

3D MODELLING OF STELLAR STRUCTURE IN CLOSE BINARIES AND TIDALLY EXCITED OSCILLATIONS

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION TO STARS, ASTEROSEISMOLOGY AND BINARY SYSTEMS

Stars are the fundamental building blocks of galaxies, and understanding their internal structure and evolution remains a central objective of modern astrophysics (Tinsley, 1980; Kippenhahn et al., 2013; Conroy, 2013). Classical observables, such as luminosity, effective temperature, and surface chemical composition, provide essential constraints on stellar models, but they mainly probe the photosphere and therefore give only indirect information about the physical processes acting in the deep interior (Gray, 2005; Hubeny & Mihalas, 2014; Aerts, 2021). Quantities such as the internal rotation profile, the efficiency of angular-momentum transport, convective-boundary mixing, and the chemical stratification consequently remain only weakly constrained by classical observations alone (Beck et al., 2012; Mosser et al., 2012; Deheuvels et al., 2014; Aerts et al., 2019; Aerts, 2021; Pedersen et al., 2021). The theoretical framework of stellar structure was established in the early twentieth century, especially through Eddington’s formulation of stellar interiors and subsequent developments in stellar-structure theory (Eddington, 1916, 1918, 1920, 1926). In particular, their quasi-static structure is linked to the mechanical balance between the inward gravitational force and the outward pressure gradient, known as hydrostatic equilibrium, a principle that remains one of the basic equations of modern stellar modelling (Eddington, 1916, 1920, 1926). Eddington also recognised that stellar variability could provide constraints on stellar interiors, notably through the pulsation problem of Cepheid variables (Eddington, 1918; Gautschi, 1997). This idea anticipated the modern use of stellar oscillations as probes of internal stellar physics. The first major seismic revolution occurred for the Sun. The discovery of the solar five-minute oscillations (Leighton et al., 1962) and their interpretation as resonant acoustic modes trapped in the solar interior (Ulrich, 1970; Leibacher & Stein, 1971; Deubner, 1975) led to the development of helioseismology, which demonstrated that acoustic-mode frequencies can be used to infer the solar internal structure with high precision (Christensen-Dalsgaard et al., 1985, 1991; Christensen-Dalsgaard, 2002; Basu, 2016; Buldgen et al., 2019, 2025). Helioseismology constrained, for example, the depth of the convection zone

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(Christensen-Dalsgaard et al., 1991; Basu & Antia, 1997), the helium abundance in the convective envelope (Basu & Antia, 1995), and the internal rotation profile (Schou et al., 1998). These results showed that seismic diagnostics can probe regions that are inaccessible to surface observables, and they remain a benchmark for stellar modelling (Christensen-Dalsgaard, 2002; Basu & Antia, 2008). More challenging diagnostics, such as searches for gravity-mode (g-mode) signatures, have also been used to try to investigate the dynamics of the deep solar core, although unambiguous detections of individual solar g modes remain difficult (García et al., 2007; Appourchaux et al., 2010; Fossat et al., 2017).

Asteroseismology extends this approach to stars other than the Sun (Aerts et al., 2010; Chaplin & Miglio, 2013; Aerts, 2021). Because the propagation cavities and frequencies of stellar oscillations depend on the sound-speed profile, density stratification, buoyancy frequency, and rotation rate of the stellar interior, oscillation spectra provide powerful constraints on the internal structure of stars (Unno et al., 1989; Christensen-Dalsgaard, 2002; Aerts et al., 2010; Aerts, 2021). When combined with stellar modelling, these observables allow stellar masses, radii, ages, and evolutionary states to be determined with much higher precision than is generally possible from classical observables alone (Bedding et al., 2011; Chaplin & Miglio, 2013; Silva Aguirre et al., 2015, 2017). Asteroseismology therefore provides a unique way to test the input physics of stellar models, including mixing processes, rotation, and transport mechanisms (Michielsen et al., 2019; Aerts et al., 2019; Aerts, 2021; Pedersen et al., 2021).

Stellar oscillations can be detected through periodic variations in radial velocity or photometric brightness (Kjeldsen & Bedding, 1995; Aerts et al., 2010). Their observational exploitation requires high precision, long time baselines, and high duty cycles, because longer and more continuous observations improve the frequency resolution, reduce aliasing, and enhance the detectability of low-amplitude modes (Aerts et al., 2010; Aerts, 2021). The advent of space photometry transformed the field. The CoRoT mission (Baglin et al., 2006) provided long, high-precision light curves for both exoplanet and stellar-oscillation studies (Michel et al., 2008; Auvergne et al., 2009). The *Kepler* (Borucki et al., 2010) mission then delivered four years of nearly continuous photometry for $\sim 1.5 \times 10^5$ stars, allowing for ensemble asteroseismology across a wide range of stellar types (Koch et al., 2010; Gilliland et al., 2010; Chaplin et al., 2011). The *TESS* (Ricker et al., 2015) mission extended high-precision photometric monitoring to nearly the whole sky, targeting brighter stars that are especially favourable for follow-up observations (Sullivan et al., 2015; Lund et al., 2017). In the coming years, *PLATO* (Rauer et al., 2014) is expected to further expand this legacy by combining exoplanet detection with high-quality asteroseismic characterisation of bright stars (Goupil et al., 2024).

Observational studies have shown that stellar multiplicity is strongly dependent on primary mass (Duchêne & Kraus, 2013; Moe & Di Stefano, 2017). In particular, O-type stars are frequently observed in close binary or higher-order multiple systems already at young ages, and the fraction of systems expected to undergo binary interaction rises steeply toward the high-mass regime (Sana et al., 2012; Chini et al., 2012; Sana et al., 2013; Moe & Di Stefano, 2017). For massive stars, binary interaction is therefore

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not a secondary effect but a dominant evolutionary channel: more than $\sim 70\%$ of O-type stars are expected to interact with a companion during their lifetime, and about one third may ultimately merge (Sana et al., 2012; de Mink et al., 2014; Marchant & Bodensteiner, 2024). Processes such as stable mass transfer, unstable Roche-lobe overflow and common-envelope evolution, tidal coupling, and stellar mergers can profoundly modify the internal structure, angular-momentum content, surface composition, and subsequent evolutionary path of the stellar components (Zahn, 1977; Hut, 1981; Wellstein et al., 2001; Langer, 2012; Ivanova et al., 2013; de Mink et al., 2013; de Mink et al., 2014; Marchant & Bodensteiner, 2024). Consequently, the evolution of close binaries can differ substantially from that of isolated stars of the same initial mass and composition (Langer, 2012; de Mink et al., 2014; Marchant & Bodensteiner, 2024). Binary systems also provide one of the most direct routes to precise stellar masses and radii, since their orbital dynamics and light curves can yield masses and radii with percent-level precision, which are fundamental parameters that are largely inaccessible for single stars (Andersen, 1991; Torres et al., 2010; Southworth, 2021). Such systems therefore constitute benchmark laboratories for testing stellar structure and evolution models (Andersen, 1991; Torres et al., 2010). In close binaries, however, the modelling becomes more complex because spherical symmetry is broken. The effective potential experienced by the stars is no longer determined solely by the star's own gravity, but also by the gravitational field of the companion and, by the centrifugal potential (Kopal, 1959; Wilson & Devinney, 1971; Hilditch, 2001). This leads naturally to the Roche geometry, whose characteristic equipotential surfaces and Roche lobes are central to the description of tidal deformation, Roche-lobe overflow, and mass transfer in interacting binaries (Paczyński, 1971; Eggleton, 1983; Hilditch, 2001; Kallrath, 2009).

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In close binary systems, tidal interactions are a key mechanism governing both the long-term orbital and rotational evolution of the system and its short-term observational properties (Zahn, 1977; Hut, 1981; Ogilvie, 2014). A tidal perturbation arises because the gravitational field of the companion is not spatially uniform across the primary star; its strength and direction vary with position, producing latitude- and longitude-dependent departures from spherical symmetry (Kopal, 1959; Wilson & Devinney, 1971; Morris, 1985; Hilditch, 2001). The impact of tides on orbital and rotational evolution is intrinsically linked to dissipation. Because the tidal response is not perfectly elastic, viscous or turbulent damping, radiative damping, and wave dissipation convert part of the mechanical energy associated with the tidal deformations into heat, while also allowing for angular-momentum exchange between stellar rotation and the orbit (Zahn, 1975, 1977; Hut, 1981; Barker & Ogilvie, 2010; Ogilvie, 2014; Barker, 2020). The resulting torques exchange angular momentum between stellar rotation and orbital motion, driving the system toward a lower-energy configuration at fixed total angular momentum (Darwin, 1880; Hut, 1981; Mazeh, 2008). Over sufficiently long timescales, this evolution tends to circularise the orbit, align the spin and orbital angular-momentum vectors, and synchronise the stellar rotation with the orbital motion (Zahn, 1977; Hut, 1981; Ogilvie, 2014). In eccentric systems,

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however, the spin may instead approach a state of pseudo-synchronisation, in which the stellar rotation is approximately matched to the orbital angular velocity near periastron, before full circularisation is reached (Hut, 1981). These tidal processes are therefore fundamental not only in close stellar binaries, but also in other astrophysical systems affected by tides, including star–planet and planet–satellite systems (Goldreich & Soter, 1966; Murray & Dermott, 1999; Ferraz-Mello, 2013; Ogilvie, 2014).

Stellar deformations also modifies the stellar surface, changes the local effective gravity, and can drive large-scale internal flows and angular-momentum redistribution (Wilson & Devinney, 1971; Moreno et al., 2005; Toledano et al., 2007; Song et al., 2013; Koenigsberger et al., 2021). In eclipsing binaries, the observed light curve depends not only on the radii and temperatures of the components, but also on their distorted shapes, projected surface areas, limb darkening, irradiation/reflection effects, and gravity darkening (Wilson & Devinney, 1971; Hilditch, 2001; Kallrath, 2009; Prša et al., 2016). Gravity darkening describes the dependence of the emergent radiative flux on the local effective gravity: regions with lower effective gravity, such as expanded equatorial or tidally elongated zones, are fainter and appear cooler, whereas compressed high-gravity regions are brighter and appear hotter (von Zeipel, 1924; Lucy, 1967; Claret, 2016). This effect was first formulated for radiative envelopes by von Zeipel (1924), for which the bolometric flux is proportional to the local effective gravity, and was later extended to convective envelopes by Lucy (1967). More recent prescriptions refine the relation between local flux, effective temperature, and effective gravity, particularly for rotating and tidally distorted stars (Claret, 2012; Espinosa Lara & Rieutord, 2011, 2012; Claret, 2016). Since these effects directly modify the morphology of eclipses and ellipsoidal variations, they can significantly affect the stellar parameters inferred from binary light-curve modelling (Wilson & Devinney, 1971; Morris, 1985; Hilditch, 2001; Kallrath, 2009; Prša et al., 2016) and therefore require an accurate treatment of stellar deformation.

In this context, accurate modelling of binary-star deformation is essential for interpreting their observational properties and predicting their subsequent evolution (Kopal, 1959; Wilson & Devinney, 1971; Hilditch, 2001; Prša et al., 2016). Many modelling procedures rely, either explicitly or implicitly, on a perturbative description in which rotation- and tide-induced distortions are treated as small departures from spherical symmetry (Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Smeyers & Willems, 2001; Willems & Aerts, 2002; Willems et al., 2003). This approach is well justified when the deformation remains weak, but its domain of validity is not strictly established, especially for highly distorted, near-contact, or overcontact systems (Fabry et al., 2022). In practice, perturbative or Roche-based descriptions are often applied even to highly distorted systems, including nearly Roche-lobe-filling and contact binaries, where the polar, side, back, and point radii can differ by several tens of percent (Kopal, 1959; Wilson & Devinney, 1971; Wilson, 1994; Kallrath, 2009; Prša et al., 2016; Fabry et al., 2022). In such regimes, the stellar structure is no longer a small perturbation of a spherical reference model, and there is no guarantee that perturbative or simplified Roche-based methods accurately recover the surface shape, gravity field, light-curve morphology, or evolutionary response of the system (Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Fabry et al., 2022). This issue constitutes the

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first major topic of this thesis.

The goal is to assess the validity of the perturbative approach by identifying the regimes in which its underlying assumptions fail to describe the actual three-dimensional deformation and observational properties of close-binary components (Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Fabry et al., 2022). To achieve this, a non-perturbative modelling tool is required to compute self-consistent three-dimensional distorted stellar structures, including surface-shape, gravity-field, and structural effects that cannot be captured by a first-order treatment (Hachisu & Eriguchi, 1984; Hachisu, 1986; Jackson et al., 2005; Lovekin, 2020). Such a tool will make it possible to quantify departures from the perturbative regime and to investigate highly distorted stars in terms of both their observable signatures and the structural effects of deformation (Hachisu, 1986; Jackson et al., 2005; Lovekin, 2020; Fabry et al., 2022). These structural effects may also influence other observables. The final objective of this part of the thesis is therefore to determine how the predicted observational properties of close binaries are modified when a fully non-perturbative treatment of stellar deformation is used and to obtain fully non-perturbative structures of highly deformed binaries.

The main challenge of the first part of this thesis is therefore the development of a new three-dimensional modelling tool. Tidal deformations are intrinsically three-dimensional and, unlike the deformation of an isolated rotating star, they are not generally axisymmetric about a single rotation axis (Kopal, 1959; Hachisu, 1986; Hilditch, 2001; Kallrath, 2009). A close binary component is instead distorted by the combined effects of its own gravity, rotation, and the non-axisymmetric gravitational field of the companion, so that its equilibrium structure and surface geometry must in general be described in three spatial dimensions (Wilson & Devinney, 1971; Hachisu, 1986; Prša et al., 2016). To address this problem, we build on numerical approaches originally developed for highly deformed and rapidly rotating stars in two dimensions (Roxburgh, 2004, 2006; Manchon, 2021; Houdayer & Reese, 2023). Adapted to binaries deformations, the basic principle is to solve iteratively the Poisson equation for the self-gravitational potential generated by the three-dimensional density distribution, while including the external tidal potential of the companion and, the centrifugal potential in the total effective potential (Kopal, 1959). The equilibrium structure is then obtained by iterating between the density distribution and gravitational potential until a self-consistent distorted configuration is reached. This approach allows the stellar structure to adjust self-consistently to the compute deformation and therefore provides a non-perturbative treatment of the problem, including effects such as the redistribution of mass induced by rotation and tides.

The structure of this part of the thesis is as follows. In the first chapter, Chapter 2, I introduce the theoretical framework describing the orbital dynamics and evolutionary processes that govern binary systems, with particular emphasis on tidal interactions and tidal dissipation. This material provides the physical basis for both parts of the thesis. The following chapter, Chapter 3, is devoted to an in-depth study of stellar deformation. In that chapter, I present the development of a new non-perturbative modelling framework that allows for fully three-dimensional calculations of close bi-

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nary stars, including tidal and rotational deformation and the associated internal mass redistribution. I also compare this new approach with Roche models in order to determine under which conditions non-perturbative calculations are required to accurately reproduce the global structural and observational properties of close binary systems. Finally, in [Chapter 4](#), I apply these non-perturbative models to the study of apsidal motion in binary systems and the perturbative approach. Apsidal motion is directly sensitive to the internal mass distribution and to the tidal and rotational deformation of the stellar components, and therefore provides an observational probe of stellar internal structure and tidal distortion in close binaries ([Sterne, 1939](#); [Kopal, 1959](#); [Claret & Gimenez, 1993](#); [Smeyers & Willems, 2001](#); [Claret & Willems, 2002](#); [Rauw et al., 2016](#); [Claret, 2019](#); [Rosu et al., 2020a,b, 2022a,b](#)). It can consequently be used to quantify the limitations of perturbative models when they are applied to strongly deformed systems and to assess the impact of non-perturbative deformations on the modelling of binary interactions.

