

Fair Allocation of Operating Envelopes for Distribution Networks Considering Voltage Unbalance

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Abstract—Operating envelopes (OEs) are increasingly used to allocate limits to distributed energy resources (DERs) while maintaining secure distribution network operation. In unbalanced low-voltage feeders, OE calculation based only on voltage magnitude and thermal constraints can yield overly optimistic limits because power quality constraints such as voltage unbalance are neglected. This paper proposes a three-phase unbalanced AC optimal power flow framework for computing coupled P-Q OEs with explicit voltage unbalance factor (VUF) constraints. In addition, two fairness mechanisms for allocating the available P-Q flexibility across multiple PV units are embedded and compared: (i) network-weighted proportional fairness and (ii) lexicographic max-min fairness. Case studies on unbalanced test feeders illustrate how VUF constraints reshape the P-Q feasible region and the impact of power quality-constrained operation. The comparison highlights the trade-off between the efficiency, equity, and practicality of fairness allocation methods.

Index Terms—distribution networks, operating envelopes, optimal power flow, three-phase systems, voltage unbalance.

NOMENCLATURE

Indices and Sets

$i, k, \mathcal{N}$	Indices of network nodes; set of all network nodes, $ \mathcal{N} = N$.
s	Stage index in the lexicographic max-min procedure.
u, v	Indices of DER units.
ϕ, μ, Φ	Phase indices; set of phases $\{a, b, c\}$.
$\mathcal{L}$	Set of network branches.
$\mathcal{U}_\phi$	Set of nodes hosting a DER on phase ϕ .
$\mathcal{U}$	Set of all DER units, $\mathcal{U} = \bigcup_{\phi \in \Phi} \mathcal{U}_\phi$.
$\mathcal{F}$	AC feasible set defined by network and operational constraints.
$\mathcal{F}_\phi^{(s)}, \mathcal{B}_\phi^{(s)}$	Free and bound unit sets at stage s of the lexicographic procedure on phase ϕ .
$\mathcal{E}_u$	Operating envelope polygon of unit u in the P-Q plane.

Network Parameters

$\mathbf{Y}$	Three-phase nodal admittance matrix, $\mathbf{Y} \in \mathbb{C}^{3N \times 3N}$.
$\mathbf{Z}$	Nodal impedance matrix, $\mathbf{Z} = \mathbf{Y}^{-1}$.
$\mathbf{z}^{(u)}$	Column of $\mathbf{Z}$ corresponding to unit u , $\mathbf{z}^{(u)} \in \mathbb{C}^{3N}$.
Z_{ik}	Branch impedance of line $(i, k) \in \mathcal{L}$.

Y_i^{sh}	Shunt admittance at node i .
$G_{\phi\mu, ik}$	Real part of the element corresponding to phase ϕ at node i and phase μ at node k in the network admittance matrix $\mathbf{Y}$.
$B_{\phi\mu, ik}$	Imaginary part of the element corresponding to phase ϕ at node i and phase μ at node k in the network admittance matrix $\mathbf{Y}$.
ℓ_u	Global column index of unit u in $\mathbf{Z}$, $\ell_u = 3(i_u - 1) + \phi_u$.

Variables

$V_{\phi, i}^r, V_{\phi, i}^i$	Real and imaginary parts of the phase- ϕ voltage phasor at node i .
V^r, V^i	Vectors of real and imaginary nodal voltage coordinates, $\in \mathbb{R}^{3N}$.
$\mathbf{v}_{\phi, i}$	Complex voltage phasor, $\mathbf{v}_{\phi, i} = V_{\phi, i}^r + jV_{\phi, i}^i$.
$V_u^{(0)}$	Baseline complex voltage at the bus of unit u .
V_i^+, V_i^-	Positive- and negative-sequence voltage phasors at node i .
$P_{\phi, i}^G, Q_{\phi, i}^G$	Real and reactive power output of the conventional generator at node i , phase ϕ .
P^G, Q^G	Vectors of real and reactive power outputs of conventional generators, $\in \mathbb{R}^{3N}$.
$P_{\phi, i}^{\text{DG}}, Q_{\phi, i}^{\text{DG}}$	Real and reactive power injection of the DER at node i , phase ϕ .
$P^{\text{DG}}, Q^{\text{DG}}$	Vectors of real and reactive power injections of DER units, $\in \mathbb{R}^{3N}$.
$\mathbf{x}$	Decision vector of the AC-OPF problem, comprising the collection of all voltage and power injection vectors ($\mathbf{x} = \{V^r, V^i, P^G, Q^G, P^{\text{DG}}, Q^{\text{DG}}\}$).
α	Shared headroom scalar optimized at each angle and stage of fairness methods.
$\alpha_m^{*(s)}$	Optimal value of α at stage s and angle θ_m .
$\alpha_{u, m}$	Final allocated headroom scalar of unit u at angle θ_m .

Constants and Parameters

c_i	Linear generation cost coefficient at node i .
$P_{\phi, i}^D, Q_{\phi, i}^D$	Active and reactive load demand at node i , phase ϕ .
$\hat{P}_u, \hat{Q}_u$	Baseline real and reactive injection of unit u from the baseline OPF solution.

$\underline{P}_i^G, \overline{P}_i^G$	Lower and upper real power limits of the conventional generator at node i .
$\underline{Q}_i^G, \overline{Q}_i^G$	Lower and upper reactive power limits of the conventional generator at node i .
$\overline{S}_{\phi,i}^{\text{DG}}$	Apparent power capacity of the DER at node i , phase ϕ .
$\underline{V}_i, \overline{V}_i$	Lower and upper voltage magnitude limits at node i .
ε	Admissible VUF limit (e.g. $\varepsilon = 2\%$ per IEC 61000-2-2).
θ_m	m -th angular sample for OE boundary tracing, $\theta_m = 2\pi m/M$.
M	Number of angular samples per OE boundary trace.
ω	Symmetrical components rotation operator, $\omega = e^{j2\pi/3}$.

