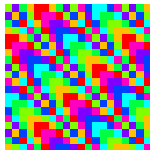


THUE–MORSE SERIES AND k -ARY PARTITIONS

M. Golafshan, M. Rigo

<http://orbi.uliege.be/>

DLT in Rouen — 30th June 2026



Don't worry, I was at a complete stop

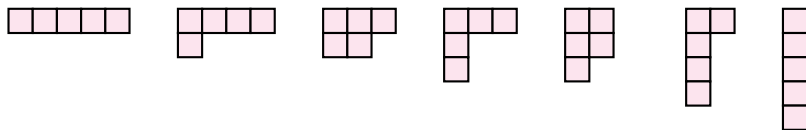


But this was a sign!

WHAT IS AN INTEGER PARTITION ?

Writing n as a sum of positive integers and the order of the terms does not matter. Knuth's TAOCP vol. 4A. 7.2.1.4.

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$



counting the number of partitions : $c(5) = 7$

$$c(n) = \#\{(i_1, i_2, i_3, \dots, i_n) \in \mathbb{N}^n \mid i_1 + 2i_2 + 3i_3 + \dots + n i_n = n\}$$

WHAT IS AN INTEGER PARTITION ?

Basic series expansion $\frac{1}{1-z^t} = \sum_{n \geq 0} z^{tn}, \quad t \in \mathbb{N}_{>0}$

Consider first a finite product

$$\begin{aligned} & \prod_{t=1}^5 \frac{1}{1-z^t} \\ &= (1+z+z^2+z^3+z^4+z^5+\dots)(1+z^2+z^4+z^6+\dots) \\ & \quad (1+z^3+z^6+\dots)(1+z^4+z^8+\dots)(1+z^5+z^{10}+\dots) \\ &= 1+z+2z^2+3z^3+5z^4+\underline{7}z^5+\dots \end{aligned}$$

WHAT IS AN INTEGER PARTITION ?

Leonhard Euler (1741)

A formal power series $c(z) = \sum_{n \geq 0} c(n) z^n = \prod_{t \geq 1} \frac{1}{1 - z^t}$ A000041

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OF INTEGER SEQUENCES®
founded in 1964 by N. J. A. Sloane

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000041 a(n) is the number of partitions of n (the partition numbers).
(Formerly M0663 N0244)

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525

(list; graph; refs; listen; history; text; internal format)

REMARK

In this talk, we make no distinction between a **sequence** $(c(n))_{n \geq 0}$ and the corresponding **formal series** $\sum_{n \geq 0} c(n) z^n \in \mathbb{C}[[z]]$.

WHAT IS AN INTEGER PARTITION ?

There is a vast literature on integer partitions:

- ▶ *unrestricted* partitions $c(z)$
- ▶ bounded partitions (sums of integers bounded by $j \geq 1$)

$$c_{\leq j}(z) = \sum_{n \geq 0} c_j(n) z^n = \prod_{\ell=1}^j \frac{1}{1 - z^\ell}$$

- ▶ k -ary partitions (sums of powers of $k \geq 2$)

$$b_k(z) = \sum_{n \geq 0} b_k(n) z^n = \prod_{r \geq 0} \frac{1}{1 - z^{k^r}}$$

- ▶ Fibonacci partitions (sums of Fibonacci numbers)

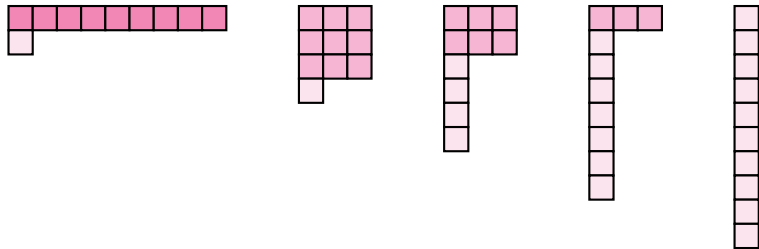
$$f(z) = \sum_{n \geq 0} f(n) z^n = \prod_{n \geq 0} \frac{1}{1 - z^{F_n}}$$

WHAT IS AN INTEGER PARTITION ?

