

Dissipation engineering of fermionic long-range order beyond the Lindblad limit

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We investigate the possibility of engineering dissipatively long-range order that is robust against heating in strongly interacting fermionic systems, relevant for atoms in cavity QED. It was previously shown [Tindall *et al.* *Phys. Rev. Lett.* **123**, 030603 (2019)] that it is possible to stabilize long-range order in a Hubbard model by exploiting a dissipative mechanism in the Lindblad limit, this latter being valid for spectrally unstructured baths. Here, we first show that this mechanism still holds when including additional spin-exchange interactions in the model, that is, for the tUJ model. Moreover, by means of a Redfield approach that goes beyond the Lindblad case, we show how the stability of the engineered state depends crucially on properties of the bath spectral density and discuss the feasibility of those properties in an experiment.

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I. INTRODUCTION

Ultracold atomic systems offer a platform for the engineering of matter phases with desired properties [1–3] and the quantum simulation of materials given their high resolution [4–6] and experimental controllability from their interaction strength and range, lattice depth, dimensionality or their temperature and, importantly, also including their dissipative coupling to their environments [7–9]. In this context, atoms inside a cavity QED setup [10], offer additional versatile tools to engineer quantum many-body Hamiltonians. Here, long-range interactions as well as phonon simulation can be engineered via coupling the atoms with a single or several cavity modes, as made possible in ring [11–14] or confocal [15,16] cavities, for example. Altogether, this allows to explore the emergence of a wide range of phenomena, such as frustration and glassiness [17], quantum liquids [18], dynamical gauge fields [19], associative memories [20–23], scrambling of information [24], (dissipative) phase transitions [25–28], including

the Mott insulator-superfluid transition [29] and the BCS-BEC crossover [30], and many others.

Describing properly the dynamics of the atoms in these setups generally requires taking into account the dissipative mechanisms and the decoherence that naturally arises, altering the behavior with respect to the one of isolated systems. Furthermore, recent studies highlight the possibility of engineering noise and dissipation in an open quantum system framework to access quantum phases of matter in a way that, at times, could be more convenient than a more conventional approach of Hamiltonian Engineering [31,32]. In particular, it has been shown [33] that long-range order robust against heating could be engineered in a Hubbard model by adopting a symmetry-based dissipative mechanism of the Lindblad form [34,35]. The final state of the above mentioned dynamics displays so-called η -pairing, predicted first by Yang [36]. The η -state is characterized by off-diagonal long-range order (ODLRO) in the correlations between a local pair of fermions (spin up, spin down on the same site) and a holon. Importantly, this mechanism is behind finite-momentum fermionic superfluid states that are also of interest in condensed matter physics [37–39] due to their exotic properties and the emergence of associated vestigial phases.

In this paper, we study how the dissipative engineering mechanism of ODLRO via η -pairing developed in [33] generalizes to the case where the fermions on the lattice possess

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spin exchange coupling, i.e., in the tUJ model, which exhibits a different symmetry structure and which could be implemented in multimode cavity QED [40–46]. In addition, we investigate the effect of a more general form of dissipation via the use of a Redfield master equation [47] that, while still requiring a weak system-bath coupling, goes beyond the Lindblad dissipation considered in [33]. We thus show that the dissipation engineering of superfluidity still holds for the tUJ model and depends crucially on properties of the spectral density of the bath coupled to the tUJ model when one considers more general system-bath coupling than the idealized Lindblad limit.

Our paper is organized as follows. In Sec. II, we present the tUJ model and its symmetries. In Sec. III, we summarize the dissipation engineering mechanism of Ref. [33] allowing for stabilizing robust superfluidity against heating and apply it to the tUJ model. In Sec. IV, we revisit the dissipative mechanism beyond Lindblad via the use of a Redfield master equation, and study the influence of the frequency cutoff of the spectral density and its symmetry around the frequency axis. In Sec. V, we elaborate on a mechanism to explain our results. In Sec. VI, we discuss possible experimental implementations of our results. Finally, in Sec. VII, we conclude and provide a few possible perspectives of our work.

II. MODEL AND ITS SYMMETRIES

We consider the tUJ model given by the following Hamiltonian

$$H = -t \sum_{i,\sigma} (c_{i\sigma}^\dagger c_{i+1\sigma} + \text{H.c.}) - U \sum_i n_{i\uparrow} n_{i\downarrow} + J \sum_i (s_i^+ s_{i+1}^- + \text{H.c.}), \quad (1)$$

where $c_{i\sigma}, c_{i\sigma}^\dagger$ are fermionic operators in second quantization, i and σ label lattice sites and spin degrees of freedom (DOF). These define the number $n_{i,\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$, and spin ladder $s_i^{+/-} = c_{i\uparrow/\downarrow}^\dagger c_{i\downarrow/\uparrow}$ operators, with t the tunneling amplitude, U the onsite interaction, and J the spin exchange coupling, that we chose, without loss of generality, to be planar, requiring an axial magnetic anisotropy. The state preparation scheme presented in this work is symmetry-based. Thus we now summarize the model properties emerging from its symmetries and highlight the connections with other similar Hamiltonians where dissipative preparation schemes have been performed. To do so, we first consider useful properties of the Hubbard Hamiltonian [33,48], which corresponds to our model in the limit $J \rightarrow 0$.

The Hubbard Hamiltonian

$$H_{HB} = -t \sum_{i,\sigma} (c_{i\sigma}^\dagger c_{i+1\sigma} + \text{H.c.}) - U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (2)$$

i.e., Eq. (1) with $J = 0$, has, at half filling, the useful symmetry structure of a double SU(2) symmetry group. The first SU(2) symmetry of this Hamiltonian can be expressed via the global representation of the spin number raising and lowering

operators $S^\pm$:

$$\begin{aligned} S^z &= \sum_i s_i^z = \sum_i (n_{i\uparrow} - n_{i\downarrow}), \\ S^+ &= \sum_i s_i^+ = \sum_i c_{i,\uparrow}^\dagger c_{i,\downarrow}, \\ S^- &= \sum_i s_i^- = \sum_i c_{i,\downarrow}^\dagger c_{i,\uparrow}, \end{aligned} \quad (3)$$

where S^z is the total magnetization. These operators form a SU(2) algebra and commute with the Hamiltonian, that is,

$$[H_{HB}, S_z] = 0, \quad (4)$$

$$[H_{HB}, S_\pm] = 0, \quad (5)$$

$$[H_{HB}, S_+ S_-] = 0, \quad (6)$$

which thus possesses a permanent spin SU(2) symmetry when defined over an arbitrary lattice. This symmetry is a manifestation of the spin-rotational invariance that characterizes the model.

The second SU(2) group corresponds to the particle-hole symmetry, via the global representation of the η number and ladder operators:

$$\begin{aligned} \eta^z &= \sum_i \eta_i^z = \sum_i (n_{i\uparrow} + n_{i\downarrow} - 1), \\ \eta^+ &= \sum_i \eta_i^+ = \sum_i f(i) c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger, \\ \eta^- &= \sum_i \eta_i^- = \sum_i f(i) c_{i,\downarrow} c_{i,\uparrow}, \end{aligned} \quad (7)$$

where $f(i)$ is a real-valued function depending on the specific lattice structure, which is $f(i) = (-1)^i$ for 1D bipartite lattice. The $\eta_\pm$ operators act on empty sites (so-called holons) and on doubly occupied sites (doublons). The η operators form a SU(2) algebra and commute with the Hamiltonian (though the commutation of $\eta_\pm$ holds only at half-filling), that is,

$$[H_{HB}, \eta_z] = 0, \quad (8)$$

$$[H_{HB} - UN/2, \eta_\pm] = 0, \quad (9)$$

$$[H_{HB}, \eta_+ \eta_-] = 0, \quad (10)$$

where $N = \sum_{i,\sigma} n_{i\sigma}$ is the total particle-number operator. The shift by $U/2$ corresponds to choosing the chemical potential appropriate for half filling, at which the Hamiltonian becomes particle-hole symmetric under the transformation $c_{i\sigma} \rightarrow (-1)^i c_{i\sigma}^\dagger$.

