Effect and its mechanism of spatial coherence of track irregularity on dynamic responses of railway vehicles

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Abstract

A running vehicle is subjected to multiple stochastic excitations due to the irregular profiles of the wheels and rails. The combined wheel-rail irregularities on each wheel can be partly incoherent and partly coherent in reality. As a result, different assumptions for the coherence of such irregularities have been adopted in vibration and noise simulations so that vehicle-track dynamic analysis can be conducted. Less attention has been paid to the effects of these assumptions on dynamic responses of railway vehicles in a wide frequency range. The aim of this study is to theoretically investigate the effects of the spatial coherence of track irregularities on the dynamic behaviors of a two-dimensional vehicle. Two frequency-domain methods that have been verified using a time-domain method are presented first to calculate the dynamic responses of the vehicle. A new expression of the power spectral density (PSD) function of the excitations comprises an admittance coefficient, a coherence matrix, and a spatial PSD function of the track irregularities. The PSD functions of the responses of the vehicle are derived analytically. It is then illustrated that the periodic fluctuations of response spectra are caused by the spatial coherence of the excitations by introducing the concept of coherence scales. Finally, it is proved that the difference between the responses under coherent and incoherent excitations become insignificant when the wavelength of the irregularity is less than the critical wavelength.

Keywords

track irregularity; vehicle dynamics; stochastic excitation; coherence scale; vibration and noise; critical wavelength

1 Introduction

Railway transportation has the advantages of low carbon emission, high speed, and large capacity; however, severe vibrations generated in the vehicle-track system [1-3] negatively affect the comfort of the ride and safety of the running trains, as well as vibration serviceability of buildings when the trains pass through residential or vibration-sensitive regions [4]. Moreover, the noise produced by the vibrations affects the physiological and psychological health of passengers. In fact, the sources of the vibrations are the irregular profiles of the wheels and rails [3], which are due to various complex random factors,

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including track dipped joints, rail corrugations, general track undulations, wheel flats, wheel surface irregularities, and wheel eccentricity [5, 6]. Among them, the general track undulations (also named as track geometry irregularities) are the most important vibration source of the vehicle-track system in a wide range of frequencies [7]. Iyengar and Jaiswal [8] suggested that both track unevenness and the absolute vertical profile could be modeled as stationary Gaussian fields. In this case, the random irregularities can be statistically characterized by a power spectral density (PSD) function [9, 10].

A running vehicle is subjected to multiple excitations from track irregularities at different wheels. It was observed that the irregularities on different wheels are generally incoherent but those on the rail are often coherent. As a result, the combined wheel-rail irregularities on each wheel can be partly incoherent and partly coherent in reality. Unfortunately, the coherence information of these irregularities is not included in the PSD functions of track irregularities. Several studies have used different assumptions for the coherence of track irregularities. For time-domain analyses, the spatial coherence of track irregularity comes into being spontaneously by a spatial lag relation [11-14]. And in frequency-domain analyses, the PSD functions are generally directly used as the excitations by assuming incoherent excitations on different wheels or coherent excitations with a delay related to distance between wheels. However, it is still unclear whether the coherence effect must be considered for the responses of different parts of the vehicle-track system in a wide frequency range. Wu and Thompson [15, 16] showed that the excitation from each wheel can be treated as incoherent in the frequency bands above 20 Hz. Li et al. [17] predicted the vibration and associated noise raised from the track-bridge system with a wavenumber domain finite element and boundary element method by assuming incoherence of excitations among all wheels. In recent years, attention has been paid to the random evolution of track irregularity due to interactions between the train and track in space [18]. Furthermore, track irregularity identification based on the dynamic responses of in-service vehicles becomes a recent trend [19]. The identification accuracy is obviously affected by the measurement noise when the vehicle dynamic responses are weakened [20] owing to the coherence excitations of track irregularities. All in all, an investigation on the coherence mechanism of multiple excitations is important for both vibration prediction and parameter identification of the vehicle-track systems.

To solve the vehicle-track dynamic interaction problem, the vehicle model that has been widely accepted in present research [21, 22] are established on the basis of the multi-body dynamic theory [23, 24]. The typical model is usually established in the vertical plane including several degrees of freedom (DOFs) that are related to the vertical and pitch motions [25]. Moreover, the wheel-rail contact model is an essential element for analyzing the dynamic performance of the entire system [26-28]. The rigid contact assumption [29, 30] is a simplified model that neglects the nonlinear components such as nonlinear geometric relationships and nonlinear characteristics of materials. It was already demonstrated that differences among different contact models arise mainly in the lateral response of the vehicle and lateral wheel-rail interaction forces [31].

Many time-domain integration methods have been applied to solve the differential equation of motion [32, 33] of the vehicle-track system. Basically, these approaches need many samples of irregularities as inputs and they are undoubtedly computationally intensive [34-37]. The expected response spectra for the system can be directly calculated using the algebraic frequency domain method [36, 38] without the Monte-Carlo approach [39], which allows much higher computational efficiency than a direct integration approach. Although the wheel-rail interaction has made the vehicle-track dynamic to be a complicated nonlinear stochastic vibration problem [28, 39], it could be very convenient to investigate the excitation mechanism with the frequency-domain approach if a reasonable equivalent linearization technique was adopted for simplifying the system as a linear stationary stochastic vibration problem [40-42]. In this way, Grassie et al. [43, 44] suggested a linearized Hertzian contact stiffness model and determined the responses subjected to various defects by Fourier transforms.

Although the random vibration of vehicle and track systems has been widely investigated for different purposes, it is still not easy to clarify the effect of spatial coherence on low frequency vibration of the car-body and high frequency vibration of the wheels. The aim of this study is to theoretically investigate the effects of the spatial coherence of track irregularities on the dynamic responses of car-bodies, bogies, wheels of railway vehicles, and wheel-rail interaction forces in a wide frequency range. To illustrate the coherence effect, the PSD function matrix for excitations on the vehicle is analytically derived and expressed as three parts: the admittance coefficient of the vehicle, coherence coefficient for the multiple wheels of the vehicle, and spatial PSD of the track irregularity. Moreover, the concept of coherence scale has been proposed, which is beneficial for understanding the coherence mechanism, as it analyzes the periodicity of the response spectrum. Additionally, a definition of the critical wavelength is proposed from the perspective of a one-third octave band spectrum, and the relation between the critical wavelength and vehicle speed is investigated. By these means, the treatment of multi-wheels as incoherent in the shortwave excitation bands is explained by theoretical derivation.