Just to be sure, count the number of *ternary partitions* for 10

$$b_3(10) = 5$$

$$9+1 = 3+3+3+1 = 3+3+\underbrace{1+\cdots+1}_4 = 3+\underbrace{1+\cdots+1}_7 = \underbrace{1+\cdots+1}_{10}$$



$$b_3(n) = \#\{(i_0, i_1, i_2, \dots) \mid i_0 \cdot 3^0 + i_1 \cdot 3^1 + i_2 \cdot 3^2 + \cdots = n\}$$

WHAT IS A k -REGULAR SEQUENCE ?

Let $k \in \mathbb{N}_{\geq 2}$. An integer sequence $(s(n))_{n \geq 0}$ is **k -regular** :

- ▶ the $\mathbb{Z}$ -module generated by its k -kernel is a finitely generated $\mathbb{Z}$ -submodule of $\mathbb{Q}^{\mathbb{N}}$ Allouche–Shallit
- ▶ it has a linear representation

$$s(n) = \lambda M_{n_0} \cdots M_{n_\ell} \gamma$$

where $\lambda \in \mathbb{Q}^{1 \times r}$, $M_0, \dots, M_{k-1} \in \mathbb{Q}^{r \times r}$, $\gamma \in \mathbb{Q}^{r \times 1}$ and

$$n = \sum_{i=0}^{\ell} n_i k^i, \quad \text{rep}_k(n) = n_\ell \cdots n_0$$

- ▶ The series $\sum_{n \geq 0} s(n) \text{rep}_k(n)$ is recognizable Berstel–Reutenauer
- ▶ computations of a weighted automaton fed with base- k expansions

WHAT IS A k -REGULAR SEQUENCE ?

Example: $(c_3(n))_{n \geq 0} = (\lfloor \frac{n^2+6n+15}{12} \rfloor)_{n \geq 0}$ is 2-regular A001399

$$\lambda = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

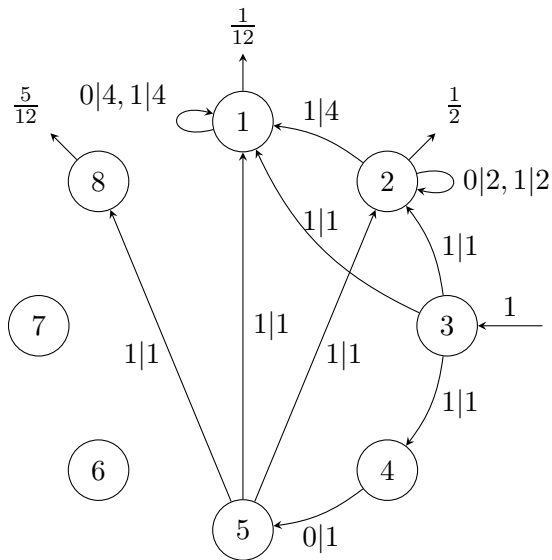
$$M_0 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = (1/12 \quad 1/2 \quad 1 \quad 5/12 \quad 2/3 \quad 3/4 \quad 2/3 \quad 5/12)^T$$

ALLOUCHE–SHALLIT (1992) EXAMPLE 18

For all $j \geq 1$, $k \geq 2$, the sequence $(c_j(n))_{n \geq 0}$ is k -regular.
The poles of $c_{\leq j}(z)$ are roots of unity.

