2\}$,*

$$\lim_{j\to+\infty} \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]} = 1, \tag{4.16}$$

*where the convergence holds almost surely. Moreover, if $(\mu_n)_{n\in\mathbb{N}}$ also satisfies assumption* (4.3)*, for any $\ell \in \{1,2\}$, we also have, almost surely,*

$$\lim_{j\to+\infty} \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]} = 1. \tag{4.17}$$

*Proof.* For all $n \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$, using the triangle inequality we get

$$\left| \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - 1 \right| \leq \left| \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]} \right| + \left| \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]} - 1 \right|.$$

Then the conclusion follows immediately from Lemmata 4.9 and 4.4. □

We now have enough materials to prove Theorem 4.1.

*Proof of Theorem 4.1.*

(i) In view of Theorem 2.6, it suffices to prove that, under the conditions (2.31) and (2.32) on the sequence $(\mu_n)_{n\in\mathbb{N}}$,

$$\lim_{j\to+\infty} \left( \widehat{H}_{j,\gamma}^{\mathrm{dis}} - \widehat{H}_{j,\gamma} \right) = 0, \tag{4.18}$$

where the convergence holds almost surely. To this end, we write, for all $j \in \mathbb{N}$,

$$\begin{aligned}
\gamma \left( \widehat{H}_{j,\gamma}^{\mathrm{dis}} - \widehat{H}_{j,\gamma} \right) &= \log_2 \left( \frac{\mathcal{V}_{2,\gamma}^{\mu_{n_j},n_j}}{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}} \right) - \log_2 \left( \frac{V_{2,\gamma}^{\mu_{n_j}}}{V_{1,\gamma}^{\mu_{n_j}}} \right) \\
&= \log_2 \left( \frac{\mathcal{V}_{2,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{n_j},n_j}]} \right) - \log_2 \left( \frac{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}]} \right) \\
&\quad + \log_2 \left( \frac{V_{1,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{1,\gamma}^{\mu_{n_j}}]} \right) - \log_2 \left( \frac{V_{2,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{2,\gamma}^{\mu_{n_j}}]} \right) \\
&\quad + \log_2 \left( \frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{n_j},n_j}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{n_j}}]} \right) - \log_2 \left( \frac{E[\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{n_j}}]} \right)
\end{aligned}$$

and the convergence (4.18) is then a direct consequence of the limits (4.16), (4.7) and (4.8).

- Similarly, for all $n \in \mathbb{N}$, we write

$$\begin{aligned}\gamma\left(\widehat{\alpha}_{j,\gamma}^{-1,\mathrm{dis}} - \widehat{\alpha}_{j,\gamma}^{-1}\right) &= \log_2\left(\frac{V_{2,\gamma}^{\mu_{n_{j+1}}}}{V_{1,\gamma}^{\mu_{n_j}}}\right) - \log_2\left(\frac{\mathcal{V}_{2,\gamma}^{\mu_{n_{j+1}},n}}{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}}\right)\\ &= \log_2\left(\frac{\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}]}\right) - \log_2\left(\frac{\mathcal{V}_{2,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{n_{j+1}},n_j}]}\right)\\ &+ \log_2\left(\frac{V_{2,\gamma}^{\mu_{n_{j+1}}}}{\mathbb{E}[V_{2,\gamma}^{\mu_{n_{j+1}}}]}\right) - \log_2\left(\frac{V_{1,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{1,\gamma}^{\mu_{n_j}}]}\right)\\ &+ \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{n_j},n_j}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{n_j}}]}\right) - \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{n_{j+1}},n_j}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{n_{j+1}}}]}\right)\end{aligned}$$

  and the conclusion follows from Theorem 2.6 (ii) and the limits (4.16), (4.17), (4.7), (4.8) and (4.9).

$\square$

We now pass to the revisit of Theorem 2.8 in the context of the estimators (4.1) and (4.2). The main ingredient is the following lemma.

**Lemma 4.11.** *Let $\underline{\alpha} \in (0,2]$ be as in Theorem 2.6 and let $\gamma \in (0, 4^{-1}\underline{\alpha})$ be arbitrary and fixed. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of integers satisfying the conditions (2.31) and (2.32) and let $(n_j)_{j\in\mathbb{N}}$ be a non-decreasing sequence of positive integers that diverges. Let $F_\gamma$ be the positive continuous function introduced in Remark 2.7. Then, for any $\ell \in \{1,2\}$, when $j$ goes to $+\infty$, the random variable*

$$\Delta_{\ell,0,\gamma}^{j} := \log(2)F_\gamma(H,\alpha^{-1})\mu_{n_j}^{1/2}\left(\log_2\left(\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]}\right) - \log_2\left(\frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]}\right)\right)$$

*converges to $0$ almost surely. Moreover, if $(\mu_n)_{n\in\mathbb{N}}$ also satisfy assumption (4.3), for any $\ell \in \{1,2\}$, when $j$ goes to $+\infty$, the random variable*

$$\Delta_{\ell,1,\gamma}^{j} := \log(2)F_\gamma(H,\alpha^{-1})\mu_{n_j}^{1/2}\left(\log_2\left(\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]}\right) - \log_2\left(\frac{V_{\ell,\gamma}^{\mu_{n_{j+1}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]}\right)\right)$$