2 Dynamics of railway vehicles

2.1 Vehicle and track model

A single railway vehicle with the rigid-body assumption [45] is adopted in this study (see Figure 1) which consists of a car body, two bogie frames, four wheelsets and several spring-damper elements connecting them with relatively small stiffness [46]. The vehicle responses due to the vibration of the rails are relatively insignificant compared with those excited by the track irregularity [47]. Therefore, the DOFs of the track structure is not included in the model to clarify the effects of track irregularity on the dynamic responses of vehicles. To illustrate the mechanism of coherence of excitations by track irregularity, the vertical vibration of the vehicle is the focus of this study. As a result, the vehicle model has a total of 10 DOFs, including seven translational and three rotational DOFs. Table 1 lists the definitions of the main parameters used to describe a general vehicle model in dynamic analysis along with their values given by Yang et al. [48] for a high-speed railway vehicle.

Based on the aforementioned assumptions, the differential equation of motion for the railway vehicle can be expressed as:

$$\mathbf{m}\ddot{\mathbf{v}} + \mathbf{c}\dot{\mathbf{v}} + \mathbf{k}\mathbf{v} = \mathbf{p} \tag{1}$$

where $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ are the mass, damping, and stiffness matrices for the vehicle, respectively; $\mathbf{p}$ is the vector of the wheel-rail interaction forces acting on the vehicle; $\mathbf{v}$ is the displacement vector of the vehicle.

$$\mathbf{v} = \{\varphi_{\mathrm{t1}} \quad v_{\mathrm{w1}} \quad v_{\mathrm{w2}} \quad v_{\mathrm{t1}} \quad v_{\mathrm{b}} \quad \varphi_{\mathrm{b}} \quad v_{\mathrm{t2}} \quad v_{\mathrm{w3}} \quad v_{\mathrm{w4}} \quad \varphi_{\mathrm{t2}}\}^{\mathrm{T}} \tag{2}$$

where $v_{\mathrm{b}}, v_{\mathrm{t}i}$, and $v_{\mathrm{w}j}$ are the vertical displacements of the car-body, the i-th bogie, and j-th wheelset, respectively; φ_{b} and $\varphi_{\mathrm{t}i}$ are the pitch displacement of the car body and i-th bogie, respectively ($i = 1,2$ and $j = 1,2,3,4$). It is noted that the elements in vector $\mathbf{v}$ are organized in a special order so that the damping and stiffness matrices obtained are as narrow-band as possible. This treatment can be used to facilitate the reverse and eigenvalue analyses of the matrices in the following calculations.

The detailed expressions of the mass matrix $\mathbf{m}$, the damping matrix $\mathbf{c}$, the stiffness matrix $\mathbf{k}$ and the force vector $\mathbf{p}$ are given in Appendix A. As expected, the damping and the stiffness matrix are both five-diagonal matrixes.

Track irregularities considered in the dynamic analysis include lateral and vertical alignment irregularities, cant deficiencies, and gauge irregularities [49]. Only the vertical track irregularities are taken into consideration in this study expressed by the US vertical alignment irregularities PSD function $S_{\mathrm{r}}(\Omega)$ [50]

$$S_{\mathrm{r}}(\Omega) = \frac{kA_{\mathrm{r}}\Omega_{\mathrm{c}}^2}{\Omega^2(\Omega^2+\Omega_{\mathrm{c}}^2)} \tag{3}$$

where Ω represents the spatial frequency of track irregularities; k represents the safety factor, which ranges from 0.25 to 1.0; A_{r} represents the roughness constant; Ω_{c} represents the cutoff frequency. The track irregularity of level six is used in this study for illustration purposes with $k = 1$, $A_{\mathrm{r}} = 0.0339\mathrm{cm}^2 \cdot \mathrm{rad/m}$, and $\Omega_{\mathrm{c}} = 0.8245\mathrm{rad/m}$. It is noted that other spectrums for track irregularity or rail roughness can be used as well.

2.2 Direct frequency domain method

The governing linear equations of motion for the vehicle can be expressed by

$$(-\mathbf{m}\omega^2 + i\omega\mathbf{c} + \mathbf{k})\mathbf{V} = \mathbf{P} \tag{4}$$

where $\mathbf{P}$ and $\mathbf{V}$ are respectively the Fourier transformation of the force vector and response vector.

The response vector $\mathbf{V}$ can be derived using the following equation

$$\mathbf{V} = \mathbf{HP} \tag{5}$$

where $\mathbf{H} = (-\mathbf{m}\omega^2 + i\omega\mathbf{c} + \mathbf{k})^{-1}$ represents the transfer function matrix of complex frequency responses.

The PSD matrices for responses $\mathbf{S_v}$ and excitations $\mathbf{S_p}$ can be expressed respectively as [51]

$$\mathbf{S_v} = \lim_{T\to\infty} \frac{\mathbf{VV}^{\mathrm{H}}}{2T} \tag{6}$$

$$\mathbf{S_p} = \lim_{T\to\infty} \frac{\mathbf{PP}^{\mathrm{H}}}{2T} \tag{7}$$

where the superscript H denotes the conjugate transpose. Substituting Eq. (5) into Eq. (6), $\mathbf{S_v}$ can be obtained as

$$\mathbf{S_v} = \lim_{T\to\infty} \frac{\mathbf{HPP}^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}}{2T} = \mathbf{H}\left[\lim_{T\to\infty} \frac{\mathbf{PP}^{\mathrm{H}}}{2T}\right]\mathbf{H}^{\mathrm{H}} = \mathbf{HS_pH}^{\mathrm{H}} \tag{8}$$

Equation (8) illustrates the relationship between the PSD matrices for the responses and excitations of a linear system.