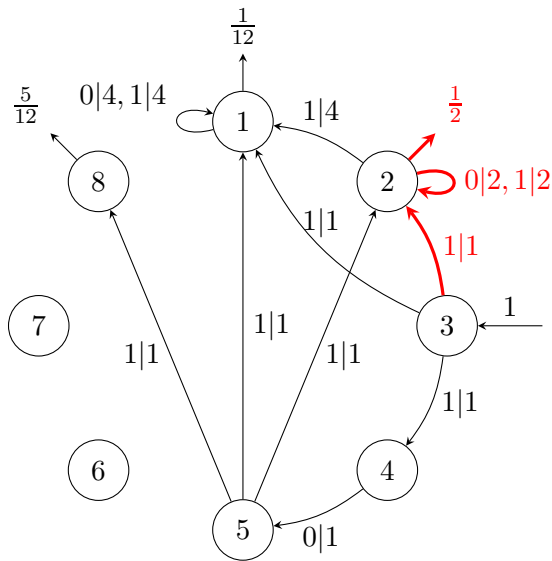
WHAT IS A k -REGULAR SEQUENCE ?



$$c_3(5) = 5$$

$$\begin{array}{r}
 1.2.2.\frac{1}{2} \\
 1.2.4.\frac{1}{12} \\
 1.4.4.\frac{1}{12} \\
 1.1.1.\frac{1}{2} \\
 1.1.1.\frac{1}{12} \\
 1.1.1.\frac{5}{12} \\
 \hline
 5
 \end{array}$$

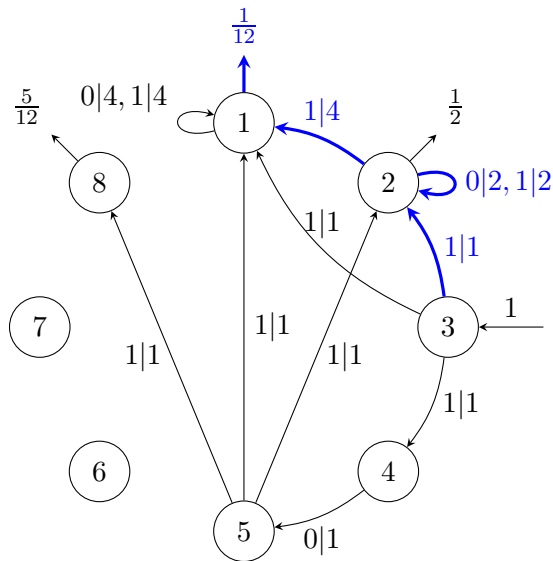
WHAT IS A k -REGULAR SEQUENCE ?



$$c_3(5) = 5$$

$$\begin{array}{r} 1.2.2.\frac{1}{2} \\ 1.2.4.\frac{1}{12} \\ 1.4.4.\frac{1}{12} \\ 1.1.1.\frac{1}{2} \\ 1.1.1.\frac{1}{12} \\ 1.1.1.\frac{5}{12} \\ \hline 5 \end{array}$$

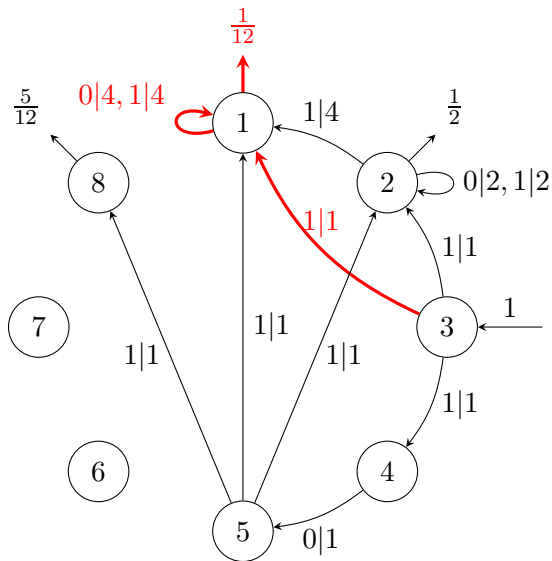
WHAT IS A k -REGULAR SEQUENCE ?



$$c_3(5) = 5$$

$$\begin{array}{r}
 1.2.2.\frac{1}{2} \\
 1.2.4.\frac{1}{12} \\
 1.4.4.\frac{1}{12} \\
 1.1.1.\frac{1}{2} \\
 1.1.1.\frac{1}{12} \\
 1.1.1.\frac{5}{12} \\
 \hline
 5
 \end{array}$$