*also converges to $0$ almost surely.*

*Proof.* Applying the mean value theorem, for all $j \in \mathbb{N}$, $\ell \in \{1,2\}$ and $k \in \{0,1\}$,

$$\left|\Delta_{\ell,k,\gamma}^{j}\right| \leq \frac{F_\gamma(H,\alpha^{-1})\mu_{n_j}^{1/2}}{\min\left\{\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]}, \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right\}}\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+k}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+k}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+k}}}]}\right|. \tag{4.19}$$

Under the conditions (2.31) and (2.32) on the sequence $(\mu_n)_{n\in\mathbb{N}}$, we use (4.14), (2.30) and (2.33) to get, for any $\ell \in \{1,2\}$,

$$\lim_{j\to+\infty} 2F_\gamma(H,\alpha^{-1})\mu_{n_j}^{1/2}\left|\frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]}\right| = 0,$$

where the convergence holds almost surely. Moreover, applying (4.16) and Lemma 4.4, we have almost surely

$$\lim_{j\to+\infty} \min\left\{ \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_j},n_j}]}, \frac{V_{\ell,\gamma}^{\mu_{n_j}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_j}}]} \right\} = 1.$$

Therefore, (4.19) implies that

$$\lim_{j\to+\infty} \Delta_{\ell,0,\gamma}^{j} = 0 \quad \text{almost surely.}$$

Now, if condition (4.3) holds as well, we use (4.15), (2.30) and (2.33) to also get, almost surely, for any $\ell \in \{1,2\}$,

$$\lim_{j\to+\infty} 2F_\gamma(H,\alpha^{-1})\mu_{n_j}^{1/2} \left| \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]} - \frac{V_{\ell,\gamma}^{\mu_{n_{j+1}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]} \right| = 0$$

Moreover, applying (4.17) and Lemma 4.4, we also have almost surely

$$\lim_{j\to+\infty} \min\left\{ \frac{\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}}{\mathbb{E}[\mathcal{V}_{\ell,\gamma}^{\mu_{n_{j+1}},n_j}]}, \frac{V_{\ell,\gamma}^{\mu_{n_{j+1}}}}{\mathbb{E}[V_{\ell,\gamma}^{\mu_{n_{j+1}}}]} \right\} = 1.$$

Then, it follows again by (4.19) that, almost surely,

$$\lim_{j\to+\infty} \Delta_{\ell,1,\gamma}^{j} = 0.$$

□

We already have enough materials to prove Theorem 4.2.

*Proof of Theorem 4.2.* One can derive from Theorem 4.1 and from the continuity property of the functions $F_\gamma$, $\tau_1$ and $\tau_2$ (see Remark 2.7 and (2.28)) that

$$\lim_{j\to+\infty} F_\gamma\big(\tau_1(\widehat{H}_{1,j,\gamma}^{\text{dis}}), \tau_2(\widehat{\alpha}_{2,j,\gamma}^{-1,\text{dis}})\big)\big(F_\gamma(H,\alpha^{-1})\big)^{-1} = 1\,, \quad \text{almost surely.}$$

Thus, for proving the theorem it is enough to show that

$$2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(\mu_{1,n_{1,j}})^{1/2}\Big(\widehat{H}_{1,j,\gamma}^{\text{dis}} - H\Big) \xrightarrow[j\to+\infty]{d} \mathcal{N}(0,1)$$

and

$$(2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(\mu_{2,n_{2,j}})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1,\text{dis}} - \alpha^{-1}\Big) \xrightarrow[j\to+\infty]{d} \mathcal{N}(0,1).$$

Then, using Theorem 2.8 and Slutsky's theorem, see e.g. [16, Page 318], that it turns out that it is even enough to show that

$$2^{-1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(\mu_{1,n_{1,j}})^{1/2}\Big(\widehat{H}_{1,j,\gamma}^{\text{dis}} - \widehat{H}_{1,j,\gamma}\Big) \xrightarrow[j\to+\infty]{\mathbb{P}} 0 \tag{4.20}$$

and

$$(2/3)^{1/2}\big(\log(2)\big)\gamma F_\gamma\big(H,\alpha^{-1}\big)(\mu_{2,n_{2,j}})^{1/2}\Big(\widehat{\alpha}_{2,j,\gamma}^{-1,\text{dis}} - \widehat{\alpha}_{2,j,\gamma}^{-1}\Big) \xrightarrow[j\to+\infty]{\mathbb{P}} 0. \tag{4.21}$$

where $\widehat{H}_{1,j,\gamma}$ and $\widehat{\alpha}_{2,j,\gamma}^{-1}$ are as in Theorem 2.8.

