

Thue–Morse Series and k -ary Partitions

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Abstract. We show that the product of the generating function of k -ary partitions with the Thue–Morse series whose coefficients are k th roots of unity defines a k -regular sequence $(g_k(n))_{n \geq 0}$, given explicitly as a product over the digits of the base- k expansions. We study this sequence and highlight similarities with classical congruence results for k -ary partitions. We provide an elementary proof that the unrestricted k -ary partition function is not k -regular. Finally, we study the inverse of the Thue–Morse series, derive explicit recurrence relations for its coefficients, and show that this inverse cannot be k -regular.

Keywords: Thue–Morse series, k -ary partition function, k -regularity, automatic sequences, Mahler equations, inverse formal power series, digit expansions, partition polynomials

1 Introduction

Automatic and regular sequences are a central object of study in combinatorics on words [1,2,4,6,8,9]. The Thue–Morse word 01101001... appears across a remarkable range of contexts, from number theory [11] and combinatorics [15,19,20] to physics, aperiodic order, and even economic modeling. To give the reader an idea, the survey paper [3] has been cited in more than 170 documents in the zbMath database!

Merca [18] recently exploited the fact that the generating function of binary partitions is the inverse of the Thue–Morse series with coefficients in $\{-1, 1\}$. Equivalently, the convolution of the first n terms of the binary Thue–Morse sequence over $\{-1, 1\}$ and the binary partition function is equal to 0 for $n > 1$. Such an identity raises a natural question: *Does a similar phenomenon occur for k -ary partitions and generalized Thue–Morse sequences when $k \geq 3$?* Let us now introduce the main concepts.

The *order* of a power series p in $\mathbb{C}[[z]]$, denoted by $\text{ord}(p)$, is the least n such that the coefficient $[z^n]p \neq 0$. Let $(p_n(z))_{n \geq 0}$ be a sequence of power series such that $\text{ord}(p_n) \rightarrow \infty$ when $n \rightarrow \infty$. Then the infinite product

$$\prod_{n \geq 0} (1 + p_n(z))$$

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is well-defined in the ring of formal series $\mathbb{C}[[z]]$. The coefficient of z^d in the partial products depends only on factors with $\text{ord}(p_n) \leq d$ and thus, there are only finitely many such factors.

Let F_n be the n th Fibonacci number with the convention $F_0 = F_1 = 1$. In the theory of partitions of integers, it is natural to encounter expressions such as

$$c(z) = \prod_{n \geq 1} \frac{1}{1 - z^n} = \sum_{n \geq 0} c(n) z^n \quad \text{or,} \quad f(z) = \prod_{n \geq 0} \frac{1}{1 - z^{F_n}} = \sum_{n \geq 0} f(n) z^n.$$

The first (resp., second) one is the generating function of the number of partitions of a non-negative integer n as a sum of integers¹ (resp., Fibonacci numbers²)—each can be used any number of times, possibly zero—and the order of the terms does not matter.

Remark 1. We use the convention $F_0 = F_1 = 1$, so the product for $f(z)$ contains the factor $(1 - z)^{-2}$. Equivalently, $f(n)$ counts Fibonacci partitions in which the part 1 occurs in two distinguishable copies (coming from F_0 and F_1). This convention is consistent with the example $f(2) = 4$ below.

As an example, $c(3) = 3$ and $f(2) = 4$ because

$$3 = 3 \cdot 1 = 1 \cdot 1 + 1 \cdot 2 = 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 \quad \text{and} \quad 2 = 2 \cdot F_0 = 1 \cdot F_0 + 1 \cdot F_1 = 2 \cdot F_1 = F_2.$$

It is worth mentioning that the sequence $(f(n))_{n \geq 0}$ is Fibonacci-regular [4, p. 452]. This is a natural generalization of the concept of a k -regular sequence recalled below where one considers Fibonacci numeration system instead of integer base numeration system.

In this article, we will not distinguish between a power series and the sequence of its coefficients. Thus, by abuse of language, we say that the sequence $(f(n))_{n \geq 0}$ or the series $f(z)$ is regular. Let R be a ring and let $k \geq 2$ be an integer. Recall that a sequence $(a_n)_{n \geq 0}$ with values in R is called k -regular if the R -module generated by its k -kernel

$$\mathcal{K}_k(a) = \{ (a_{k^\ell n + r})_{n \geq 0} \mid \ell \geq 0, 0 \leq r < k^\ell \}$$

is finitely generated. Equivalently, it can be produced by a finite linear representation over R depending on the base- k expansion of n . We refer the reader to [2, 4, 7]. Regular sequences form a natural generalization of automatic sequences. They arise naturally in combinatorics on words, number theory, and the study of formal power series, where many arithmetically defined sequences exhibit regular behavior with respect to their base- k expansions. This concept has been generalized to other numeration systems [1, 8, 9].