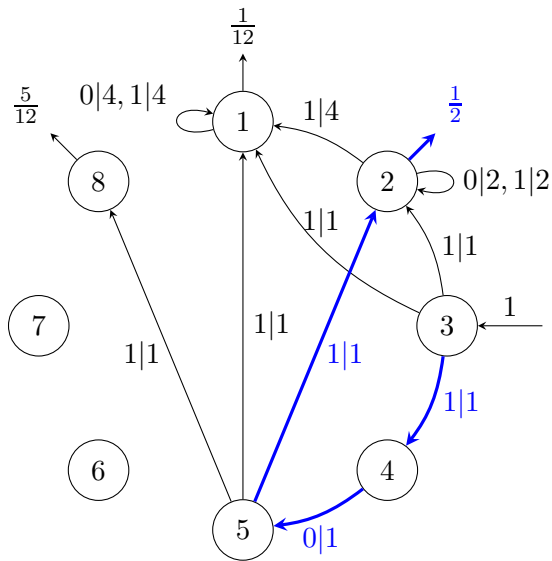
WHAT IS A k -REGULAR SEQUENCE ?



$$c_3(5) = 5$$

$$\begin{array}{r}
 1.2.2.\frac{1}{2} \\
 1.2.4.\frac{1}{12} \\
 1.4.4.\frac{1}{12} \\
 1.1.1.\frac{1}{2} \\
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 1.1.1.\frac{5}{12} \\
 \hline
 5
 \end{array}$$

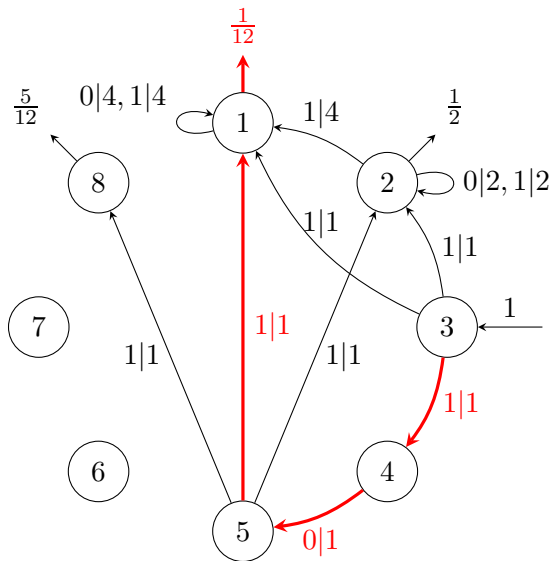
WHAT IS A k -REGULAR SEQUENCE ?



$$c_3(5) = 5$$

$$\begin{array}{r}
 1.2.2.\frac{1}{2} \\
 1.2.4.\frac{1}{12} \\
 1.4.4.\frac{1}{12} \\
 1.1.1.\frac{1}{2} \\
 1.1.1.\frac{1}{12} \\
 1.1.1.\frac{5}{12} \\
 \hline
 5
 \end{array}$$

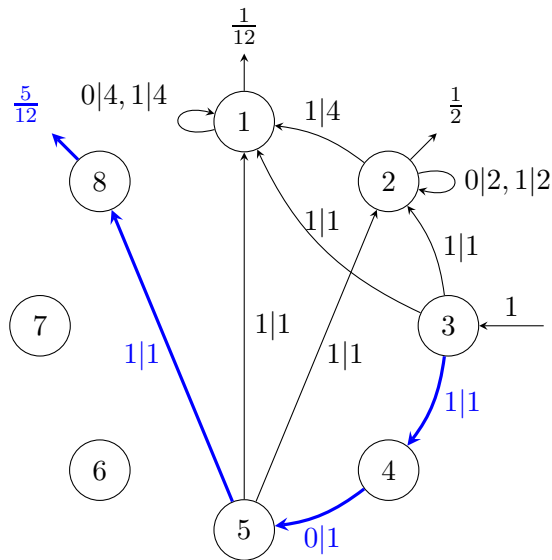
WHAT IS A k -REGULAR SEQUENCE ?