Sensitivity and Fairness Quantities

σ_u	Aggregate voltage impact of unit u , $\sigma_u = \ \partial \mathbf{V} /\partial P_u\ ^2 + \ \partial \mathbf{V} /\partial Q_u\ ^2$.
$\tilde{w}_u$	Raw inverse-sensitivity weight of unit u , $\tilde{w}_u = 1/(\sqrt{\sigma_u} + \varepsilon_0)$.
w_u	Phase-normalised sensitivity weight of unit u , $w_u \in (0, 1]$.
δ_u	Step magnitude of unit u from its baseline, $\delta_u = \ (\Delta P_u, \Delta Q_u)\ $.
$P_{u,m}^{\text{OE}}, Q_{u,m}^{\text{OE}}$	Real and reactive components of the m -th boundary point of the OE of unit u .

I. INTRODUCTION

High penetrations of distributed energy resources (DERs), particularly inverter-interfaced photovoltaic (PV) generation, are reshaping the operational paradigm of medium- and low-voltage distribution networks. The resulting bidirectional and highly time-varying power flows can trigger voltage magnitude excursions and thermal congestion, thereby motivating network-aware mechanisms that communicate permissible DER operating limits without requiring the distribution system operator (DSO) to directly dispatch behind-the-meter devices. In this context, operating envelopes (OEs) have emerged as a practical interface between DSOs and DER owners/aggregators by specifying feasible import/export ranges that preserve network integrity [1]–[3]. Compared with static connection limits designed for worst-case conditions, the dynamic assignment of OEs can unlock latent hosting capacity by adapting limits to prevailing operating states, forecasts, and controllable network assets [1], [4]–[6].

A large body of OE research adopts optimization-based computation, most commonly via optimal power flow (OPF), to embed voltage and thermal constraints into the envelope definition [1]–[3]. Because distribution feeders are inherently unbalanced due to single- and two-phase laterals, untransposed lines, and unevenly distributed single-phase loads and DER connections, three-phase unbalanced modeling is often essential for envelope accuracy and for preventing phase-specific violations [1], [7]. Recent advances have further expanded

envelope concepts beyond scalar import/export limits to multi-dimensional feasible regions in the active–reactive (P–Q) plane, enabling explicit utilization of inverter reactive power capability for voltage management and service provision [8]. Nevertheless, computation and allocation of P–Q envelopes remain challenging because they must balance network security, inverter capability constraints, and the equitable sharing of scarce network capacity among multiple DERs [2], [8].

In parallel with voltage magnitude and thermal constraints, power quality constraints—most notably voltage unbalance (VU)—are receiving renewed attention. Increased single-phase DER uptake and uneven phase loading can exacerbate VU, which may elevate losses and adversely impact three-phase equipment [9], [10]. Multiple standards define and quantify VU differently (e.g., IEC, IEEE, and NEMA), and the associated metrics are not fully consistent, which complicates the use of VU in optimization-based operation [11], [12]. In practice, the IEC definition based on symmetrical components is widely used through the voltage unbalance factor (VUF), i.e., the ratio of negative- to positive-sequence voltage magnitudes [13], [14]. Importantly, OPF-based coordination that neglects VUF constraints can inadvertently aggravate unbalance even when voltage magnitude limits are satisfied, motivating the explicit inclusion of VUF constraints in distribution optimization [9], [15].

Another critical aspect of OE deployment is fairness. Envelopes effectively allocate a limited network resource, namely hosting capacity, among many DERs. Purely efficiency-driven allocations can therefore produce systematically unequal curtailment or flexibility access, undermining social acceptance and regulatory feasibility [2], [5]. Existing OE allocation approaches have therefore begun to incorporate equitable perspectives and to analyze fairness properties, including proportional-fair allocations under uncertainty and explicit equitable allocation strategies [3], [16].

The intersection of these three domains, P–Q flexibility, VUF regulation, and fairness, remains largely unexplored [17]. To the best of the authors’ knowledge, this work presents a novel and comprehensive framework that, for the first time, jointly: (i) constructs P–Q operating envelopes in unbalanced three-phase networks, (ii) enforces explicit VUF constraints during the envelope computation, and (iii) compares alternative fairness paradigms under a common three-phase AC-OPF framework. This gap is particularly consequential because VUF constraints can reshape the feasible P–Q region, and fairness choices can significantly alter how remaining flexibility is distributed across DERs.

This paper proposes a unified framework for computing VU-aware operating envelopes for PV inverters in unbalanced three-phase distribution networks. The framework is based on a three-phase unbalanced AC-OPF formulation augmented with nodal VUF constraints and inverter capability limits. To characterize the feasible P–Q operating region, a directional OPF sweep is employed to trace the envelope boundary in the P–Q plane. Two fairness mechanisms are embedded and compared: (i) a network-weighted proportional fairness formulation that balances efficiency and equity through weighted share of utilities [16], [18]–[20], and (ii) a lexicographic max–

TABLE I: Positioning of this paper relative to representative OE and VU literature

Reference	Network model	P-Q	VUF	Fairness / allocation
Liu <i>et al.</i> [1]	3 ϕ , linear OPF	P-only	No	Not emphasized
Alam <i>et al.</i> [3]	3 ϕ , linear OPF	P-only	No	Technical vs. (soft-)equitable
Liu & Braslavsky [16]	3 ϕ , linear robust OPF	P-only	No	Proportional fairness property (implicit)
Gao <i>et al.</i> [8]	3 ϕ , linear OPF	Yes	No	Equitable allocation
Fan & Kockar [15]	3ϕ, AC OPF	–	Yes	Not emphasized
Russell <i>et al.</i> [24]	3 ϕ , linear robust OPF	P-only	Yes	Not emphasized
This paper	3ϕ, AC OPF	Yes	Yes	Proportional fairness vs. lexicographic max-min

min fairness formulation that prioritizes the worst-off DER in a hierarchical manner [18], [21]–[23]. The proposed framework is evaluated on two unbalanced test feeders to quantify (a) the impact of VUF constraints on envelope size and shape, and (b) the operational and distributive differences induced by the two fairness paradigms. Table I positions the proposed framework relative to representative OE and VU studies.

The remainder of the paper is structured as follows: Section II introduces the three-phase VUF-constrained AC-OPF model, explains the envelope tracing procedure, and elaborates on the fairness formulations. Section III presents the case studies along with comparative results. Section IV offers a discussion and comparative analysis of the findings; and finally, Section V summarizes and concludes the paper.