If we consider the addition of a spin exchange interaction between neighboring fermions (XY-type anisotropy), as it is present in the tUJ model (1), the full SU(2) spin symmetry of the Hubbard model does not hold anymore. In this case,

$$[H, S_z] = 0, \quad (11)$$

$$[H, S_\pm] \neq 0, \quad (12)$$

$$[H, S_+ S_-] \neq 0. \quad (13)$$

As only the commutation $[H, S^z]$ vanishes, the system retains only a $U(1)$ spin symmetry corresponding to rotations about the z axis. However, the $SU(2)$ η (charge) symmetry at half-filling remains valid, indeed,

$$[H, \eta_z] = 0, \quad (14)$$

$$[H - UN/2, \eta_{\pm}] = 0, \quad (15)$$

$$[H, \eta_+ \eta_-] = 0. \quad (16)$$

Thus, while the Hubbard model possesses both spin and charge $SU(2)$ symmetries at half-filling, the tUJ model preserves only a $U(1)$ spin symmetry together with the $SU(2)$ charge symmetry.

In the following section, we describe how the underlying double symmetry structure present both in the Hubbard and tUJ models will be crucial for the realization of the fermionic superfluidity through the symmetry-based dissipative mechanism [33].

III. DISSIPATIVE STATE PREPARATION SCHEME

Our aim is to investigate the possibility of preparing a state with long-range order in the charge sector of the tUJ model by engineering the dissipative dynamics in our system, as in [33], to target a single symmetry sector so to generate a highly correlated state in another one. To achieve this, the idea is to couple the system governed by the Hamiltonian in Eq. (1) to an environment that induces spin-dependent dephasing. This could be realized by, e.g., immersion of the fermionic species in a bosonic one [49]. In this section, we first model such a dissipative preparation scheme in the simplest weak system-reservoir coupling limit, as in Ref. [33]. In the next section, we study how going beyond this idealized scenario modifies the system dynamics.

Hence, let us first consider that the evolution of the reduced density matrix ρ of our system is described by the GSKL (Gorini-Kossakowski-Sudarshan-Lindblad) master equation

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho = -i[H, \rho] + \frac{\gamma}{2} \sum_j ([L_j \rho, L_j^\dagger] + [L_j, \rho L_j^\dagger]), \quad (17)$$

where γ is a constant dissipative, and L_j the jump operators given by

$$L_j = s_j^\pm = n_{\uparrow,j} - n_{\downarrow,j}. \quad (18)$$

In Fig. 1, we show that it is possible to stabilize a η -superfluid under such a GSKL dissipative dynamics for the tUJ model shown in Fig. 1(a). The key aspect of the mechanism is pictorially illustrated in Fig. 1(b); we see that the spin sector is driven to infinite temperature through the coupling to the external environment that induces heating, while the order in the charge sector is preserved, due to the symmetry:

$$[L_j, \eta^\pm] = [L_j, \eta^+ \eta^-] = 0. \quad (19)$$

This can be assessed via the reduced density matrix of the system in the respective symmetry sectors shown in Fig. 1(c), where we can see that the steady state density matrix is diagonal in the spin sector, while it exhibits off-diagonal elements in

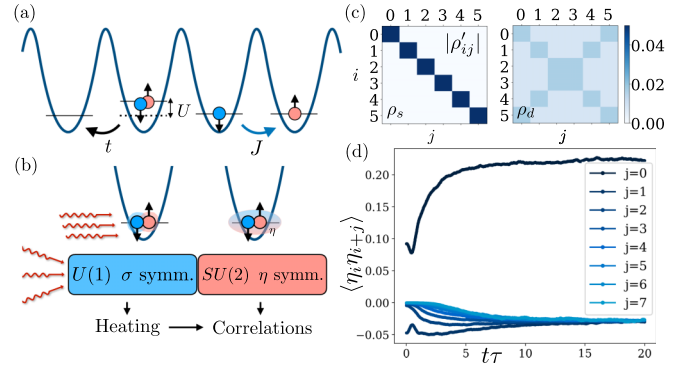


FIG. 1. The concept. Superfluid formation in the tUJ model under dissipative dynamics. (a) Scheme of a fermionic chain whose dynamics are governed by the tUJ model: fermions can tunnel between neighboring sites with rate t and experience an energy offset U when two opposite spins are in the same site, and spin exchange interaction with nearest-neighbours at rate J . (b) Dissipative engineering of fermionic states. Pictorial representation of the dual $SU(2) \times U(1)$ symmetry in the tUJ model and the dissipative heating mechanism targeting the spin sector and leading to correlations in the η sector. (c) Projections of the steady-state density matrix in the spin $\rho_s = P_s \rho P_s$ and charge $\rho_d = P_d \rho P_d$ sectors for $M = 6$ particles at half-filling, after a quench with onsite coupling strength $U = 4t$ and dissipative rate $\gamma = 2$. For the formal definition of the projector operators $P_{s,d}$, see [33,48]. (d) Evolution of the two-point superfluid pairing correlation function $\langle \eta_i^+ \eta_{i+j}^- \rangle$ for varying distances j as in the legend, with $t = 1$, $U = 4t$, and $U/J = -0.1$ at half-filling. Here $M = 8$ sites. Time and energy units are in units of the tunneling rate t unless stated otherwise. The pairing correlations reach the steady state approximately after $t\tau \sim 13$, with a robust condensate fraction as large as $\approx 27\%$. The simulation is performed with $N = 2000$ trajectories and time step $\delta t = 0.01$.

the charge sector. The fact that the steady state density matrix is diagonal in the spin sector can be viewed as a consequence of the hermiticity of the jump operator. This can indeed be easily shown that $\rho \propto \mathbb{1}$ is always a valid steady state of a Lindblad master equation of the form (17), as the map is unital (that is, $\mathcal{L}(\mathbb{1}) = 0$). Finally, in Fig. 1(d), we assess how the system relaxes to the steady state via the time-evolution of the two-point correlations $\langle \eta_i^+ \eta_{i+j}^- \rangle$ for varying distances $|i - j|$. We observe that the stationary value of the correlations is finite and constant as a function of distance, highlighting the establishment of a long-range order with correlation values compatible with a condensate fraction as large as $n \approx 0.27$. Note that a nonzero, distance-independent steady-state value of $\langle \eta_i^+ \eta_{i+j}^- \rangle$ is a sufficient condition for off-diagonal long-range order (ODLRO), although it is not a necessary and sufficient condition in the sense of the Penrose-Onsager criterion. The spreading of the η correlations is visible by the decrease of the short-range correlations to favor the emergence of a distance-invariant long-range order. Hence, our results show that the dissipative mechanism to engineer long-range order described in [33] is still valid in our tUJ model despite the presence of the spin exchange energy term. This result generalizes the findings of [33] to the paradigmatic tUJ model, that is, amenable to simulate the presence of tunable-range interactions. However, implementations in

experiment requires more realistic scenarios. In the following, we investigate how going beyond the weak-coupling limit approximation assumed above and in [33] reflects in the feasibility of the preparation schemes. To this aim, we consider reservoirs with spectral structures.

IV. DISSIPATIVE PREPARATION SCHEME BEYOND GSKL

In this section, we investigate whether the same SF state could be realized in more general scenarios, namely, when the system is coupled to a structured bath rather than the idealized case of an unstructured environment. In general, most materials and experiments are coupled to spectrally structured environments [8], and it is thus fundamental to investigate their effects on the system dynamics and properties, even perturbatively, leading to different behaviors and phenomena, notably in the context of dissipative phase transitions [50–54]. It is therefore natural to ask the extent to which the results in Sec. III hold in the presence of more realistic colored baths.