In [2, Example 18], it is shown that the sequence $(c_{\leq j}(n))_{n \geq 0}$ that counts the number of partitions of n as a sum of integers $\leq j$ is k -regular, for all $j \geq 1$ and

¹ A000041: 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, ...

² A007000: 1, 2, 4, 7, 11, 17, 25, 35, 49, 66, 88, 115, 148, ...

all $k \geq 2$,

$$c_{\leq j}(z) = \prod_{\ell=1}^j \frac{1}{1 - z^\ell}. \quad (1)$$

This follows from [2, Thm. 3.3] because $c_{\leq j}(z)$ is a rational function whose poles are roots of unity. For the z -adic topology, $c_{\leq j}(z) \rightarrow c(z)$ when $j \rightarrow \infty$. However, the sequence $(c(n))_{n \geq 0}$ counting *unrestricted* partitions is not k -regular because it grows too quickly (see [4, Thm. 16.3.1]).

Let $k \geq 2$. Motivated by the recent paper [18], we are interested in the following type of partition. A k -ary partition of the non-negative integer n is a way of writing n as a sum of powers of k , where each power can be used any number of times (possibly zero), and the order of the terms does not matter. It is well-known [5] and it follows from the uniqueness of base- k expansions that

$$b_k(z) = \sum_{n \geq 0} b_k(n) z^n = \prod_{r \geq 0} (1 + z^{k^r} + z^{2k^r} + \dots) = \prod_{r \geq 0} \frac{1}{1 - z^{k^r}}. \quad (2)$$

For instance, $b_3(10) = 5$ because

$$10 = 10 \cdot 3^0 = 7 \cdot 3^0 + 1 \cdot 3^1 = 4 \cdot 3^0 + 2 \cdot 3^1 = 1 \cdot 3^0 + 3 \cdot 3^1 = 1 \cdot 3^0 + 1 \cdot 3^2.$$

and $b_3(z)$ encodes the sequence [A062051](#) in the OEIS.

$$1 + z + z^2 + 2z^3 + 2z^4 + 2z^5 + 3z^6 + 3z^7 + 3z^8 + 5z^9 + 5z^{10} + 5z^{11} + \dots$$

Mahler, and later de Bruijn were already interested in estimating $b_k(n)$ [10,17].

Having the Fibonacci sequence and the series f , which is Fibonacci-regular, on the one side, and a base- k numeration system on the other, we seek analogues of Fibonacci-regularity phenomena in base- k systems. It is therefore natural to ask whether the sequence $(b_k(n))_{n \geq 0}$ is k -regular.

Similarly to (1), the sequence counting restricted k -ary partitions in which only powers k^r with $0 \leq r \leq j$ are allowed forms an ℓ -regular sequence, for all $j \geq 1$ and all $\ell \geq 2$.

1.1 Some Results on k -ary Partitions

Let us recall some basic results. See, for instance, [5].

Lemma 1. *Let $k \geq 2$. For all $n \geq 0$, we have*

$$b_k(kn) = b_k(kn + 1) = \dots = b_k(kn + k - 1)$$

and $b_k(kn) = b_k(kn - 1) + b_k(n)$, for all $n \geq 1$. Therefore, $(1 - z) b_k(z) = b_k(z^k)$.

Proof. Based on (2), we have

$$(1 - z) b_k(z) = \prod_{r \geq 1} \frac{1}{1 - z^{k^r}} = \prod_{r \geq 0} \frac{1}{1 - (z^k)^{k^r}} = b_k(z^k).$$

Comparing coefficients of z^m gives

$$b_k(m) - b_k(m-1) = \begin{cases} b_k(m/k), & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m; \end{cases}$$

(with the convention $b_k(-1) = 0$). For $m = kn + r$ with $1 \leq r \leq k-1$ this yields $b_k(kn + r) = b_k(kn + r - 1)$, hence $b_k(kn) = b_k(kn + 1) = \dots = b_k(kn + k - 1)$. For $m = kn$ with $n \geq 1$ this yields $b_k(kn) = b_k(kn - 1) + b_k(n)$. $\square$

Andrews et al. showed that $b_k(kn) \equiv 0$ modulo k if the base- k expansion of n contains the digit $k-1$. Their main result is the following [5].

