

Moment equations-driven discovery of stochastic systems with coloured noise

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ABSTRACT

This study presents a stochastic system identification framework based on one/two-time moment equations for discovering the governing nonlinear stochastic differential equations (SDEs) directly from data. The framework is complemented by a Selective Segmented Sampling data-acquisition strategy that enables reliable estimation of the required statistical moments/correlation functions, even from limited experimental measurements.

Reduced-order modeling of nonlinear dynamical systems from first principles remains a central challenge across science and engineering [1]. Recent advances in data-driven discovery and machine learning [2] have enabled direct identification of governing equations from data. However, most existing approaches are restricted to deterministic systems [3], providing no means to discover the underlying stochastic dynamics other than additive measurement noise.

Inspired by sparse regression [4], the proposed method seeks to identify stochastic systems under white/coloured noise excitations from observed data. The drift and diffusion functions are represented by a linear combination of candidate nonlinear functions with sparse matrices of coefficients. By deriving deterministic one-time [5] and two-time moment equations [6] from the SDEs, the identification problem reduces to a sparse nonlinear regression, in which the sparse coefficients are identified by minimizing the discrepancy between predicted and observed time derivatives of the moments and correlation functions.

The proposed framework is validated on several representative examples of increasing complexity, including stochastic Duffing oscillators, the van der Pol oscillator under additive coloured noise, and the stochastic Mathieu equation. The results demonstrate excellent accuracy and robustness in recovering both deterministic and stochastic components of nonlinear dynamics, highlighting the method's potential for data-driven discovery of stochastic systems in practical engineering applications.

References

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