In order to do so, we consider a more general form of the system evolution than that provided by Eq. (17), namely, a Redfield master equation of the form

$$\dot{\rho} = -i[H, \rho] + \sum_j ([\bar{L}_j(\tau)\rho, L_j^\dagger] + [L_j, \rho\bar{L}_j^\dagger(\tau)]), \quad (20)$$

with

$$\bar{L}_j(\tau) = \int_0^\tau \alpha(s) e^{-iHs} L_j e^{iHs} ds, \quad (21)$$

and the zero-temperature bath correlation function

$$\alpha(s) = \int_0^\infty J(\omega) e^{-i\omega s} d\omega \quad (22)$$

written in terms of the bath spectral density

$$J(\omega) = \frac{\pi}{\hbar} \sum_k g_k^2 \delta(\omega - \omega_k), \quad (23)$$

where g_k denotes the system coupling with the mode k of frequency ω_k of the bath. The Redfield master equation is completely determined from the knowledge of H , L_j , and the bath correlation function $\alpha(s)$, or equivalently, the bath spectral density $J(\omega)$. One can directly see from Eq. (21) that (20) reduces to (17) if the bath spectral density is completely flat, and either one can consider neglecting the imaginary part or extending the lower bound of the integration to $-\infty$, which indeed yields a bath correlation function of the form $\alpha(s) \propto \gamma \delta(s)$. By contrast, if the bath has a nonflat spectral density, the bath correlation function departs from such an instantaneous decaying form. Hence, the jump operators L_j appearing in the integral of Eq. (21) have time to rotate according to the system Hamiltonian before the decay of the correlation function, which makes the dissipative terms in the master equation dependent on the system spectrum and eigenstates. Already from that fact, we can foresee that this is likely to break the hermiticity of the jump operator, preventing the infinite temperature state from appearing.

Here, we investigate the effects of various kinds of spectral densities $J(\omega)$ on the dissipative phase transition discussed in Sec. III. To interpolate between the weak-coupling limit and

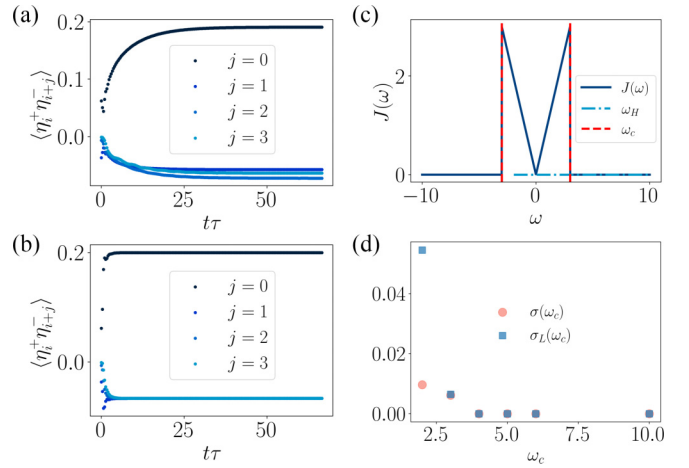


FIG. 2. Analysis of dissipative dynamics in the tUJ model with spectral structured baths. (a) Time evolution of $\langle \eta_i^+ \eta_{i+j}^- \rangle$ under the dissipative dynamics in Eq. (20) beyond GSKL with an effective linear spectral density specified in (c) with cutoff frequency $\omega_c = 3/t$. (b) Time evolution of $\langle \eta_i^+ \eta_{i+j}^- \rangle$ for a system with the same linear spectral density but with a cutoff $\omega_c = 10/t$. (c) Corresponding form of the bath correlation function $J(\omega) = |\omega|$ for an exemplary cutoff $\omega_c = 3/t$ in the interval of the Hamiltonian eigenfrequencies ω_H . Simulation with $M = 4$ sites for $U = 4t$ and $J = -0.1U$. (d) Comparison of $\sigma(\omega_c)$ and σ_L as a function of the cutoff ω_c , where $\sigma(\omega_c) = 0$ implies a convergence to ODLRO, while $\sigma_L = 0$ implies a convergence of the off-diagonal elements to the value reached in the fully Markovian limit.

Redfield cases, we extend the lower bound of the integration appearing in Eq. (22) to $-\infty$ and consider

$$\alpha(s) = \int_{-\infty}^{\infty} J(\omega) e^{-i\omega s} d\omega, \quad (24)$$

so that the limit of $J(\omega) \rightarrow \gamma/(2\pi) \forall \omega$ (i.e., flat spectral density) recovers Eq. (17). As we will see below, the main qualitative features of our results do not depend on the exact shape of the spectral density but rather on its (i) frequency cutoff and (ii) symmetry with respect to the zero-frequency axis. Thus, for the sake of conciseness, we present the case of an effective linear spectral density, while other cases are summarized in Appendix A. We solve the master equation (20) in its Bloch-Redfield form, employing the QuTip Python library [55]. Numerical details can be found in Appendix B.

A. Impact of the frequency cutoff of the spectral density

First, let us highlight the relevance of the frequency cutoff of the spectral density for the state preparation protocol to be effective. To understand this mechanism, we chose $J(\omega) \propto |\omega|$, for $-\omega_c \leq \omega \leq \omega_c$ where ω_c is a low frequency cutoff as compared with the differences between eigenfrequencies of the Hamiltonian, i.e., the transition energies of the system. In Fig. 2(a), we observe that with a sufficiently low frequency cutoff, we find no value of the bath parameters for which we obtain a steady state with constant η correlations $\langle \eta_i \eta_{i+j} \rangle$ over distance. By contrast, for a larger cutoff frequency $\omega_c \sim 10$, which encompasses all possible frequency differences between the system Hamiltonian eigenfrequencies, we observe

in Fig. 2(b) long-range order in the long-time limit for the two-point η correlations. In the lower cutoff case, there is a lack of off-diagonal long-range order in the steady state—the state is still thermal in the spin sector—and the η correlations are no longer able to organize themselves in a translation-invariant state. The corresponding spectral density $J(\omega)$ used in both preparation schemes is included in Fig. 2(c). While here we include only illustrative examples, we find that this discrepancy from ODLRO becomes smaller with increasing cutoff. We observe that both the time needed to converge to the steady state and the possibility of creating long-range order depend strongly on the chosen cutoff frequency to study the convergence. Thus we introduce a quantifier for the variation in off-diagonal correlations, defined as

$$\sigma(\omega_c) = \sqrt{\frac{1}{N} \sum_{j=1}^N (\langle \eta_i^+ \eta_{i+j}^- \rangle - \mu(\omega_c))^2}, \quad (25)$$

where $\mu(\omega_c) = \sum_{j=1}^N \frac{1}{N} \langle \eta_i^+ \eta_{i+j}^- \rangle$ represents the mean value of the correlations, depending on the cutoff frequency ω_c through $\langle \eta_i^+ \eta_{i+j}^- \rangle$. Here, we choose the central site $i = N/2 + 1$ without loss of generality, as the results are independent of the chosen i . We evaluate this quantifier for the case of an effective linear spectral density as a function of ω_c in Fig. 2(d). We include as well the comparison with the fully Markovian case Eq. (17) via

$$\sigma_L(\omega_c) = \sqrt{\frac{1}{N} \sum_{j=1}^N (\langle \eta_i^+ \eta_{i+j}^- \rangle - \mu_L)^2}. \quad (26)$$

Here, μ_L corresponds to the mean value of the ODLRO correlations in the Lindblad case, allowing to quantify how $\mu(\omega_c)$ departs from μ_L . From these quantifiers, we confirm that while the support of the spectral density, determined by the frequency cutoff ω_c , increases and encompasses more transition energies of the system Hamiltonian, the long-range order is developed, as characterized by $\sigma(\omega_c) \rightarrow 0$. We also notice that while $\sigma_L(\omega_c) \neq 0$, the mean value of the ODLRO correlations in the Lindblad case does not correspond to that of the Redfield case.

To deepen our understanding of the bath properties, we study the spectrum of the Redfield Liouvillian and compare it to that of the Lindblad Liouvillian. Our results are shown in Fig. 3(a) for a cutoff frequency $\omega_c = 3/t$, corresponding to the evolution depicted in Fig. 2(a). We observe that a small cutoff leads to the appearance of imaginary eigenvalues with small but nonzero real parts and causes the spectrum to deviate from the standard Liouvillian case. In addition, we compare in Fig. 3(b) the Redfield Liouvillian spectrum for the two different cutoffs investigated in Figs. 2(a) and 2(b). For the large cutoff $\omega_c = 10/t$ (violet dots), the spectra are already comparable to the Lindblad case, and we observe the ORDLO.

B. Impact of the symmetry of the spectral density

We now turn to studying how the symmetry of the spectral density with respect to the zero-energy axis affects the dynamics. To this end, we use a steplike spectral density, shown in Fig. 4(a), where the discontinuity occurs at a cutoff frequency ω_c , which we vary from 0 to increasingly negative values.