Theorem 1. *If the base- k expansion of n is $d_\ell \dots d_0$ (with $0 \leq d_i < k$), then $b_k(kn) \equiv (d_\ell + 1) \dots (d_0 + 1) \pmod{k}$.*

Lemma 1 is a standard result showing that the k -ary partition function is k -Mahler. A power series $q \in \mathbb{C}[[z]]$ is k -Mahler if it satisfies an equation of the form

$$\sum_{i=0}^d p_i(z) q(z^{k^i}) = 0$$

where $d \geq 1$, $p_0, \dots, p_d$ are polynomials in $\mathbb{C}[z]$ and $p_0(z)p_d(z) \neq 0$.

Becker proved that all k -regular power series are k -Mahler [6]. Conversely, if a power series $q \in \overline{\mathbb{Q}}[[z]]$ satisfies

$$q(z) = \sum_{i=1}^d p_i(z) q(z^{k^i})$$

for some positive d and polynomials $p_1, \dots, p_d$ in $\overline{\mathbb{Q}}[z]$, not all zero, then it is k -regular. However, for the Mahler relation given by Lemma 1, $b_k(z)$ has a non-constant polynomial coefficient. So Becker's result does not directly apply.

1.2 Formal Series for Thue–Morse Words

Let $k, m \geq 2$. Let $\omega_m = \exp(2i\pi/m)$. As a consequence of Lemma 1, the convolution of $(b_k(n))_{0 \leq n < k^\ell}$ —i.e., prefixes whose length is a multiple of k — with consecutive powers of ω_k does not provide any information:

$$\sum_{n=0}^{k^\ell-1} b_k(n) \omega_k^{k^{\ell-1}-n} = \sum_{j=0}^{\ell-1} \sum_{r=0}^{k-1} b_k(kj+r) \omega_k^{-1-kj-r} = \sum_{j=0}^{\ell-1} b_k(kj) \sum_{r=0}^{k-1} \omega_k^{-1-r} = 0.$$

It always vanishes, and hence cannot distinguish prefixes.

As the value of $b_k(n)$ depends on the base- k digits of n , it is thus relevant to consider the generalized Thue–Morse word $\mathbf{t}_{k,m} = (t_{k,m}(n))_{n \geq 0}$ over the alphabet $\{0, \dots, m-1\}$ where $t_{k,m}(n)$ is the base- k sum-of-digits $\sigma_k(n)$ of n reduced modulo m , i.e.,

$$t_{k,m}(n) = \sigma_k(n) \pmod{m}. \quad (3)$$

See, for instance, [19,20].

It is a standard technique, see [11], to consider the formal series in $\mathbb{C}[[z]]$

$$t_{k,m}(z) := \prod_{r \geq 0} \left(\sum_{d=0}^{k-1} \omega_m^d z^{d \cdot k^r} \right) = \sum_{n \geq 0} \omega_m^{t_{k,m}(n)} z^n.$$

The trick is based on the uniqueness of base- k expansions. Indeed, if the base- k expansion of n is $d_\ell \cdots d_0$, then the coefficient of z^n is obtained from

$$\prod_{i=0}^{\ell} \omega_m^{d_i} z^{d_i \cdot k^i} = \omega_m^{\sigma_k(n)} z^n$$

and by Euclidean division, the sum-of-digits $\sigma_k(n)$ is $q \cdot m + t_{k,m}(n)$ for some $q \geq 0$ (with $0 \leq t_{k,m}(n) < m$). Thus the series $t_{k,m}(z)$ encodes the word $\mathbf{t}_{k,m}$. As an example, the first terms of $t_{3,3}(z)$ are

$$1 + \omega_3 z + \omega_3^2 z^2 + \omega_3 z^3 + \omega_3^2 z^4 + z^5 + \omega_3^2 z^6 + z^7 + \omega_3 z^8 + \cdots.$$

In what follows, we fix $k = m$ and, for simplicity, write ω and t_k instead of ω_k and $t_{k,k}$.

1.3 Our Contributions

The starting point of this note is to consider variations on Merca’s paper [18]: we multiply the k -ary partition generating function b_k by the series t_k , obtaining (by Theorem 2) an identity of the form

$$t_k(z) \cdot b_k(z) = \sum_{n \geq 0} g_k(n) z^n. \quad (4)$$

Such a convolution is a standard way to measure how “digit-structured” the sequence $(b_k(n))_{n \geq 0}$ may be, the ideas being similar to the discrete Fourier transform. In another direction, compositional inverses are discussed in [14].