II. FAIR OPERATING ENVELOPE COMPUTATION

This section formulates a three-phase unbalanced AC-OPF framework for computing coupled P-Q operating envelopes under explicit VUF constraints, and it embeds two alternative fairness mechanisms for allocating feasible flexibility across multiple DERs.

A. Network Model and Problem Setting

Consider a three-phase unbalanced radial distribution network with node set $\mathcal{N} = \{1, \dots, N\}$ and branch set $\mathcal{L}$. Node 1 is the slack bus. Voltages are expressed in rectangular coordinates $\mathbf{v}_{\phi,i} = V_{\phi,i}^r + jV_{\phi,i}^i \in \mathbb{C}$ for phase $\phi \in \Phi = \{a, b, c\}$ and node $i \in \mathcal{N}$. The three-phase nodal admittance matrix $\mathbf{Y} \in \mathbb{C}^{3N \times 3N}$ is assembled from branch impedances Z_{ij} and shunt admittances Y_i^{sh} .

Let $\mathcal{U}_{\phi} \subseteq \mathcal{N}$ denote the set of nodes hosting a DER on phase ϕ and $\mathcal{U} = \bigcup_{\phi \in \Phi} \mathcal{U}_{\phi}$. For each unit $u \in \mathcal{U}_{\phi}$, let i_u be its bus index and $(\hat{P}_u, \hat{Q}_u)$ its baseline real and reactive injection. The $\mathcal{E}_u$ of unit u is a polygon in the P-Q plane whose boundary is traced at M angular samples $\theta_m = 2\pi m/M$, $m = 0, \dots, M-1$ representing its OE.

B. Baseline Three-Phase AC Optimal Power Flow

The baseline dispatch determines the optimal decision vector $\mathbf{x} = \{V^r, V^i, P^G, Q^G, P^{\text{DG}}, Q^{\text{DG}}\}$, comprising nodal voltage coordinates in rectangular form and active/reactive power injections for both conventional generation and DERs. The objective is to minimize the total cost of conventional generation (e.g., feeder bus imports/exports):

$$\min_{\mathbf{x}} \sum_{i \in \mathcal{N}} \sum_{\phi \in \Phi} c_i P_{\phi,i}^G \quad (1)$$

where c_i represents the cost coefficient assigned to conventional sources. The optimization is subject to the following constraints for all $i \in \mathcal{N}$ and $\phi \in \Phi$:

a) *Power balance*: The nodal injections must satisfy the three-phase AC power flow equations for each phase:

$$P_{\phi,i}^G + P_{\phi,i}^{\text{DG}} - P_{\phi,i}^D = \sum_{k \in \mathcal{N}} \sum_{\mu \in \Phi} \left[V_{\phi,i}^r (G_{\phi\mu,ik} V_{\mu,k}^r - B_{\phi\mu,ik} V_{\mu,k}^i) + V_{\phi,i}^i (G_{\phi\mu,ik} V_{\mu,k}^i + B_{\phi\mu,ik} V_{\mu,k}^r) \right], \quad (2)$$

$$Q_{\phi,i}^G + Q_{\phi,i}^{\text{DG}} - Q_{\phi,i}^D = \sum_{k \in \mathcal{N}} \sum_{\mu \in \Phi} \left[V_{\phi,i}^i (G_{\phi\mu,ik} V_{\mu,k}^r - B_{\phi\mu,ik} V_{\mu,k}^i) - V_{\phi,i}^r (G_{\phi\mu,ik} V_{\mu,k}^i + B_{\phi\mu,ik} V_{\mu,k}^r) \right]. \quad (3)$$

b) *Voltage magnitude bounds*: Nodal voltages must remain within regulatory limits to ensure network stability:

$$\underline{V}_i^2 \leq (V_{\phi,i}^r)^2 + (V_{\phi,i}^i)^2 \leq \overline{V}_i^2. \quad (4)$$

c) *Generation limits*:

$$\underline{P}_i^G \leq P_{\phi,i}^G \leq \overline{P}_i^G, \quad \underline{Q}_i^G \leq Q_{\phi,i}^G \leq \overline{Q}_i^G. \quad (5)$$

d) *DER apparent power capacity*:

$$(P_{\phi,i}^{\text{DG}})^2 + (Q_{\phi,i}^{\text{DG}})^2 \leq (\overline{S}_{\phi,i}^{\text{DG}})^2. \quad (6)$$

e) *Voltage unbalance factor constraint*: VU is quantified using the symmetrical component definition, i.e., the ratio of negative- to positive-sequence voltage magnitudes [13]. Using $\omega = e^{j2\pi/3}$, the positive- and negative-sequence phasors at bus i are:

$$V_i^- = \frac{1}{3} (\mathbf{v}_{a,i} + \omega^2 \mathbf{v}_{b,i} + \omega \mathbf{v}_{c,i}), \quad (7)$$

$$V_i^+ = \frac{1}{3} (\mathbf{v}_{a,i} + \omega \mathbf{v}_{b,i} + \omega^2 \mathbf{v}_{c,i}). \quad (8)$$

The nodal VUF limit is enforced via

$$|V_i^-|^2 \leq \varepsilon^2 |V_i^+|^2, \quad \forall i \in \mathcal{N}, \quad (9)$$

where ε denotes the admissible VUF threshold (e.g., $\varepsilon = 0.02$ corresponds to 2% VUF). Let $\mathcal{F}$ denote the feasible set defined by (2)–(9). Note that (9) is quadratic in the voltage variables and, when coupled with the nonlinear three-phase AC power-flow constraints, yields a nonconvex nonlinear program for OE tracing.

C. Voltage Sensitivity Weights

In this section, the foundations of weight calculations for proportional fairness are presented. Let $\mathbf{Z} = \mathbf{Y}^{-1} \in \mathbb{C}^{3N \times 3N}$ denote the system impedance matrix. For each unit $u \in \mathcal{U}_\phi$, we define its global column index ℓ_u as:

$$\ell_u = 3(i_u - 1) + \phi_u \quad (10)$$

where i_u is the node index and $\phi_u \in \{1, 2, 3\}$ represents the integer phase index of unit u , so that $\mathbf{z}^{(u)} \triangleq \mathbf{Z}_{:, \ell_u}$ denotes the ℓ_u -th column of $\mathbf{Z}$.