2.3 Complex modal method

To obtain the frequency response transfer function matrix, a matrix inversion operation is required; however, it is inefficient in practice particularly for a system with large number of DOFs. In addition, the inversion operation leads to inaccurate, even erroneous, results when the transfer function matrix is morbid.

With regard to this problem, a complex modal analysis method [52, 53] is used to improve the computational efficiency and accuracy. The detailed derivation and corresponding variables declaration are given in Appendix B. Therefore, the modal formulation of the frequency response transfer function is established as:

$$\mathbf{H} = \sum_{i=1}^{n} \left(\frac{\boldsymbol{\psi}_i \boldsymbol{\psi}_i^{\mathrm{T}}}{j\omega a_i + b_i} + \frac{\boldsymbol{\psi}_i^* \boldsymbol{\psi}_i^{\mathrm{H}}}{j\omega a_i^* + b_i^*} \right) \tag{9}$$

The advantage of the complex modal analysis method for a frequency transfer function lies in the fact that the inverse operation of the matrix can be avoided. As $\mathbf{m}, \mathbf{c},$ and $\mathbf{k}$ of the system are invariant, the modal parameters of each order, including the modal vectors $\boldsymbol{\psi}_i$ and a_i, b_i, can be obtained and stored after the complex modal analysis performed once. In other words, $\mathbf{H}$ can be simply obtained by substituting the relevant parameters into the modal expansion when $\mathbf{H}$ is about to be calculated at a certain frequency point, thereby greatly improving the efficiency.

2.4 Excitations PSD and coherence matrix

As the external excitations of the vehicle system are caused by track irregularities, it is essential to establish the bridge between the excitations spectrum $\mathbf{S_p}$ and the spatial-PSD function for track irregularities. Assume that the discrete force vector $\mathbf{p}$ represents a stationary ergodic random process characterized by its PSD matrix:

$$\mathbf{S_p} = \begin{bmatrix} S_{p_1p_1} & S_{p_1p_2} & \cdots & S_{p_1p_N} \\ S_{p_2p_1} & S_{p_2p_2} & \cdots & S_{p_2p_N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{p_Np_1} & S_{p_Np_2} & \cdots & S_{p_Np_N} \end{bmatrix} \tag{10}$$

where $S_{p_ip_j}$ is the cross-PSD function for the excitation p_i and excitation p_j that becomes the self-PSD function under the condition $i = j$. If the initial position of the first wheelset is set as the origin of the x-axis coordinate, $\mathbf{p}$ can be expressed as

$$\mathbf{p} = \begin{bmatrix} 0 \\ k_\mathrm{a} r(ut) + uc_\mathrm{a} r'(ut) \\ k_\mathrm{a} r(ut-2b) + uc_\mathrm{a} r'(ut-2b) \\ 0 \\ 0 \\ 0 \\ 0 \\ k_\mathrm{a} r(ut-2a) + uc_\mathrm{a} r'(ut-2a) \\ k_\mathrm{a} r(ut-2a-2b) + uc_\mathrm{a} r'(ut-2a-2b) \\ 0 \end{bmatrix} \tag{11}$$

where $r'(x)$ is the derivation of the spatial parameter x. According to $\mathbf{S_p} = \mathcal{F}\left(\mathbf{R_p}(\tau)\right) = \mathcal{F}(\mathrm{E}[\mathbf{p}(t)\mathbf{p}^\mathrm{T}(t+\tau)])$, only the intersections of rows 2, 3, 8, and 9, and columns 2, 3, 8, and 9 in $\mathbf{S_p}$ are nonzero elements. Take the non-diagonal element $S_{p2,3}$ as an example, where the derivation process is universal and applicable to the other elements. The cross-correlation function for the excitations p_2 and p_3 becomes

$$\begin{aligned} R_{p_2p_3}(\tau) = {} & k_\mathrm{a}^2 \mathrm{E}[r(ut)r(ut+u\tau-2b)] + u^2 c_\mathrm{a}^2 \mathrm{E}[r'(ut)r'(ut+u\tau-2b)] \\ & + uk_\mathrm{a}c_\mathrm{a}\{\mathrm{E}[r(ut)r'(ut+u\tau-2b)] + \mathrm{E}[r'(ut)r(ut+u\tau-2b)]\} \end{aligned} \tag{12}$$

It can be seen that the cross correlation function $R_{p_2p_3}(\tau)$ has been divided into three parts: the first part is defined as the stiffness term, which is caused by the vertical displacement, the second part is defined as the damping term caused by the vertical velocity, and the last part is defined as the cross term, which can be proved to be equal to zero. Therefore, the cross-correlation function $R_{p_2p_3}(\tau)$ becomes

$$R_{p_2p_2}(\tau) = k_\mathrm{a}^2 R_\mathrm{r}(u\tau - 2b) - c_\mathrm{a}^2 \frac{d^2 R_\mathrm{r}(u\tau-2b)}{d\tau^2} \tag{13}$$

Taking the Fourier transform of the $R_{p_2p_3}(\tau)$ gives the cross-PSD function for the excitations p_2 and p_3:

$$S_{p_2p_3} = \int_{-\infty}^{\infty} R_{p_2p_3}(\tau)\mathrm{e}^{-i\omega\tau}d\tau = \left[\frac{k_{\mathrm{a}}^2 + c_{\mathrm{a}}^2\omega^2}{u}\right]\left(\mathrm{e}^{-i\frac{\omega}{u}2b}\right)S_{\mathrm{r}}\left(\frac{\omega}{u}\right)$$
$$= \chi(\omega,u)\beta(\omega,u,2b)S_{\mathrm{r}}\left(\frac{\omega}{u}\right) \quad (14)$$

Note that the cross-PSD function $S_{p_2p_3}$ is derived by multiplying three parts.