$$\begin{array}{r}
 1.2.2.\frac{1}{2} \\
 1.2.4.\frac{1}{12} \\
 1.4.4.\frac{1}{12} \\
 1.1.1.\frac{1}{2} \\
 \textcolor{red}{1.1.1.\frac{1}{12}} \\
 \hline
 1.1.1.\frac{5}{12} \\
 5
 \end{array}$$

$$c_3(5) = 5$$

WHAT IS A k -REGULAR SEQUENCE ?



$$1.2.2.\frac{1}{2}$$

$$1.2.4.\frac{1}{12}$$

$$1.4.4.\frac{1}{12}$$

$$1.1.1.\frac{1}{2}$$

$$1.1.1.\frac{1}{12}$$

$$1.1.1.\frac{5}{12}$$

$$5$$

$$c_3(5) = 5$$

WHAT IS A k -REGULAR SEQUENCE ?

ALLOUCHE–SHALLIT (1992) THM. 2.10

If a sequence is k -regular, then it is $\mathcal{O}(n^c)$ for some c .

Even though $c_{\leq j}(z) \rightarrow c(z)$ as $j \rightarrow \infty$,
the number of *unrestricted* partitions grows too quickly:

KNUTH TAOCP VOL. 4A P. 399

$$c(n) = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \left(1 + \mathcal{O}(n^{-1/2})\right)$$

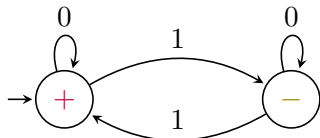
Hence, $(c(n))_{n \geq 0}$ is k -regular for NO $k \geq 2$.

WHAT IS THE THUE–MORSE WORD?

The **sum-of-digits function** (in base 2)

$$\sigma_2(n) = \sum_{i=0}^{\ell} n_i, \quad \text{rep}_2(n) = n_{\ell} \cdots n_0 \in \{0, 1\}^*, \quad n = \sum_{i=0}^{\ell} n_i 2^i$$

$$\mathbf{t}_2 = (-1)^{\sigma_2(n)}, \quad \mathbf{t}_2 = + - + - ++ - ++ - + - + \cdots$$



$$+ \mapsto + -$$

$$- \mapsto - +$$

We define a formal power series

$$\mathbf{t}_2(z) = \sum_{n \geq 0} (-1)^{\sigma_2(n)} z^n = 1 - z - z^2 + z^3 - z^4 + z^5 + \cdots$$

WHAT IS THE THUE–MORSE WORD?

Easy to generalize, in base $k \geq 2$

Let $\omega = \omega_k = \exp(2i\pi/k)$

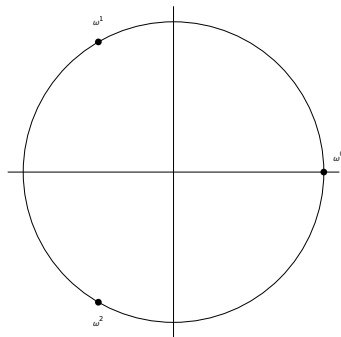
$$t_k(n) = \omega^{\sigma_k(n)}$$

We define a formal power series

$$t_k(z) = \sum_{n \geq 0} \omega^{\sigma_k(n)} z^n$$

$$t_3(z) = \omega^0 + \omega^1 z + \omega^2 z^2 + \omega^1 z^3 + \omega^2 z^4 + \omega^0 z^5 + \omega^2 z^6 + \dots$$

generalized Thue–Morse word: 012120201120...