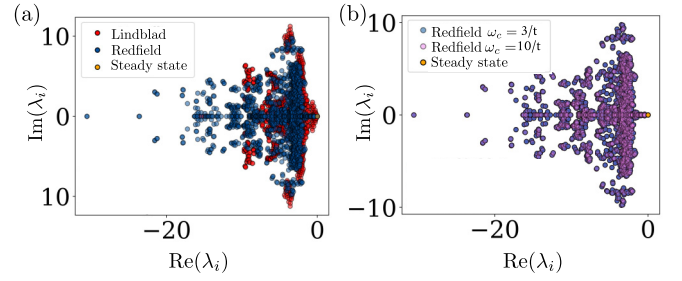


FIG. 3. (a) Comparison between the Liouvillian spectrum of the Redfield tensor in the case of a linear spectral density $J(\omega) = |\omega|$ with a cutoff $\omega_c = 3/t$ to that of the Lindblad Liouvillian (corresponding to the case of a flat spectral density with infinite support). Both Liouvillians are restricted to the half-filling symmetry sector. (b) Comparison between the Liouvillian spectrum of the Redfield tensor in the case of a linear spectral density $J(\omega) = |\omega|$ with two different cutoffs, $\omega_c = 3/t$ and $10/t$. The parameters used are the same as in Fig. 2.

In this analysis, we study how the reduced density matrix projected onto the target η sector changes as a function of the position of the step in the spectral density.

For a small (negative) cutoff [panel (b) of Fig. 4], the reduced η state does not exhibit translationally invariant order. In contrast, for a large (negative) cutoff [panel (d) of Fig. 4], the long-range order is established, and it is possible to stabilize the η -superfluid. Intermediate cutoffs [panel (c)] show both the initial buildup of correlations at the boundaries and their redistribution across the entire chain as ω_c is varied. The increasing overlap between the Lindblad and Redfield Liouvillian spectra as the cutoff frequency acquires larger negative values [right plots in panels (b)–(d)] confirms the convergence of the model to the known Lindblad scenario associated with the stable η -superfluid.

We interpret the lack of identical η -correlations at small ω_c as a consequence of the asymmetry of the spectral density over the transition energy range of the Hamiltonian. Indeed, in this regime, the rotated jump operator in Eq. (21) becomes non-Hermitian, which breaks the unitality of the quantum map, which is a condition to obtain equal η -correlations [33]. More explicitly,

$$\begin{aligned} \mathcal{L}(\mathbb{1}) &= \sum_j [\bar{L}_j(\tau) - \bar{L}_j^\dagger(\tau), L_j] \\ &= \sum_{n,n'} [\Gamma(\omega_{nn'}) - \Gamma^*(-\omega_{nn'})] \langle n | L_j | n' \rangle |n\rangle \langle n'|, L_j], \end{aligned} \quad (27)$$

where $H|n\rangle = \omega_n|n\rangle$, $\omega_{nn'} = \omega_n - \omega_{n'}$ and $\Gamma(\omega_{nn'}) = \int_0^\infty ds \alpha(s) e^{i\omega_{nn'} s}$. From this expression, we see that $\mathcal{L}(\mathbb{1}) = 0$ whenever the spectral density is symmetric, $J(\omega) = J(-\omega)$, over the transition frequencies $\omega_{nn'}$ for which $\langle n | L_j | n' \rangle \neq 0$. For a small cutoff, we may have $\mathcal{L}(\mathbb{1}) \neq 0$ because $J(\omega_{nn'}) - J(-\omega_{nn'}) \neq 0$. As the cutoff becomes large in modulus, the symmetry over the relevant range of frequencies of the spectral density is recovered, and so the unitality of the map. These outcomes can be extended to other spectral structures, as discussed in Appendix A.

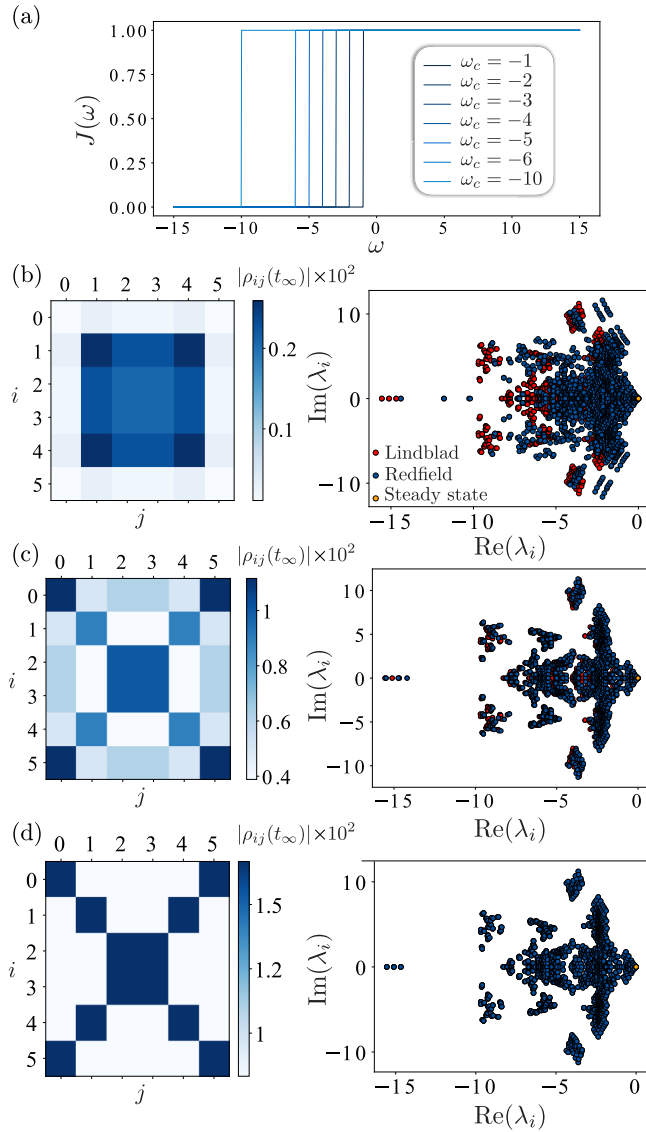


FIG. 4. In (a), we present the spectral density used to study the dynamics: $\theta(\omega - \omega_c)$. (b)–(d) display the projections of the final state density matrix in the charge sector for cutoffs $\omega_c = \{-1, -5, -10\}$ (right column), respectively. The associated Redfield tensor eigen-spectra (blue dots) are overlaid with the Liouvillian eigen-spectra (red dots) (left column). This allows us to trace the evolution of correlations between doublons, from their absence to the emergence of off-diagonal long-range order (ODLRO) in the sector. The parameters used are the same as in Fig. 2.

V. DISCUSSION

From the analysis in Sec. IV we extract two essential indications: (i) the spectral density has to be symmetrical with respect to the zero-energy axis for the convergence to ODLRO to occur, and (ii) the cutoff of the spectral density is required to be large enough so that it covers, symmetrically, the relevant transition energy range determined by H . Overall, this can be viewed as the requirement that the bath must be able to connect the eigenstates of the system to the target correlated state via relevant dissipative channels. Closing some dissipative channels via the vanishing of the spectral density or

breaking the unitality of the quantum map via its asymmetry can thus be detrimental to the emergence of the target state.

Note that the observed breaking of the emergence of identical η -correlations is not due to the breaking of the strong symmetry condition, which was studied in Ref. [33] via the addition of Lindblad dissipative terms that explicitly break it. In fact, it has already been predicted in Ref. [56] that going beyond Lindblad does not break strong symmetries under general assumptions. In the case of our Redfield master equation, one can indeed compute $[\bar{L}_j(\tau), \eta^\pm]$ and $[\bar{L}_j(\tau), \eta^+ \eta^-] = 0$ and show that these commutators remain zero even for asymmetric spectral density or small cutoff frequency. Interestingly, however, our results show that even without breaking the strong symmetry, going beyond Lindblad can still alter the steady state. Although this does not prevent the emergence of η -correlations, it hinders their convergence towards an identical value. This behavior can be understood as a consequence of the modification of the steady-state structure in the presence of a spectrally structured bath. In Appendix C, we provide the derivation of the steady state density matrix in the case $M = 2$, which allows us to draw in a very transparent way the following conclusions: for any spectral density, the strong symmetry is preserved and splits the Liouvillian into invariant symmetry sectors labels by two quantum numbers S and η for spin and charge, respectively. Consequently, the steady state is block-diagonal and depends on the initial weights in each sector. In the pure Lindblad limit (completely flat spectral density), the maximally mixed state is the unique possible steady state in each symmetry sector. If one considers structured spectral densities but applies the secular approximation, the steady state in each sector becomes incoherent, typically diagonal in the energy basis (Gibbs-like if detailed balance holds; maximally mixed within the sector if the map is unital). Finally, in the Redfield case studied here, the steady state in each sector can be nontrivial in the energy basis and may retain stationary coherences. In short, as soon as the maximally mixed state is not the unique steady state in a symmetry sector, η -correlations cannot be uniform. Appendix C presents the analytical formulas for the steady state and for the η -correlations in all three cases discussed above, clearly illustrating the impact of structured environments on the emergence of long-range order.