$$\chi(\omega,u) = \frac{k_{\mathrm{a}}^2+c_{\mathrm{a}}^2\omega^2}{u} \quad (15)$$

$$\beta(\omega,u,l) = \mathrm{e}^{-i\frac{\omega}{u}l} \quad (16)$$

The first term $\chi(\omega,u)$ is defined as the admittance coefficient, which is used to convert spatial geometric statistics into physical mechanics statistics. The second term $\beta(\omega,u,l)$ is defined as the coherence coefficient, which reflects the spatial lag relationship of excitations at different positions. For example, the spatial lag of p_3 and p_2 in $S_{p_3p_2}$ is $2b$; thus, l can be taken as $-2b$. The third term is the spatial-PSD function of track irregularities, which is the spatial geometric statistic. Accordingly, the PSD function matrix for excitations can be expressed as

$$\mathbf{S_p} = \chi(\omega,u)\boldsymbol{\mathcal{B}}(\omega,u)S_{\mathrm{r}}\left(\frac{\omega}{u}\right) \quad (17)$$

$$\boldsymbol{\mathcal{B}}(\omega,u) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \mathrm{e}^{-i\frac{\omega}{u}2b} & 0 & 0 & 0 & 0 & \mathrm{e}^{-i\frac{\omega}{u}2a} & \mathrm{e}^{-i\frac{\omega}{u}2(a+b)} & 0 \\ 0 & \mathrm{e}^{i\frac{\omega}{u}2b} & 1 & 0 & 0 & 0 & 0 & \mathrm{e}^{-i\frac{\omega}{u}2(a-b)} & \mathrm{e}^{-i\frac{\omega}{u}2a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{i\frac{\omega}{u}2a} & \mathrm{e}^{i\frac{\omega}{u}2(a-b)} & 0 & 0 & 0 & 0 & 1 & \mathrm{e}^{-i\frac{\omega}{u}2b} & 0 \\ 0 & \mathrm{e}^{i\frac{\omega}{u}2(a+b)} & \mathrm{e}^{i\frac{\omega}{u}2a} & 0 & 0 & 0 & 0 & \mathrm{e}^{i\frac{\omega}{u}2b} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (18)$$

where $\boldsymbol{\mathcal{B}}(\omega,u)$ is defined as the coherence matrix, describing the coherence of the excitations.

2.5 Response PSD function

The PSD function matrix for displacement responses can be directly obtained using Eq. (12). There are eight physical quantities of interest in this study: (1) centroid vertical acceleration of the car body $\ddot{v}_{\mathrm{bc}}$; (2) rotational acceleration of the car body $\ddot{\varphi}_{\mathrm{b}}$; (3) vertical acceleration of the front end of the car body $\ddot{v}_{\mathrm{bf}}$; (4) centroid vertical acceleration of the bogie frame $\ddot{v}_{\mathrm{t1c}}$; (5) rotational acceleration of the bogie frame $\ddot{\varphi}_{\mathrm{t1}}$; (6)vertical acceleration of the front end of the bogie frame $\ddot{v}_{\mathrm{t1f}}$; (7) vertical acceleration of the first wheelset $\ddot{v}_{\mathrm{w1}}$;

and (8) wheel-rail interaction force F_{w1}.

The PSD function of these physical quantities cannot be obtained using $\mathbf{S}_{\mathrm{v}}$ directly and needs to be derived by conversion. Table 2 lists the PSD function conversion relationship of these eight physical quantities. To make the expressions more concise, the results are simplified using the Hermite matrix self-conjugate property of the response PSD matrix.

3 Validation

To illustrate the reliability of the proposed frequency-domain methods, the direct frequency-domain method, a complex modal method, and a time-domain method are applied to analyze the dynamics of a high-speed railway vehicle. The standard fourth-order four-segment Runge–Kutta method is adopted to analyze the response of the vehicle in the time domain. The track irregularity samples for the time domain analysis are generated from the PSD of the track irregularity. Figure 2 shows the PSD functions of vertical acceleration at the front end of the car body, the front end of the first bogie frame, and centroid of the first wheelset, calculated from three analysis methods at passage speeds of $120\,\mathrm{km/h}$ and $240\,\mathrm{km/h}$. All the accelerations obtained using the three methods are in good agreement, particularly for accelerations of the car body and wheelset. The results of the bogie frame from complex modal analysis (Figure 2c, d) show slight fluctuations, which are due to the accumulation of initial errors in the complex eigenvalue analysis. Although this method has lost some accuracy, it has high computational efficiency, which is especially suitable for a system with high DOFs. The direct frequency-domain analysis method is adopted in the following sections to guarantee the accuracy in the analysis of a vehicle with only 10 DOFs.

4 Coherence mechanism

4.1 Four coherence scales

It can be observed from Figure 2 that there are several peaks and troughs with a certain periodicity in the response spectrum of the car body. Moreover, if we divide the frequencies corresponding to these peaks or troughs by passage speed, the results (spatial frequencies) are constant values. This finding suggests that the peaks and troughs are related to the spatial coherence of excitations rather than the resonance caused by the structural natural frequency [37]. To understand this, four coherence scales are proposed in the following analysis.

In the frequency-domain method, the coherence between different excitations can be easily controlled by modifying the related elements in the matrix $\boldsymbol{\mathcal{B}}(\omega, u)$. This is because the nonzero elements in the matrix $\boldsymbol{\mathcal{B}}(\omega, u)$ represent the spatial lag relationship of the corresponding excitations. There are four kinds of coherence coefficients in $\boldsymbol{\mathcal{B}}(\omega, u)$:

$$\beta(\omega, u, 2b) = \mathrm{e}^{-i\frac{\omega}{u}2b} \tag{19}$$

$$\beta(\omega, u, 2a-2b) = \mathrm{e}^{-i\frac{\omega}{u}2(a-b)} \tag{20}$$

$$\beta(\omega, u, 2a) = \mathrm{e}^{-i\frac{\omega}{u}2a} \tag{21}$$

$$\beta(\omega, u, 2a + 2b) = \mathrm{e}^{-i\frac{\omega}{u}2(a+b)} \tag{22}$$

The coherence matrix can be rewritten to take different coherence scales into consideration by zeroing some elements and leaving the remaining ones. Therefore, four kinds of incoherent excitations types are defined: Type I , where only the coherence between the first and fourth wheelsets is considered; Type II, where the coherences between both the first and third wheelsets as well as the second and fourth wheelsets are considered; Type III, where only the coherence between the second and third wheelsets is considered; Type IV, where the coherences between both the first and second wheelsets as well as the third and fourth wheelsets are considered.