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To Rigo Michel 

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THEOREM (MERCA 2026)

$t_2(z)$ is the inverse of $b_2(z)$ in $\mathbb{C}[[z]]$

$$(1 - z - z^2 + z^3 - z^4 + z^5 + \dots) \cdot (1 + z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + \dots) = 1$$

→ What will happen in base $k > 2$?

$$t_3(z) = \omega^0 + \omega^1 z + \omega^2 z^2 + \omega^1 z^3 + \omega^2 z^4 + \omega^0 z^5 + \omega^2 z^6 + \dots$$

$$b_3(z) = 1 + z + z^2 + 2z^3 + 2z^4 + 2z^5 + 3z^6 + 3z^7 + 3z^8 + 5z^9 + 5z^{10} + \dots$$

$$t_3(z).b_3(z) = 1 + (1+\omega)z + \underbrace{(1+\omega+\omega^2)}_0 z^2 + \underbrace{(2+2\omega+\omega^2)}_{1+\omega} z^3 + \dots$$

THEOREM (GOLAFSHAN, R.)

There exists a k -regular sequence $(g_k(n))_{n \geq 0}$ such that

$$t_k(z).b_k(z) = \sum_{n \geq 0} g_k(n) z^n = g_k(z)$$

We can quite well describe $g_k(z) \dots$

$$t_3(z) = \omega^0 + \omega^1 z + \omega^2 z^2 + \omega^1 z^3 + \omega^2 z^4 + \omega^0 z^5 + \omega^2 z^6 + \dots$$

$$b_3(z) = 1 + z + z^2 + 2z^3 + 2z^4 + 2z^5 + 3z^6 + 3z^7 + 3z^8 + 5z^9 + 5z^{10} + \dots$$

$$t_3(z).b_3(z) = 1 + (1+\omega)z + \underbrace{(1+\omega+\omega^2)}_0 z^2 + \underbrace{(2+2\omega+\omega^2)}_{1+\omega} z^3 + \dots$$

THEOREM (GOLAFSHAN, R.)

There exists a k -regular sequence $(g_k(n))_{n \geq 0}$ such that

$$t_k(z).b_k(z) = \sum_{n \geq 0} g_k(n) z^n = g_k(z)$$

We can quite well describe $g_k(z) \dots$

A USEFUL POLYNOMIAL

Let $k \geq 2$ and $\omega = \omega_k = \exp(2i\pi/k)$.

$$\frac{1}{1-z} \sum_{i=0}^{k-1} \omega^i z^i = -\omega^{k-1} \prod_{j=1}^{k-2} (z - \omega^j)$$

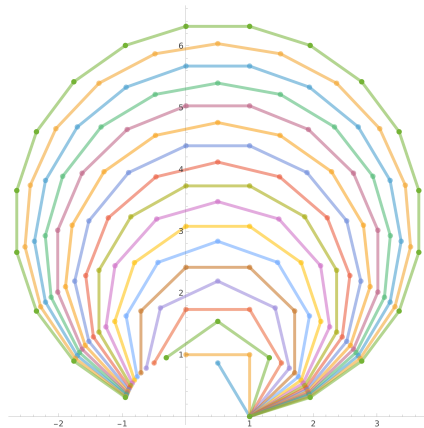
is a polynomial of degree $k-2$

$$P_k(z) = c_{k-2}^{(k)} z^{k-2} + \cdots + c_0^{(k)}.$$

The coefficients $c_j^{(k)}$ are all non-zero. More precisely, for $0 \leq j \leq k-2$, we have

$$c_j^{(k)} = \frac{1 - \omega^{j+1}}{1 - \omega} \quad \text{and} \quad 1 \leq \left| c_j^{(k)} \right| \leq \frac{1}{\sin(\pi/k)}.$$

Coefficients $c_j^{(k)}$ of $P_k(z)$ for $3 \leq k \leq 20$



“(g_k(n))_{n≥0} is a digital sequence”

$$\mathbf{t}_k(z) \cdot \mathbf{b}_k(z) = \sum_{n \geq 0} g_k(n) z^n = \mathbf{g}_k(z)$$

- ▶ g_k(n) = 0 if and only if k − 1 appears in rep_k(n)
- ▶ otherwise, rep_k(n) = d_ℓ ⋯ d₀ ∈ {0, ..., k − 2}^{*}

$$g_k(n) = \prod_{j=0}^{\ell} c_{d_j}^{(k)}$$

COROLLARY

(g_k(n))_{n≥0} is trivially k-regular (rank-1 representation)