This information suggests that the ODLRO we aim to prepare under more general dissipation conditions needs to be engineered around a Lindblad-type description. Specifically, states asymptotically converging to the target state can be achieved, requiring longer preparation times constrained by the conditions on the bath structure. These findings open up the possibility of realizing an SF state in some realistic experiments.

VI. EXPERIMENTAL IMPLEMENTATION

1D fermionic models can be realized by tightly trapping an ultracold fermionic gas of atoms in optical lattices along the longitudinal axis of a cavity QED [40–42,45,46,57], where the spin-exchange interactions of the tUJ models could be implemented elaborating, e.g., in Refs. [43,44]. Multimode cavity setups could also be useful for controlling the range of interactions between the particles [16].

Regarding the dissipation, previous proposals [33] for spin-dependent dephasing have been inspired by reproducing phenomenologically spin fluctuation theory [58] via superfluid immersion [59,60] tuning appropriately the corresponding scattering lengths via Feshbach resonances [61]. Alternative approaches could make use of spin-dependent potentials, as in Refs. [62,63], in order to induce spin-dependent coupling to the environment in a fully optical setup. We mention the proposal in [64], where it is shown that it is possible to engineer pair coherence over long-distances through incoherent local environments using a probing beam tuned exactly between the spin states, in a protocol inspired by phase-contrast imaging [65].

The development of these schemes in cold-atom platforms would provide a new testing ground for current relevant models in condensed matter. Indeed, the superfluid state here prepared dissipatively corresponds to a superconducting state that breaks time reversal symmetry, and it is characterized by finite momentum Cooper pairing [37], in a condensed matter context, these states are exotic with the only experimental observation, in the absence of an external magnetic field, reported in EuRbFe₄As₄ [38]. Interestingly, in real materials, this state has been theorized in the presence of magnetic fluctuations, in particular driven by the presence of altermagnetism [39]. Thus the possibility of realizing such a state in an ultracold atom platform would give access to the study of its properties and, potentially, to the understanding of its governing mechanism by engineering different kinds of “spin baths” in a controlled experimental setup, therefore, employing the ultracold atom platform as a quantum simulator for a condensed matter problem.

VII. CONCLUSION

In this work, we have shown that it is possible to prepare a state with finite momentum by means of dissipation in a tUJ model compatible with current fermionic multimode cavity QED setups. We have analyzed the importance of the Markovian limit in the state preparation, which requires a bath that has a complete overlap of the system energy spectrum and is symmetric around low frequencies. We have then identified the conditions for the preparation scheme in the presence of dissipation preserving the target symmetry, but not following the full Lindblad description. Thus we can conclude that this state preparation requires dissipation engineering around a Lindblad dynamics in the relevant frequency range. Open questions remain regarding the effects of other forms of non-Markovianity and the role of the bath properties in more general dissipative state preparation proposals based on symmetries, which we leave for future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: OTHER BATHS

We present here the results obtained considering different kinds of spectral functions for the bath.

We focus in particular on a Lorentzian spectral density $J(\omega) = \eta\gamma/(\gamma^2 + (\omega - \omega_c)^2)$, which can be considered to approximate well an environment made up of a single harmonic oscillator with frequency ω_c , dissipating into the vacuum at rate γ .

As can be seen in Fig. 5, moving away from $\omega_c = 0$ (or if γ is not large enough) slows down the convergence towards the steady state.

As a structured spectral density with several peaks is a generic feature of real materials, we also show results of a spectral density built as the sum of two shifted Lorentzians in Fig. 6. As can be seen, depending on the behavior near low frequencies, we can again drastically slow down the convergence towards the steady state.

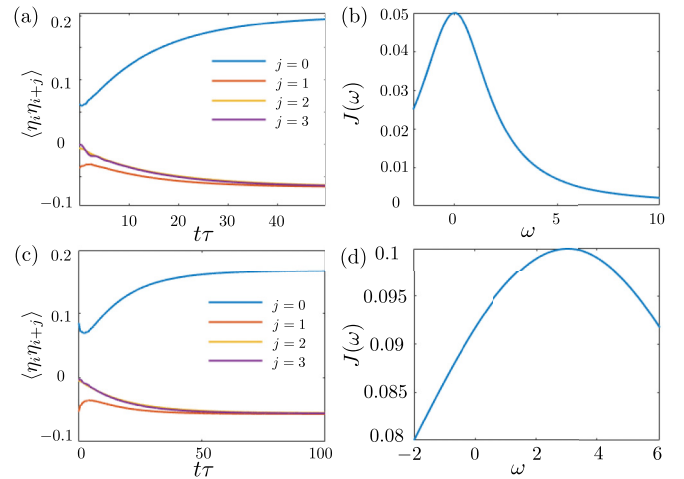


FIG. 5. Time evolution of $\langle \eta_i^+ \eta_{i+j}^- \rangle$ under the dissipative dynamics in Eq. (20) beyond GSKL with Lorentzian spectral density $J(\omega) = \eta\gamma/(\gamma^2 + (\omega - \omega_c)^2)$ (see text). [(a)–(c)] $\langle \eta_i^+ \eta_{i+j}^- \rangle$ at different distances j as in the legend. [(b)–(d)] Corresponding form of the bath correlation function of the form $J(\omega)$ in the interval of the Hamiltonian eigenfrequencies. [(a) and (b)] $\eta = 1$, $\gamma = 2$, and $\omega_c = 0$. [(c) and (d)] Shifted Lorentzian $\omega_c = 3$, the values of $\gamma = 10$ and $\eta = 1$ being tuned to ensure the same homogeneity of the spectral function in the low frequency spectrum. We observe that in the case of a centered Lorentzian, we could not find any value breaking the convergence to the steady state. Simulation with $M = 4$ sites for $U = 4t$ and $J = -0.1U$.

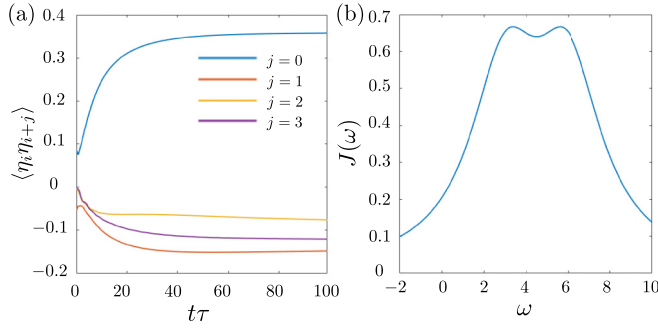


FIG. 6. Time evolution of $\langle \eta_i^+ \eta_{i+j}^- \rangle$ under the dissipative dynamics in Eq. (20) with the spectral density $J(\omega) = \eta\gamma/(\gamma^2 + (\omega - \omega_c)^2) + \eta'\gamma'/(\gamma'^2 + (\omega - \omega'_c)^2)$ given by the sum of two Lorentzians (see text). (a) $\langle \eta_i^+ \eta_{i+j}^- \rangle$ at different distances j as in the legend; (b) the corresponding form of the bath correlation function of the form $J(\omega)$ in the interval of the Hamiltonian eigenfrequencies. The parameter values are $\eta = 1$, $\gamma = 2$, $\omega_c = 3$, $\eta' = 1$, $\gamma' = 2$, and $\omega'_c = 6$. While it is of course possible to achieve convergence by fine tuning of the values until a sufficiently flat function is obtained, we observe that it is hard to obtain convergence also for a slightly structured bath if the frequencies involved fall in the relevant range of the Hamiltonian spectrum. Simulation with $M = 4$ sites for $U = 4t$ and $J = -0.1U$.