Linearizing the bus voltage equation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ around the baseline and converting via $\Delta I_u \approx \Delta S_u^* / \bar{V}_u^{(0)}$ gives the $\mathbf{Z}$ -bus voltage sensitivity [25], [26]:

$$\frac{\partial |\mathbf{V}|}{\partial P_u} \approx \text{Re} \left(\frac{\mathbf{z}^{(u)}}{\bar{V}_u^{(0)}} \right), \quad (11)$$

$$\frac{\partial |\mathbf{V}|}{\partial Q_u} \approx \text{Im} \left(\frac{\mathbf{z}^{(u)}}{\bar{V}_u^{(0)}} \right). \quad (12)$$

The sensitivities (11)–(12) follow from [25], [26]. Based on these, we define an aggregate voltage-impact score and a normalized weighting scheme as follows. The aggregate network-wide voltage impact of unit u is:

$$\sigma_u = \left\| \frac{\partial |\mathbf{V}|}{\partial P_u} \right\|^2 + \left\| \frac{\partial |\mathbf{V}|}{\partial Q_u} \right\|^2. \quad (13)$$

The sensitivity weight is normalized within each phase group independently:

$$\tilde{w}_u = \frac{1}{\sqrt{\sigma_u} + \varepsilon_0}, \quad w_u = \frac{\tilde{w}_u}{\max_{v \in \mathcal{U}_\phi} \tilde{w}_v}, \quad \forall u \in \mathcal{U}_\phi, \quad (14)$$

where $\varepsilon_0 > 0$ is a regularization constant. Phase-local normalization ensures $w_u \in (0, 1]$ with $w_u = 1$ assigned to the least voltage-sensitive unit on each phase, so that allocation ratios reflect intra-phase network heterogeneity rather than inter-phase impedance differences.

D. Fair OE Computation Methods

Having established the network model and the voltage sensitivity weights, two methods of computing fair allocated OEs that embed these ingredients are presented. Method I pursues a closed-form proportional allocation, offering computational simplicity and transparency. Method II formulates a lexicographic max–min optimization that prioritizes equity across units within each phase.

1) *Network-Weighted Proportional Fairness*: Method I extends classical proportional fairness by weighting allocations with w_u , so that units with lower grid impact receive proportionally larger OEs. The resulting allocation is closed-form and computationally lightweight.

a) *Formulation*: Proportional fairness assigns headroom according to each unit's ability to withstand loading variations in bus voltage, without breaching network constraints [19], [20]. Here, the capacity is quantified by the sensitivity weight

$w_u \in (0, 1]$. For each phase ϕ and angle θ_m , a single shared scalar $\alpha \geq 0$ is maximized:

$$\alpha_m^* = \max_{\alpha \geq 0, \mathbf{x} \in \mathcal{F}} \alpha \quad (15)$$

subject to:

$$P_{\phi, i_u}^{\text{DG}} = \hat{P}_u + w_u \alpha \cos \theta_m, \quad \forall u \in \mathcal{U}_\phi, \quad (16)$$

$$Q_{\phi, i_u}^{\text{DG}} = \hat{Q}_u + w_u \alpha \sin \theta_m, \quad \forall u \in \mathcal{U}_\phi, \quad (17)$$

and the full network feasibility constraints (2)–(9).

This constitutes a single-level nonconvex nonlinear program in which (16)–(17) augment $\mathcal{F}$ with a shared proportionality structure across all units on phase ϕ . The boundary point of unit u at angle θ_m is:

$$(P_{u,m}^{\text{OE}}, Q_{u,m}^{\text{OE}}) = (\hat{P}_u + w_u \alpha_m^* \cos \theta_m, \hat{Q}_u + w_u \alpha_m^* \sin \theta_m). \quad (18)$$

b) *Fairness Properties*: The allocation ratio between any two units on the same phase is determined solely by their weights:

$$\frac{\|\Delta \mathbf{s}_u\|}{\|\Delta \mathbf{s}_v\|} = \frac{w_u}{w_v} = \frac{\sqrt{\sigma_u}}{\sqrt{\sigma_v}}, \quad \forall u, v \in \mathcal{U}_\phi, \quad (19)$$

where $\Delta \mathbf{s}_u = (w_u \alpha_m^* \cos \theta_m, w_u \alpha_m^* \sin \theta_m)$ denotes the shift of unit u relative to the baseline. This ratio is angle-independent and fully determined by the linearized sensitivity (13) evaluated at the baseline operating point. The allocation is therefore Pareto-efficient and satisfies the proportional fairness axioms of [19]: no reallocation can increase one unit's headroom by more than a proportional factor without reducing another unit's by at least the same factor.

A structural limitation follows directly from (19): the allocation ratios are fixed before the OE optimization is solved and do not adapt to the actual feasible headroom at each angle θ_m . Consequently, if the network constraint that limits α_m^* originates from a unit with large weight (electrically remote, large w), a more sensitive unit with small w may receive less headroom than the feasible set $\mathcal{F}$ could support for it, a conservatism that the method does not adapt within the current formulation.

2) *Lexicographic Max-Min Fairness*: Lexicographic max-min fairness is a classical fairness criterion in resource allocation and network optimization [22]. It has recently been applied to fair allocation problems in power and energy systems, including distribution-level resource management and EV charging coordination [23], [27].

a) *Fairness Properties*: A feasible allocation $\delta = (\delta_1, \dots, \delta_n)$ is lexicographically max-min optimal if and only if its sorted version $\delta_{(1)} \leq \delta_{(2)} \leq \dots \leq \delta_{(i)} \leq \dots \leq \delta_{(n)}$ is lexicographically maximum over all feasible allocations: no reallocation can increase $\delta_{(i)}$ without decreasing some $\delta_{(k)}$ with $k < i$. This criterion implies both Pareto efficiency and proportional fairness: any lexicographic max–min optimal allocation is also Pareto efficient and proportionally fair, but the opposite does not hold. The distinction is that the worst-off unit is protected unconditionally, irrespective of gains provided to others.

b) *Formulation*: For a fixed phase ϕ and angle θ_m , the algorithm proceeds through at most $|\mathcal{U}_\phi|$ sequential stages. Let $\mathcal{F}_\phi^{(s)}$ and $\mathcal{B}_\phi^{(s)}$ denote the free and bound unit sets at stage s , initialised as $\mathcal{F}_\phi^{(1)} = \mathcal{U}_\phi$ and $\mathcal{B}_\phi^{(1)} = \emptyset$.