For all $j \in \mathbb{N}$, we write

$$\begin{aligned}(\mu_{1,n_{1,j}})^{1/2}\Big(\widehat{H}^{\mathrm{dis}}_{1,j,\gamma} - \widehat{H}_{1,j,\gamma}\Big) =& (\mu_{1,n_{1,j}})^{1/2}\left(\log_2\left(\frac{\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}}{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}\right) - \log_2\left(\frac{V_{2,\gamma}^{\mu_{1,n_{1,j}}}}{\mathbb{E}[V_{2,\gamma}^{\mu_{1,n_{1,j}}}]}\right)\right)\\ &+ (\mu_{1,n_{1,j}})^{1/2}\left(\log_2\left(\frac{V_{1,\gamma}^{\mu_{1,n_{1,j}}}}{\mathbb{E}[V_{1,\gamma}^{\mu_{1,n_{1,j}}}]}\right) - \log_2\left(\frac{\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}}{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}\right)\right)\\ &+ (\mu_{1,n_{1,j}})^{1/2}\left(\log_2\left(\frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{1,n_{1,j}}}]}\right) - \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{1,n_{1,j}}}]}\right)\right).\end{aligned}$$

First, using (4.12), we have, for all $j \in \mathbb{N}$,

$$\begin{aligned}(\mu_{1,n_{1,j}})^{1/2}&\left|\log_2\left(\frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{1,n_{1,j}}}]}\right) - \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{1,n_{1,j}}}]}\right)\right|\\ &\le 2c\,\mu_{1,n_{1,j}}^{\gamma(H+1/\alpha+1/2)}\max\left\{2^{-n_{1,j}H\gamma}, T_{n_{1,j}}^{-(L-1-H)\gamma}\right\}.\end{aligned}$$

Then, since $H\gamma > 0$ and $(L-1-H)\gamma > 0$, using (2.30) and (2.33), we get

$$(\mu_{1,n_{1,j}})^{1/2}\left|\log_2\left(\frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{1,n_{1,j}}}]}\right) - \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{1,n_{1,j}}}]}\right)\right| \xrightarrow[j\to+\infty]{} 0.$$

Moreover, we know from the first part of Lemma 4.11 that the two random variables

$$(\mu_{1,n_{1,j}})^{1/2}\left(\log_2\left(\frac{\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}}{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}\right) - \log_2\left(\frac{V_{2,\gamma}^{\mu_{1,n_{1,j}}}}{\mathbb{E}[V_{2,\gamma}^{\mu_{1,n_{1,j}}}]}\right)\right)$$

and

$$(\mu_{1,n_{1,j}})^{1/2}\left(\log_2\left(\frac{V_{1,\gamma}^{\mu_{1,n_{1,j}}}}{\mathbb{E}[V_{1,\gamma}^{\mu_{1,n_{1,j}}}]}\right) - \log_2\left(\frac{\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}}{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{1,n_{1,j}},n_{1,j}}]}\right)\right)$$

converge to 0 almost surely. As a consequence, (4.20) holds and thus

$$D^{\mathrm{dis}}_{1,j,\gamma} \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,1).$$

In the same vein, for all $j \in \mathbb{N}$, we write

$$\begin{aligned}(\mu_{2,n_{2,j}})^{1/2}\Big(\widehat{\alpha}^{-1,\mathrm{dis}}_{2,j,\gamma} - \widehat{\alpha}^{-1}_{2,j,\gamma}\Big) =& (\mu_{2,n_{2,j}})^{1/2}\left(\log_2\left(\frac{V_{2,\gamma}^{\mu_{2,n_{2,j+1}}}}{\mathbb{E}[V_{2,\gamma}^{\mu_{2,n_{2,j+1}}}]}\right) - \log_2\left(\frac{\mathcal{V}_{2,\gamma}^{\mu_{2,n_{2,j+1}},n_{2,j}}}{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{2,n_{2,j+1}},n_{2,j}}]}\right)\right)\\ &+ (\mu_{2,n_{2,j}})^{1/2}\left(\log_2\left(\frac{\mathcal{V}_{1,\gamma}^{\mu_{2,n_{2,j}},n_{2,j}}}{\mathbb{E}[\mathcal{V}_{1,\gamma}^{\mu_{2,n_{2,j}},n_{2,j}}]}\right) - \log_2\left(\frac{V_{1,\gamma}^{\mu_{2,n_{2,j}}}}{\mathbb{E}[V_{1,\gamma}^{\mu_{2,n_{2,j}}}]}\right)\right)\\ &+ (\mu_{2,n_{2,j}})^{1/2}\left(\log_2\left(\frac{E[\mathcal{V}_{1,\gamma}^{\mu_{2,n_{2,j}},n_{2,j}}]}{\mathbb{E}[V_{1,\gamma}^{\mu_{2,n_{2,j}}}]}\right) - \log_2\left(\frac{\mathbb{E}[\mathcal{V}_{2,\gamma}^{\mu_{2,n_{2,j+1}},n_{2,j}}]}{\mathbb{E}[V_{2,\gamma}^{\mu_{2,n_{2,j+1}}}]}\right)\right)\end{aligned}$$

and we conclude in the same way, using inequality (4.13) and the second part of Lemma 4.11, that the convergence (4.21) holds true and thus

$$D_{2,j,\gamma}^{\text{dis}} \xrightarrow[j\to+\infty]{d} \mathcal{N}(0,1).$$

□

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