4.2 Acceleration of the bogie frame

To show the effects of four kinds of incoherent excitations, the PSD functions of vertical and rotational accelerations for the first bogie centroid are presented in Figure 3 and Figure 4 respectively, at a passage speed of $120\ \mathrm{km/h}$ under coherent excitation and four incoherent excitations.

It can be seen that the bogie accelerations under completely coherent excitations show periodic troughs. The spatial frequencies (or wavelengths) of these minima can be defined as the vibration suppression frequencies (or wavelengths). In other words, no matter what the passage speed is, the frame has no vertical (or rotational) vibration excited by track irregularities with these wavelengths. It is also observed that only under $2b$ scale coherent excitations can the frame accurately reproduce the periodic peaks and troughs. This implies that the vibration suppression wavelength of the bogie is related to the coherence between the first and second wheelsets, as well as the third and fourth wheelsets.

When taking the $2b$ coherence scale into account, the front and rear wheelsets of each bogie form a stable spatial phase difference. If the wavelength of a particular component of the track irregularity coincidently excites the front and rear wheelsets with a phase difference of π, the vertical vibration of the bogie is suppressed, as shown in Figure 5.

The situation of the front and rear wheelsets receiving opposite excitations can be expressed as

$$\left(n\lambda_s - \frac{\lambda_s}{2}\right) = 2b,\ n = 1,2,\cdots \tag{23}$$

Therefore, the suppression frequency for vertical movement of the bogie can be written as

$$f_s = \frac{1}{\lambda_s} = \frac{2n-1}{4b},\ n = 1,2,\cdots \tag{24}$$

Similarly, the wavelengths of rotational suppression for the bogie (Figure 6) can be expressed as

$$n\lambda_s = 2b,\ n = 1,2,\cdots \tag{25}$$

Therefore, the rotational suppression frequencies for the bogie can be written as

$$f_s = \frac{1}{\lambda_s} = \frac{n}{2b},\ n = 1,2,\cdots \tag{26}$$

From the theoretical analysis above, the vertical and rotational vibration suppression frequencies of the frame are calculated as $0.2\ \mathrm{m}^{-1}$, $0.6\ \mathrm{m}^{-1}$, and $1.0\ \mathrm{m}^{-1}$; and $0.4\ \mathrm{m}^{-1}$ and $0.8\ \mathrm{m}^{-1}$, respectively, which are perfectly consistent with the numerical analysis results shown in Figure 3 and Figure 4.

This phenomenon of the troughs appearing by $1/2b$ period can also be explained from the perspective of the coherence matrix $\boldsymbol{\mathcal{B}}(\omega, u)$ with the periodicity expressed as

$$\boldsymbol{\mathcal{B}}\left(f + \frac{n}{2b}, u\right) = \boldsymbol{\mathcal{B}}(f, u) \tag{27}$$

where $f = \omega/(2\pi u)$.

4.3 Acceleration of car body

Similar to the analysis above, Figure 7 and Figure 8 respectively show the PSD functions of the vertical and rotational accelerations of the car body at a passage speed of $120\ \mathrm{km/h}$ obtained using various excitation types.

The accelerations of the car body under completely coherent excitations exhibits abundant vibration suppression frequencies than those of the bogies. The distribution of vibration suppression frequencies shows a strict periodicity for vertical vibrations. However, for the rotational vibrations, the suppression frequencies present a more complicated distribution with several periods. It can be seen that the responses under $2a$ coherence scale excitation reproduce the phenomenon of vibration suppression. This implies that the vibration suppression phenomenon of the car body is related to the coherence between the first and third wheelsets, as well as the second and fourth wheelsets.

As illustrated in Figure 9, the center of the car body has no movement subjected to a certain wavelength of track irregularity; thus, the suppression wavelength and frequency of the car body can be expressed as

$$\left(n\lambda_s - \frac{\lambda_s}{2}\right) = 2a,\ n = 1,2,\cdots \tag{28}$$

$$f_s = \frac{1}{\lambda_s} = \frac{2n-1}{4a},\ n = 1,2,\cdots \tag{29}$$

Similarly, when a particular wavelength of track irregularity coincidently excites the centroids of two bogie frames with the same phase, all the points of the car body will be always at the same height, and no rotational movement will occur, as shown in Figure 10.

Therefore, the rotational suppression frequencies for the car body can then be obtained by

$$n\lambda_s = 2a,\ n = 1,2,\cdots \tag{30}$$

$$f_s = \frac{1}{\lambda_s} = \frac{n}{2a},\ n = 1,2,\cdots \tag{31}$$

The car body is equivalent to be vibration isolated when the two bogies are vertically suppressed. In this case, both vertical and rotational vibrations in the car body are suppressed, as shown in Figure 5. As a result, the vertical vibration suppression wavelengths of the bogie are also the absolute vibration suppression wavelengths for the car body. The suppression frequencies for rotational movement of the car body can then be summed up as

$$\begin{cases} f_s = \frac{n}{2a} & \text{or} \\ f_s = \frac{2n-1}{4b}, & n = 1,2,\cdots \end{cases} \tag{32}$$

The suppression frequencies for vertical and rotational vibration of the car body are entirely in agreement with the numerical analysis results. Moreover, the response PSD values of the car body are much lower at the absolute vibration suppression frequencies owing to the effect of vibration isolation.

4.4 Wheel-rail interaction forces

Wheel-rail interaction forces have an essential influence on the riding safety, wear in the wheels and rails, and the rolling noise. In this section, we investigate the influence of different incoherent excitations on the wheel-rail interaction forces. Figure 11 shows the PSD functions of the first wheel-rail interaction force at a passage speed of $120\ \mathrm{km/h}$. The wavelength range of track irregularities is extended to $0.01\ \mathrm{m} - 100\ \mathrm{m}$ to cover the frequency range for the problems of ride safety and rolling noise.