At stage s , the following optimization is solved:

$$\alpha_m^{*(s)} = \max_{\alpha \geq 0, \mathbf{x} \in \mathcal{F}} \alpha \quad (20)$$

subject to, the network feasibility constraints (2)–(9), and for all free units $u \in \mathcal{F}_\phi^{(s)}$:

$$P_{\phi, i_u}^{\text{DG}} = \hat{P}_u + \alpha \cos \theta_m, \quad (21)$$

$$Q_{\phi, i_u}^{\text{DG}} = \hat{Q}_u + \alpha \sin \theta_m, \quad (22)$$

and for all bound units $u \in \mathcal{B}_\phi^{(s)}$:

$$P_{\phi, i_u}^{\text{DG}} = \hat{P}_u + \alpha_m^{*(s_u)} \cos \theta_m, \quad (23)$$

$$Q_{\phi, i_u}^{\text{DG}} = \hat{Q}_u + \alpha_m^{*(s_u)} \sin \theta_m, \quad (24)$$

where $s_u < s$ is the stage at which unit u was fixed. Every free unit moves by the same scalar α with no external weight factor; the allocation ratios are not predetermined but emerge from the feasibility of $\mathcal{F}$ at each stage.

After solving, the binding unit set is identified as those units whose DER apparent-power constraint (6) is active at $\alpha_m^{*(s)}$:

$$\mathcal{B}^{\text{new}} = \left\{ u \in \mathcal{F}_\phi^{(s)} \mid \bar{S}_{\phi, i_u}^2 - (\hat{P}_u + \alpha_m^{*(s)} \cos \theta_m)^2 - (\hat{Q}_u + \alpha_m^{*(s)} \sin \theta_m)^2 \leq \tau \right\}, \quad (25)$$

where $\tau > 0$ is a numerical tolerance. These units are moved to $\mathcal{B}_\phi^{(s+1)}$ with $\alpha_{u,m} = \alpha_m^{*(s)}$, and the procedure repeats until $\mathcal{F}_\phi^{(s)} = \emptyset$.

The boundary point of each unit is:

$$(P_{u,m}^{\text{OE}}, Q_{u,m}^{\text{OE}}) = (\hat{P}_u + \alpha_{u,m} \cos \theta_m, \hat{Q}_u + \alpha_{u,m} \sin \theta_m). \quad (26)$$

E. Comparative Analysis

1) *Allocation structure*: The two methods share the same feasible set $\mathcal{F}$, the same baseline $(\hat{P}_u, \hat{Q}_u)$, and the same angular tracing framework. They differ in one fundamental respect: in Method I the allocation ratios are fixed by the linearized sensitivity (19) and do not depend on the angle θ_m or on which constraint in $\mathcal{F}$ is active. In Method II the ratios vary with θ_m and are determined by the nonlinear feasibility of $\mathcal{F}$ at each stage.

This difference has a direct geometric interpretation. In Method I the OE polygon $\mathcal{E}_u$ is a scaled copy of a common reference polygon, with the scaling factor w_u fixed for all u . In Method II the shape of $\mathcal{E}_u$ varies across units in a way that reflects the actual headroom available at each location and direction: units that are electrically constrained in one direction but not another will have asymmetric envelopes that Method I cannot reproduce.

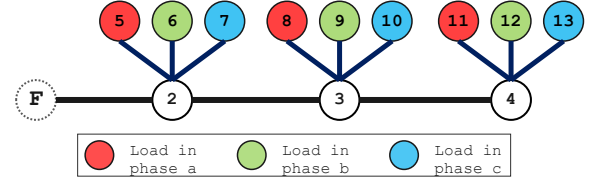


Fig. 1: Network diagram for Test Case 1.

2) *Sensitivity to linearisation accuracy*: Method I depends critically on the accuracy of the Z-bus sensitivity (11)–(12), which is a first-order linearisation around the baseline operating point. If the true operating point deviates significantly from the baseline, as occurs under high DER penetration or heavy loading, the weights w_u become inaccurate and the allocation ratios (19) are systematically biased. Method II imposes no such linearisation: the sequential stage solves are full AC-OPF problems over $\mathcal{F}$, so the binding constraints are evaluated at the true nonlinear operating point. Method II is therefore more robust to baseline deviation at the cost of increased computational complexity.

3) *Fairness criteria satisfied*: Both methods satisfy Pareto-efficiency: no unit's envelope can be enlarged without shrinking another's, because the objective in both (15) and (20) is the maximum feasible α over $\mathcal{F}$. Method I additionally satisfies the proportional fairness axiom [19]. Method II satisfies lex. max-min optimality, which is logically independent of proportional fairness.

The summary of these conceptual comparisons is included in Table II.

The simulation results in Section III quantify these differences for two different networks under varying DER penetration and loading levels. The metrics reported are the OE polygon area per unit and the VUF constraint activity across the feeder.

III. TEST CASES AND SIMULATION SETUP

This section introduces the two test systems, describes the simulation setup, and presents the main numerical outcomes used in the subsequent discussion. The first test system is a small unbalanced feeder intended to illustrate the impact of embedding VUF constraints in OE computation and to reveal the resulting changes in the feasible P–Q regions at individual nodes. The second test system is based on a simplified electrically equivalent representation of the IEEE European LV feeder [28] and is used to evaluate the proposed framework under a larger and more realistic setting, with particular attention to the comparison between the two fairness inclusion methods.

The first test case considers the 13-node network shown in Fig. 1. The feeder includes nine nodes with single-phase loads and rooftop PV units rated at 2.5 kVA. The total network demand is 18 kWh and the nominal PV capacity is 22.5 kVA.