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The second major stellar response to tidal forcing is the excitation of stellar oscillations, usually referenced as the dynamical tide in the dissipative context ([Zahn, 1975, 1977](#); [Ogilvie, 2014](#)). Unlike the quasi-static deformation associated with the equilibrium tide, this response corresponds to forced oscillations and wave propagation inside the star, driven by the time-dependent component of the companion's tidal potential ([Willems & Aerts, 2002](#); [Willems, 2003](#); [Willems et al., 2003](#); [Willems & Claret, 2003](#)). In eccentric and/or asynchronous binaries, the tidal potential can be decomposed with Fourier series into a sum of harmonic components whose forcing frequencies are linear combinations of the orbital frequency and the stellar rotation frequency ([Smeyers & Willems, 2001](#); [Willems, 2003](#); [Fuller & Lai, 2012](#); [Fuller, 2017](#)). When one of these forcing frequencies lies close to a natural eigenfrequency of the star, the corresponding oscillation mode can be resonantly excited ([Willems, 2003](#); [Willems et al., 2003](#); [Fuller & Lai, 2012](#); [Fuller, 2017](#)). These eigenfrequencies are the same seismic quantities studied in classical asteroseismology, but in this case, the modes are externally forced by the companion rather than being stochastically excited by near-surface convection or self-excited by internal heat-engine mechanisms ([Aerts et al., 2010](#); [Chaplin & Miglio, 2013](#); [Aerts, 2021](#); [Guo, 2021](#)). Consequently, tidally excited oscillations provide a binary-driven form of asteroseismology, in which the orbital forcing selects the excited modes and the stellar interior determines the resonant response ([Fuller & Lai, 2012](#); [Fuller, 2017](#); [Guo, 2021](#)). Such resonances can produce high-amplitude tidally excited oscillations, particularly in eccentric binaries where the tidal forcing contains many orbital harmonics ([Fuller & Lai, 2012](#); [Fuller, 2017](#); [Cheng et al., 2020](#)). These systems therefore provide a form of tidal asteroseismology, in which the observed amplitudes, phases, and orbital-harmonic structure of the oscillations constrain both stellar interiors and binary properties ([Hambleton et al., 2013](#); [Fuller, 2017](#); [Guo, 2020](#); [Cheng et al., 2020](#); [Guo, 2021](#)).

As in the case of equilibrium tides, dynamical tides can exchange energy and angular momentum between stellar rotation and orbital motion ([Zahn, 1975, 1977](#); [Ogilvie,](#)

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2014). In stars with radiative envelopes and no deep convective envelope, typically above $1.4 M_{\odot}$, dissipation of tidally excited internal gravity waves and gravity modes is expected to be the dominant mechanism for driving spin–orbit synchronisation and orbital circularisation (Zahn, 1975, 1977; Witte & Savonije, 1999a,b, 2001; Ogilvie, 2014). In intermediate- and high-mass main-sequence stars, the situation can be more complex because self-excited pulsations may coexist with tidally forced oscillations (Balona, 2018; Guo, 2021; Fuller, 2021). Stars in the approximate mass range 1.4 – $8 M_{\odot}$ include several classes of coherent pulsators, such as γ -Doradus, δ -Scuti, slowly pulsating B, and β -Cephei stars, whose modes may be driven by mechanisms such as convective blocking or the κ -mechanism (Pamyatnykh, 1999; Guzik et al., 2000; Aerts et al., 2010; Aerts, 2021). If the tidal forcing resonates with an unstable self-excited mode, the mode can in principle amplify the tidal response rather than damp it, potentially modifying both the amplitude and the sign of the tidal exchange (Balona, 2018; Fuller, 2021). In this case, thermal energy supplied by the stellar heat engine may be transferred to the orbit, a process sometimes referred to as inverse tides (Fuller, 2021). Thus, in pulsating binaries, ordinary dissipative dynamical tides and inverse tidal effects may coexist and compete, making the net orbital and rotational evolution sensitive to the evolving mode spectrum, including the mode frequencies, growth or damping rates, nonlinear saturation amplitudes, and tidal coupling strengths (Zahn, 1975; Ogilvie, 2014; Fuller, 2017; Balona, 2018; Guo, 2021; Fuller, 2021).

For the modelling of dynamical tides, substantial progress has been made in describing the excitation of individual tidally forced modes, resonant tidal response, and resonance locking in eccentric binaries (Witte & Savonije, 1999a, 2001; Willems, 2003; Fuller & Lai, 2012; Fuller, 2017). In eccentric systems, the tidal potential contains many orbital harmonics, so resonances may occur repeatedly during the main-sequence lifetime rather than as isolated events (Willems, 2003; Fuller & Lai, 2012; Fuller, 2017; Guo, 2021). However, long-term binary-evolution calculations still often rely on simplified tidal prescriptions, typically based on constant-lag or weak-friction formalisms, or reduced descriptions of equilibrium and dynamical tides (Hut, 1981; Zahn, 1977; Hurley et al., 2002; Ogilvie, 2014; Fragos et al., 2023). As a result, the detailed frequency dependence of the tidal response and the evolving spectrum of resonantly excited modes are usually not followed explicitly in secular binary-evolution calculations. Explicit mode-by-mode treatments, including the excitation and damping of individual oscillation modes by multiple tidal-forcing harmonics, have been carried out only for very restricted periods of time, or with a focus on specific phenomena such as resonance locking (Willems, 2003; Willems et al., 2003; Fuller & Lai, 2012; Fuller, 2017; Sun et al., 2026). Therefore, the self-consistent time-dependent coupling between stellar evolution, orbital evolution, multiple forcing harmonics, resonant mode excitation, mode damping, and changes in the stellar oscillation spectrum remains incompletely explored (Witte & Savonije, 1999a, 2001; Ogilvie, 2014; Fuller, 2017; Kołaczek-Szymański & Różański, 2023). This is particularly important in intermediate-mass stars, where radiative envelopes favour dynamical tides and where damped tidally forced oscillations pulsations may coexist with inverse tides (Zahn, 1975, 1977; Aerts et al., 2010; Balona, 2018; Aerts, 2021; Guo, 2021; Fuller, 2021).

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The central question of this part is therefore how dynamical tides drive the secular evolution of binary systems in regimes where many tidal forcing frequencies may interact with the evolving stellar eigenmode spectrum (Witte & Savonije, 1999a, 2001; Fuller, 2017; Kołaczek-Szymański & Różański, 2023). The objective of this part of the thesis is therefore to develop a new tool capable of integrating the cumulative effect of resonant dynamical tides over long evolutionary timescales in intermediate-mass binary systems (Witte & Savonije, 1999a, 2001; Savonije, 2008; Fuller, 2017). This tool should follow the simultaneous contribution of all relevant tidal forcing components, rather than restricting the calculation to a single resonance or to a prescribed effective dissipation law (Ogilvie, 2014; Fuller, 2017; Guo, 2021). It will make it possible to quantify the limitations of commonly used tidal prescriptions, determine how multi-frequency integrations differ from single-resonance treatments, and assess the cumulative impact of resonant dynamical tides on spin–orbit synchronisation, circularisation, eccentricity pumping, and angular-momentum redistribution (Zahn, 1977; Hurley et al., 2002; Mirouh et al., 2023; Fuller, 2021). On a broader scale, the same framework will provide a basis for predicting the population-level evolution of intermediate-mass binaries in the regime where resonant dynamical tides dominates their orbital and rotational evolution (Balona, 2018; Guo, 2021; Fuller, 2021).

Two main approaches are commonly used to study dynamical tides. The first, usually referred to as modal decomposition, expands the tidal perturbation of the secondary onto the free oscillation eigenmodes of the primary (Zahn, 1975; Unno et al., 1989; Willems, 2003; Fuller & Lai, 2012; Fuller, 2017). This approach provides a clear physical interpretation of resonant tidal excitation, because each contribution can be associated with a particular stellar eigenmode. However, the situation becomes more subtle when the objective is to follow the secular orbital evolution, because the non-adiabatic effects, governing the damping or driving of the oscillations and hence the sign and amplitude of the associated tidal torques (Willems et al., 2003; Savonije, 2008; Ogilvie, 2014; Fuller, 2021), cannot be obtained directly with the assumptions of the modal decomposition and requires supplementary assumptions. The second approach consists of solving the forced stellar-oscillation equations directly for each tidal forcing frequency. In this formulation, the perturbing potential of the companion is introduced as an explicit inhomogeneous forcing term in the linear oscillation equations, and the stellar response is computed using the same non-adiabatic formalism as in classical stellar-pulsation calculations (Willems et al., 2003; Valsecchi et al., 2013; Townsend & Teitler, 2013; Sun et al., 2025). This approach is adopted in this thesis because it naturally includes the non-adiabatic driving and damping mechanisms implemented in the oscillation code without supplementary assumption, and because it can be easily coupled to an orbital integrator to study the long-term impact of dynamical tides on binary evolution (Witte & Savonije, 1999a, 2001; Ogilvie, 2014; Fuller, 2017; Sun et al., 2025, 2026).

Since this part of the thesis is closely connected to stellar oscillations, the first chapter, [Chapter 5](#), introduces the general formalism of non-adiabatic asteroseismology used to describe stellar pulsations. Particular attention is given to intermediate-mass main-sequence stars and to the physical mechanisms responsible for mode excitation and damping, such as κ -mechanism and convective-blocking (Dziembowski & Pamy-

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atnykh, 1993; Pamyatnykh, 1999; Guzik et al., 2000; Dupret et al., 2005; Aerts et al., 2010). In Chapter 6, I then apply this framework to develop a stellar-oscillation code capable of computing the response of a star to the tidal forcing induced by a companion (Willems et al., 2003; Townsend & Teitler, 2013; Sun et al., 2025). By including non-adiabatic effects, this approach makes it possible to determine whether tidally excited oscillations are damped or driven, and therefore to quantify their contribution to the exchange of energy and angular momentum between the star and the orbit (Witte & Savonije, 1999a,b, 2001; Willems, 2003; Willems et al., 2003; Valsecchi et al., 2013; Townsend & Teitler, 2013; Sun et al., 2023, 2025). Finally, in Chapter 7, I develop the numerical framework and orbital integrator required to follow the effect of dynamical tides throughout the main-sequence evolution of intermediate-mass stars. This framework couples the new forced-oscillation calculations to the orbital evolution of the binary system, allowing the cumulative effect of many simultaneous tidal forcing frequencies to be followed over long timescales (Witte & Savonije, 1999a, 2001; Fuller, 2017). This chapter therefore explores in detail the orbital evolution of intermediate-mass binaries when the frequency-dependent non-adiabatic response to dynamical tides is included, rather than relying on a single-resonance treatment or on simplified tidal prescriptions.

PART I

MODELLING OF BINARIES DEFORMATIONS

Binary-star systems provide some of the most interesting laboratories in stellar astrophysics. Observational studies of massive O-type stars have shown that a large fraction are members of close binaries or higher-order multiple systems already at young ages, while broader multiplicity surveys indicate that the companion fraction increases strongly with primary mass (Chini et al., 2012; Sana et al., 2012; Duchêne & Kraus, 2013; Sana et al., 2014; Moe & Di Stefano, 2017). For massive stars in particular, binary interaction is expected to affect the evolution of a substantial majority of systems: more than $\sim 70\%$ of O-type stars are predicted to exchange mass with a companion, with a significant fraction eventually merging (Sana et al., 2012; de Mink et al., 2014; Marchant & Bodensteiner, 2024). Processes such as stable and unstable mass transfer, common-envelope evolution, and stellar mergers can profoundly alter the internal structure, rotation, surface composition, and observable properties of the stellar components (Wellstein et al., 2001; Langer, 2012; Ivanova et al., 2013; de Mink et al., 2013; de Mink et al., 2014; De Marco & Izzard, 2017; Marchant & Bodensteiner, 2024). As a result, the evolutionary pathways of close binaries may differ markedly from those of isolated stars (Davis et al., 2013; Siess et al., 2013; De Marco & Izzard, 2017; Marchant & Bodensteiner, 2024). In sufficiently compact systems, the combined action of gravity, rotation, and tides induces strong perturbations of the stellar interiors and significant departures from spherical symmetry in the external geometry of the stars (Kopal, 1959; Wilson & Devinney, 1971; Zahn, 1977; Hut, 1981; Hilditch, 2001; Fabry et al., 2022). These tidal and rotational distortions are central to the secular spin–orbit evolution of close binaries. Tidal friction exchanges angular momentum between the stellar spins and the orbit, driving the system toward spin–orbit synchronisation, alignment, and circularisation of initially eccentric orbits (Zahn, 1977; Hut, 1981; Zahn, 2008; Ogilvie, 2014).

In the simplest approximation, an isolated star in hydrostatic equilibrium remains nearly spherical because the inward pull of self-gravity is balanced by the outward pressure-gradient force. In a close binary, however, this symmetry is broken: the potential experienced by each component is no longer determined solely by its own gravity, but is modified by the tidal field of the companion and, in the corotating frame, by the centrifugal potential (Kopal, 1959; Wilson & Devinney, 1971; Hilditch, 2001; Kallrath, 2009). The resulting effective potential defines the Roche equipotential surfaces and the associated Roche lobe of each star, so that the equilibrium structure departs from sphericity (Paczynski, 1971; Eggleton, 1983; Hilditch, 2001; Prša et al., 2016). Figure 1.1 illustrate the characteristic surface deformation of a star subjected to strong tides in a close binary system. These distortions become stronger as the orbital separation decreases (Hut, 1981; Eggleton, 1983) and grow particularly important as one of the stars approaches Roche-lobe filling (Eggleton, 1983; de Mink et al., 2009; Siess et al., 2013). Their amplitude depends on the mass ratio and orbital separation through the Roche geometry (Eggleton, 1983), on eccentricity and spin-orbit configuration (Hut, 1981; Moreno et al., 2011), and on the internal density stratification that governs the stellar response to external forcing (Zahn, 1975; Claret & Gimenez, 1993; Feiden & Dotter, 2013). Even in detached systems, where both components still underfill their Roche lobes, quasi-static tides can already modify the surface layers and the global stellar structure (Song et al., 2009; de Mink et al., 2009; Siess et al., 2013). Tides and rotation can drive large-scale flows, tidal shear, and meridional circulations

(Zahn, 1977, 1992; Moreno et al., 2011; Ogilvie, 2014), which redistribute angular momentum and interact with mixing processes and shear instabilities (Pinsonneault, 1997; de Mink et al., 2009, 2013). These effects progressively couple the stellar spins to the orbit (Zahn, 1977; Hut, 1981; Ogilvie, 2014) and help set the background on which later mass transfer and long-term orbital evolution take place (de Mink et al., 2009; Siess et al., 2013; Moe & Di Stefano, 2017).

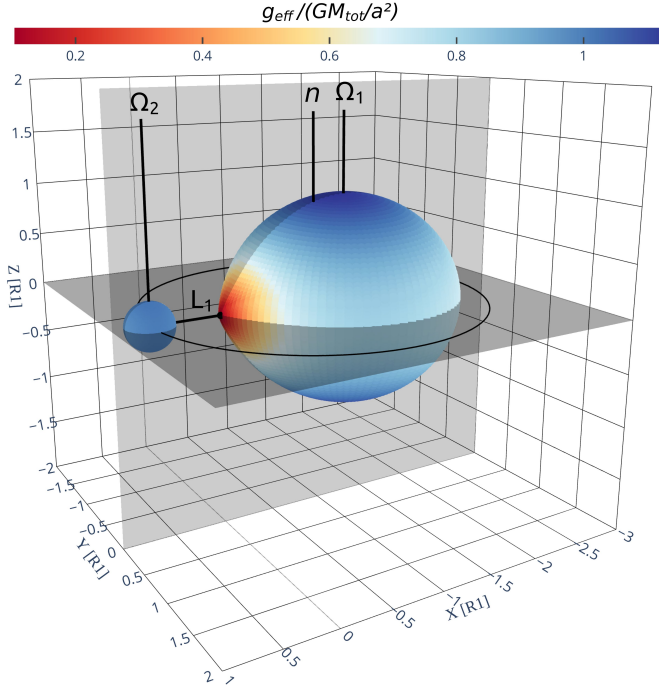


Figure 1.1: Illustration of a binary system configuration with non-synchronised rotations. The system is composed of $1 M_{\odot}$ MS primary star and a $0.2 M_{\odot}$ MS companion separated by $r = 2R_1$. The two grey planes are the planes of symmetry, each rotation axis is marked with a black line. The orbital rotation is originating from the system centre of mass, located inside the primary. The black curve is the secondary orbital path, a black axis is joining the centre of each star with and emphasis on the Lagrangian point L_1 . Finally, the colour code corresponds to the surface effective gravity, the local flux emitted by a star being a function of this quantity.