From Figure 11, it can be seen that, unlike for the response PSD of the bogie or the car body, no periodic peaks or troughs exist. The responses under $2b$ coherence scale excitations are approximately consistent with the completely coherent excitation. This implies that the dynamic response of a wheelset is only related to the coherence between two wheelsets under the same bogie frame.

Furthermore, there is a noticeable peak in the response PSD of the wheel-rail interaction force. The peak frequency reflects the natural vibration characteristic of the vehicle rather than track irregularities. To illustrate this, PSD functions (see Figure 12) of the wheel-rail interaction force are given in terms of the vibration frequency at passage speeds of $120\ \mathrm{km/h}$, $240\ \mathrm{km/h}$, and $360\ \mathrm{km/h}$. As can be seen from Figure 12, the peak frequency of the wheel-rail interaction force remains constant (44 Hz) at different passage speeds. The natural frequency of the wheelsets is obtained as $45.14\ \mathrm{Hz}$ using complex modal analysis, which is in accord with the peak frequency of the wheel-rail interaction force.

5 Critical wavelength

5.1 Responses in one-third octave band

The concept of a one-third octave band is usually used in the measurement and assessment of vibration and noise, which can average the responses in different frequency bands. Figure 13 presents the root mean square (RMS) of accelerations or forces in terms of the one-third octave band, comparing the results of coherence excitations and incoherent excitations at a passage speed of $120\ \mathrm{km/h}$. The wavelength range of track irregularities is extended to $0.01\ \mathrm{m} - 100\ \mathrm{m}$ and divided into 41 octave bands where center frequencies are fixed at $10^{-\frac{20}{10}}\mathrm{m}^{-1}, 10^{-\frac{19}{10}}\ \mathrm{m}^{-1}, \cdots, 10^{\frac{19}{10}}\ \mathrm{m}^{-1}, \mathrm{and} 10^{\frac{20}{10}}\ \mathrm{m}^{-1}$.

It can be seen from Figure 13a, Figure 13c, and Figure 13e that the difference between the responses of coherent excitations and incoherent excitations decreases as the wavelength of the irregularity reduces. A practical critical wavelength is defined if the level difference between the responses of two coherence excitations at this wavelength is less than 1 dB.

$$\left|20\log_{10}\frac{R_{\mathrm{C}}}{R_{\mathrm{I}}}\right| \leq 1,\ \lambda \leq \lambda_{\mathrm{c}} \tag{33}$$

Here, $\mathrm{R_C}$ represents the RMS response obtained from coherent excitations and R_{I} represents the RMS response from incoherent excitations. As can be seen from Figures 13b, 13d, and 13f, the critical wavelength λ_{c} for each response of the vehicle is generally different.

5.2 Relationship between the critical wavelength and passage speed

To investigate the relationship between the critical wavelength and passage speed, the critical wavelengths λ_{c} of vertical accelerations for the car body front end, bogie frame front end, and first wheel-rail interaction force have been analyzed in a passage speed range of $30 - 360\ \mathrm{km/h}$, as shown in Figure 14.

It can be seen from Figure 14a that the critical wavelengths of the acceleration of the car body change insignificantly with variation in the passage speed. This indicates that the vibration of the car body is significantly affected by the coherence of excitations. The critical wavelengths of the acceleration of the bogie frame show a slight upward trend with increasing passage speed, but it remains at approximately $1 - 2\ \mathrm{m}$, as can be seen from Figure 14b. However, from Figure 14c, the critical wavelengths of the wheel-rail interaction forces shift toward longer wavelength side with increasing passage speed, implying that the wheelsets are less sensitive to irregularity coherence but more influenced by their own vibration properties.

5.3 Theoretical explanation for critical wavelengths

In this section, an in-depth discussion is presented on internal factors causing the appearance of critical wavelengths. Note that there are many intersections in the PSD functions as shown in Figures 3–6 and Figures 7–10. The existence of these intersections is not a coincidence. A bogie model with a total of four DOFs (see Figure 15) is proposed to explain the mechanism of these intersections and the relationship between them and the critical wavelength.

By changing the distance between the two wheelsets of each bogie, the responses of the centroid of the frame are investigated under completely coherent excitations and incoherent excitations. Figure 16 presents the PSD functions of the vertical and rotational acceleration for the bogie frame with $b = 0.25\ \mathrm{m}, 0.5\ \mathrm{m}, \mathrm{and}\ 1.25\ \mathrm{m}$ at a passage speed of $120\ \mathrm{km/h}$.

Interestingly, the number of intersections increases with the lengthening of the wheelbase as shown in Figure 16. Let us define the corresponding frequencies (or wavelengths) of these intersections as stagnation frequencies (or wavelengths). It means that the response on the stagnation points is not affected by the coherence of excitations. Furthermore, there is an apparent functional relationship between the stagnation frequency and wheelbase. The stagnation frequency equals $(2n+1)/8b$, which is proved in Appendix C.

The definition of the stagnation frequency has a significant effect on the mechanism analysis of the critical frequency. If the number of stagnation points increase, the RMS difference between coherent and incoherent excitations will get small. When determining the critical wavelength in a one-third octave band domain, the band of short wavelengths will be wider than that of a long wavelength, therefore, the more stagnation points will be covered. This explains why the difference between the responses under two coherence excitations is diminished in the high-frequency region.

6 Conclusions

Two frequency-domain methods have been presented to investigate the dynamic responses of railway vehicles produced by random track irregularities and the mechanism of spatial coherence of excitations generated from multiple wheels. A new expression of the PSD of the excitations has been proposed to enable the coherence information to be separated from all the influence factors on vibration generation. Furthermore, four coherence scales have been introduced to explain the suppression of vehicle responses excited by track irregularity of specific wavelengths. The stagnation wavelengths have been then derived to illustrate the rationality of considering excitation from each wheel as incoherent in the high-frequency region. The major conclusions can be drawn as follows.

(1) The PSD functions of the responses of the car body and bogie frames show periodic fluctuations under the excitations of vertical track irregularities. It is the excitation coherence that leads to the periodicity of vibration suppression frequencies.