The second test case is based on the feeder described in [28], but modified to incorporate eight rooftop PV panel units per phase, each rated at 7.5 kVA, together with a three-phase solar generator rated at 22.5 kVA, and four three-phase loads representing a larger motor-type demand, resulting in a total

TABLE II: Comparison of fairness mechanisms used for P-Q operating-envelope allocation

Aspect	Method I: Network-weighted proportional fairness	Method II: Lexicographic max-min fairness
Fairness criterion	Proportional fairness via utility-based rationale [19], [20]	Lexicographic max-min optimality (worst-off protection) [21], [23]
Allocation structure	Predefined ratios through network weights w_u ; angle-independent scaling within a sweep direction	Ratios emerge from feasibility; may vary with θ_m due to binding constraints
Required modeling ingredients	Z-bus sensitivity for w_u (linearization) + AC feasibility for OE tracing	AC feasibility throughout; no sensitivity linearization required
OPF solves per angle and phase	Single solve per θ_m and phase group	Multiple staged solves; worst-case $\leq \mathcal{U}_\phi $ per θ_m and phase group
Efficiency-equity trade-off	May undermine equity and utilization	Strongest guarantee for equity and utilization
Sensitivity to baseline point	Dependent on linearization accuracy of w_u around baseline	Less dependent on baseline; constraints evaluated at nonlinear operating point

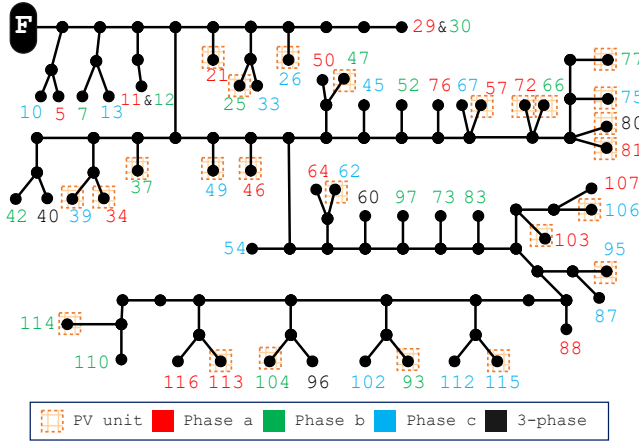


Fig. 2: Network diagram for Test Case 2.

feeder demand of 165 kWh. Fig. 2 shows the network and modification applied on it.

In both test systems, the phase loading follows the order $a > b > c$, so that the network operates under intentionally asymmetric conditions. These two cases provide complementary evidence: the first one highlights the local and phase-specific effect of VUF-aware OE construction, whereas the second one is better suited to assessing the scalability and consistency of the two fairness formulations.

All simulations were implemented in Julia using JuMP and solved with Ipopt. The simulations were performed on a laptop with an Intel Core i7-1255U CPU at 1.70 GHz and 16 GB of RAM. The implementation is publicly available online.¹ The OEs were obtained through the directional OPF sweep described in the Subsection II-D, first without and then with explicit VUF constraints. Two fairness methods were evaluated in each case: Method I based on network-weighted proportional fairness, and Method II based on lexicographic max-min fairness.

From a computational perspective, Method II is more intensive because it requires a sequence of solves for each angular direction, whereas Method I needs only a single solve per direction. The runtimes in Table III corroborate this behavior for both networks. For the smaller feeder NW1 (13 nodes, 9 DERs), Method II is roughly $5\times$ slower than Method I, with runtimes of 56.14 s and 11.66 s, respectively. This disparity

TABLE III: Computational Time of Proposed Methods over Two Networks with 5° (72 steps) Angular Resolution.

Network name	Number of nodes	Number of DERs	Method	without VUF (sec)	With VUF (sec)
NW1	13	9	M-I	11.7	12.3
			M-II	56.1	45.5
NW2	116	27	M-I	217.5	296.1
			M-II	1273.5	1579.9

becomes even larger in the bigger feeder NW2 (116 nodes, 27 DERs), where Method II takes 1274 s compared to 218 s for Method I, nearly a $6\times$ difference. This indicates that the scalability gap between the two methods grows as the network size and the number of DER units increase. VUF enforcement introduces a steady but secondary computational overhead for both methods and both networks, leading to runtime increases between $\approx 5\%$ and $\approx 36\%$. The only exception is Method II in NW1, where a slight decrease in runtime is observed, which is probably attributed to a solver artifact.

The resulting OEs are shown in Fig. 3 for the 13-node feeder and in Fig. 4 for the larger LV feeder. In both cases, the introduction of VUF limits reduces the feasible operating region, although the magnitude and direction of this reduction vary across phases and nodes. The figures also show that the differences between Method I (M1) and Method II (M2) are system- and node-dependent: in some subplots, the two methods lead to similar envelopes, while in others they produce visibly different feasible regions. Rather than interpreting these differences in detail here, the next section examines their physical and operational meaning, with particular focus on the role of voltage unbalance and on the practical trade-off between the two fairness mechanisms.

IV. DISCUSSION AND COMPARATIVE ANALYSIS

This section discusses the OEs in Figs. 3 and 4. The discussion is organized as follows. First, the effect of the VUF constraint is examined. Then, M1 and M2 are compared. Next, the phase- and location-dependent structure of the envelopes is analyzed. Finally, the main implications are summarized.

A. Effect of the VUF Constraint

Across both test cases, incorporating the VUF constraint visibly reduces the OE of every DER unit, though the degree

¹GitHub Repository: https://github.com/alireza33zz/OE_VU-F2-v1.0.git

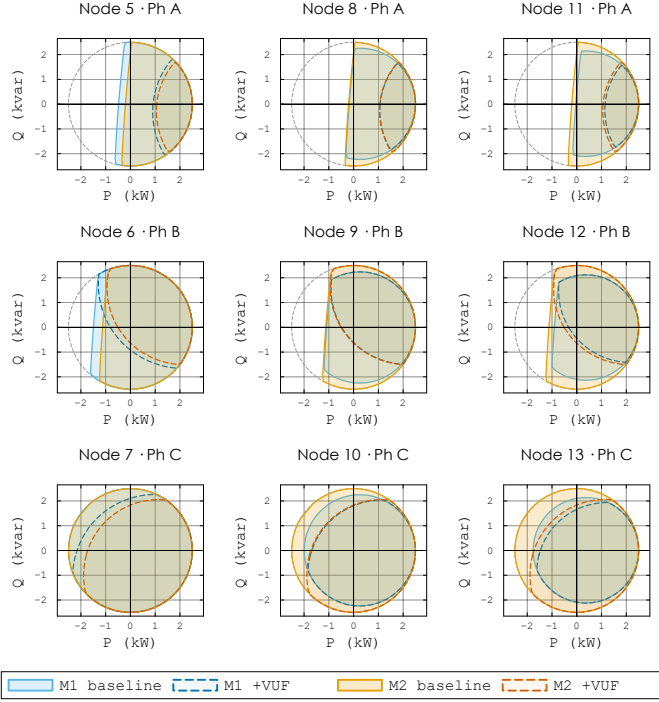


Fig. 3: OE of Test Case 1 (Guide: M1:= Method I, M2:= Method II).

of reduction varies considerably depending on which phase the unit is connected to. The reduction is not uniform in the P–Q plane either, it is concentrated in the direction of maximum real power export, where additional injection on an already unbalanced phase most aggressively amplifies the negative-sequence voltage component. The curtailment and reactive power directions are comparatively less affected.