In Fig. 7, we summarize the results for all the other baths¹ by representing on a scatter plot the mean value of the off-diagonal correlations reached in η and the time τ_{ss} at which convergence is reached by the system coupled to the corresponding bath. It is apparent that there exist regimes for each kind of bath in which we still manage to reach the steady state. These regimes are characterized by homogeneity of the bath correlation functions, which are approximately symmetric in frequency at least in the neighborhood of the low frequency part of the Hamiltonian spectrum.

In general, the farther the functional form of the spectral density moves away from the Markovian bath, the smaller is the fraction of superfluid obtained in the final state, while the convergence time increases. The value at the steady state and the time required to get there are both seen to strongly depend on the parameters of the spectral function.

APPENDIX B: NUMERICAL SOLUTION OF THE BLOCH-REDFIELD EQUATION

In order to solve the Redfield Eq. (20), we use the Bloch-Redfield master equation integrator provided by the

¹In Fig. 7, super-Ohmic, sub-Ohmic, and Lorentzian spectral functions are interchangeable, depending on the functional form adopted, meaning that the values are specific and do not represent a general trend. For example, in the Lorentzian spectral function, whose width can be tuned until it is completely flat in the relevant region, we can obtain results consistent with the Markovian case. We decide here to represent the values obtained by adopting the minimal parameter in the functional form of the spectral density, to achieve convergence without excessively deforming $J(\omega)$.

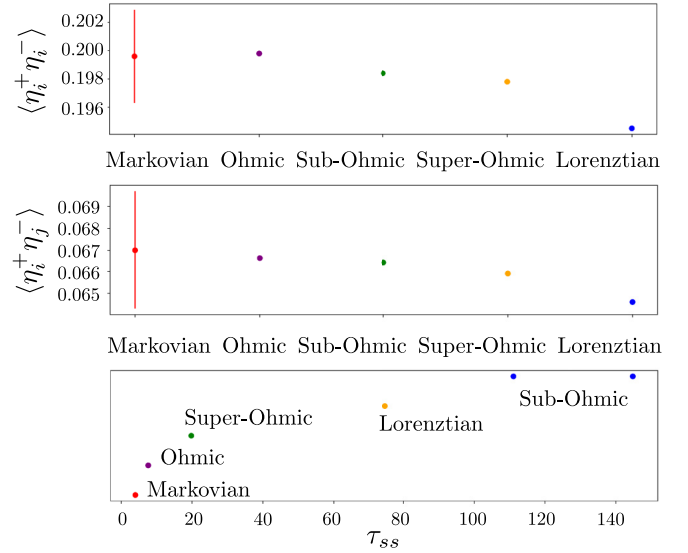


FIG. 7. Summary of the beyond-GSKL analysis using different spectral densities $J(\omega)$. The values associated with each spectral density are obtained with specific and minimal parameters to achieve a steady state. For the Markovian case, calculations are performed for $M = 4$ sites, consistent with the system size used for studying the non-Markovian regime and based on 1000 trajectories, resulting in an associated error of approximately 4%. (Top) Scatter plot of the doublon occupancy in the steady state for different spectral functions. A general decrease is observed, with the Ohmic-like (i.e., linear spectral density) case performing the best. Except for the Lorentzian case, the values obtained are consistent within error bars with the Markovian limit. (Center) Off-diagonal terms of the η -correlations, showing that the Ohmic bath better reproduces the Markovian limit. For the other spectral functions, a decrease in the condensate fraction is observed, though it remains consistent with the results in the Markovian limit. (Bottom) Trend of the steady-state convergence time τ_{ss} in different cases, defined as the first instant at which the final value is reached. A progressive and significant increase in the convergence time is observed as we depart from the Markovian case.

Python package QuTiP [55]. Given the complexity of the numerical calculation related to the exponential growth of the Bloch-Redfield tensor, which slows down the numerical integrator, we present results for a maximum of $M = 8$ sites.

APPENDIX C: ANALYTICAL RESULTS FOR $M = 2$

For $M = 2$ and for the half-filling scenario considered here with an equal number of spin up and down fermions (sector $S^z = 0$), we can restrict the analysis to the four-dimensional Hilbert space spanned by the following four states of the occupation basis: $|\uparrow, \downarrow\rangle$, $|\downarrow, \uparrow\rangle$, $|\uparrow\downarrow, 0\rangle$, $|0, \uparrow\downarrow\rangle$. because of the $U(1)$ -spin and $SU(2)$ -charge symmetries of the Hamiltonian, i.e., $[H_J, S^z] = [H_J, S^+ S^-] = [H_J, \eta^z] = [H_J, \eta_+ \eta_-] = 0$, it is useful to define the quantum numbers S and η associated with the operators $S^2 = (S^z)^2 + (S^+ S^- + S^- S^+)/2$ and $\eta^2 = (\eta^z)^2 + (\eta^+ \eta^- + \eta^- \eta^+)/2$ and work in their eigenstates basis, i.e., in the basis of states satisfying

$S^2|S, \eta\rangle = S(S+1)|S, \eta\rangle$ and $\eta^2|S, \eta\rangle = \eta(\eta+1)|S, \eta\rangle$. They are

$$\begin{aligned} |t\rangle &= |S=1, \eta=0\rangle = \frac{1}{\sqrt{2}}(|\downarrow, \uparrow\rangle + |\uparrow, \downarrow\rangle), \\ |s\rangle &= |S=0, \eta=0\rangle = \frac{1}{\sqrt{2}}(|\downarrow, \uparrow\rangle - |\uparrow, \downarrow\rangle), \\ |d_+\rangle &= |S=0, \eta=0\rangle = \frac{1}{\sqrt{2}}(|0, \uparrow\downarrow\rangle + |\uparrow\downarrow, 0\rangle), \\ |d_-\rangle &= |S=0, \eta=1\rangle = \frac{1}{\sqrt{2}}(|0, \uparrow\downarrow\rangle - |\uparrow\downarrow, 0\rangle), \end{aligned} \quad (C1)$$

where $|t\rangle$ and $|s\rangle$ are the triplet and singlet spin states, and where $|d_+\rangle$ and $|d_-\rangle$ are the doublon-holon symmetric and antisymmetric states, respectively. In the basis $\{|t\rangle, |s\rangle, |d_+\rangle, |d_-\rangle\}$, the Hamiltonian is block-diagonal and

reads

$$H_I = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & -J & 2t & 0 \\ 0 & 2t & -U & 0 \\ 0 & 0 & 0 & -U \end{pmatrix}. \quad (C2)$$

To go further, it is useful to work in the Hamiltonian basis $\{|t\rangle, |\psi_+\rangle, |\psi_-\rangle, |d_-\rangle\}$ where

$$|\psi_+\rangle = \cos\theta|s\rangle + \sin\theta|d_+\rangle, \quad (C3)$$

$$|\psi_-\rangle = -\sin\theta|s\rangle + \cos\theta|d_+\rangle \quad (C4)$$

with $\cos 2\theta = (U - J)/\Delta$, $\Delta = \sqrt{(U - J)^2 + 16t^2}$, $H|n\rangle = \omega_n|n\rangle$ with $\omega_t = J$, $\omega_{\pm} = -(U + J)/2 \pm \Delta/2$ and $\omega_{d_-} = -U$.