When one component expands until it fills its Roche lobe, Roche-lobe overflow (RLOF) begins and matter is transferred through the inner Lagrange point L_1 (Eggleton, 1983; Podsiadlowski et al., 2002). This marks a major transition in the evolution of the system, because the ensuing mass exchange modifies both stellar structures and the orbital configuration itself (Packet, 1981; de Mink et al., 2007; Podsiadlowski et al., 2002; Marchant & Bodensteiner, 2024). The donor generally loses a large fraction of its envelope once it leaves the main sequence or evolves through later giant phases

(Podsiadlowski et al., 2002; Langer, 2012; Marchant & Bodensteiner, 2024), while the accretor can be spun up efficiently by the angular momentum of the transferred material and may approach critical rotation after accreting only a relatively low amount of mass (Packet, 1981; de Mink et al., 2013). If the transferred stream carries sufficiently high specific angular momentum, accretion proceeds through a disc around the gainer rather than by direct impact (Packet, 1981; Van Rensbergen & De Greve, 2016); the structure, observability, and lifetime of such discs depend on the binary geometry, tidal torques, and dissipative processes within the flow (Deschamps et al., 2015; Van Rensbergen & De Greve, 2016). The redistribution of mass and orbital angular momentum during RLOF may either widen or shrink the orbit, depending mainly on the mass ratio and on the degree to which mass and angular momentum are conserved (de Mink et al., 2007; Podsiadlowski et al., 2002; Deschamps et al., 2015; Marchant & Bodensteiner, 2024). Roche-lobe overflow, together with wind-fed accretion in wider systems, therefore shapes the diverse phenomenology of interacting binaries, including Algol-type systems (Deschamps et al., 2015; Van Rensbergen & De Greve, 2016), cataclysmic variables (Knigge et al., 2011), and X-ray binaries (Podsiadlowski et al., 2002), whose present-day properties keep the imprint of their earlier interaction histories. Long before these strongly interacting phases are reached, however, the components are already deformed by mutual gravity, tides, and rotation (Hut, 1981; Song et al., 2009; Siess et al., 2013); these distortions help precondition the system for the subsequent onset of mass transfer and angular-momentum exchange.

The influence of these deformations extends well beyond the morphology of the stellar surface and affects the dynamics of the orbit itself. In eccentric binaries, departures from spherical symmetry modify the stellar gravitational potentials and perturb the Keplerian two-body motion, so that the line of apsides undergoes a secular precession (Sterne, 1939; Kopal, 1959; Giménez, 2007). This phenomenon, known as apsidal motion, arises because the orbit is no longer governed solely by the inverse-square force between two point masses, but also by higher-order multipole terms in the gravitational potential, with the leading classical contribution associated with the quadrupolar distortion produced by tides and rotation (Claret & Giménez, 2010; Sirotkin & Kim, 2009; Claret, 2019). When averaged over many orbital cycles, these corrections generate a measurable advance of the line of apsides (Sterne, 1939; Kopal, 1959; Claret & Giménez, 2010). In sufficiently compact systems, general relativity contributes an additional term to the secular apsidal advance, so that the observed precession reflects the combined action of classical tidal and rotational distortions and relativistic effects (Giménez & Margrave, 1985; Baroch et al., 2021). Because the classical apsidal-motion rate depends on how centrally concentrated the stellar mass is, it is commonly expressed in terms of internal-structure or apsidal-motion constants, most commonly the quadrupolar apsidal constant k_2 (Kopal, 1959; Claret & Giménez, 2010; Claret, 2019). This constant quantifies the response of the stellar gravitational potential to tidal and rotational perturbations and therefore provides a direct link between apsidal motion and the internal mass distribution of the stars (Claret & Willems, 2003; Claret, 2019). Accurate apsidal-motion measurements in well-characterised eclipsing binaries therefore provide stringent tests of stellar-structure and stellar-evolution models, because the observed precession rate depends on the internal mass concentration of the stellar components through the apsidal-motion constant (Sterne, 1939; Kopal,

1959; Claret & Giménez, 2010; Claret, 2019). Such tests have been applied across a broad mass range, from massive O- and early B-type binaries to lower-mass eclipsing systems (Claret & Giménez, 2010; Rosu et al., 2020a,b; Claret et al., 2021; Rosu et al., 2022a,b). They continue to offer valuable constraints when combined with modern high-precision photometry from space missions such as *Kepler* and *TESS*, which provide precise eclipse timings and light-curve morphologies (Southworth, 2021; Baroch et al., 2021; Claret et al., 2021; Zasche et al., 2023).

Observationally, apsidal motion manifests itself through slow but measurable shifts in the times of primary and secondary eclipses (Wolf et al., 2002; Wilson & Van Hamme, 2014; Hong et al., 2015; Kim et al., 2018; Zasche et al., 2020; Baroch et al., 2021). Because the precession of the line of apsides affects the two eclipse branches differently, these timing measurements provide a direct estimate of the apsidal-motion rate (Wolf et al., 2002; Wilson & Van Hamme, 2014). The study of apsidal motion has therefore remained a classical tool in binary-star astrophysics, since the inferred internal-structure constants can be confronted with stellar-evolution calculations that include rotation, mass loss, convective overshooting, and chemical-composition gradients (Claret & Giménez, 1993; Claret & Giménez, 2010; Claret et al., 2021; Rosu et al., 2020a,b, 2022a,b, 2023). In this sense, apsidal motion constrains not only the present-day internal structure of the stars, but also aspects of their evolutionary history, especially in massive binaries whose internal stratification may have been modified by tides, rotation, mass exchange, or prior binary interaction (Rauw et al., 2016; Rosu et al., 2020a,b, 2022a,b; Marchant & Bodensteiner, 2024). For systems that are approaching mass transfer, however, the interpretation becomes more delicate: redistribution of mass and angular momentum can modify both the stellar quadrupole moments and the orbital elements, and therefore the apsidal-motion rate itself (Hut, 1981; Mayer et al., 2016; Tauris & van den Heuvel, 2023). Reliable constraints from apsidal-motion measurements therefore require a detailed theoretical and observational treatment of both tidal distortion and rotational flattening (Claret & Willems, 2003; Claret & Giménez, 2010; Claret, 2019).

The theoretical modelling of deformations and apsidal motion in binary stars is most often carried out within two approximate, but widely used, frameworks: Roche-geometry models and perturbative treatments of stellar structure (Kopal, 1959; Wilson & Devinney, 1971; Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Hilditch, 2001). In Roche-type descriptions, the stellar surfaces are identified with equipotential surfaces generated by the gravitational fields of point-mass components and the centrifugal potential (Eggleton, 1983; Kallrath, 2009; Prša et al., 2016). Perturbative treatments, by contrast, describe the effects of tides and rotation as small deviations from an underlying spherically symmetric or spherically averaged stellar model (Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Smeyers & Willems, 2001; Claret & Giménez, 2010). For detached systems that are only weakly or moderately distorted, these assumptions are generally adequate and allow detailed analyses of the stellar shape, effective gravity, apsidal motion, and some aspects of the internal response to tides and rotation (Claret & Willems, 2003; Sirotkin & Kim, 2009; Claret, 2019). In more strongly distorted configurations, however, the redistribution of mass produced by tidal and rotational forces is expected to no longer be modelled with the same accu-

racy. As a result, the simplifying assumptions built into these classical formalisms could lead to significant discrepancies, especially as the stars approach Roche-lobe filling or depart substantially from spherical symmetry. The first part of this thesis is therefore devoted to assessing the domain of validity of Roche and perturbative approaches in close binary systems, and to identifying the regimes in which more complete, non-perturbative treatments become necessary to model stellar deformation.

In the first chapter of this part, [Chapter 2](#), I introduce the theoretical framework for the orbital dynamics and evolutionary processes that govern binary systems, with particular emphasis on tidal interactions and tidal dissipation. In the second chapter, [Chapter 3](#), I present the development of a new non-perturbative modelling framework that enables fully three-dimensional calculations of close binary stars, including their tidal and rotational deformations as well as the effects of internal mass redistribution. In that chapter, I also compare this new approach with the classical Roche model and assess under which conditions non-perturbative models are required to reproduce the global properties of binary systems accurately. Finally, in the last chapter, [Chapter 4](#), I apply these non-perturbative models to the study of apsidal motion in binary systems. In particular, I develop the formalism required to compute apsidal-motion rates within the non-perturbative framework and quantify the discrepancies between perturbative models and the new approach. This makes it possible to identify the regimes in which non-perturbative modelling becomes necessary for interpreting the observed apsidal motion of binary stars. I then apply this framework to observed systems by modelling them under the constraints provided by their measured apsidal-motion rates.

CHAPTER 2

CLASSICAL DYNAMICS AND ORBITAL MECHANICS

Orbital mechanics provides the fundamental framework for describing the motion of gravitating bodies, from planets and satellites to stars in binary and multiple systems. In the absence of external perturbations, the relative motion of two point masses follows a Keplerian orbit whose geometry is fully characterised by a small set of orbital elements, while its evolution is constrained by the conservation of energy and angular momentum. This idealised problem forms the foundation on which more realistic descriptions of astrophysical systems are built. Since the core of this thesis relies on the deformations and orbital evolution of binaries, and on describing dissipative processes as perturbations of an otherwise Keplerian orbit, it is essential to establish the orbital-mechanics foundations that will later be perturbed and modified in multiple ways. In [Section 2.1](#), I introduce Kepler’s laws and the two-body problem, which will be used throughout this thesis.

In reality, the gravitational potential is the central quantity that underlies both orbital motion and tidal interactions in binary systems. In the ideal Keplerian two-body problem, the stars are treated as point masses and the motion is governed by a purely central Newtonian potential, leading to the conservation of energy and angular momentum. In real binaries, however, the stars are extended and deformable bodies, so the potential is modified by tidal, rotational, and centrifugal effects. The resulting effective potential determines not only the orbital dynamics, but also the stellar shapes, Roche lobes, and the perturbations responsible for secular effects such as apsidal motion, synchronisation, and mass transfer. A proper description of the gravitational potential is therefore necessary. In [Section 2.2](#), I provide a general introduction to the gravitational potential and derive its general expression by solving the Poisson equation, which will be used throughout the thesis.

One of the dominant effects perturbing the Keplerian description of close binary systems is tides. Tides arise because the gravitational field produced by one component of a binary is not spatially uniform across its companion. This differential forcing

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deforms the stars, modifies their external gravitational field, and couples their spins to the orbit (Zahn, 1977; Ogilvie, 2014). In close systems, tides are therefore a key ingredient of the long-term evolution, since they can drive synchronisation, circularisation, and apsidal motion. In Section 2.3, I provide a general introduction to the physics of tides and tidal dissipation. Then, in Section 2.4, I introduce the Gauss planetary equations, which link non-Keplerian perturbations to the secular evolution of the orbital elements, and which will be used in both parts of this thesis.

2.1 NEWTONIAN/LAGRANGIAN TREATMENT OF THE TWO-BODY PROBLEM

In classical mechanics, a binary star system can be described as two masses orbiting their common centre of mass under their mutual gravitational attraction. According to Newton's universal law of gravitation, the force between two bodies of masses M_1 and M_2 , separated by a distance r , is

$$\mathbf{F} = G \frac{M_1 M_2}{r^2} \mathbf{e}_r, \quad (2.1)$$

where bold symbols denote vectors, $\mathbf{e}_r$ is the unit vector pointing along the line joining the two bodies (i.e. in the direction of the separation vector $\mathbf{r}$), and G is the universal gravitational constant. We consider a primary star of mass M_1 located at position vector $\mathbf{r}_1$ in an inertial frame, and a secondary star of mass M_2 located at position vector $\mathbf{r}_2$ in the same frame. The relative position vector is

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \quad (2.2)$$

with $r = \|\mathbf{r}\|$ and $\mathbf{e}_r = \mathbf{r}/r$. The gravitational forces acting on the two bodies are then given by:

$$\mathbf{F}_1 = m_1 \ddot{\mathbf{r}}_1 = G \frac{M_1 M_2}{r^3} \mathbf{r}, \quad (2.3)$$

and,

$$\mathbf{F}_2 = m_2 \ddot{\mathbf{r}}_2 = -G \frac{M_1 M_2}{r^3} \mathbf{r}, \quad (2.4)$$

The relative acceleration of the two bodies is given by

$$\ddot{\mathbf{r}} = -\frac{G(M_1 + M_2)}{r^3} \mathbf{r}. \quad (2.5)$$

The vector product $\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}$ implies that the acceleration is always parallel to $\mathbf{r}$ (i.e. the force is central). This relation can be integrated to obtain the fundamental property of orbital dynamics

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \quad (2.6)$$

where $\mathbf{h}$ is a constant of motion called the specific angular momentum. Equation 2.6 is the angular-momentum integral; physically, it implies that the motion is confined to a fixed orbital plane perpendicular to $\mathbf{h}$. By writing the motion in polar coordinates centred on the primary, we can express the position, velocity, and acceleration as

$$\mathbf{r} = r \mathbf{e}_r, \quad (2.7)$$

2.1. NEWTONIAN/LAGRANGIAN TREATMENT OF THE TWO-BODY PROBLEM

$$\dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r\dot{\theta} \mathbf{e}_\theta, \quad (2.8)$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + \left(\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \right) \mathbf{e}_\theta, \quad (2.9)$$

where θ is the polar angle in the orbital plane, $\mathbf{e}_\theta$ is the associated transverse unit vector, and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are orthonormal. With these equations, the specific angular momentum can be written as

$$\mathbf{h} = r^2 \dot{\theta} \mathbf{e}_z, \quad (2.10)$$

and its magnitude,

$$h = r^2 \dot{\theta}, \quad (2.11)$$

is a constant of the motion. Consequently, the angular component of the equation of motion (e.g. [Equation 2.9](#)) vanishes. The area element dA swept by the radius vector in a time interval dt is

$$dA = \frac{1}{2} r^2 d\theta, \quad (2.12)$$

so that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{const.} \quad (2.13)$$

This is Kepler's second law: equal areas are swept out in equal times. By equating the gravitational acceleration with the polar-coordinate expression of the acceleration, we obtain the following differential equation:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G(M_1 + M_2)}{r^2}, \quad (2.14)$$

and we wish to solve this equation to obtain the separation as a function of θ . To do so, we introduce the auxiliary variable $u = 1/r$. Differentiating $r = 1/u$ with respect to time, applying the chain rule, and using the definition of the specific angular momentum $h = r^2 \dot{\theta}$, one obtains

$$\ddot{r} = -h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u \right). \quad (2.15)$$

Substituting into [Equation 2.14](#) yields the standard Binet equation,

$$\frac{d^2 u}{d\theta^2} + u = \frac{G(M_1 + M_2)}{h^2}. \quad (2.16)$$

The general solution is

$$u(\theta) = \frac{G(M_1 + M_2)}{h^2} [1 + e \cos(\theta - \omega)], \quad (2.17)$$

where e and ω are constants of the orbit (specified below). The orbital separation can therefore be written as

$$r(\theta) = \frac{\ell}{1 + e \cos(\theta - \omega)}, \quad (2.18)$$

where

$$\ell = \frac{h^2}{G(M_1 + M_2)}. \quad (2.19)$$

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This is the polar equation of a conic section; for bound motion ($0 \leq e < 1$) it describes an ellipse. Thus, the relative orbit of two point masses under their mutual gravity is an ellipse, as stated by Kepler's first law. Although e , ω , and ℓ were introduced as integration constants, they have clear geometrical meanings: e is the orbital eccentricity, ω is the argument of periastron, and ℓ is the semi-latus rectum. The latter is related to the semi-major axis a through

$$\ell = a(1 - e^2), \quad (2.20)$$

so that

$$r = \frac{a(1 - e^2)}{1 + e \cos(\theta - \omega)}. \quad (2.21)$$

We now derive Kepler's third law, i.e. the relation between the semi-major axis a , the orbital period P_{orb} , and the total mass of the system. Integrating the areal velocity over one full orbital period gives the total area swept:

$$A = \int_0^{P_{\text{orb}}} \frac{dA}{dt} dt = \frac{1}{2} h P_{\text{orb}}. \quad (2.22)$$

The area of an ellipse is

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}, \quad (2.23)$$

where $b = a \sqrt{1 - e^2}$ is the semi-minor axis. Equating the two expressions for A and squaring yields

$$\pi^2 a^4 (1 - e^2) = \frac{1}{4} h^2 P_{\text{orb}}^2. \quad (2.24)$$

Using $\ell = h^2/[G(M_1 + M_2)]$ together with $\ell = a(1 - e^2)$ gives

$$h^2 = G(M_1 + M_2) a(1 - e^2). \quad (2.25)$$

Substituting this into the previous expression and cancelling the common factor $(1 - e^2)$ leads to

$$\frac{a^3}{P_{\text{orb}}^2} = \frac{G(M_1 + M_2)}{4\pi^2}, \quad (2.26)$$

which is Kepler's third law for a two-body system. In [Figure 2.1](#), I illustrate the typical orbital trajectory of a binary system with an eccentricity of $e = 0.2$, showing the different orbital elements that were introduced in this section.