In Test Case 1, the effect is most pronounced on Phase A and Phase B, which carry the heaviest and intermediate loads respectively. At baseline, these phases are already operating closer to the VUF limit, so the constraint becomes active earlier during the angular tracing and clips a larger portion of the envelope boundary. Phase C, being the least loaded, shows some reduction, the dashed and solid boundaries are nearly coincident for all three Phase C nodes, indicating that Phase C injections have negligible influence on the network’s voltage unbalance at the baseline operating point.

In Test Case 2, the situation is changed to some extent. With a larger number of DER units and higher aggregate penetration, the baseline operating point already produces a more significant unbalance level, and the VUF constraint bites noticeably across all three phases. The overall reduction in envelope area is larger than in Test Case 1, and no phase is entirely exempt from the effect. This suggests that as DER penetration grows, the VUF constraint transitions from a phase-selective restriction to a network-wide limiting factor that must be accounted for in OE computation regardless of which phase a unit is connected to.

The results show that a considerable part of the apparent DER flexibility is not usable once VU limits are imposed. Therefore, OEs computed without VUF constraints would overestimate the admissible operating region in unbalanced

feeders.

B. Comparison of M1 and M2 Across the Feeder

Perhaps the most informative aspect of the results is how the relative performance of the two methods changes depending on where a unit sits in the network. Rather than one method being uniformly better, the figures reveal a clear spatial gradient along the feeder that tells a consistent and physically meaningful story.

Units located close to the substation tend to receive larger envelopes under M1 than under M2. These units have low voltage impact on the network and therefore receive high sensitivity weights under M1, which translates directly into a large share of the phase-wide headroom allocation. Under M2 the same units receive a more moderate allocation because the lexicographic procedure does not use predetermined ratios, but vice versa it determines each unit’s share from the actual network feasibility at each stage. In Test Case 1 this is most visible at Node 5 on Phase A and Node 6 on Phase B, where the M1 envelope is noticeably larger than M2. In Test Case 2 the same pattern appears at nodes 21 and 34 on Phase A, 25 and 37 on Phase B, and 26 and 39 on Phase C.

Moving toward the middle of the feeder, the two methods converge. At these intermediate nodes, the proportional weights that M1 assigns happen to be a reasonable approximation of the allocation that M2 produces through its stage procedure, so the resulting envelopes are nearly identical. This convergence is visible at Node 8 on Phase A and Node 9 on Phase B in Test Case 1, and at several mid-feeder nodes in Test Case 2.

At the far end of the feeder the situation reverses. Deep-feeder units have high voltage impact across the network and consequently receive small weights under M1, which caps their headroom at a small fraction of the phase-wide allocation and results in noticeably suppressed envelopes. Under M2, these units are not subject to a predetermined cap, but the stage procedure allocates them whatever headroom the network can genuinely support for them, independently of their sensitivity weight. The result is that M2 envelopes are larger than M1 at the far end of the feeder, as seen at Node 11 and Node 12 in Test Case 1 and at nodes 113, 114, and 115 in Test Case 2.

Taken together, this spatial gradient reveals a fundamental characteristic of proportional fairness in radial networks. M1 allocates more headroom to near-slack units and less to deep-feeder units, in proportion to their sensitivity weights computed at the baseline. M2 does not impose this structure, it discovers the allocation from the network physics at each stage, which in practice gives deep-feeder units more headroom than M1 would allow and near-slack units moderately less. For applications where equitable treatment of all units regardless of feeder position is the priority, M2 is the appropriate choice. For applications where speed matters and the weight-based allocation is acceptable, M1 provides a practical, scalable, and computationally efficient alternative. Further computation results are already provided in greater detail in Table III.

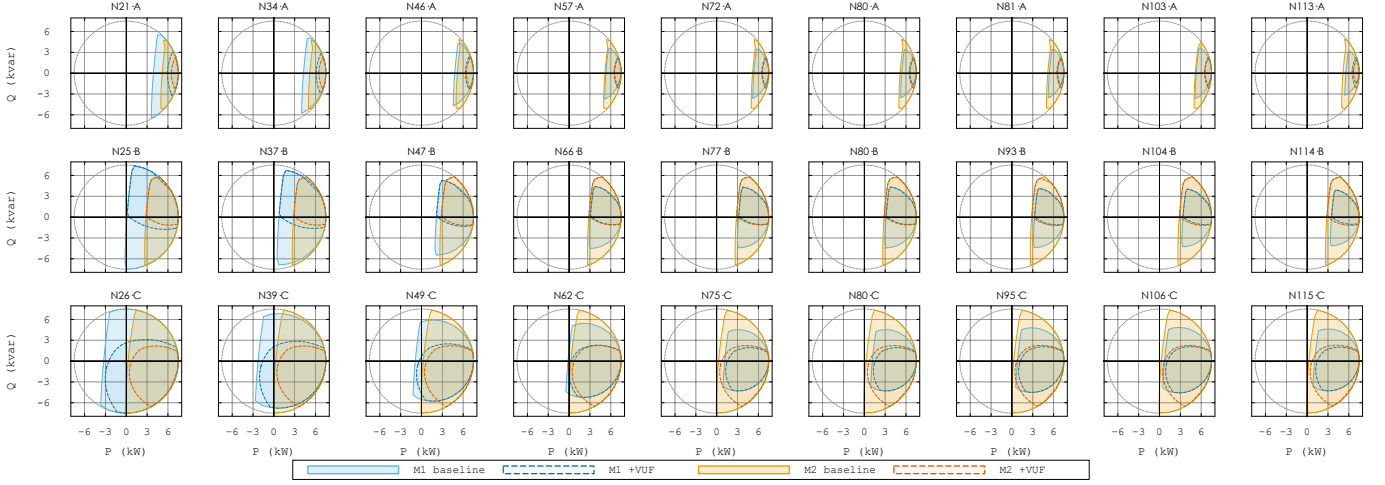


Fig. 4: OE of Test Case 2 (Guide: M1:= Method I, M2:= Method II).