“($g_k(n)$) $_{n \geq 0}$ is a digital sequence”

$$\mathbf{t}_k(z) \cdot \mathbf{b}_k(z) = \sum_{n \geq 0} g_k(n) z^n = \mathbf{g}_k(z)$$

- ▶ $g_k(n) = 0$ if and only if $k - 1$ appears in $\text{rep}_k(n)$
- ▶ otherwise, $\text{rep}_k(n) = d_\ell \cdots d_0 \in \{0, \dots, k - 2\}^*$

$$g_k(n) = \prod_{j=0}^{\ell} c_{d_j}^{(k)}$$

COROLLARY

($g_k(n)$) $_{n \geq 0}$ is trivially k -regular (rank-1 representation)

Vanishing when the digit $k - 1$ appears is not (totally) suprising

THEOREM ANDREWS–FRAENKEL–SELLERS (2015)

If $\text{rep}_k(n) = d_\ell \cdots d_0$, then

$$b_k(kn) \equiv (d_\ell + 1) \cdots (d_0 + 1) \pmod{k}.$$

LEMMA

For all $n \geq 0$, we have

$$b_k(kn) = b_k(kn + 1) = \cdots = b_k(kn + k - 1)$$

and $b_k(kn) = b_k(kn - 1) + b_k(n)$, for all $n \geq 1$.

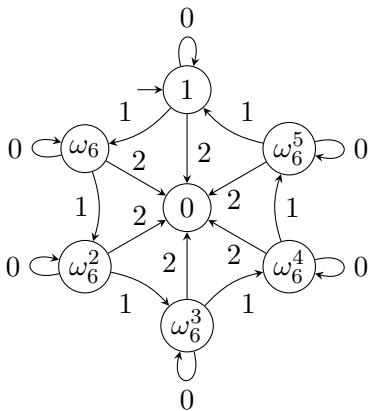
Therefore, $(1 - z) \mathbf{b}_k(z) = \mathbf{b}_k(z^k)$.

The special case $k = 3$; $P_3(z) = \omega_6 z + 1$ where $\omega_6 = \exp(2i\pi/6)$

ALLOUCHE–SHALLIT (1992)

A k -regular seq. taking finitely many values is k -automatic.

$P_3 = \omega_6 z + 1$, so g_3 takes its values in $\{0\} \cup \{\omega_6^i \mid 1 \leq i \leq 5\}$



APPLICATIONS

$$\mathbf{t}_k(z) \cdot \mathbf{b}_k(z) = \mathbf{g}_k(z)$$

i.e., if $\text{rep}_k(n) = d_\ell \cdots d_0$, then

$$\sum_{j=0}^n \omega^{t_k(j)} b_k(n-j) = g_k(n) = \prod_{j=0}^{\ell} c_{d_j}^{(k)}.$$

Triangular system to count k -ary partitions when **knowing $\mathbf{t}_k$, $\mathbf{g}_k$** :

example $k = 3$

$$b_3(0) = g_3(0) = 1 \quad \Rightarrow \quad b_3(0) = 1$$

$$b_3(1) + \omega b_3(0) = g_3(1) = -\omega^2 \quad \Rightarrow \quad b_3(1) = 1$$

$$b_3(2) + \omega b_3(1) + \omega^2 b_3(0) = g_3(2) = 0 \quad \Rightarrow \quad b_3(2) = 1$$

$$b_3(3) + \omega b_3(2) + \omega^2 b_3(1) + \omega b_3(0) = g_3(3) = -\omega^2 \quad \Rightarrow \quad b_3(3) = 2$$

$\vdots$

APPLICATIONS

Conversely, we can recover the prefix of length $n + 1$ of the Thue–Morse word $\mathbf{t}_k$, when **knowing** $\mathbf{b}_k, \mathbf{g}_k$.