In the case $k = 2$ and the classical Thue–Morse sequence over $\{0, 1\}$, it is known that $t_2(z)$ is the inverse of $b_2(z)$ in $\mathbb{C}[[z]]$, i.e., $g_2(0) = 1$ and $g_2(n) = 0$ for all $n > 0$ [18]. In this paper, we observe that this fails for $k \geq 3$ and we quantify the deviation via explicit digit-products. The function $g_k : \mathbb{N} \rightarrow \mathbb{C}$ is non-zero infinitely often and k -regular (see Proposition 1, where we explain that it is digit-wise multiplicative). For $k = 3$, the sequence g_3 takes only finitely many values and is therefore 3-automatic; a deterministic finite automaton with output (DFAO) with 7 states is provided in Fig. 1b. For $k \geq 3$, we briefly discuss its summatory function $N \mapsto \sum_{n < N} g_k(n)$ using classical procedures.

In Section 3, using elementary arguments on the growth of $b_k(n)$, we show that $b_k(z)$ is not k -regular (Theorem 3). Since g_k is k -regular, as a consequence of Theorem 2, the inverse of $t_k(z)$ cannot be k -regular. We nevertheless describe the inverse of $t_k(z)$ in $\mathbb{C}[[z]]$: In Section 3.2, we obtain recurrence relations for its coefficients. Finally, we reconsider Ulas–Žmija partition polynomials [21] in the context of this article and extend the function $g_k : \mathbb{N} \rightarrow \mathbb{C}$ to $n \mapsto g_{k,n}(t) \in \mathbb{C}[t]$, where $g_{k,n}(1) = g_k(n)$, opening the way to new developments.

2 The Product of the Two Series

We need a useful description of $g_k(n)$ given in (4). First, consider the product of two factors, in general form, from the two series $t_k(z)$ and $b_k(z)$. Let us start with some classical cyclotomic manipulations.

Lemma 2. *Let $k \geq 2$ and $\omega = \exp(2i\pi/k)$. We have*

$$\frac{1}{1-z} \sum_{i=0}^{k-1} \omega^i z^i = -\omega^{k-1} \prod_{j=1}^{k-2} (z - \omega^j) =: P_k(z)$$

which is thus a polynomial of degree $k-2$. In particular, we have $P_2(z) = 1$.

As an example, we have

$$P_3(z) = -\omega_3^2 z + 1 \quad \text{and} \quad P_4(z) = iz^2 + (1+i)z + 1.$$

Proof. The numerator in the l.h.s. evaluated at $z = 1$ is the sum of the k th roots of unity and thus vanishes. Hence $1-z$ divides the numerator whose leading term has coefficient ω^{k-1} which is non zero. So the quotient has a leading term $-\omega^{k-1} z^{k-2}$.

The numerator is also a geometric sum, so we can write

$$\frac{1}{1-z} \sum_{i=0}^{k-1} \omega^i z^i = \frac{1-z^k}{(1-z)(1-\omega z)}. \quad (5)$$

The numerator of the r.h.s. vanishes for all k -th roots of unity. The denominator of the r.h.s. vanishes for $z = 1$ and $z = \omega^{k-1}$. The conclusion follows from the fundamental theorem of algebra. $\square$

Lemma 3. *Let $k \geq 2$ and $\omega = \exp(2i\pi/k)$. The coefficients c_j of the polynomial $P_k(z) = c_{k-2}^{(k)} z^{k-2} + \dots + c_0^{(k)}$ (depicted in Fig. 1a) are all non-zero. More precisely, for $0 \leq j \leq k-2$, we have*

$$c_j^{(k)} = \frac{1 - \omega^{j+1}}{1 - \omega} \quad \text{and} \quad 1 \leq |c_j^{(k)}| \leq \frac{1}{\sin(\pi/k)}.$$

Proof. Using (5), we get

$$-\omega P_k(z) = \prod_{j=1}^{k-2} (z - \omega^j) = \frac{z^k - 1}{(z-1)(z - \omega^{-1})}.$$

Hence

$$-\omega(z - \omega^{-1}) \sum_{j=0}^{k-2} c_j^{(k)} z^j = \frac{z^k - 1}{z - 1} = \sum_{i=0}^{k-1} z^i.$$

We then compare the coefficients, $-\omega c_{k-2}^{(k)} = 1$, for $1 \leq j \leq k-2$, $c_j^{(k)} = 1 + \omega c_{j-1}^{(k)}$ and $c_0^{(k)} = 1$. The result follows from the recurrence relation. $\square$