2.2 GRAVITATIONAL POTENTIAL

The gravitational force is a conservative force: the work required to move a test particle under gravity from a point a to a point b depends only on the endpoints and not on the path taken. Equivalently, the work done by gravity around any closed curve vanishes,

$$\oint \mathbf{F} \cdot d\boldsymbol{\ell} = 0, \quad (2.27)$$

2.2. GRAVITATIONAL POTENTIAL

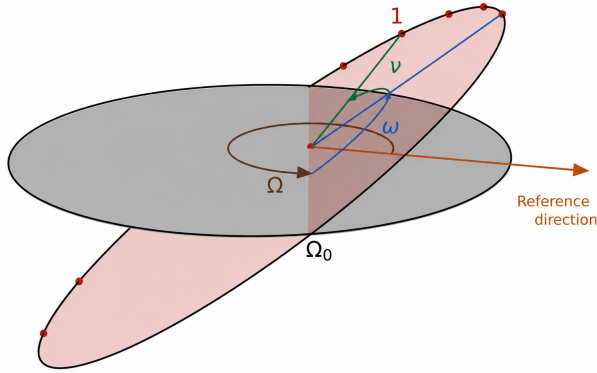


Figure 2.1: Illustration of the orbital elements of an orbit in a binary system. The orbital plane is in red, the grey plane is a the reference plane (i.e. the plane of the sky), i is the inclination of the system compared to the reference plane, Ω is the longitude of the ascending node, Ω_0 is the ascending node, ω is the argument of the periastron, and ν is the true anomaly (noted θ in this section).

which implies that the force field can be written as the gradient of a scalar potential. We therefore introduce the gravitational potential $\Psi(\mathbf{r})$ (per unit mass) such that

$$\mathbf{g}(\mathbf{r}) \equiv \frac{\mathbf{F}}{m} = -\nabla\Psi(\mathbf{r}), \quad (2.28)$$

where $\mathbf{g}$ and Ψ are respectively the gravitational acceleration and potential. The potential is defined up to an arbitrary additive constant; a common convention in astrophysics is to choose $\Psi \rightarrow 0$ as $r \rightarrow \infty$. The operator ∇ denotes the gradient (nabla) operator. In Cartesian coordinates it reads

$$\nabla = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z, \quad (2.29)$$

while in standard spherical coordinates (r, θ, ϕ) it is given by

$$\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{e}_\phi. \quad (2.30)$$

From Newton's law of universal gravitation, the gravitational acceleration $\mathbf{g}$ experienced at position $\mathbf{r}$ due to a point mass M' located at $\mathbf{r}'$ is

$$\mathbf{g}(\mathbf{r}) = GM' \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}. \quad (2.31)$$

The corresponding gravitational potential (per unit mass) follows from $\mathbf{g} = -\nabla\Psi$ and is

$$\Psi(\mathbf{r}) = -\frac{GM'}{|\mathbf{r}' - \mathbf{r}|}, \quad (2.32)$$

up to an arbitrary additive constant. Gravity obeys the superposition principle: the total field produced by a collection of point masses is the sum of the individual contributions. Hence, the gravitational potential produced at $\mathbf{r}$ by N point masses M'_i located

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at $\mathbf{r}'_i$ is

$$\Psi(\mathbf{r}) = -G \sum_{i=1}^N \frac{M'_i}{|\mathbf{r}'_i - \mathbf{r}|}. \quad (2.33)$$

For an extended body, the mass distribution is described by a continuous density field $\rho(\mathbf{r}')$. The mass contained in an infinitesimal volume element $d^3\mathbf{r}'$ around $\mathbf{r}'$ is $\rho(\mathbf{r}') d^3\mathbf{r}'$. Taking the continuum limit of the discrete sum leads to the integral representation

$$\Psi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}', \quad (2.34)$$

which is the general expression for the Newtonian gravitational potential $\Psi(\mathbf{r})$ generated by an extended body with density $\rho(\mathbf{r}')$, evaluated at the field point $\mathbf{r}$. The integral in Equation 2.34 is taken over the region of space where $\rho(\mathbf{r}') \neq 0$.

2.2.1 SPECTRAL EXPANSION OF THE GRAVITATIONAL POTENTIAL.

For convenience, in the following we adopt the standard spherical polar coordinate system (r, θ, ϕ) centered on the star studied where θ is the colatitude and ϕ is the longitude with respect to axe linking center of the two stars, which is related to Cartesian coordinates by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{bmatrix}. \quad (2.35)$$

Let us consider the general case of a non-spatially symmetric density distribution $\rho(\mathbf{r}) = \rho(r, \theta, \phi)$. According to Equation 2.34, the gravitational potential $\Psi(r, \theta, \phi)$ generated by such a mass distribution is

$$\Psi(r, \theta, \phi) = -G \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\rho(r', \theta', \phi')}{|\mathbf{r}' - \mathbf{r}|} r'^2 \sin \theta' d\phi' d\theta' dr'. \quad (2.36)$$

The denominator in this integral can be written as

$$|\mathbf{r}' - \mathbf{r}|^{-1} = (r^2 + r'^2 - 2\mathbf{r}' \cdot \mathbf{r})^{-1/2}, \quad (2.37)$$

where

$$\begin{aligned} 2\mathbf{r}' \cdot \mathbf{r} &= 2(xx' + yy' + zz') \\ &= 2rr'(\sin \theta \cos \phi \sin \theta' \cos \phi' + \sin \theta \sin \phi \sin \theta' \sin \phi' + \cos \theta \cos \theta') \\ &= 2rr'(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')) \equiv 2rr' \cos \varphi, \end{aligned} \quad (2.38)$$

and φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$. Exploiting the definition of the trigonometric functions in the complex space we can write

$$\cos \phi = \frac{e^{i\phi} + e^{-i\phi}}{2}, \quad (2.39)$$

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and,

$$\sin \phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}, \quad (2.40)$$

that we can insert in Equation 2.38 to obtain a complex representation of $\cos \varphi$:

$$\cos \varphi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \frac{e^{i\phi} e^{-i\phi'} + e^{-i\phi} e^{i\phi'}}{2}. \quad (2.41)$$

Then, $|\mathbf{r}' - \mathbf{r}|^{-1}$ can be rewritten as

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{(r^2 + r'^2 - 2rr' \cos \varphi)^{1/2}} = \frac{1}{r} \left[1 - 2 \left(\frac{r'}{r} \right) \cos \varphi + \left(\frac{r'}{r} \right)^2 \right]^{-1/2}. \quad (2.42)$$

Assuming $r > r'$, the previous expression can be expanded as a convergent power series:

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r} \left[1 + \left(\frac{r'}{r} \right) \cos \varphi + \frac{1}{2} \left(\frac{r'}{r} \right)^2 (3 \cos^2 \varphi - 1) + \mathcal{O} \left(\left(\frac{r'}{r} \right)^3 \right) \right] = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r} \right)^{\ell} P_{\ell}(\cos \varphi), \quad (2.43)$$

where P_{ℓ} denotes the Legendre polynomial of spherical degree ℓ . The Legendre polynomials are directly connected to spherical harmonics through the spherical-harmonic addition theorem:

$$P_{\ell}(\cos \varphi) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell}^m(\theta, \phi) Y_{\ell}^{m*}(\theta', \phi').$$

With the Condon–Shortley phase convention, the spherical harmonics are defined by

$$Y_{\ell}^m(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell}^m(\cos \theta) e^{im\phi}, \quad (2.44)$$

where P_{ℓ}^m are the associated Legendre polynomials, given by

$$P_{\ell}^m(x) = \frac{(-1)^m}{2^{\ell} \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^{\ell}. \quad (2.45)$$

In the present context, the addition theorem is especially useful because it rewrites the angular dependence on the relative angle φ in terms of functions of the two directions (θ, ϕ) and (θ', ϕ') separately. This makes it possible to express the gravitational potential in a complete angular basis adapted to the stellar coordinate system. As will also be seen throughout this thesis, spherical harmonics are now used in a wide range of applications well beyond their original role in potential theory. The lowest-order

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spherical-harmonic terms are listed below.

$$Y_0^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{1}{\pi}} \quad (2.46)$$

$$Y_1^{-1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi} \quad (2.47)$$

$$Y_1^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta \quad (2.48)$$

$$Y_1^1(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi} \quad (2.49)$$

$$Y_2^{-2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{-2i\phi} \quad (2.50)$$

$$Y_2^{-1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{-i\phi} \quad (2.51)$$

$$Y_2^0(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{2\pi}} (3 \cos^2 \theta - 1) \quad (2.52)$$

$$Y_2^1(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\phi} \quad (2.53)$$

$$Y_2^2(\theta, \phi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} \quad (2.54)$$

To illustrate the validity of the spherical-harmonic addition theorem, the first terms of the developments of $|\mathbf{r}' - \mathbf{r}|^{-1}$, one we may write

$$\cos \varphi = \cos \theta \cos \theta' + \frac{1}{2} \sin \theta \sin \theta' e^{i(\phi' - \phi)} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\phi - \phi')}. \quad (2.55)$$

and we obtain

$$\cos \varphi = \frac{4\pi}{3} \sum_{m=-1}^1 Y_1^m(\theta, \phi) Y_1^{m*}(\theta', \phi'). \quad (2.56)$$

Similarly, expanding $(\cos \varphi)^2$ gives

$$\cos^2 \varphi = \cos^2 \theta \cos^2 \theta' + 2 \cos \theta \cos \theta' \sin \theta \sin \theta' \cos(\phi - \phi') + \sin^2 \theta \sin^2 \theta' \cos^2(\phi - \phi') \quad (2.57)$$

$$= \cos^2 \theta \cos^2 \theta' + \sin^2 \theta \sin^2 \theta' \left[\frac{1}{2} + \frac{1}{4} e^{2i(\phi - \phi')} + \frac{1}{4} e^{-2i(\phi - \phi')} \right] \\ + \cos \theta \cos \theta' \sin \theta \sin \theta' \left[e^{i(\phi - \phi')} + e^{-i(\phi - \phi')} \right].$$

Therefore,

$$\frac{1}{2} (3 \cos^2 \varphi - 1) = \frac{4\pi}{5} \sum_{m=-2}^2 Y_2^m(\theta, \phi) Y_2^{m*}(\theta', \phi') = P_2(\cos \varphi). \quad (2.58)$$

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Therefore, when $r > r'$, $|\mathbf{r}' - \mathbf{r}|^{-1}$ can be written as

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r} \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \left(\frac{r'}{r}\right)^{\ell} Y_{\ell}^m(\theta, \phi) Y_{\ell}^{m*}(\theta', \phi'). \quad (2.59)$$

In this “external” expansion the small parameter is $r'/r < 1$, which guarantees convergence of the series. Truncating the sum at $\ell = L$ corresponds to retaining multipoles up to order L ; physically, the first terms describe the monopole ($\ell = 0$), dipole ($\ell = 1$), quadrupole ($\ell = 2$), and so on. Similarly, when $r < r'$ one obtains the complementary “internal” expansion,

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r'} \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \left(\frac{r}{r'}\right)^{\ell} Y_{\ell}^m(\theta, \phi) Y_{\ell}^{m*}(\theta', \phi'). \quad (2.60)$$

Here the expansion parameter is $r/r' < 1$, and the roles of r and r' are effectively exchanged. Substituting these expansions into the integral definition of the gravitational potential,

$$\Psi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}',$$

and splitting the radial integration domain into $0 < r' < r$ and $r < r' < \infty$ (so that the appropriate expansion is used in each region), one finds that the potential can be written as a sum over multipoles. In this representation, the dependence on (θ, ϕ) is entirely carried by the spherical harmonics, while all information about the mass distribution is contained in weighted integrals of ρ projected onto the same harmonic basis:

$$\begin{aligned} \Psi(r, \theta, \phi) = & \quad (2.61) \\ & - \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \frac{4\pi G}{(2\ell+1)r^{\ell+1}} Y_{\ell}^m(\theta, \phi) \int_0^r \int_0^{\pi} \int_0^{2\pi} \rho(r', \theta', \phi') Y_{\ell}^{m*}(\theta', \phi') \sin \theta' r'^{\ell+2} d\phi' d\theta' dr' \\ & - \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \frac{4\pi G r^{\ell}}{2\ell+1} Y_{\ell}^m(\theta, \phi) \int_r^{\infty} \int_0^{\pi} \int_0^{2\pi} \rho(r', \theta', \phi') Y_{\ell}^{m*}(\theta', \phi') \sin \theta' r'^{1-\ell} d\phi' d\theta' dr'. \end{aligned}$$

The first term includes the contribution from the mass enclosed within radius r and decays as $r^{-(\ell+1)}$, while the second term accounts for mass located outside r and scales as r^{ℓ} . For a field point outside a finite-size body (i.e. for r larger than the body’s outer radius), the second contribution vanishes because $\rho(r') = 0$ for $r' > R$. Since the spherical harmonics form a complete orthonormal basis on the unit sphere, the potential can also be expanded at fixed radius as

$$\begin{aligned} \Psi(r, \theta, \phi) &= \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} Y_{\ell}^m(\theta, \phi) \int_0^{\pi} \int_0^{2\pi} \Psi(r, \theta', \phi') Y_{\ell}^{m*}(\theta', \phi') \sin \theta' d\phi' d\theta' \quad (2.62) \\ &\equiv \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} Y_{\ell}^m(\theta, \phi) \Psi_{\ell}^m(r), \end{aligned}$$

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where the radial coefficient $\Psi_\ell^m(r)$ is the angular projection of the potential onto the mode (ℓ, m) :

$$\Psi_\ell^m(r) \equiv \int_0^\pi \int_0^{2\pi} \Psi(r, \theta', \phi') Y_\ell^{m*}(\theta', \phi') \sin \theta' d\phi' d\theta'. \quad (2.63)$$

In the same way, the density can be decomposed into spherical harmonics,

$$\rho(r, \theta, \phi) = \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta, \phi) \rho_\ell^m(r), \quad (2.64)$$

with radial coefficients

$$\rho_\ell^m(r) \equiv \int_0^\pi \int_0^{2\pi} \rho(r, \theta', \phi') Y_\ell^{m*}(\theta', \phi') \sin \theta' d\phi' d\theta'. \quad (2.65)$$

Finally, combining the harmonic expansions of Ψ and ρ yields an explicit radial expression for the projected potential:

$$\Psi_\ell^m(r) = -\frac{4\pi G}{(2\ell+1)r^{\ell+1}} \int_0^r \rho_\ell^m(r') r'^{\ell+2} dr' - \frac{4\pi G r^\ell}{2\ell+1} \int_r^\infty \rho_\ell^m(r') r'^{1-\ell} dr'. \quad (2.66)$$

This form makes the separation between interior ($r' < r$) and exterior ($r' > r$) contributions explicit, and it is the starting point for constructing the structure of deformed stars in the remainder of the thesis. In practice, since we work in real space, we often restrict to a real spherical-harmonic basis. In that case the projected quantities (denoted $\Psi_{\ell m}$ and $\rho_{\ell m}$) are obtained with

$$\Psi_{\ell m}(r) \equiv \int_0^\pi \int_0^{2\pi} \Psi(r, \theta', \phi') Y_{\ell m}(\theta', \phi') \sin \theta' d\phi' d\theta', \quad (2.67)$$

and

$$\rho_{\ell m}(r) \equiv \int_0^\pi \int_0^{2\pi} \rho(r, \theta', \phi') Y_{\ell m}(\theta', \phi') \sin \theta' d\phi' d\theta', \quad (2.68)$$

where $Y_{\ell m}$ denotes the chosen real form of the spherical harmonics. The corresponding real-space version of [Equation 2.66](#) is obtained by using the same basis consistently for both $\Psi_{\ell m}$ and $\rho_{\ell m}$.