(2) If the irregularity component of a specific wavelength excites the front and rear wheelsets with a phase difference of π, the vertical motions of the frame are suppressed. The spatial frequencies of suppression in the vertical vibrations of the bogie frames can be obtained by $f_s = (2n-1)/4b$. When the wheelsets are in the same phase, the rotational movement of the bogie frame is suppressed; and the rotational suppression frequency f_s is $n/2b$.

(3) There are more abundant vibration suppression frequencies of the car body than of the bogies. For vertical vibrations of the car body, the suppression frequency shows a strict periodic distribution as $f_s = (2n-1)/4a$. For rotational vibrations, the suppression frequency presents a complicated periodicity represented by both $f_s = n/2a$ and $f_s = (2n-1)/4b$.

(4) The peak frequencies of the PSD functions of wheel-rail interaction forces reflect the natural vibration characteristics of the wheelset rather than the properties of track irregularities.

(5) Stagnation frequencies exist when there is no difference between the responses under coherent and incoherent excitations. More stagnation points are covered in the frequency range excited by the short wavelength irregularities. As a result, the influence of coherence can be neglected if the wavelength is less than a critical value. This justifies the assumption of the existence of incoherent excitations in the simulation of the vibration and noise induced by moving vehicles.

The wheel-track interactions and non-stationary character of track irregularities have not been considered in this study. Nonetheless, this study has provided an essential theoretical basis for understanding the coherence mechanism of vehicle vibration produced by track irregularities. The effect of wheel-track interactions and non-stationary irregularities on the responses of the vehicle and track can be investigated in detail in future work.

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Appendix A. Matrixes $\mathbf{m}, \mathbf{c}, \mathbf{k}$ and force vector $\mathbf{p}$

The mass matrix $\mathbf{m}$ of the vehicle is a diagonal matrix, which is expressed as:

$$\mathbf{m} = \mathrm{diag}\{J_\mathrm{t} \quad m_\mathrm{w} \quad m_\mathrm{w} \quad m_\mathrm{t} \quad m_\mathrm{b} \quad J_\mathrm{b} \quad m_\mathrm{t} \quad m_\mathrm{w} \quad m_\mathrm{w} \quad J_\mathrm{t}\} \tag{A1}$$

The damping matrix $\mathbf{c}$ is a five-diagonal matrix expressed as:

$$\mathbf{c} = \begin{bmatrix} 2b^2c_1 & bc_1 & -bc_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ bc_1 & c_1+c_\mathrm{a} & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -bc_1 & 0 & c_1+c_\mathrm{a} & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_1 & -c_1 & c_2+2c_1 & -c_2 & ac_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_2 & 2c_2 & 0 & -c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & ac_2 & 0 & 2a^2c_2 & -ac_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & -ac_2 & c_2+2c_1 & -c_1 & -c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_1 & c_1+c_\mathrm{a} & 0 & bc_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & c_1+c_\mathrm{a} & -bc_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & bc_1 & -bc_1 & 2b^2c_1 \end{bmatrix} \tag{A2}$$

The stiffness matrix $\mathbf{k}$ is similarly expressed as:

$$\mathbf{k}=\begin{bmatrix} 2b^2k_1 & bk_1 & -bk_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ bk_1 & k_1+k_\mathrm{a} & 0 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -bk_1 & 0 & k_1+k_\mathrm{a} & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_1 & -k_1 & k_2+2k_1 & -k_2 & ak_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_2 & 2k_2 & 0 & -k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & ak_2 & 0 & 2a^2k_2 & -ak_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_2 & -ak_2 & k_2+2k_1 & -k_1 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_1 & k_1+k_\mathrm{a} & 0 & bk_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_1 & 0 & k_1+k_\mathrm{a} & -bk_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & bk_1 & -bk_1 & 2b^2k_1 \end{bmatrix} \quad \text{(A3)}$$

The force vector $\mathbf{p}$ is expressed as:

$$\mathbf{p}=\{0 \quad p_1 \quad p_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad p_3 \quad p_4 \quad 0\}^\mathrm{T} \quad \text{(A4)}$$

where p_j represents $k_\mathrm{a}r_j(x)+c_\mathrm{a}\dot{r}_j(x)$, and $r_j(x)$ denotes the irregularity function in the position of the j-th wheel-track contact point ($j=1,2,3,4$).

Appendix B. Derivation for the modal formulation of H

A system of n second order differential equations (4) can be reduced to a system of $2n$ first order differential equations [42]

$$\mathbf{R}\dot{\mathbf{v}}'+\mathbf{S}\mathbf{v}'=\mathbf{p}' \quad \text{(A5)}$$

where

$$\mathbf{R}=\begin{bmatrix}\mathbf{c} & \mathbf{m}\\ \mathbf{m} & \mathbf{0}\end{bmatrix} \qquad \mathbf{S}=\begin{bmatrix}\mathbf{k} & \mathbf{0}\\ \mathbf{0} & -\mathbf{m}\end{bmatrix}$$

$$\mathbf{v}'=[\mathbf{v}^\mathrm{T} \quad \dot{\mathbf{v}}^\mathrm{T}]^\mathrm{T} \qquad \mathbf{p}'=[\mathbf{p}^\mathrm{T} \quad \mathbf{0}]^\mathrm{T} \quad \text{(A6)}$$

The $2n$ characteristic values λ_i and their associated complex characteristic vectors $\boldsymbol{\psi}_i'$ can be obtained using well-established methods [52, 53]. The upper n elements of $\boldsymbol{\psi}_i'$ represent the desired modal displacements, i.e., $\boldsymbol{\psi}_i$, and the lower n elements represent the corresponding modal velocities, i.e., $\lambda_i\boldsymbol{\psi}_i$. When the damping is below a critical value, λ_i and $\boldsymbol{\psi}_i'$ appear in conjugate pairs. Therefore, a $2n\times 2n$ complex modal matrix $\boldsymbol{\Psi}'$ can be constructed as

$$\boldsymbol{\Psi}'=[\boldsymbol{\psi}_1' \quad \boldsymbol{\psi}_2' \quad \cdots \quad \boldsymbol{\psi}_n' \quad \boldsymbol{\psi}_1'^* \quad \boldsymbol{\psi}_2'^* \quad \cdots \quad \boldsymbol{\psi}_n'^*] \quad \text{(A7)}$$