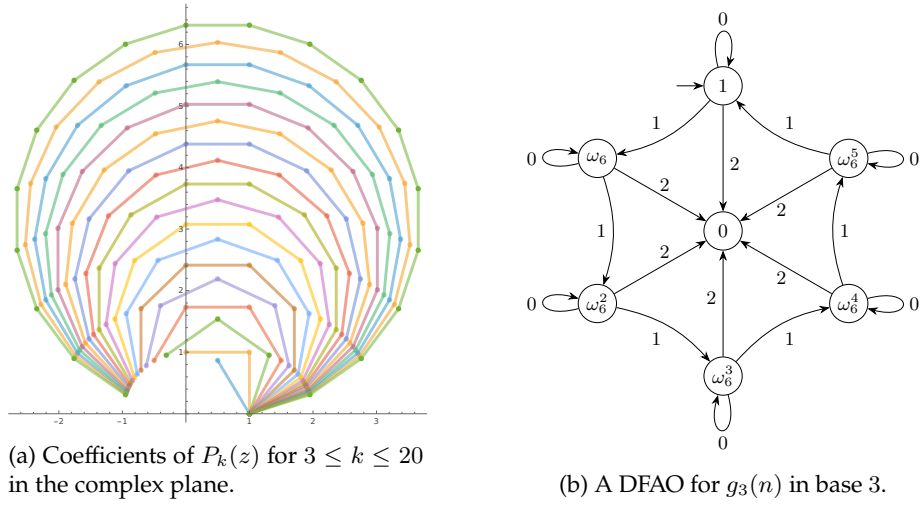


Fig. 1: The coefficients of P_k (left) and a DFAO for g_3 (right).

As a consequence of Lemma 2, we have

$$\mathbf{t}_k(z) \cdot \mathbf{b}_k(z) = \prod_{r \geq 0} P_k(z^{k^r}) = \sum_{n \geq 0} g_k(n) z^n.$$

Compared with the case $k = 2$, studied in [18] where the r.h.s. is equal to the constant 1, the situation is a bit more involved. We have $g_k(n) = 0$ if and only if $k - 1$ appears in the base- k expansion of $n \geq 0$. Indeed, inspecting the form of

$$P_k(z^{k^r}) = \sum_{i=0}^{k-2} c_i^{(k)} z^{i \cdot k^r},$$

there is no term of the form $z^{(k-1) \cdot k^r}$. One may notice the similarity with Theorem 1.

If the base- k expansion of n is of the form $d_\ell \cdots d_0 \in \{0, \dots, k-2\}^*$, then

$$g_k(n) = \prod_{j=0}^{\ell} c_{d_j}^{(k)} \quad (6)$$

and from Lemma 3, all factors have modulus at least one. We state this observation.

Proposition 1. *The function $g_k : \mathbb{N} \rightarrow \mathbb{C}$ is k -regular.*

Proof. There is a 1-dimensional linear representation associating with each digit $d \neq k-1$ the complex number $c_d^{(k)}$ and associating $k-1$ with 0. $\square$

Remark 2. In the case $k = 3$, g_3 is 3-automatic since its values lie in $\{0\} \cup \{\omega_6^i \mid 1 \leq i \leq 5\}$. Since $c_1^{(3)} = -\omega_3^2 = \omega_6$, we get the DFAO generating $g_3(n)$ when fed with base-3 expansion of n depicted in Fig. 1b. Since the coefficients of $P_3(z)$ are $c_0^{(3)} = 1$ and $c_1^{(3)} = \omega_6$, we get the loops with labels 0 and reading a digit 1 corresponds to a multiplication by ω_6 .

Theorem 2. Let $k \geq 2$. The following relations hold, for all $n \geq 0$,

$$\sum_{j=0}^n \omega^{t_k(j)} b_k(n-j) = g_k(n)$$

where $\omega = \exp(2i\pi/k)$, $(t_k(j))_{j \geq 0} = (t_{k,k}(j))_{j \geq 0}$ is the k -letter Thue–Morse word defined in (3) and g_k is the k -regular function defined in (6).