2.2.2 POISSON EQUATION

An equivalent way to obtain the gravitational potential is to solve the so-called Poisson equation. In this thesis, several developments rely on solving Poisson's equation in different setups (in particular when dealing with stellar structure and oscillations). It is therefore useful to introduce the equation in its most basic form and show how it is solved in a spherical coordinate system. Starting from the superposition principle, the gravitational acceleration generated by a continuous density distribution $\rho(\mathbf{r}')$ can be written as

$$\mathbf{g}(\mathbf{r}) = -G \int \rho(\mathbf{r}') \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} d^3 \mathbf{r}'. \quad (2.69)$$

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Taking the divergence of this expression and using the classical identity

$$\nabla \cdot \left[(\mathbf{r}' - \mathbf{r}) / |\mathbf{r}' - \mathbf{r}|^3 \right] = 4\pi \delta(\mathbf{r}' - \mathbf{r}), \quad (2.70)$$

leads to Gauss's law for gravity,

$$\nabla \cdot \mathbf{g}(\mathbf{r}) = -4\pi G \rho(\mathbf{r}). \quad (2.71)$$

Since $\mathbf{g} = -\nabla\Psi$, this is equivalently written as the Poisson equation for the gravitational potential,

$$\Delta\Psi(\mathbf{r}) = 4\pi G \rho(\mathbf{r}), \quad (2.72)$$

where $\Delta \equiv \nabla^2$ is the Laplacian. Depending on the context, one may work either with $\nabla \cdot \mathbf{g} = -4\pi G \rho$ (more common in celestial-mechanics applications) or directly with Poisson's equation for Ψ (standard in stellar structure and asteroseismology). In what follows we adopt the latter, which will be used repeatedly in later parts of the thesis. To proceed, we project the equation onto a coordinate system adapted to (nearly) spherical stars. In spherical coordinates (r, θ, ϕ) , Poisson's equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} = 4\pi G \rho(\mathbf{r}). \quad (2.73)$$

It is convenient to separate the Laplacian into a purely radial operator and an angular operator. Defining

$$\Delta_r \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right), \quad (2.74)$$

and the angular (Legendre) operator

$$\mathcal{L}^2 \equiv -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \quad (2.75)$$

Poisson's equation can be rewritten in the compact form

$$\Delta_r \Psi(\mathbf{r}) - \frac{1}{r^2} \mathcal{L}^2 \Psi(\mathbf{r}) = 4\pi G \rho(\mathbf{r}). \quad (2.76)$$

This form highlights why spherical harmonics are natural for the angular dependence: they are eigenfunctions of $\mathcal{L}^2$ and therefore diagonalise the angular part of the operator. Specifically, spherical harmonics satisfy

$$\mathcal{L}^2 Y_\ell^m(\theta, \phi) = \ell(\ell + 1) Y_\ell^m(\theta, \phi). \quad (2.77)$$

We therefore expand the potential and density in spherical harmonics,

$$\Psi(r, \theta, \phi) = \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \Psi_\ell^m(r) Y_\ell^m(\theta, \phi), \quad \rho(r, \theta, \phi) = \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} \rho_\ell^m(r) Y_\ell^m(\theta, \phi), \quad (2.78)$$

and project Poisson's equation onto a given mode (ℓ, m) using the scalar product on the unit sphere, noted $\langle \dots | \dots \rangle$. The orthonormality relation

$$\langle Y_\ell^m | Y_{\ell'}^{m'} \rangle = \int_0^\pi \int_0^{2\pi} Y_\ell^m(\theta, \phi) Y_{\ell'}^{m'*}(\theta, \phi) \sin \theta \, d\phi \, d\theta = \delta_{\ell\ell'} \delta_{mm'}, \quad (2.79)$$

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ensures that different (ℓ, m) components decouple. One obtains, for each (ℓ, m) ,

$$\Delta_r \Psi_\ell^m(r) - \left\langle Y_\ell^m(\theta, \phi) \left| \frac{1}{r^2} \mathcal{L}^2 \sum_{\ell'=0}^L \sum_{m'=-\ell'}^{\ell'} \Psi_{\ell'}^{m'}(r) Y_{\ell'}^{m'}(\theta, \phi) \right. \right\rangle = 4\pi G \rho_\ell^m(r), \quad (2.80)$$

$$\Delta_r \Psi_\ell^m(r) - \left\langle Y_\ell^m(\theta, \phi) \left| \frac{1}{r^2} \sum_{\ell'=0}^L \sum_{m'=-\ell'}^{\ell'} \ell'(\ell' + 1) \Psi_{\ell'}^{m'}(r) Y_{\ell'}^{m'}(\theta, \phi) \right. \right\rangle = 4\pi G \rho_\ell^m(r), \quad (2.81)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi_\ell^m}{dr} \right) - \frac{\ell(\ell + 1)}{r^2} \Psi_\ell^m(r) = 4\pi G \rho_\ell^m(r). \quad (2.82)$$

This is an ordinary differential equation in r for each spherical-harmonic component of the potential. In other words, the angular structure is handled entirely by the eigenvalue $\ell(\ell + 1)$, while the remaining problem reduces to solving a set of radial equations, one for each multipole.

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The tidal force arises because real astrophysical bodies have a finite spatial extent: even if the companion is approximated as a point mass, its gravitational field is not uniform across them. This spatially differential forcing produces a deformation of the body and a redistribution of mass, which in turn perturbs the body's own gravitational potential (Zahn, 1975; Ogilvie, 2014). A convenient way to formalise this in a simple binary system is to expand the companion's gravitational potential about the centre of the tidally distorted body in spherical harmonics (Zahn, 1975; Press & Teukolsky, 1977; Smeyers & Willems, 2001; Ogilvie, 2014). In this expansion, the $\ell = 0$ contribution is a spatially constant monopole term; it is associated with the usual point-mass orbital interaction but does not produce an internal force or deformation. The $\ell = 1$ term describes the uniform acceleration of the body's centre of mass; when working in the co-accelerating frame associated with the orbital motion, this dipole term is cancelled by the corresponding inertial acceleration and therefore does not generate any physical distortion of the body (see Subsection 4.A.3 for a derivation). The leading term responsible for tidal deformation is therefore the quadrupolar contribution, $\ell = 2$. Because the quadrupolar tidal coefficient scales as r^{-3} , where r is the instantaneous separation between the two bodies, the associated tidal acceleration becomes rapidly stronger as the separation decreases. This explains why tides are particularly important in compact binary systems and in close-in exoplanetary systems (Zahn, 1975; Ogilvie, 2014). Higher-order multipoles also contribute, but the ℓ -th component of the tide-raising potential evaluated near the stellar surface scales as $r^{-(\ell+1)}$, so that higher-order terms are usually small unless the system is very close or strongly distorted (Press & Teukolsky, 1977; Remus et al., 2012; Ogilvie, 2014).

In the low forcing frequency regime, the stellar response $\Psi_{\ell,k}^m$ evaluated at the surface, to a tide-raising potential component $\Psi_{\ell,k}^m$ with $\ell > 1$ and forcing frequency $\omega_{k,m}$

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may be written schematically as

$$\frac{\Psi'_{\ell,k}}{\Psi_{\ell,k}} = 2k_{\ell} e^{i\alpha(\sigma_{k,m})} + E_{\ell} \sigma_{k,m}^{8/3} [q_{\ell}(\sigma_{k,m}) + i p_{\ell}(\sigma_{k,m})], \quad (2.83)$$

where

$$\sigma_{k,m} = \omega_{k,m} \left(\frac{R^3}{GM} \right)^{1/2}$$

is the dimensionless forcing frequency, the k indices being related to the orbital frequency harmonics (the multiples of the orbital harmonics). The quantities entering this expression will be defined in the next two sections; here the important point is the physical separation that the formula makes explicit. The first term corresponds to the large-scale, quasi-hydrostatic adjustment of the star to the external potential. In dissipative models, this contribution is associated with the equilibrium tide (Zahn, 1975; Ogilvie, 2014), because it describes a deformation that closely follows hydrostatic balance. The second term corresponds to the dynamical tide, i.e. the excitation and damping of wave-like motions and oscillation modes driven by the tidal forcing (Zahn, 1977; Ogilvie, 2014; Ahuir et al., 2021). These two components will play distinct roles in this thesis: the equilibrium tide is the focus of the first part (introduced in Subsection 2.3.1), while the dynamical tide is central to the second part (introduced in Subsection 2.3.2).

The goal of this section is not to provide a complete mathematical derivation of the tidal force or of its consequences for stellar structure and orbital evolution, these topics are developed in detail in Chapter 3, Chapter 4 and Section II. Instead, the aim is to introduce tides and tidal dissipation in a way that provides a common physical framework for the different effects explored throughout this work: how tides create a deformation, how dissipation introduces a phase lag, and how that lag leads to secular exchanges of energy and angular momentum.

2.3.1 EQUILIBRIUM TIDES

Tidal dissipation drives the secular evolution of a binary by modifying several orbital and rotational parameters, including the semi-major axis, eccentricity, stellar obliquities, and spin periods of both bodies (Darwin, 1880; Zahn, 1977; Hut, 1981; Ogilvie, 2014). The key mechanism is that dissipative tides introduce a phase lag between the tidal deformation and the instantaneous line of centres. The companion then exerts a net torque on the misaligned tidal bulge, transferring angular momentum between the stellar spins and the orbit and thereby changing both the spin and orbital angular momenta (Goldreich & Soter, 1966; Hut, 1981; Mazeh, 2008; Ogilvie, 2014). In many systems, this evolution tends toward spin–orbit synchronisation, circularisation, and damping of any initial obliquity, although the timescales and even the detailed evolutionary path depend sensitively on the dominant dissipation mechanism and on the internal structure of each body (Zahn, 1977; Hut, 1981; Barker & Ogilvie, 2009; Ogilvie, 2014; Barker, 2020). In eccentric systems, the spin may first approach a pseudo-synchronous state before full circularisation is reached (Hut, 1981). Finally, the non-spherical tidal and rotational distortions also modify the gravitational field

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of the components and therefore contribute to apsidal motion and orbital precession (Sterne, 1939; Kopal, 1959; Claret & Giménez, 2010; Claret, 2019).

For a circular, coplanar binary, the perturbing potential raised by body 2 on body 1 can be expanded into harmonic components associated with the orbital motion as seen in the rotating frame of the deformed body (Darwin, 1880; Kaula, 1964; Murray & Dermott, 1999; Ogilvie, 2014). Restricting to the dominant quadrupolar contribution, the relevant forcing depends primarily on the orbital mean motion n and on the spin rate Ω_1 of body 1. In the frame corotating with body 1, the dominant quadrupolar tide is the semidiurnal component, with forcing frequency $2(n - \Omega_1)$ or $2(\Omega_1 - n)$ up to the adopted sign convention (Hut, 1981; Ogilvie, 2014; Lanza, 2020). Although the Earth–Moon system provides the historical prototype for this description, the same formalism applies to stellar and planetary binaries, with the amplitude, phase lag, and dissipation rate of the response depending on each body’s spin and internal structure (Goldreich & Soter, 1966; Zahn, 1977; Hut, 1981; Ogilvie, 2014).

In linear tidal theory, the response of a body to a perturbing potential is described by a proportionality between the forcing potential and the induced perturbation of the body’s gravitational potential or surface shape (Darwin, 1880; Love, 1911). This proportionality is quantified by the Love numbers (or apsidal-motion constants) k_2 , which describe both the amplitude of the deformation and its phase relative to the forcing. For a given Fourier component, it is convenient to introduce a complex Love number such that

$$\frac{\Psi'_{\ell,k}}{\Psi_{\ell,k}} = 2\tilde{k}_\ell(\omega_{k,m}) = 2|k_\ell(\omega_{k,m})|\exp[i\epsilon_\ell(\omega_{k,m})], \quad (2.84)$$

or equivalently

$$\frac{\Psi'_{\ell,k}}{\Psi_{\ell,k}} = 2|k_\ell(\omega_{k,m})|\cos[\epsilon_\ell(\omega_{k,m})] + 2i|k_\ell(\omega_{k,m})|\sin[\epsilon_\ell(\omega_{k,m})]. \quad (2.85)$$

The phase lag $\epsilon_\ell(\omega_{k,m})$ measures the delay between the applied tidal forcing and the resulting deformation; it depends on the internal structure of the body and on the dissipative mechanism (turbulent friction, radiative damping, nonlinear wave breaking, etc.) (Ogilvie, 2014). This complex response naturally separates the instantaneous (elastic or hydrostatic) deformation from the dissipative component that drives secular evolution:

- $\text{Re}[\Psi'_{\ell,k}/\Psi_{\ell,k}]$ corresponds to the in-phase part of the response and sets the shape of the tidally distorted body. It therefore contributes to conservative effects such as apsidal motion (Claret & Willems, 2002; Ogilvie, 2014).
- $\text{Im}[\Psi'_{\ell,k}/\Psi_{\ell,k}]$ corresponds to the out-of-phase part produced by dissipation. This term is responsible for transfers of energy and angular momentum between spin and orbit, and thus for synchronisation, circularisation, and orbital migration (Zahn, 1977; Ogilvie, 2014).

In weak-friction models, the deformation responds with a small time delay $\Delta\tau_\ell$ to the forcing. For small lag angles, the phase lag is approximately proportional to the

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forcing frequency (Darwin, 1880),

$$\epsilon_\ell(\omega_{k,m}) \simeq \omega_{k,m} \Delta\tau_\ell(\omega_{k,m}), \quad (2.86)$$

which defines the constant time-lag model when $\Delta\tau_\ell$ is taken to be frequency-independent. In that approximation,

$$\left| \operatorname{Im} \left[\frac{\Psi'_{\ell,k}}{\Psi_{\ell,k}} \right] \right| = 2 |k_\ell(\omega_{k,m})| \sin(\omega_{k,m} \Delta\tau_\ell), \quad (2.87)$$

and $\Delta\tau_\ell$ is treated as a free parameter (Darwin, 1880). A second widely used approximation is the constant phase-lag model, in which the lag angle is taken to be frequency-independent (Goldreich & Soter, 1966). In that framework, dissipation is commonly parameterised through the tidal quality factor Q_ℓ , defined by

$$Q_\ell^{-1} = |\sin[\epsilon_\ell(\omega_{k,m})]|, \quad (2.88)$$

or through the reduced quality factor

$$Q'_\ell = \frac{3}{2|k_\ell|} Q_\ell. \quad (2.89)$$

Using Equation 2.84, one obtains

$$\left| \operatorname{Im} \left[\frac{\Psi'_{\ell,k}}{\Psi_{\ell,k}} \right] \right| = \frac{2 |k_\ell(\omega_{k,m})|}{Q_\ell}, \quad (2.90)$$

so that either Q_ℓ or Q'_ℓ can be used as an effective free parameter controlling the efficiency of tidal dissipation (Goldreich & Soter, 1966; Goldreich & Nicholson, 1977, 1989; Hut, 1981). Although the constant time-lag and constant phase-lag models are especially popular in planetary applications, they are not really employed for stars, as stellar structures are more constrained and less uncertain allowing for physically based prescriptions. In stars, dissipation typically depends on specific physical processes such as turbulent friction acting on the equilibrium tide or wave damping associated with the dynamical tide (Zahn, 1977; Ogilvie, 2014). Nevertheless, the Q - and $\Delta\tau$ -based parameterisations remain useful as effective descriptions and as convenient tools for comparing different systems and tidal regimes.

In binary stars, ordinary microscopic viscosity is generally not the dominant source of dissipation for the equilibrium tide. In stars with deep convective envelopes, the main mechanism is instead turbulent friction: the large-scale tidal flow interacts with convective turbulence, converting tidal kinetic energy into heat (Zahn, 1966, 1977; Goldreich & Nicholson, 1977; Ogilvie, 2014). This process is particularly effective in low-mass stars, typically $M \lesssim 1.3 M_\odot$, because they possess extended outer convective envelopes (Zahn, 1989; Mazeh, 2008; Barker, 2020). By contrast, more massive main-sequence stars usually have radiative envelopes and convective cores; in that case turbulent damping of the equilibrium tide is weaker and tidal dissipation is more often dominated by the dynamical tide, where internal gravity waves are excited and damped by radiative diffusion (Zahn, 1975, 1977; Ogilvie, 2014). In Zahn-type prescriptions, the efficiency of equilibrium-tide dissipation is therefore controlled mainly

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by the properties of the convective envelope, whereas the efficiency of radiative damping of the dynamical tide in massive stars depends strongly on the structure near the convective core, often parametrised through the coefficient E_2 (Zahn, 1975, 1977, 1989), see Subsection 2.3.2.