example $k = 3$

$$\omega^{t_0} \cdot 1 = g_3(0) = 1 \quad \Rightarrow t_0 = 0$$

$$\omega^{t_0} \cdot 1 + \omega^{t_1} \cdot 1 = g_3(1) = -\omega^2 \quad \Rightarrow t_1 = 1$$

$$\omega^{t_0} \cdot 1 + \omega^{t_1} \cdot 1 + \omega^{t_2} \cdot 1 = g_3(2) = 0 \quad \Rightarrow t_2 = 2$$

$$\omega^{t_0} \cdot 2 + \omega^{t_1} \cdot 1 + \omega^{t_2} \cdot 1 + \omega^{t_3} \cdot 1 = g_3(3) = -\omega^2 \quad \Rightarrow t_3 = 1$$

$\vdots$

“That’s a fun way to compute Thue–Morse words!”

(NON)-REGULARITY

ALLOUCHE–SHALLIT (1992) THM. 3.1

The set of k -regular sequences is closed under convolution.

$$t_k \cdot b_k = \underbrace{t_k}_{\text{automatic}} \star \underbrace{b_k}_{\text{status-?}} = \underbrace{g_k}_{\text{regular}}$$

PROPOSITION

b_k grows too quickly, it is not regular.

But, for *bounded* k -ary partitions, $b_{k,\leq j}$ is ℓ -regular, for all ℓ, j .

INVERSE?

RECALL

- ▶ $b_2(z)$ is the inverse of $t_2(z)$ in $\mathbb{C}[[z]]$. (Merca 2026)
- ▶ $b_k(z)$ is not the inverse of $t_k(z)$, for $k \geq 3$.

$t_k(z) = 1 + \dots$ is invertible in $\mathbb{C}[[z]]$. Let $u_k(z)$ s.t. $t_k \cdot u_k = 1$.

We get

$$u_n = u_{n-k} + \begin{cases} u_{n/k}, & \text{if } n \equiv 0 \pmod{k}; \\ -\omega u_{(n-1)/k}, & \text{if } n \equiv 1 \pmod{k}; \\ 0, & \text{otherwise;} \end{cases}$$

with initial values $(u_0, u_1, \dots, u_{k-1}) = (1, -\omega, 0, \dots, 0)$.

$$t_k \cdot b_k = g_k \Rightarrow \underbrace{b_k}_{\text{non-regular}} = u_k \cdot \underbrace{g_k}_{k\text{-regular}}$$

So u_k cannot be k -regular.

A SIDE NOTE

There exists also a notion of **formal inverse**:

Let $F(z) = \sum_{n \geq 0} f_n z^n \in \mathbb{F}_p[[z]]$. If $f_0 = 0$ and $g_1 \neq 0$, there exists a unique $G(z) = \sum_{n \geq 0} g_n z^n \in \mathbb{F}_p[[z]]$ such that

$$F(G(z)) = G(F(z)) = z.$$

- ▶ M. Gawron, M. Ulas, On formal inverse of the Prouhet-Thue-Morse sequence, Disc. Math. 339 (2016).
- ▶ N. Rampersad, M. Stipulanti, The formal inverse of the period-doubling sequence, J. Integer Seq. 21 (2018).
- ▶ Ł. Merta, Formal inverses of the generalized Thue-Morse sequences and variations of the Rudin-Shapiro sequence, DMTCS 22 (2020–2021)

GOING FURTHER

Introducing **partition polynomials**

$$b_k(t, z) := \prod_{r \geq 0} \frac{1}{1 - t z^{k^r}} = \sum_{n \geq 0} b_{k,n}(t) z^n.$$

M. Ulas, B. Żmija. On arithmetic properties of binary partition polynomials. Adv. in Appl. Math. (2019)

INTEREST

$$b_{k,n}(t) = \sum_{j=0}^n a_{k,n,j} t^j$$

$a_{k,n,j}$ counts partitions of n with exactly j parts

$$b_{k,n}(1) = b_k(n).$$