C. Phase-Dependent Envelope Structure

Beyond simply comparing methods, the figures show a consistent and physically meaningful trend in how envelope shapes change across phases.

Phase A envelopes are anchored in the right half-plane, with real power export being the dominant direction of available headroom. The curtailment direction is partially constrained by the voltage lower bound on the most heavily loaded phase, which is already near its limit at the baseline dispatch. Phase B envelopes shift downward, with more headroom available in the inductive reactive absorption direction than in the capacitive injection direction, reflecting the intermediate voltage sensitivity of this phase. Phase C envelopes are by far the largest in both test cases, approaching the full inverter capacity circle at several nodes. This confirms that network constraints are nearly non-binding for the least-loaded phase, and that DERs of Phase C have substantially more operating freedom than their Phase A and Phase B counterparts at the same physical location.

This phase-dependent structure is reproduced consistently by both M1 and M2, which confirms that it is a property of the network's physical operating point rather than an artifact of the fairness formulation.

D. Summary of Implications

The results across both test cases point to several practically relevant conclusions.

The VUF constraint cannot be ignored in OE computation, particularly as DER penetration increases. In Test Case 1 its effect is phase-selective and concentrated on the most-loaded phases, but in Test Case 2 it affects all phases and reduces envelope areas to a degree that would be consequential in a real operational setting. Formulations that omit voltage unbalance will systematically overestimate the safe operating region.

An important and perhaps unexpected finding is that the VUF constraint significantly narrows the gap between M1 and M2. Without VUF, the two methods produce noticeably different envelopes, particularly at near-slack nodes where

M1 allocates substantially more headroom than M2, and at deep-feeder nodes where the opposite holds. Once the VUF constraint is enforced, the larger envelopes of same unit through different methods are clipped more aggressively, since they extend further into the regions where voltage unbalance becomes problematic. The net effect is that the VUF-constrained envelopes of M1 and M2 are considerably closer to each other than their unconstrained counterparts, as is clearly visible in both figures when comparing the dashed boundaries across methods.

This has a direct practical implication for the choice of method. M2 is the gold standard for fairness and delivers the strongest theoretical guarantee, no unit's headroom can be improved without reducing that of a less well-served unit. However, when VUF constraints are active, M1 produces envelopes that are nearly as fair as M2 in practice, while requiring only a single optimization solve per angle rather than a sequential multi-stage procedure. Combined with its superior numerical performance — nearly ten times faster than M2 in Test Case 2 — M1 emerges as a particularly attractive choice for operational settings where VUF constraints are enforced and computational speed is a priority. M2 remains the preferred option for offline planning and regulatory assessments where the strongest possible fairness guarantee is required and the computation time is not the limiting factor.

E. Special Considerations

Several aspects related to the three-phase network structure and the choice of baseline operating point warrant clarification after presenting the simulation results.

1) *Phase Interdependence and Baseline Operating Point:* Although the OE computation is performed independently for each phase, the phases remain electrically coupled through neutral voltage shifts and mutual impedance effects. Consequently, the units connected to phases not under analysis must be assigned fixed operating points. Three assumptions of decreasing physical fidelity are possible:

- **Baseline OPF (adopted in this work):** units on non-analyzed phases are fixed at the dispatch points obtained

from a baseline OPF solution. This assumption provides the most accurate representation of inter-phase interactions under normal operating conditions.

- **Nominal values:** units on non-analyzed phases are set to their rated power. This option is appropriate when the DSO has incomplete network information, as it partially compensates for data gaps while reducing computational burden.
- **Full curtailment** ($P = Q = 0$): all units on non-analyzed phases are assumed to be de-energized. Although this assumption permits the model to produce results under conditions of extreme data scarcity, it effectively decouples the phases, reducing the three-phase formulation to behavior approaching a single-phase OPF. This results in significantly larger OEs and a considerable overestimation of available network headroom.

2) *Operational Validity of Computed OEs:* A question arises regarding operational compliance when a unit produces zero output at a given time interval—for example, a PV unit during periods of low or zero irradiance. In such cases, the actual operating point ($P = Q = 0$) lies strictly within the computed OE boundary and therefore satisfies the envelope constraints by definition. The OE represents an outer boundary of permissible operation rather than a prescribed setpoint; any operating point within the envelope is inherently network-safe.

It is noted that the mathematical formulation is independent of the specific baseline assumption employed and yields valid results under all three scenarios described above. The baseline OPF assumption is adopted throughout this work as it provides the tightest and most operationally representative envelopes. The systematic quantification of OE sensitivity to the choice of baseline assumption, particularly under conditions of partial or unavailable network data, is left for future investigation.

V. CONCLUSION

This paper presented a unified framework for computing voltage-unbalance-aware operating envelopes for PV inverters in unbalanced three-phase distribution networks. The framework combines a three-phase AC-OPF model with voltage magnitude, inverter capability, and nodal VUF constraints, and uses a directional OPF sweep to trace the coupled P-Q operating region. Two allocation methods were embedded in the framework: M1, based on network-weighted proportional fairness, and M2, based on lexicographic max-min fairness.

The case studies showed that VUF constraints materially reduce and reshape the feasible operating region. The effect is phase- and node-dependent and is especially strong where the unconstrained envelopes are broad. In several subplots, the VUF-constrained regions of M1 and M2 become much closer than their unconstrained counterparts. This shows that voltage unbalance can be the dominant limiting factor in unbalanced feeders and should be modeled explicitly when deriving DER OEs.

The comparison between M1 and M2 showed that their difference is not uniform across the network. In some nodes the difference is still significant, but VUF integration generally makes M1 and M2 regions much closer to each other. Overall,

M1 remains attractive for scalable implementation because of its much lower computational cost, while M2 remains valuable when a stricter fairness notion or a more detailed allocation study is required.

ACKNOWLEDGMENT

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