From a modelling standpoint, turbulent dissipation is difficult to predict from first principles because it requires a reliable description of the interaction between convection and large-scale tidal flows (Zahn, 1977; Ogilvie, 2014). Direct hydrodynamical simulations of tidally perturbed convection can in principle calibrate this interaction, but such calculations remain theoretically and computationally challenging, especially when one seeks prescriptions applicable across stellar masses, evolutionary stages, and tidal forcing frequencies (Ogilvie & Lesur, 2012; Duguid et al., 2020; Barker, 2020). For this reason, binary-evolution calculations generally rely on approximate prescriptions based on an effective eddy viscosity, a tidal quality factor, or an effective friction timescale, often implemented within the weak-friction or equilibrium-tide framework (Hut, 1981; Hurley et al., 2002; Siess et al., 2013). In stars with deep convective envelopes, a commonly used characteristic timescale is the eddy turnover timescale (Rasio et al., 1996; Villaver & Livio, 2009) defined as

$$t_{\text{conv}} \sim \left(\frac{M_{\text{env}} (R - R_{\text{env}})^2}{3L} \right)^{1/3}, \quad (2.91)$$

where M_{env} is the mass in the convective envelope, R_{env} is its position, R is the radius of the star and L its luminosity. This timescale roughly estimates how long a turbulent convective structure takes to travel through the convective zone. It is then used to construct an effective turbulent viscosity or the corresponding tidal lag in weak-friction models (Zahn, 1977; Hut, 1981; Ogilvie, 2014). Despite their simplicity, such prescriptions reproduce the main empirical trends of synchronisation and circularisation reasonably well in low-mass binaries (Zahn, 1977; Hut, 1981).

To gain physical insight into the direction of tidal angular-momentum exchange, it is useful to introduce the concept of a corotation (synchronisation) separation. Consider a binary with mean motion n and stellar spin angular velocities Ω_1 and Ω_2 . For each component $i \in \{1, 2\}$, the corotation separation $a_{\text{cor},i}$ is defined as the orbital distance for which

$$n(a_{\text{cor},i}) = \Omega_i. \quad (2.92)$$

Using Kepler's third law, this may be written as

$$a_{\text{cor},i} = \left[\frac{G(M_1 + M_2)}{\Omega_i^2} \right]^{1/3}. \quad (2.93)$$

Whether the orbit lies inside or outside corotation determines the sign of the tidal torque acting on each component (Hut, 1981; Ogilvie, 2014). In practice, the secular orbital evolution results from the combined torques raised on both bodies, which may reinforce or partially counteract each other if Ω_1 and Ω_2 lie on opposite sides of n :

- $\Omega_1 = \Omega_2 = n$: the binary is doubly synchronous. If the orbit is also circular and the spin axes are aligned, the tidal bulges are stationary in the corotating

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frame, dissipation vanishes, and secular evolution effectively stops (Hut, 1981; Ogilvie, 2014).

- $\Omega_i > n$: component i rotates faster than the orbit. Its tidal bulge leads the line of centres, so angular momentum is transferred from spin to orbit. Component i spins down while the orbit expands (Hut, 1981; Ogilvie, 2014); the Earth–Moon system provides the classical example.
- $\Omega_i < n$: component i rotates more slowly than the orbit. Its tidal bulge lags behind the line of centres, so angular momentum is transferred from orbit to spin. The orbit shrinks while component i spins up (Hut, 1981; Ogilvie, 2014). In sufficiently close binaries this inward evolution may lead to Roche-lobe overflow, mass transfer, merger, or a common-envelope phase.

The corotation separations $a_{\text{cor},i}$ evolve because the spins Ω_i evolve. Stellar evolution plays an important role: as a star expands, its moment of inertia typically increases and its surface rotation decreases, shifting the corotation radius and modifying both the magnitude and the direction of the tidal torque (Zahn, 1977; Hut, 1981; Siess et al., 2013). Because in most binaries the orbital angular momentum greatly exceeds the spin angular momentum of either component, tides usually synchronise spins faster than they change the semi-major axis. Spin evolution is therefore often the most immediate tidal effect, even though significant orbital evolution can still occur in sufficiently close, pre-main-sequence, evolved, or compact systems (Hut, 1981; Ogilvie, 2014; Marchant & Bodensteiner, 2024).

2.3.2 DYNAMICAL TIDES

Tidal forcing can also excite eigenmodes of oscillation in the distorted body; this wave-like response is referred to as the dynamical tide (Zahn, 1975, 1977). Several classes of waves may couple to the tidal potential depending on the dominant restoring force: acoustic waves, gravity waves, inertial waves, and (in magnetised regions) Alfvén waves (Ogilvie, 2014). Efficient excitation occurs when one or more forcing frequencies of the tidal potential are close to the natural mode frequencies, which depend sensitively on stellar structure, rotation, and stratification (Zahn, 1977; Ogilvie, 2014; Ahuir et al., 2021).

For stars with deep convective envelopes, typically $M \lesssim 1.3 M_\odot$, acoustic modes usually have frequencies that are too high to be efficiently excited by tidal forcing in ordinary binaries (Ogilvie, 2014). Gravity modes, by contrast, are trapped primarily in the radiative core and are not directly observable in solar-type main-sequence stars, although their tidal excitation and damping may still contribute to orbital evolution and remain an active topic of research (Goodman & Dickson, 1998; Barker & Ogilvie, 2010; Ahuir et al., 2021). Inertial waves can be excited in the convective envelope when the forcing frequency in the rotating frame satisfies

$$|\omega| \leq 2\Omega_1, \quad (2.94)$$

These waves are restored by the Coriolis force and are believed to provide an efficient channel for tidal dissipation in rotating convective envelopes. They may therefore

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contribute to orbital circularisation and spin synchronisation, although the efficiency of this process depends sensitively on the stellar rotation rate, internal structure, differential rotation, and nonlinear damping mechanisms (Ogilvie, 2014; Guenel et al., 2016; Astoul & Barker, 2022; Barker, 2022). When the frequency criterion is satisfied, both the equilibrium tide and the dynamical tide may contribute significantly to the secular evolution; otherwise, the equilibrium tide usually dominates (Ogilvie, 2014). In solar-type stars, dissipation by tidally excited inertial waves is expected to be especially important during the pre-main-sequence and early main-sequence phases, when the star rotates rapidly (Ogilvie, 2014).

For higher-mass stars without deep outer convective envelopes, the picture is different. Their radiative envelopes support low-frequency gravity waves, and, when rotation is important, gravito-inertial waves, which can couple efficiently to the time-dependent tidal potential (Zahn, 1975, 1977; Ogilvie, 2014). Because the characteristic periods of these modes can be comparable to the orbital periods of close binaries, tidal excitation and dissipation may be efficient in this regime, especially when the forcing frequency lies close to a stellar eigenfrequency or when travelling internal gravity waves are efficiently damped (Witte & Savonije, 1999a, 2001; Fuller, 2017; Su & Lai, 2022). In the classical Zahn picture, the dominant dissipation mechanism is radiative damping of tidally excited gravity waves in the radiative envelope (Zahn, 1975, 1977). In Zahn’s asymptotic low-frequency formalism, the imaginary part of the tidal response may be written schematically as

$$\text{Im}[\Psi'_{\ell,k}/\Psi_{\ell,k}^m] = E_\ell \sigma_{k,m}^{8/3}, \quad (2.95)$$

where one has neglected the factor $p_\ell(\sigma_{k,m})$, which tends to unity in the low-frequency limit (Zahn, 1977). The coefficient E_ℓ plays a role analogous to an apsidal-motion constant for the dynamical tide and depends strongly on the internal stellar structure, scaling approximately as

$$E_\ell \propto \left(\frac{R_c}{R}\right)^{2\ell+4}, \quad (2.96)$$

where R_c is the radius of the convective core (Zahn, 1977). This radiative-envelope dynamical-tide framework is therefore central to interpreting tidal dissipation in intermediate- and high-mass close binaries (Zahn, 1977; Ogilvie, 2014; Ahuir et al., 2021). In the second part of this work, however, we are interested in a different regime: the frequency-dependent resonant regime, in which individual tidal forcing frequencies can approach the eigenfrequencies of stellar oscillation modes. In that case, the response can become much stronger than predicted by the smooth low-frequency asymptotic limit, giving rise to tidally excited oscillations and, at sufficiently large amplitudes (Willems, 2003; Willems et al., 2003; Fuller & Lai, 2012; Fuller, 2017; Guo, 2021).

2.4 GAUSS PLANETARY EQUATIONS

In Section 2.1 we developed the classical orbital equations in the simplest configuration, using a Cartesian coordinate system embedded in the orbital plane and neglecting the inclination of the system. In practice, however, orbital systems are generally

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inclined, and their long-term evolution (called secular evolution) is driven by weak non-Keplerian perturbations, in particular those arising from tides. In this section we derive the analytical framework used to compute the secular evolution of the orbital and rotational properties of the components of a binary system. This framework, based on the Gauss planetary equations, will be used throughout the thesis whenever tidal forces (or any other small perturbation) produce cumulative changes in the orbital elements. Before proceeding with the derivations, we define a new inertial Cartesian coordinate system centred on the center of mass of the binary. The reference (x, y) plane is fixed in space, while the orbital plane is inclined with respect to it by an angle i ; the orientation of the line of nodes is described by the longitude of the ascending node Ω (see Figure 2.1). In this geometry the specific angular momentum $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to the orbital plane and written as

$$\mathbf{h} = h \sin \Omega \sin i \mathbf{e}_x - h \cos \Omega \sin i \mathbf{e}_y + h \cos i \mathbf{e}_z, \quad (2.97)$$

with the norm of $\mathbf{h}$ noted h , which remains unchanged in the Keplerian problem and is expressed as

$$h^2 = G(M_1 + M_2) a(1 - e^2), \quad (2.98)$$

while the specific orbital energy (denoted here by $\mathcal{E}$ to avoid confusion with the eccentric anomaly E) is

$$\mathcal{E} = -\frac{G(M_1 + M_2)}{2a}. \quad (2.99)$$

Using these definitions of the energy and angular momentum, we can rewrite the orbital elements a, e, Ω, i as

$$a = -\frac{G(M_1 + M_2)}{2\mathcal{E}}, \quad (2.100)$$

$$e = \left(\frac{2\mathcal{E}h^2}{G^2(M_1 + M_2)^2} + 1 \right)^{1/2}, \quad (2.101)$$

$$\cos i = h_z/h, \quad (2.102)$$

$$\tan \Omega = -h_x/h_y, \quad (2.103)$$

where h_x, h_y and h_z are the Cartesian components of $\mathbf{h}$. We now add a perturbation to the orbit, so that the acceleration is expressed as

$$\ddot{\mathbf{r}} + \frac{G(M_1 + M_2)}{r^3} \mathbf{r} = \mathbf{F}_P, \quad (2.104)$$

where $\mathbf{F}_P$ is the perturbing acceleration. It is convenient to decompose it along the instantaneous orbital triad:

$$\mathbf{F}_P = F_R \mathbf{e}_r + F_S \mathbf{e}_\theta + F_W \mathbf{e}_\phi, \quad (2.105)$$

where, if the perturbation derives from a potential $\Psi_{R_{12}}$, the components can be written as

$$F_R = \left(\frac{\partial \Psi_{R_{12}}}{\partial r} \right)_{r, \phi=0, \theta=0}, \quad (2.106)$$

$$F_S = \left(\frac{1}{r} \frac{\partial \Psi_{R_{12}}}{\partial \phi} \right)_{r, \phi=0, \theta=0}, \quad (2.107)$$

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and

$$F_W = - \left(\frac{1}{r} \frac{\partial \Psi_{R_{12}}}{\partial \theta} \right)_{r, \phi=0, \theta=0}, \quad (2.108)$$

where $\Psi_{R_{12}}$ is the perturbing potential. At this step, we note that we do not use the most common notation (R, S, W) for the perturbing components, because these symbols conflict with other quantities used throughout the thesis (e.g. radii, entropy, and work). With the usual notation one would have $F_R \equiv R$, $F_S \equiv S$ and $F_W \equiv W$. When the perturbation is introduced, the Keplerian relations are satisfied only instantaneously and the orbital parameters become time dependent (osculating elements). Their variations are obtained by differentiating the expressions of the orbital elements with respect to time. For the semi-major axis we have

$$\frac{da}{dt} = \frac{2a^2}{G(M_1 + M_2)} \frac{d\mathcal{E}}{dt}, \quad (2.109)$$

where $d\mathcal{E}/dt$ is the work per unit mass done per unit time by the perturbing force:

$$\frac{d\mathcal{E}}{dt} = \dot{\mathbf{r}} \cdot \mathbf{F}_P = \dot{r} F_R + r \dot{\theta} F_S, \quad (2.110)$$

with no ϕ component because the unperturbed motion lies in the orbital plane. Using $h = r^2 \dot{\theta}$ and introducing the mean motion $n = \sqrt{G(M_1 + M_2)/a^3}$, we obtain

$$\frac{da}{dt} = \frac{2}{n \sqrt{1 - e^2}} \left[F_R e \sin \nu + \frac{a(1 - e^2)}{r} F_S \right]. \quad (2.111)$$

For the eccentricity, differentiating the expression linking e to $\mathcal{E}$ and h gives

$$\frac{de}{dt} = \frac{1}{2} \frac{e^2 - 1}{e} \left(\frac{2\dot{h}}{h} + \frac{\dot{\mathcal{E}}}{\mathcal{E}} \right) \quad (2.112)$$

The rate of change of the angular momentum is

$$\frac{d}{dt}(\mathbf{h}) = \mathbf{r} \times \mathbf{F}_P = r F_S \mathbf{e}_\phi - r F_W \mathbf{e}_\theta, \quad (2.113)$$

which implies that the magnitude varies as

$$\frac{dh}{dt} = r F_W, \quad (2.114)$$

as the component along $\mathbf{e}_\phi$ primarily changes the direction of $\mathbf{h}$ and thus the orientation of the orbital plane but not the norm. The time variation of the eccentricity can finally be written as

$$\frac{de}{dt} = \frac{\sqrt{1 - e^2}}{na} [F_R \sin \nu + (\cos \nu + \cos E) F_S], \quad (2.115)$$

where ν is the true anomaly and E is the eccentric anomaly. For the next two orbital parameters i and Ω , we need to relate the time derivatives of the Cartesian components of $\mathbf{h}$ to the unit vectors of the co-planar frame. Rather than relying on purely geometric arguments, we use successive 3D rotation matrices (extrinsic Euler rotations with angles Ω, i, ν) to transform from the inertial Cartesian frame to the co-planar frame. With

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this representation, the angular unit vectors of the co-planar frame can be expressed in the Cartesian frame as

$$\mathbf{e}_\theta = -(\sin \nu \cos i \cos \Omega + \cos \nu \sin \Omega) \mathbf{e}_x - (\cos \nu \cos i \cos \Omega - \sin \nu \sin \Omega) \mathbf{e}_y + \cos \nu \sin i \mathbf{e}_z, \quad (2.116)$$

$$\mathbf{e}_\phi = \sin \Omega \sin i \mathbf{e}_x - \cos \Omega \sin i \mathbf{e}_y + \cos i \mathbf{e}_z. \quad (2.117)$$

Using $\dot{\mathbf{h}} = rF_S \mathbf{e}_\phi - rF_W \mathbf{e}_\theta$, the components of the time derivative of the angular momentum are then identified as

$$\frac{dh_x}{dt} = rF_S \sin \Omega \sin i + rF_W (\sin \nu \cos i \cos \Omega + \cos \nu \sin \Omega), \quad (2.118)$$

$$\frac{dh_y}{dt} = -rF_S \cos \Omega \sin i + rF_W (\cos \nu \cos i \cos \Omega - \sin \nu \sin \Omega), \quad (2.119)$$

and

$$\frac{dh_z}{dt} = rF_S \cos i - rF_W \cos \nu \sin i. \quad (2.120)$$

The variation of the inclination can be obtained by differentiating $\cos i = h_z/h$, which gives

$$\frac{di}{dt} = \frac{\dot{h}/h - \dot{h}_z/h_z}{\sqrt{(h/h_z)^2 - 1}}, \quad (2.121)$$

and may be rewritten in the standard Gauss form

$$\frac{di}{dt} = \frac{r \cos u}{na^2 \sqrt{1 - e^2}} F_W, \quad (2.122)$$

where

$$u = \omega + \nu. \quad (2.123)$$

For the node angle we obtain, by differentiating $\tan \Omega = -h_x/h_y$,

$$\frac{d\Omega}{dt} = \frac{\sin \Omega \dot{h}_y - \cos \Omega \dot{h}_x}{h \sin i}, \quad (2.124)$$

which can be rewritten as

$$\frac{d\Omega}{dt} = \frac{r \sin u}{na^2 \sqrt{1 - e^2} \sin i} F_S. \quad (2.125)$$

For the two last orbital elements the derivation is not as straightforward because $\bar{\omega} = \Omega + \omega$ and $\epsilon = \Omega + \omega - nt_0$ are not direct functions of the energy and angular momentum. To obtain those equations one has to rewrite the classical orbital equation given in [Section 2.1](#) to explicitly isolate those elements before doing the time differentiation. Without reproducing the full derivation here, the Gauss-variation procedure yields

$$\frac{d\bar{\omega}}{dt} = -\frac{\sqrt{1 - e^2}}{nae} \left[F_R \cos \nu + \left(1 + \frac{r}{a(1 - e^2)} \right) F_S \sin \nu \right] + \frac{F_W r \sin u}{na^2 \sqrt{1 - e^2}} \tan \frac{i}{2}, \quad (2.126)$$

and,

$$\frac{d\epsilon}{dt} = -\frac{1}{an} \left[\frac{2r}{a} - \frac{1 - e^2}{e} \cos \nu \right] F_R - \frac{1 - e^2}{nae} \left[1 + \frac{r}{a(1 - e^2)} \right] F_S \sin \nu. \quad (2.127)$$

Full derivation of those equations can be found for example in [Fitzpatrick \(2012\)](#).