The complex modes satisfy the following orthogonality relations:

$$\boldsymbol{\Psi}'^\mathrm{T}\mathbf{R}\boldsymbol{\Psi}'=\mathrm{diag}\{a_i,a_i^*\} \quad \text{(A8)}$$

$$\boldsymbol{\Psi}'^\mathrm{T}\mathbf{S}\boldsymbol{\Psi}'=\mathrm{diag}\{b_i,b_i^*\} \quad \text{(A9)}$$

The orthogonality properties of the modal matrix are used to obtain the state vector in the frequency domain as follows:

$$\mathbf{V}' = \begin{bmatrix} \mathbf{V} \\ j\omega\mathbf{V} \end{bmatrix} = \mathbf{\Psi}'\mathrm{diag}\left\{\frac{1}{j\omega a_i + b_i}, \frac{1}{j\omega a_i^* + b_i^*}\right\}\mathbf{\Psi}'^{\mathrm{T}}\begin{Bmatrix} \mathbf{P} \\ \mathbf{0} \end{Bmatrix} \tag{A10}$$

where $\mathbf{P}$, $\mathbf{V}$, and $\mathbf{V}'$ are respectively the Fourier transformation of the force vector, response vector, and state vector. Eventually, a modal formulation of the frequency response transfer function is established as:

$$\mathbf{H} = \sum_{i=1}^{n}\left(\frac{\mathbf{\psi}_i\mathbf{\psi}_i^{\mathrm{T}}}{j\omega a_i + b_i} + \frac{\mathbf{\psi}_i^*\mathbf{\psi}_i^{\mathrm{H}}}{j\omega a_i^* + b_i^*}\right) \tag{A11}$$

Appendix C. Proof of the stagnation frequency

For the simple bogie model, the coherence matrix $\boldsymbol{\mathcal{B}}(f)$ can be expressed as

$$\boldsymbol{\mathcal{B}}(f) = \begin{bmatrix} 0 & & & \\ & 1 & \mathrm{e}^{-i2\pi f 2b} & \\ & \mathrm{e}^{i2\pi f 2b} & 1 & \\ & & & 0 \end{bmatrix} \tag{A12}$$

where f represents the spatial frequency. When the excitation frequency equals the stagnation frequency, $f = (2n+1)/8b$, the coherence matrix becomes

$$\boldsymbol{\mathcal{B}}(f) = \begin{bmatrix} 0 & & & \\ & 1 & -i & \\ & i & 1 & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 1 & 0 & \\ & 0 & 1 & \\ & & & 0 \end{bmatrix} + i \times \begin{bmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{bmatrix} = \mathbf{A} + i \times \mathbf{B} \tag{A13}$$

where $\mathbf{A}$ is the coherence matrix of completely incoherent excitations and $\mathbf{B}$ is an antisymmetric matrix. Similarly, the complex frequency response transfer function matrix $\mathbf{H}$ can be decomposed as

$$\mathbf{H} = \mathbf{C} + i \times \mathbf{D} \tag{A14}$$

where $\mathbf{C}$ and $\mathbf{D}$ are both symmetric matrices. Therefore, the response PSD matrix can then be obtained as follows:

$$\begin{aligned} \mathbf{S}_{\mathbf{v}} = {} & [(\mathbf{C} + i \times \mathbf{D}) \times \mathbf{A} \times (\mathbf{C} - i \times \mathbf{D})]\chi(\omega, u)\mathbf{S}_{\mathrm{r}}(2\pi f) \\ & + [i(\mathbf{C} + i \times \mathbf{D}) \times \mathbf{B} \times (\mathbf{C} - i \times \mathbf{D})]\chi(\omega, u)\mathbf{S}_{\mathrm{r}}(2\pi f) \end{aligned} \tag{A15}$$

where the first term of the result represents the response caused by completely incoherent excitations and the second term is called the remainder term.

If a certain diagonal element in the remainder term equals zero, it means that the response of the corresponding coordinate cannot be affected by the coherence of excitations at the stagnation frequency. To prove that the first and fourth diagonal elements, which are the rotational and vertical vibrations of the frame respectively, are both zero, the remainder term is expanded as

$$i(\mathbf{C} + i \times \mathbf{D}) \times \mathbf{B} \times (\mathbf{C} - i \times \mathbf{D}) = i(\mathbf{CBC} + \mathbf{DBD}) + [\mathbf{CBD} + (\mathbf{CBD})^{\mathrm{T}}] \tag{A16}$$

Here, the diagonal elements of matrix $\mathbf{CBD}$ are

$$\mathrm{diag}(\mathbf{CBD}) = \begin{bmatrix} c_{13}d_{12} - c_{12}d_{13} \\ c_{23}d_{22} - c_{22}d_{23} \\ c_{33}d_{23} - c_{23}d_{33} \\ c_{43}d_{42} - c_{42}d_{43} \end{bmatrix} \tag{A17}$$

where c_{ij} and d_{ij} denote the (i,j)-th entry of the matrices $\mathbf{C}$ and $\mathbf{D}$, respectively. In consideration of the structural symmetry, there should be $H_{12} = H_{13} = c_{12} + id_{12} = c_{13} + id_{13}$. Therefore, the first diagonal element $c_{13}d_{12} - c_{12}d_{13}$ and fourth diagonal element $c_{43}d_{42} - c_{42}d_{43}$ in matrix $\mathbf{CBD}$ both equal zero. Additionally, the matrices $\mathbf{CBC}$ and $\mathbf{DBD}$ are both antisymmetric; therefore, their diagonal elements are all zero. In the end, the first and fourth diagonal elements in the remainder term is proved to be zero. This explains why the responses of the frame under coherent excitations and incoherent excitations are both same on the stagnation points.

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