Given this, the jump and rotated jump operators have, in the energy basis $\{|t\rangle, |\psi_+\rangle, |\psi_-\rangle, |d_-\rangle\}$, the matrix representations

$$L_1 = \begin{pmatrix} 0 & -\cos\theta & \sin\theta & 0 \\ -\cos\theta & 0 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -L_2, \quad (C5)$$

$$\bar{L}_1 = \begin{pmatrix} 0 & -\cos\theta\Gamma(\omega_{t+}) & \sin\theta\Gamma(\omega_{t-}) & 0 \\ -\cos\theta\Gamma(-\omega_{t+}) & 0 & 0 & 0 \\ \sin\theta\Gamma(-\omega_{t-}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (C6)$$

with $\Gamma(\omega) = \pi J(-\omega) - i\text{P.V.} \int_{-\infty}^{\infty} J(v)/(v + \omega)dv$. From this, the matrix representation of $\mathcal{L}(\mathbb{1})$ reads

$$\mathcal{L}(\mathbb{1}) = 2 \begin{pmatrix} 2\pi[\cos^2\theta\Delta J(\omega_{t+}) + \sin^2\theta\Delta J(\omega_{t-})] & 0 & 0 & 0 \\ 0 & -2\pi\cos^2\theta\Delta J(\omega_{t+}) & \cos\theta\sin\theta(\Delta\Gamma^*(\omega_{t+}) + \Delta\Gamma(\omega_{t-})) & 0 \\ 0 & \cos\theta\sin\theta(\Delta\Gamma(\omega_{t+}) + \Delta\Gamma^*(\omega_{t-})) & -2\pi\sin^2\theta\Delta J(\omega_{t-}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (C7)$$

where $\Delta\Gamma(\omega) = \Gamma(\omega) - \Gamma^*(-\omega) = \pi(J(-\omega) - J(\omega)) - i\text{P.V.} \int_{-\infty}^{\infty} (J(v) - J(-v))/(v + \omega)dv$, and $\Delta J(\omega) = J(-\omega) - J(\omega)$, which shows again how the unitality of the map depends on the spectral density of the bath.

Using the expressions above, we can get an analytical solution for the steady state of the Redfield master equation, but the expression is cumbersome. It has, however, the following general structure in the energy basis $\{|t\rangle, |\psi_+\rangle, |\psi_-\rangle, |d_-\rangle\}$

$$\rho_{ss} = (1 - p_{d_-}) \begin{pmatrix} c_t & 0 & 0 & 0 \\ 0 & c_+ & \chi & 0 \\ 0 & \chi^* & c_- & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + p_{d_-} |d_-\rangle\langle d_-| \quad (C8)$$

with p_{d_-} fixed by the initial condition, and c_t , $c_{\pm}$, c_- , and χ the steady states populations and coherences that depend on the spectral density of the bath. If one secularizes the master

equation, we have $\chi = 0$ (diagonal state in the energy basis). In the pure Lindblad limit (flat spectral density), we have $c_t = c_+ = c_- = 1/3$. Hence, as soon as the bath has spectral structures, the steady state is no longer a convex combination of maximally mixed states in each symmetry sector; the response of the bath tends to redistribute the populations unevenly. This result can be extended to larger system sizes and explains why the η correlations are, in general, not uniform.

Note that from this, we can deduce the general form of the η correlations

$$\begin{aligned} \langle \eta_1^+ \eta_2^- \rangle &= -p_{d_-} + (1 - p_{d_-})(c_- \cos^2\theta + c_+ \sin^2\theta \\ &\quad + \text{Re}(\chi) \sin 2\theta), \end{aligned} \quad (C9)$$

which does not depend on the population of the triplet state. For the secularized master equation, we thus have $\langle \eta_1^+ \eta_2^- \rangle = -p_{d_-} + (1 - p_{d_-})(c_- \cos^2\theta + c_+ \sin^2\theta)$, while for the pure Lindblad limit, we have $\langle \eta_1^+ \eta_2^- \rangle = (1 - 4p_{d_-})/3$.

- [1] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Cold bosonic atoms in optical lattices, *Phys. Rev. Lett.* **81**, 3108 (1998).
- [2] D. Jaksch and P. Zoller, The cold atom hubbard toolbox, *Ann. Phys.* **315**, 52 (2005).
- [3] J. Yago Malo, L. Lepori, L. Gentini, and M. L. M. Chiofalo, Atomic quantum technologies for quantum matter and fundamental physics applications, *Technologies* **12**, 64 (2024).
- [4] C. Gross and W. S. Bakr, Quantum gas microscopy for single atom and spin detection, *Nat. Phys.* **17**, 1316 (2021).
- [5] W. S. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, A quantum gas microscope for detecting single atoms in a hubbard-regime optical lattice, *Nature (London)* **462**, 74 (2009).
- [6] E. Haller, J. Hudson, A. Kelly, D. A. Cotta, B. Peaudecerf, G. D. Bruce, and S. Kuhr, Single-atom imaging of fermions in a quantum-gas microscope, *Nat. Phys.* **11**, 738 (2015).
- [7] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
- [8] I. de Vega and D. Alonso, Dynamics of non-Markovian open quantum systems, *Rev. Mod. Phys.* **89**, 015001 (2017).
- [9] H. Weimer, A. Kshetrimayum, and R. Orús, Simulation methods for open quantum many-body systems, *Rev. Mod. Phys.* **93**, 015008 (2021).
- [10] T. D. Farokh Mivehvar, F. Piazza, and H. Ritsch, Cavity QED with quantum gases: new paradigms in many-body physics, *Adv. Phys.* **70**, 1 (2021).
- [11] D. Schmidt, H. Tomczyk, S. Slama, and C. Zimmermann, Dynamical instability of a Bose-Einstein condensate in an optical ring resonator, *Phys. Rev. Lett.* **112**, 115302 (2014).
- [12] M. Gangl and H. Ritsch, Cold atoms in a high-q ring cavity, *Phys. Rev. A* **61**, 043405 (2000).
- [13] F. Mivehvar, S. Ostermann, F. Piazza, and H. Ritsch, Driven-dissipative supersolid in a ring cavity, *Phys. Rev. Lett.* **120**, 123601 (2018).
- [14] J. Klinner, M. Lindholdt, B. Nagorny, and A. Hemmerich, Normal mode splitting and mechanical effects of an optical lattice in a ring cavity, *Phys. Rev. Lett.* **96**, 023002 (2006).
- [15] A. J. Kollár, A. T. Papageorge, K. Baumann, M. A. Armen, and B. L. Lev, An adjustable-length cavity and Bose-Einstein condensate apparatus for multimode cavity QED, *New J. Phys.* **17**, 043012 (2015).
- [16] V. D. Vaidya, Y. Guo, R. M. Kroeze, K. E. Ballantine, A. J. Kollár, J. Keeling, and B. L. Lev, Tunable-range, photon-mediated atomic interactions in multimode cavity qed, *Phys. Rev. X* **8**, 011002 (2018).
- [17] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Frustration and glassiness in spin models with cavity-mediated interactions, *Phys. Rev. Lett.* **107**, 277201 (2011).
- [18] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Emergent crystallinity and frustration with bose-einstein condensates in multimode cavities, *Nat. Phys.* **5**, 845 (2009).
- [19] K. E. Ballantine, B. L. Lev, and J. Keeling, Meissner-like effect for a synthetic gauge field in multimode cavity qed, *Phys. Rev. Lett.* **118**, 045302 (2017).
- [20] V. Torggler, S. Krämer, and H. Ritsch, Quantum annealing with ultracold atoms in a multimode optical resonator, *Phys. Rev. A* **95**, 032310 (2017).
- [21] P. Rotondo, M. Marcuzzi, J. P. Garrahan, I. Lesanovsky, and M. Müller, Open quantum generalisation of hopfield neural networks, *J. Phys. A: Math. Theor.* **51**, 115301 (2018).
- [22] E. Fiorelli, M. Marcuzzi, P. Rotondo, F. Carollo, and I. Lesanovsky, Signatures of associative memory behavior in a multimode dicke model, *Phys. Rev. Lett.* **125**, 070604 (2020).
- [23] B. P. Marsh, Y. Guo, R. M. Kroeze, S. Gopalakrishnan, S. Ganguli, J. Keeling, and B. L. Lev, Enhancing associative memory recall and storage capacity using confocal cavity QED, *Phys. Rev. X* **11**, 021048 (2021).
- [24] G. Bentsen, T. Hashizume, A. S. Buyskikh, E. J. Davis, A. J. Daley, S. S. Gubser, and M. Schleier-Smith, Treelike interactions and fast scrambling with cold atoms, *Phys. Rev. Lett.* **123**, 130601 (2019).
- [25] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Dicke quantum phase transition with a superfluid gas in an optical cavity, *Nature (London)* **464**, 1301 (2010).
- [26] J. Klinder, H. Keßler, M. Wolke, L. Mathey, and A. Hemmerich, Dynamical phase transition in the open dicke model, *Proc. Natl. Acad. Sci. USA* **112**, 3290 (2015).
- [27] J. Benary, C. Baals, E. Bernhart, J. Jiang, M. Röhrle, and H. Ott, Experimental observation of a dissipative phase transition in a multi-mode many-body quantum system, *New J. Phys.* **24**, 103034 (2022).
- [28] F. Ferri, R. Rosa-Medina, F. Finger, N. Dogra, M. Soriente, O. Zilberberg, T. Donner, and T. Esslinger, Emerging dissipative phases in a superradiant quantum gas with tunable decay, *Phys. Rev. X* **11**, 041046 (2021).
- [29] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, *Nature (London)* **415**, 39 (2002).
- [30] Q. Chen, J. Stajic, S. Tan, and K. Levin, BCS-BEC crossover: From high temperature superconductors to ultracold superfluids, *Phys. Rep.* **412**, 1 (2005).
- [31] J. Jin, A. Biella, O. Viyuela, L. Mazza, J. Keeling, R. Fazio, and D. Rossini, Cluster mean-field approach to the steady-state phase diagram of dissipative spin systems, *Phys. Rev. X* **6**, 031011 (2016).
- [32] T. E. Lee, S. Gopalakrishnan, and M. D. Lukin, Unconventional magnetism via optical pumping of interacting spin systems, *Phys. Rev. Lett.* **110**, 257204 (2013).
- [33] J. Tindall, B. Buča, J. R. Coulthard, and D. Jaksch, Heating-induced long-range η pairing in the hubbard model, *Phys. Rev. Lett.* **123**, 030603 (2019).
- [34] G. Lindblad, On the generators of quantum dynamical semigroups, *Commun. Math. Phys.* **48**, 119 (1976).
- [35] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of n-level systems, *J. Math. Phys.* **17**, 821 (1976).
- [36] C. N. Yang, η pairing and off-diagonal long-range order in a hubbard model, *Phys. Rev. Lett.* **63**, 2144 (1989).
- [37] D. F. Agterberg, J. S. Davis, S. D. Edkins, E. Fradkin, D. J. Van Harlingen, S. A. Kivelson, P. A. Lee, L. Radzihovsky, J. M. Tranquada, and Y. Wang, The physics of pair-density waves: Cuprate superconductors and beyond, *Annu. Rev. Condens. Matter Phys.* **11**, 231 (2020).
- [38] H. Zhao, R. Blackwell, M. Thinel, T. Handa, S. Ishida, X. Zhu, A. Iyo, H. Eisaki, A. N. Pasupathy, and K. Fujita, Smectic