2.4. GAUSS PLANETARY EQUATIONS

2.4.1 PHYSICAL INTERPRETATION OF THE GAUSS PLANETARY EQUATIONS

The Gauss planetary equations are particularly useful because they make the physical origin of the secular evolution especially transparent. In particular, when written in terms of the radial, transverse, and normal components of the perturbing acceleration, they show directly which broken symmetries are responsible for the long-term evolution of each orbital element (Zahn, 1977; Hut, 1981; Ogilvie, 2014). In this sense, they provide an intuitive link between the geometry of the perturbation and the dynamical response of the binary.

In the notation adopted here, F_R is the radial component of the perturbing acceleration, F_S is the transverse component in the orbital plane, and F_W is the component normal to the orbital plane. In a coplanar problem, $F_W = 0$, so the inclination and the longitude of the ascending node remain constant. More generally, F_W measures the breaking of symmetry with respect to the orbital plane and is therefore the only component that can drive the secular evolution of i and Ω . By averaging the Gauss equations over one orbital period, denoted by $\langle \cdots \rangle_P$, one obtains

$$\left\langle \frac{d\Omega}{dt} \right\rangle_P = \frac{\langle r \sin u F_W \rangle_P}{na^2 \sqrt{1-e^2} \sin i}, \quad (2.128)$$

$$\left\langle \frac{di}{dt} \right\rangle_P = \frac{\langle r \cos u F_W \rangle_P}{na^2 \sqrt{1-e^2}}, \quad (2.129)$$

$$\left\langle \frac{da}{dt} \right\rangle_P = \frac{2}{n \sqrt{1-e^2}} [e \langle F_R \sin \nu \rangle_P + \langle F_S (1 + e \cos \nu) \rangle_P], \quad (2.130)$$

$$\left\langle \frac{de}{dt} \right\rangle_P = \frac{\sqrt{1-e^2}}{na} [\langle F_R \sin \nu \rangle_P + \langle (\cos \nu + \cos E) F_S \rangle_P], \quad (2.131)$$

$$\left\langle \frac{d\bar{\omega}}{dt} \right\rangle_P = -\frac{\sqrt{1-e^2}}{nae} \left[\langle F_R \cos \nu \rangle_P - \left\langle \left(1 + \frac{r}{a(1-e^2)} \right) F_S \sin \nu \right\rangle_P \right] + \frac{\langle r \sin u F_W \rangle_P}{na^2 \sqrt{1-e^2}} \tan \frac{i}{2}, \quad (2.132)$$

$$\left\langle \frac{d\epsilon}{dt} \right\rangle_P = -\frac{1}{an} \left\langle \left[\frac{2r}{a} - \frac{1-e^2}{e} \cos \nu \right] F_R \right\rangle_P - \frac{1-e^2}{nae} \left\langle \left[1 + \frac{r}{a(1-e^2)} \right] F_S \sin \nu \right\rangle_P. \quad (2.133)$$

In this thesis, given the assumptions adopted in most applications, in particular coplanar systems with negligible obliquity, we focus mainly on the secular evolution of the semi-major axis, the eccentricity, and the apsidal motion, namely da/dt , de/dt , and $d\bar{\omega}/dt$. In that limit, the most important terms are those involving F_R .

The apsidal-motion equation is especially interesting, its dominant contribution is generally the orbit-averaged term $\langle F_R \cos \nu \rangle_P$, which corresponds the asymmetry of the perturbing force with respect to the cartesian x axis in the orbital plane during an orbit. In eccentric binaries, the tidal distortion depends strongly on the instantaneous

2.A. SPECTRAL SOLUTION OF THE POISSON EQUATION

orbital separation, which varies asymmetrically between periastron and apastron, naturally giving rise to a secular precession of the line of apsides (Claret & Willems, 2002; Claret et al., 2021). Physically, this is why apsidal motion can exist even in the absence of dissipation: it is primarily produced by the instantaneous non-spherical deformation of the stars and by the associated departure from a purely Keplerian inverse-square potential.

By contrast, the secular evolution of the semi-major axis and eccentricity is dominated by the orbit-averaged terms $\langle F_R \sin \nu \rangle_P$ which corresponds to the average of the tidal force y axis asymmetry along an orbit. However, a purely conservative tidal deformation does not generate such a phase-shifted component, so its orbit-averaged contribution to a and e vanishes. Secular evolution of the orbit therefore requires dissipation, that is, a finite lag between the application of the tidal forcing and the response of the star (Zahn, 1977; Hut, 1981; Ogilvie, 2014). This phase lag is precisely what gives rise to equilibrium and dynamical tides and ultimately drives synchronisation, circularisation, and orbital migration.

This interpretation directly motivates the structure of the present thesis. In Section I, we investigate how non-perturbative modelling modifies the instantaneous tidal deformation of close binaries; the most direct application is therefore the modelling of apsidal motion. In Section II, we focus on dissipative processes, in particular tidally excited oscillations and their damping or growth, which corresponds to studying the origin of the phase lag that controls the secular evolution of the semi-major axis and eccentricity.

2.A SPECTRAL SOLUTION OF THE POISSON EQUATION

For each spherical-harmonic component of spherical degree ℓ and azimuthal order m , Poisson's equation reduces to the radial ordinary differential equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi_\ell^m(r)}{dr} \right) - \frac{\ell(\ell+1)}{r^2} \Psi_\ell^m(r) = 4\pi G \rho_\ell^m(r). \quad (2.134)$$

This is a second-order linear equation. In vacuum ($\rho_\ell^m = 0$) it reduces to Laplace's equation, whose two linearly independent solutions are r^ℓ and $r^{-(\ell+1)}$. Inside a star we require the potential to remain finite at the centre, which selects the regular behaviour $\Psi_\ell^m \propto r^\ell$ as $r \rightarrow 0$. This motivates writing the solution in the form

$$\Psi_\ell^m(r) \equiv r^\ell k_\ell^m(r), \quad (2.135)$$

where the unknown function $k_\ell^m(r)$ accounts for the deviation from the purely harmonic solution due to the source term $\rho_\ell^m(r)$. Substituting $\Psi_\ell^m = r^\ell k_\ell^m$ into the radial equation and simplifying yields

$$\frac{d^2 k_\ell^m}{dr^2} + \frac{2\ell+2}{r} \frac{dk_\ell^m}{dr} = \frac{4\pi G}{r^\ell} \rho_\ell^m(r). \quad (2.136)$$

Introducing

$$v_\ell^m(r) \equiv \frac{dk_\ell^m(r)}{dr}, \quad (2.137)$$

2.A. SPECTRAL SOLUTION OF THE POISSON EQUATION

reduces the problem to a first-order linear differential equation,

$$\frac{dv_\ell^m}{dr} + \frac{2\ell + 2}{r} v_\ell^m(r) = \frac{4\pi G}{r^\ell} \rho_\ell^m(r). \quad (2.138)$$

This equation has the standard form $v' + a(r)v = b(r)$ with $a(r) = (2\ell + 2)/r$ and $b(r) = 4\pi G \rho_\ell^m(r) r^{-\ell}$. Using an integrating factor

$$\mu(r) = \exp\left(\int_0^r a(r) dr\right) = \exp((2\ell + 2) \ln r) = r^{2\ell+2}, \quad (2.139)$$

one finds that the differential equation can be rewritten as

$$\frac{d}{dr} \left(r^{2\ell+2} v_\ell^m(r) \right) = 4\pi G \rho_\ell^m(r) r^{\ell+2}. \quad (2.140)$$

Imposing regularity at the centre (which removes the singular homogeneous contribution) gives the interior solution

$$v_\ell^m(r) = r^{-(2\ell+2)} \int_0^r 4\pi G \rho_\ell^m(r') r'^{\ell+2} dr'. \quad (2.141)$$

Integrating once more provides $k_\ell^m(r)$,

$$k_\ell^m(r) = k_\ell^m(R) + \int_R^r v_\ell^m(r') dr' = k_\ell^m(R) + \int_R^r r'^{-(2\ell+2)} \left[\int_0^{r'} 4\pi G \rho_\ell^m(r'') r''^{\ell+2} dr'' \right] dr', \quad (2.142)$$

and therefore the potential component can be written as

$$\Psi_\ell^m(r) = r^\ell \int_R^r r'^{-(2\ell+2)} \left[\int_0^{r'} 4\pi G \rho_\ell^m(r'') r''^{\ell+2} dr'' \right] dr' + r^\ell k_\ell^m(R). \quad (2.143)$$

At this stage $k_\ell^m(R)$ is an integration constant that must be fixed by a boundary condition at the stellar surface $r = R$. To determine this constant, we match the interior solution to the exterior (vacuum) solution. Outside the star $\rho = 0$ and the physically acceptable solution (vanishing at infinity) is proportional to $r^{-(\ell+1)}$. Continuity of Ψ_ℓ^m and of its radial derivative at $r = R$ implies

$$\left. \frac{d\Psi_\ell^m}{dr} \right|_R = -\frac{\ell+1}{R} \Psi_\ell^m(R), \quad \text{or equivalently} \quad \frac{\ell+1}{R} \Psi_\ell^m(R) + \left. \frac{d\Psi_\ell^m}{dr} \right|_R = 0. \quad (2.144)$$

Using $\Psi_\ell^m(R) = R^\ell k_\ell^m(R)$ and $v_\ell^m(R) = k_\ell^{m'}(R)$ together with the expression for v_ℓ^m , one obtains

$$\frac{2\ell+1}{R} \Psi_\ell^m(R) + \frac{4\pi G}{R^{\ell+2}} \int_0^R \rho_\ell^m(r') r'^{\ell+2} dr' = 0, \quad (2.145)$$

which yields

$$k_\ell^m(R) = -\frac{4\pi G}{(2\ell+1)R^{2\ell+1}} \int_0^R \rho_\ell^m(r') r'^{\ell+2} dr'. \quad (2.146)$$

2.A. SPECTRAL SOLUTION OF THE POISSON EQUATION

Replacing this constant in the previous expression gives a closed semi-analytic solution for the Poisson equation:

$$\Psi_\ell^m(r) = r^\ell \int_R^r r'^{-(2\ell+2)} \left[\int_0^{r'} 4\pi G \rho_\ell^m(r'') r''^{\ell+2} dr'' \right] dr' - \frac{4\pi G r^\ell}{(2\ell+1)R^{2\ell+1}} \int_0^R \rho_\ell^m(r') r'^{\ell+2} dr'. \quad (2.147)$$

This result is equivalent to the standard multipole form derived from the Green-function expansion. Indeed, by performing an integration by parts, [Equation 2.147](#) can be recast into the more familiar expression

$$\Psi_\ell^m(r) = -\frac{4\pi G}{2\ell+1} \left[\frac{1}{r^{\ell+1}} \int_0^r \rho_\ell^m(r') r'^{\ell+2} dr' + r^\ell \int_r^R \rho_\ell^m(r') r'^{1-\ell} dr' \right], \quad (r \leq R),$$

with the exterior continuation $\Psi_\ell^m(r) = -\frac{4\pi G}{2\ell+1} \frac{1}{r^{\ell+1}} \int_0^R \rho_\ell^m(r') r'^{\ell+2} dr'$ for $r \geq R$.

CHAPTER 3

NON-PERTUBATIVE MODELLING OF BINARIES DEFORMATIONS

When two extended bodies orbit at sufficiently small separations, the combined action of mutual gravity and rotation drives departures from spherical symmetry that can no longer be treated as minor corrections. In this regime, the equilibrium figure of the stars, their external gravitational field, and the tidal and rotational flows within their interiors become coupled to the orbital dynamics (Kopal, 1959; Zahn, 1977; Hut, 1981; Ogilvie, 2014). These tidal and rotational deformations underlie a wide range of observable and evolutionary phenomena, including ellipsoidal variability and gravity-darkening signatures in light curves, apsidal motion, tidal synchronisation and orbital circularisation, and eventually the onset of Roche-lobe overflow and mass transfer (Wilson & Devinney, 1971; Morris, 1985; Eggleton, 1983; Hilditch, 2001; Prša et al., 2016). Accurate modelling of these effects is therefore essential both for interpreting high-precision photometric and spectroscopic observations of close binaries and for predicting the long-term outcomes of interacting systems, from detached configurations to contact systems and mergers (Langer, 2012; de Mink et al., 2014; Fabry et al., 2022; Marchant & Bodensteiner, 2024).

From a physical point of view, binary deformation is governed by the competition between self-gravity, which tends to restore sphericity, and external perturbations that distort the body. The tidal field of the companion introduces a spatially varying gravitational force across the star and tends to elongate it approximately along the line of centres, while rotation produces centrifugal flattening about the spin axis (Kopal, 1959; Zahn, 1977; Ogilvie, 2014). A major modelling difficulty is that the problem spans regimes of increasing non-linearity as the orbital separation decreases. In wide and moderately distorted systems, approximate approaches are often sufficient: Roche-type descriptions provide a useful representation of the distorted surface geometry, while perturbative treatments of stellar structure describe the tidal and rotational response as small departures from a spherically symmetric model (Wilson & Devinney, 1971; Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Hilditch, 2001; Prša et al., 2016). In this regime, the deformation can often be described in terms

of low-order multipoles, and the stellar response may be summarised through effective coefficients linking the perturbing potential to the induced gravitational response (Smeyers & Willems, 2001; Claret & Willems, 2003; Claret & Giménez, 2010). In close systems, however, higher-order multipoles should become non-negligible as the surface departs strongly from spherical symmetry, and the redistribution of mass can modify the gravitational potential itself.

For global deformations, that is, for the study of the stellar surface shape and its observable properties, the Roche model remains the most commonly used framework (Kopal, 1959; Wilson & Devinney, 1971; Hilditch, 2001; Prša et al., 2016). In this approximation, the companion is treated as a point mass, and the deformation of the star under study is obtained from the corresponding effective potential. This approach has proved remarkably successful for many detached and moderately distorted systems, and it remains widely used in the modelling of eclipsing and interacting binaries. Nevertheless, important questions remain open in the strongly deformed regime: what is the quantitative contribution of higher-order terms in the potential, how does the internal redistribution of mass modify the gravitational field and the resulting surface shape, and to what extent does the point-mass approximation remain reliable for highly distorted stars? The aim of this chapter is to address these questions by developing a new three-dimensional, non-perturbative modelling framework in which the gravitational potential is computed self-consistently from the distorted density distribution, the multipolar structure of the potential is not restricted to the lowest-order terms, and the redistribution of mass is explicitly taken into account. This new tool, MoBiDICT, therefore provides a direct test of the validity limits of Roche-type models in close binary systems, despite their widespread use even in highly deformed and overcontact configurations.