Proof. For all $n \geq 0$, the coefficient of z^n in $\mathbf{t}_k \cdot \mathbf{b}_k$ is

$$\sum_{j=0}^n [z^j] \mathbf{t}_k \cdot [z^{n-j}] \mathbf{b}_k = \sum_{j=0}^n \omega^{t_k(j)} b_k(n-j) = g_k(n). \quad (7)$$

□

So (7) gives a triangular system that allows one to compute the values of the k -ary partition function $b_k(0), \dots, b_k(n)$ from the Thue–Morse word $(t_k(j))_{j \geq 0}$ over $\{0, \dots, k-1\}$ and the values $g_k(0), \dots, g_k(n)$ of the k -regular function g . As an example, for $k = 3$ and $\omega = \exp(2i\pi/3)$:

$$\begin{aligned} b_3(0) &= g_3(0) = 1 && \Rightarrow b_3(0) = 1 \\ b_3(1) + \omega b_3(0) &= g_3(1) = -\omega^2 && \Rightarrow b_3(1) = 1 \\ b_3(2) + \omega b_3(1) + \omega^2 b_3(0) &= g_3(2) = 0 && \Rightarrow b_3(2) = 1 \\ b_3(3) + \omega b_3(2) + \omega^2 b_3(1) + \omega b_3(0) &= g_3(3) = -\omega^2 && \Rightarrow b_3(3) = 2 \\ &\vdots \end{aligned}$$

Conversely, from $b_k(0), \dots, b_k(n)$ and $g_k(0), \dots, g_k(n)$, we can recover the prefix of length $n+1$ of the Thue–Morse word $(t_k(j))_{j \geq 0}$. Again, for $k = 3$, we get an unconventional way to recover Thue–Morse prefixes from k -ary partition data:

$$\begin{aligned} \omega^{t_0} \cdot 1 &= g_3(0) = 1 && \Rightarrow t_0 = 0 \\ \omega^{t_0} \cdot 1 + \omega^{t_1} \cdot 1 &= g_3(1) = -\omega^2 && \Rightarrow t_1 = 1 \\ \omega^{t_0} \cdot 1 + \omega^{t_1} \cdot 1 + \omega^{t_2} \cdot 1 &= g_3(2) = 0 && \Rightarrow t_2 = 2 \\ \omega^{t_0} \cdot 2 + \omega^{t_1} \cdot 1 + \omega^{t_2} \cdot 1 + \omega^{t_3} \cdot 1 &= g_3(3) = -\omega^2 && \Rightarrow t_3 = 1 \\ &\vdots \end{aligned}$$

Consider the summatory function $S_k(N) := \sum_{n=0}^{N-1} g_k(n)$. Its asymptotic behavior follows directly from classical results, see [4, Thm. 3.5.1] or [16,12],

$$S_k(N) = N^{\log_k |A_k|} \Phi(\log_k N) + \mathcal{O}(N^\beta)$$

where Φ is a 1-periodic Hölder continuous function, $\beta < \log_k |A_k|$ and

$$A_k := \sum_{r=0}^{k-2} c_r^{(k)} \text{ and } |A_k| = \frac{k}{2 \sin(\pi/k)}.$$

One has to sum the matrices of the linear representation, which is straightforward in this case. The reader may consult [13] to get precise asymptotic expansions on a larger family of series. We have $\log_3 |A_3| = 1/2$, $\log_4 |A_4| = 3/4$, $\log_5 |A_5| \simeq 0.899$, $\log_6 |A_6| = 1$ and $1 < \log_k |A_k| < 2$ for all $k \geq 7$.

We briefly highlight an easy computation to get an intuition about Fig. 2. Using the digit-wise definition of g_k , first observe that

$$\sum_{r=0}^{k-1} g_k(kn + r) = \sum_{r=0}^{k-2} c_r^{(k)} g_k(n) = A_k g_k(n).$$

Note that $S_k(k) = \sum_{r=0}^{k-2} c_r^{(k)} = A_k$. Hence

$$S_k(kN) = \sum_{n=0}^{N-1} \sum_{r=0}^{k-1} g_k(kn + r) = \sum_{n=0}^{N-1} A_k g_k(n) = A_k S_k(N)$$

and for all ℓ , along the corresponding subsequence, we have $|S_k(k^\ell)| = |A_k|^\ell$.

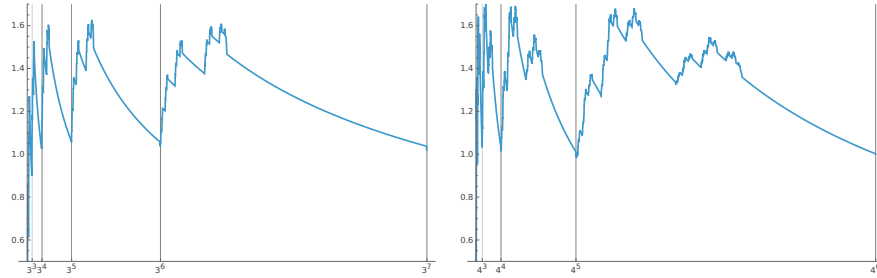


Fig. 2: Summatory functions $S_3(N)/\sqrt{N}$ and $S_4(N)/N^{3/4}$.