- pair-density-wave order in $\text{EuRbFe}_4\text{As}_4$, *Nature (London)* **618**, 940 (2023).
- [39] S.-B. Zhang, L.-H. Hu, and T. Neupert, Finite-momentum cooper pairing in proximitized altermagnets, *Nat. Commun.* **15**, 1801 (2024).
- [40] E. Colella, R. Citro, M. Barsanti, D. Rossini, and M.-L. Chiofalo, Quantum phases of spinful fermi gases in optical cavities, *Phys. Rev. B* **97**, 134502 (2018).
- [41] F. Schlawin and D. Jaksch, Cavity-mediated unconventional pairing in ultracold fermionic atoms, *Phys. Rev. Lett.* **123**, 133601 (2019).
- [42] A. Camacho-Guardian, R. Paredes, and S. F. Caballero-Benítez, Quantum simulation of competing orders with fermions in quantum optical lattices, *Phys. Rev. A* **96**, 051602(R) (2017).
- [43] E. J. Davis, G. Bentsen, L. Homeier, T. Li, and M. H. Schleier-Smith, Photon-mediated spin-exchange dynamics of spin-1 atoms, *Phys. Rev. Lett.* **122**, 010405 (2019).
- [44] M. A. Norcia, R. J. Lewis-Swan, J. R. K. Cline, B. Zhu, A. M. Rey, and J. K. Thompson, Cavity-mediated collective spin-exchange interactions in a strontium superradiant laser, *Science* **361**, 259 (2018).
- [45] S. Baier, D. Petter, J. H. Becher, A. Patscheider, G. Natale, L. Chomaz, M. J. Mark, and F. Ferlaino, Realization of a strongly interacting fermi gas of dipolar atoms, *Phys. Rev. Lett.* **121**, 093602 (2018).
- [46] A. Mazurenko, C. S. Chiu, G. Ji, M. F. Parsons, M. Kanász-Nagy, R. Schmidt, F. Grusdt, E. Demler, D. Greif, and M. Greiner, A cold-atom fermi-hubbard antiferromagnet, *Nature (London)* **545**, 462 (2017).
- [47] A. G. Redfield, On the theory of relaxation processes, *IBM J. Res. Dev.* **1**, 19 (1957).
- [48] J. Tindall, Realising complex quantum states of matter via symmetries and heating, Ph.D. thesis, University of Oxford, 2021.
- [49] A. Klein, M. Bruderer, S. R. Clark, and D. Jaksch, Dynamics, dephasing and clustering of impurity atoms in bose-einstein condensates, *New J. Phys.* **9**, 411 (2007).
- [50] F. Damanet, A. J. Daley, and J. Keeling, Atom-only descriptions of the driven-dissipative dicke model, *Phys. Rev. A* **99**, 033845 (2019).
- [51] R. Palacino and J. Keeling, Atom-only theories for $U(1)$ symmetric cavity-QED models, *Phys. Rev. Res.* **3**, L032016 (2021).
- [52] B. Debecker, J. Martin, and F. Damanet, Controlling matter phases beyond markov, *Phys. Rev. Lett.* **133**, 140403 (2024).
- [53] B. Debecker, J. Martin, and F. Damanet, Spectral theory of non-markovian dissipative phase transitions, *Phys. Rev. A* **110**, 042201 (2024).
- [54] B. Debecker, L. Pausch, J. Louvet, T. Bastin, J. Martin, and F. Damanet, The role of non-markovian dissipation in quantum phase transitions: tricriticality, spin squeezing, and directional symmetry breaking, *Phys. Rev. A* **112**, 012210 (2025).
- [55] J. Johansson, P. Nation, and F. Nori, QuTiP: An open-source python framework for the dynamics of open quantum systems, *Comput. Phys. Commun.* **183**, 1760 (2012).
- [56] B. Buča, J. Tindall, and D. Jaksch, Non-stationary coherent quantum many-body dynamics through dissipation, *Nat. Commun.* **10**, 1730 (2019).
- [57] L. Lucchesi and M. L. Chiofalo, Many-body entanglement in short-range interacting fermi gases for metrology, *Phys. Rev. Lett.* **123**, 060406 (2019).
- [58] S. Kruchinin, H. Nagao, and S. Aono, *Modern Aspects of Superconductivity* (World Scientific, New Jersey, London, 2010).
- [59] A. J. Daley, P. O. Fedichev, and P. Zoller, Single-atom cooling by superfluid immersion: A nondestructive method for qubits, *Phys. Rev. A* **69**, 022306 (2004).
- [60] M. Bruderer, A. Klein, S. R. Clark, and D. Jaksch, Polaron physics in optical lattices, *Phys. Rev. A* **76**, 011605(R) (2007).
- [61] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, *Rev. Mod. Phys.* **82**, 1225 (2010).
- [62] O. Mandel, M. Greiner, A. Widera, T. Rom, T. W. Hänsch, and I. Bloch, Coherent transport of neutral atoms in spin-dependent optical lattice potentials., *Phys. Rev. Lett.* **91**, 010407 (2003).
- [63] B. Yang, H.-N. Dai, H. Sun, A. Reingruber, Z.-S. Yuan, and J.-W. Pan, Spin-dependent optical superlattice, *Phys. Rev. A* **96**, 011602(R) (2017).
- [64] J.-S. Bernier, P. Barnettler, D. Poletti, and C. Kollath, Emergence of spatially extended pair coherence through incoherent local environmental coupling, *Phys. Rev. A* **87**, 063608 (2013).
- [65] Y. Shin, M. W. Zwierlein, C. H. Schunck, A. Schirotzek, and W. Ketterle, Observation of phase separation in a strongly interacting imbalanced fermi gas, *Phys. Rev. Lett.* **97**, 030401 (2006).