3 Going Further

3.1 (Non)-Regularity

We have already mentioned in the introduction that the number of restricted k -partitions where appear only powers of k bounded by k^j , for some fixed j ,

form a ℓ -regular sequence, for all $j \geq 1$ and all $\ell \geq 2$.

$$b_{k,\leq j}(z) := \prod_{r=0}^j \frac{1}{1 - z^{k^r}} = \sum_{n \geq 0} b_{k,\leq j}(n) z^n.$$

For instance, for $k = 2$ and $j = 3$, we found the sequence [A008643](#) in the OEIS for the number of partitions of n into parts 1, 2, 4 and 8. The sequence [A008642](#) ($k = 2, j = 2$) seems to have several combinatorial interpretations.

However, the unrestricted sequence $(b_k(n))_{n \geq 0}$ grows too quickly to be k -regular (see [4, Thm. 16.3.1]).

Theorem 3. *The series $b_k(z)$ is not k -regular.*

Proof. From Lemma 1, we have, for all $m \geq 0$ and $r \in \{0, \dots, k-1\}$,

$$b_k(km + r) = \sum_{j=0}^m b_k(j). \quad (8)$$

For each $\ell \geq 0$, we consider the sequence $(s_\ell(n))_{n \geq 0}$ defined by

$$s_\ell(n) := b_k(k^\ell n),$$

i.e., it is a particular sequence of the k -kernel. Using (8), we have

$$s_{\ell+1}(n) = b_k(k^{\ell+1}n) = \sum_{j=0}^{k^\ell n} b_k(j) \geq \sum_{m=0}^n b_k(k^\ell m) = \sum_{m=0}^n s_\ell(m).$$

Roughly speaking, each time we multiply the argument by k (to extract an element from the k -kernel), we get (at least) a discrete integral of the previous element.

We prove by induction on ℓ that the sequence satisfies $s_\ell(n) \geq \binom{n+\ell}{\ell}$ and thus $s_\ell(n)$ grows at least like n^ℓ as $n \rightarrow \infty$. The base case is clear: $s_0(n) = b_k(n) \geq 1$. Assuming inequality holds for ℓ , we have

$$s_{\ell+1}(n) \geq \sum_{m=0}^n s_\ell(m) \geq \sum_{m=0}^n \binom{m+\ell}{\ell} = \binom{n+\ell+1}{\ell+1}$$

where the last equality is the classical hockey-stick identity (also called Christmas stocking theorem).

However, if a sequence $(r(n))_{n \geq 0}$ is k -regular, then there exist constants C, D such that $|r(n)| \leq Cn^D$, for all $n \geq 1$. Moreover, all elements of the finitely generated module spanned by the k -kernel share this property.

But we have shown that $s_\ell(n)$ grows at least as n^ℓ , contradicting the above property of k -regular sequences. $\square$

3.2 Inverse of the k -automatic Series $\mathbf{t}_k(z)$

Since the constant term of $\mathbf{t}_k(z)$ is 1, the series is invertible in $\mathbb{C}[[z]]$. Let $\mathbf{u}_k(z) = \sum_{n \geq 0} u_n^{(k)} z^n$ denote the inverse of $\mathbf{t}_k(z)$. Since k is fixed, we simply write u_n instead of $u_n^{(k)}$ for the sake of notation. Note that $\mathbf{u}_k(z)$ cannot be k -regular: if it were the case, then $\mathbf{b}_k(z) = \mathbf{u}_k(z) \cdot \mathbf{g}_k(z)$ would be k -regular: convolution of k -regular sequences is k -regular [2].

From the definition of $\mathbf{t}_k$, it is straightforward to see that

$$\mathbf{t}_k(z) = \left(\sum_{d=0}^{k-1} \omega^d z^d \right) \mathbf{t}_k(z^k).$$

Hence, we obtain the following Mahler-type functional equations:

$$(1 - \omega z) \mathbf{t}_k(z) = (1 - z^k) \mathbf{t}_k(z^k)$$

and

$$(1 - z^k) \mathbf{u}_k(z) = (1 - \omega z) \mathbf{u}_k(z^k).$$

From this, for $n \geq k$, we get

$$u_n = u_{n-k} + \begin{cases} u_{n/k}, & \text{if } n \equiv 0 \pmod{k}; \\ -\omega u_{(n-1)/k}, & \text{if } n \equiv 1 \pmod{k}; \\ 0, & \text{otherwise;} \end{cases}$$

with initial values $(u_0, u_1, \dots, u_{k-1}) = (1, -\omega, 0, \dots, 0)$. These recurrence relations directly yield, for all $n \geq 0$ and $r \in \{2, \dots, k-1\}$ when $k \geq 3$,

$$u_{kn} = \sum_{j=0}^n u_j, \quad u_{kn+1} = -\omega \sum_{j=0}^n u_j, \quad u_{kn+r} = 0.$$

3.3 Partition Polynomials

In [21], *binary partition polynomials* are considered. We consider their base- k analogue, a family of polynomials in t , defined by

$$\mathbf{b}_k(t, z) := \prod_{r \geq 0} \frac{1}{1 - tz^{kr}} = \sum_{n \geq 0} b_{k,n}(t) z^n.$$

The interest is that $b_{k,n}(t) = \sum_{j=0}^n a_{k,n,j} t^j$, where $a_{k,n,j}$ counts partitions of n with exactly j parts. So $b_{k,n}(t)$ records the number of parts (powers of k , counted with multiplicity), not only the total number of partitions. It is known that $(1 - tz) \mathbf{b}_k(t, z) = \mathbf{b}_k(t, z^k)$. Two paths can therefore be followed.

We multiply, as in the first part of the paper, the above series in $\mathbb{C}[t][[z]]$ by $\mathbf{t}_k(z)$

$$\mathbf{t}_k(z) \cdot \mathbf{b}_k(t, z) = \sum_{n \geq 0} g_{k,n}(t) z^n \tag{9}$$

and we can therefore study this sequence $(g_{k,n}(t))_{n \geq 0}$ of polynomials.

n	$g_{2,n}(t)$	n	$g_{2,n}(t)$
0	1	5	$t^5 - t^3 - t + 1$
1	$t - 1$	6	$t^6 - t^4 + t^3 - t^2 - t + 1$
2	$t^2 - 1$	7	$t^7 - t^5 + t^4 - 2t^3 - t^2 + 3t - 1$
3	$t^3 - 2t + 1$	8	$t^8 - t^6 + t^5 - t^4 - t^3 + 2t^2 - 1$
4	$t^4 - t^2 + t - 1$	9	$t^9 - t^7 + t^6 - t^5 - 2t^4 + 2t^3 + t^2 - 2t + 1$

For $k = 2$, $g_{2,0}(t) = 1$ and

$$g_{2,2n}(t) = t g_{2,2n-1}(t) + g_{2,n}(t) \text{ and } g_{2,2n+1}(t) = t g_{2,2n}(t) - g_{2,n}(t).$$

Observe that, from these relations, the sequence $(g_{2,n}(0))_{n \geq 0}$ is the Thue–Morse word over $\{-1, 1\}$.

Note that in [21], this is not the series $t_2(z)$ whose coefficient of z^n is $(-1)^{\sigma_2(n)}$ which is considered, but a series whose coefficient of z^n is $(-t)^{\sigma_2(n)}$. So an alternative generalization is to consider the series

$$t_k(t, z) := \prod_{r \geq 0} \left(\sum_{d=0}^{k-1} (t \omega_k)^d z^{d \cdot k^r} \right) = \sum_{n \geq 0} (t \omega_k)^{\sigma_k(n)} z^n.$$

In this context, we recover that $t_2(t, z) \cdot b_2(t, z) = 1$ [21, Thm. 2.1]. As observed in the first part of the paper, we obtain a new relation for $k \geq 3$:

$$t_k(t, z) \cdot b_k(t, z) = \sum_{n \geq 0} g_k(n) t^{\sigma_k(n)} z^n$$

where g_k is the k -regular function defined in (6). So, the result is quite similar to Theorem 2.

It is therefore more interesting to study the relation (9). For a uniform presentation, we set $g_2(n) = \delta_{n,0}$. The polynomials $g_{k,n}(t)$ satisfy recurrences mirroring those of $b_{k,n}(t)$. Due to space limitation, the proof is not given here.

Proposition 2. *Let $k \geq 2$ and let $\omega_k = \exp(2i\pi/k)$. Then $g_{k,0}(t) = 1$ and*

$$\begin{aligned} g_{k,kn}(t) &= t g_{k,kn-1}(t) + g_{k,n}(t) & (n \geq 1), \\ g_{k,kn+r}(t) &= t g_{k,kn+r-1}(t) + \omega_k^r g_{k,n}(t) & (n \geq 0, 1 \leq r \leq k-1). \end{aligned}$$

In particular, for all $n \geq 0$, we have $g_{k,n}(1) = g_k(n)$, i.e., the sum of coefficients of the polynomial $g_{k,n}(t)$ is equal to $g_k(n)$.

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