In this chapter, I first introduce the general properties and assumptions adopted to model the deformation of close binaries in Section 3.1. I then present the general modelling procedure used to describe binary deformation, in particular the Roche model and the assumptions on which it relies, in Section 3.2. Next, in Section 3.3, I introduce the general non-perturbative modelling procedure, before describing the post-processing applied to the models in Section 3.4. In Section 3.5, I discuss one of the main limitations of the non-perturbative method, namely the choice of a critical direction used to reconstruct the three-dimensional structure of the models. In Section 3.6, I describe the numerical optimisation and implementation of MoBiDICT, which make the computation of efficient three-dimensional models possible. Finally, in Section 3.7, I compare the surface properties predicted by the three-dimensional non-perturbative models and by Roche models for highly distorted stars, with the aim of determining the regime in which non-perturbative modelling becomes necessary to reproduce the surface properties of strongly deformed binaries.

3.1. GENERAL MODELLING AND PROPERTIES OF BINARY DEFORMATIONS

3.1 GENERAL MODELLING AND PROPERTIES OF BINARY DEFORMATIONS

This section introduces the general framework used to model deformed stars in close binary systems. The aim is to highlight the physical assumptions that make the problem tractable and to summarise the main geometrical and structural properties that follow from them. These ideas will be used throughout the rest of the chapter, both in simplified perturbative descriptions and in our new three-dimensional non-perturbative framework (Endal & Sofia, 1976; Song et al., 2009; Siess et al., 2013; Fabry et al., 2022; Fellay et al., 2024).

A particularly useful simplification for modelling stellar deformation is the barotropic case. Under this condition, the pressure can be written solely as a function of density, so that the equation of hydrostatic equilibrium can be integrated provided that the remaining accelerations are conservative, that is, derivable from scalar potentials. (Ostriker & Mark, 1968; Hachisu, 1986; Rieutord, 2006). For a synchronously rotating binary in a corotating frame, the total effective potential then includes the self-gravitational potential, the external tidal potential of the companion, and the centrifugal potential (Kopal, 1959; Hachisu & Eriguchi, 1984). The stellar structure is organised by surfaces of constant total potential, which coincide with surfaces of constant pressure and density in the barotropic limit. This greatly simplifies the computation of the stellar shape, internal stratification, and effective gravity. This property is central to self-consistent two-dimensional and three-dimensional models of rotational and tidal deformation (Roxburgh, 2004, 2006; Manchon, 2021; Houdayer & Reese, 2023), and to the derivation of reliable corrections to observables such as surface gravity, emergent flux, and apsidal-motion rates (Kippenhahn & Thomas, 1970; Endal & Sofia, 1976; Espinosa Lara & Rieutord, 2013; Fabry et al., 2022; Fellay et al., 2024).

If all the forces derive from potentials, then the hydrostatic structure of a rotating component i of the binary is governed by

$$\frac{\nabla P}{\rho} = -\nabla(\Psi_1 + \Psi_2 + \Psi_c) = \mathbf{g}_{\text{eff}}, \quad (3.1)$$

together with the Poisson equation, Equation 2.72. Here, $\rho(r, \mu, \phi)$ and $P(r, \mu, \phi)$ denote the density and pressure in spherical polar coordinates (r, μ, ϕ) , with $\mu = \cos \theta$, θ the polar angle, and ϕ the azimuthal angle. The direction joining the centre of the star to its companion is taken to be $(\mu = 0, \phi = 0)$. The self-gravitational potential of the chosen star is denoted by $\Psi_1(r, \mu, \phi)$, the potential of the companion by $\Psi_2(r, \mu, \phi)$, and the centrifugal contribution by Ψ_c .

A key assumption that makes this formulation possible is that of solid-body rotation. Under rigid rotation, the centrifugal acceleration derives from a scalar potential, so that the total force remains conservative (Endal & Sofia, 1976; Hut, 1981; Maeder & Meynet, 2000; Song et al., 2009). If differential rotation is present, additional terms appear because of spatial gradients in the angular velocity. These terms are generally non-conservative, are associated with meridional flows and angular-momentum redistribution, and lead to a baroclinic structure in which isobars no longer coincide with

3.1. GENERAL MODELLING AND PROPERTIES OF BINARY DEFORMATIONS

equipotential surfaces (Maeder & Meynet, 2000; Aerts et al., 2019). In that regime, the geometry of the stellar structure becomes substantially more complicated.

Assuming solid-body rotation, all the forces considered here derive from a potential and the problem is conservative. Taking the curl of Equation 3.1 gives

$$\nabla \times \left(\frac{\nabla P}{\rho} \right) = - \frac{\nabla \rho \times \nabla P}{\rho^2} = - \nabla \times \nabla (\Psi_1 + \Psi_2 + \Psi_c) = \mathbf{0}, \quad (3.2)$$

which implies

$$\nabla \rho \times \nabla P = \mathbf{0}. \quad (3.3)$$

Hence, the density and pressure are constant on the same equipotential surfaces, and the stellar structure is barotropic (Maeder & Meynet, 2000; Song et al., 2009). At fixed chemical composition, the temperature is likewise constant on equipotentials, so that the structure is also thermotropic (Song et al., 2009).

In the present work, the barotropic approximation is especially convenient because the tidal deformation is generated by a gravitational potential. For a uniformly rotating star, the centrifugal acceleration can also be written as the gradient of a scalar potential. Thus, the hydrostatic balance can be expressed in terms of a single total potential combining self-gravity, the tidal potential of the companion, and the centrifugal potential (Kopal, 1959; Hachisu, 1986). This provides a mathematically consistent framework for constructing equilibrium models in which pressure, density, and total potential surfaces coincide.

Physically, the assumption of quasi-uniform rotation may be viewed as an idealised limiting case motivated by angular-momentum transport processes: tidal torques tend to synchronise the stellar spin with the orbit, while convection, magnetic stresses, waves, and other internal transport mechanisms can reduce radial shear (Zahn, 1977; Hut, 1981; Ogilvie, 2014; Aerts et al., 2019). This picture is broadly compatible with the evidence for efficient angular-momentum redistribution inferred from asteroseismology, although the degree of internal differential rotation depends strongly on stellar mass and evolutionary state (Beck et al., 2012; Mosser et al., 2012; Deheuvels et al., 2014; Aerts, 2021). In the low-mass regime, deep convective envelopes are expected to maintain nearly homogeneous rotation (Barnes, 2003; Gallet & Bouvier, 2015). Once a radiative core forms, however, radial differential rotation can in principle develop as the core and envelope evolve on different timescales (Maeder & Meynet, 2000; Spruit, 2002; Eggenberger et al., 2019; Gallet & Bouvier, 2015). Direct observational constraints remain limited, except in evolved red-giant stars, where mixed-mode asteroseismology has shown that the core generally rotates faster than the envelope (Beck et al., 2012; Mosser et al., 2012; Eggenberger et al., 2012; Deheuvels et al., 2014; Aerts et al., 2019; Fellay et al., 2021). Around $1.4 M_{\odot}$, corresponding roughly to the γ -Doradus instability strip, stars possess convective cores, radiative envelopes, and only shallow outer convective layers. Gravity-mode asteroseismology of such stars provides direct constraints on near-core rotation and generally points to low internal differential rotation, consistent with efficient angular-momentum redistribution (Van Reeth et al., 2016; Van Reeth et al., 2018; Li et al., 2019, 2020a).

3.2. ROCHE MODEL

For more massive stars, stable internal magnetic fields in their extended radiative envelopes is believed to efficiently redistribute angular momentum through magnetic stresses and thus favour nearly solid-body rotation (Ferraro, 1937; Maeder & Meynet, 2003; Braithwaite & Nordlund, 2006; Briquet et al., 2010). Nevertheless, asteroseismic studies of some β -Cephei stars show that differential rotation can occur, with the near-core region rotating faster than the outer envelope in cases such as HD 129929 and ν Eridani (Aerts et al., 2003; Pamyatnykh et al., 2004; Bowman, 2020).

In this thesis, we adopt solid-body rotation as the reference approximation, while keeping in mind that it is an idealisation. Throughout this work, we also assume that the stellar rotation axes are perpendicular to the orbital plane and therefore aligned with the orbital angular-momentum vector. This assumption is motivated by the tendency of tidal torques to damp spin–orbit misalignment in close binaries, although the alignment timescale depends on the stellar structure and dissipation mechanism (Hut, 1981; Ogilvie, 2014). The limitations of this assumption will be discussed in the conclusion of this thesis.

When the stellar spin axes are aligned with the orbital angular-momentum vector, two planes of symmetry arise naturally. The first is the orbital plane. The second is the meridional plane containing the two stellar spin axes and the orbital angular-momentum axis. Exploiting these two symmetries, it is sufficient to compute only one quarter of each star in order to reconstruct the full three-dimensional structure of the system. These symmetry planes are illustrated in Figure 1.1. In spherical-like coordinates (r, μ, ϕ) , with $\mu = \cos \theta$, the computational domain may therefore be restricted to

$$r \in [0, r_{L1}], \quad \mu \in [1, 0], \quad \phi \in [0, \pi],$$

where r_{L1} is the distance to the inner Lagrange point L_1 in the direction $(\mu = 0, \phi = 0)$.

The general properties described in this section are common to all the modelling approaches considered in this chapter. Their main difference lies in how the gravitational potential of each stellar component is expressed and computed. In the next section, we introduce the Roche model, which relies on a highly simplified expression for that potential (Eggleton, 1983; Siess et al., 2013). In the subsequent sections, we then develop our non-perturbative method for computing the gravitational potential of close binary stars in a more self-consistent manner.

3.2 ROCHE MODEL

The Roche model, classically associated with the work of Edouard Roche, provides one of the simplest and most widely used theoretical descriptions of close binary systems (Kopal, 1959; Hilditch, 2001; Kallrath, 2009). Its enduring success rests on a set of idealised assumptions that remain useful in many regimes. In its classical formulation, the gravitational field of each component is represented by the monopole field of a point mass located at its centre of mass, or equivalently by that of a spherically symmetric body outside its surface. The two masses are assumed to move on a circular orbit, and the stellar surfaces are described in a frame corotating synchronously with

3.2. ROCHE MODEL

the orbital motion (Wilson & Devinney, 1971; Eggleton, 1983; Prša et al., 2016).

As discussed in the previous section, under the assumption of solid-body rotation, the structure of the system can be described in terms of equipotential surfaces of a total potential that includes self-gravity, the gravity of the companion, and the centrifugal contribution in the corotating frame (Zahn, 1975, 1977; Hut, 1981; Eggleton, 1983; Song et al., 2009; Siess et al., 2013). A surface of particular importance is the critical equipotential that passes through the inner Lagrange point L_1 . Located between the two stars, L_1 is the point at which the effective force vanishes in the corotating frame, that is, where the gravitational and centrifugal contributions exactly balance (Eggleton, 1983; Sepinsky et al., 2007). The corresponding equipotential divides the system into two distinct gravitational domains, the Roche lobes, each enclosing the region within which material remains gravitationally bound to one of the components (Eggleton, 1983; Sepinsky et al., 2007). The geometry and size of the Roche lobes depend sensitively on the mass ratio and orbital separation, and the degree to which a star approaches or fills its Roche lobe is therefore a key indicator of its evolutionary state (Paczynski, 1971; Eggleton, 1983; Hilditch, 2001). When stellar expansion or orbital contraction causes the stellar surface to reach the critical Roche equipotential, matter is no longer confined within the donor's Roche lobe and can flow through the inner Lagrange point, L_1 , toward the companion. This process, known as Roche-lobe overflow (RLOF), is one of the principal channels of mass transfer in close binaries (Paczynski, 1971; Tauris & van den Heuvel, 2006). It strongly influences the subsequent evolution of the system by modifying the component masses, orbital separation, angular-momentum budget, and observable properties (Soberman et al., 1997; Wellstein et al., 2001; Langer, 2012; de Mink et al., 2014). Its onset, stability, and long-term behaviour depend intricately on the binary mass ratio, the response of the donor star to mass loss, the accretion efficiency of the companion, and the time-dependent evolution of the orbital configuration (Soberman et al., 1997; Tauris & van den Heuvel, 2006; Temmink et al., 2023).

Although the classical Roche model assumes point-mass gravitational sources, the effective potential it defines naturally predicts tidal and rotational distortions of stellar surfaces in close binaries (Kopal, 1959; Wilson & Devinney, 1971; Hilditch, 2001). In a synchronously rotating binary, each star is subject not only to its own self-gravity, but also to the gravitational field of its companion and to the centrifugal acceleration in the corotating frame. If the stellar surface is approximated by an equipotential surface, then spatial variations in this effective potential immediately imply departures from spherical symmetry (Kallrath, 2009; Prša et al., 2016). These distortions become increasingly significant as a star approaches its Roche lobe: the surface is progressively elongated along the line joining the two components and compressed in the transverse directions, so that the morphology evolves from nearly spherical to markedly ellipsoidal and, near the critical equipotential passing through L_1 , to the characteristic Roche-lobe-filling, teardrop-like configuration (Eggleton, 1983; Hilditch, 2001; Kallrath, 2009). In Figure 3.1, we illustrate the classical topology of the equipotential surfaces in the Roche model and the associated deformation regimes.

The Roche model is therefore widely used to compute the effective gravitational po-

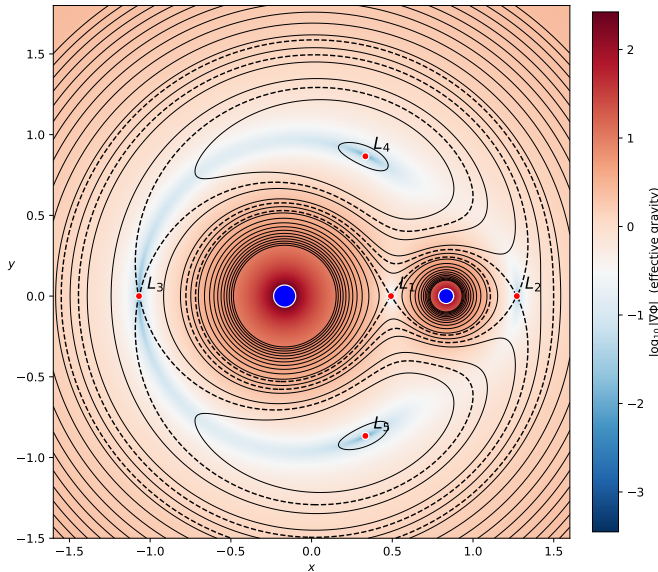


Figure 3.1: Illustration of the topology of the equipotential surfaces in the Roche model. The black curves show the equipotential surfaces, and the red dots indicate the Lagrange points, where the effective gravity vanishes. The colour scale represents the magnitude of the effective gravity, $|\mathbf{g}_{\text{eff}}|$, with red regions corresponding to high values and blue regions to low values.

tential of distorted binaries, both at the stellar surface and in the surrounding space (Eggleton, 1983; Budaj, 2011; Siess et al., 2013; Fabry et al., 2022). In the most extreme applications, this potential is even used as a proxy for the structural deformation of the stellar components through correction factors such as f_p and f_T , effectively replacing the full internal response of the stars by the external Roche geometry (Fabry et al., 2022). In its classical formulation, the total Roche potential at a given point in space is written as

$$\Psi_{\text{tot},i} = \Psi_1 + \Psi_2 + \Psi_c, \quad (3.4)$$

where, in the point-mass approximation,

$$\Psi_i = -\frac{GM_i}{r_i}. \quad (3.5)$$

The centrifugal contribution is simplest in the classical case of circular, synchronised motion, where the corotating frame is uniquely defined (Hut, 1981; Eggleton, 1983; Siess et al., 2013). More general formulations can be constructed for asynchronous rotation and eccentric orbits by introducing a more complicated expression for the effective potential and for the positions of the Lagrange points (Sepinsky et al., 2007). A more detailed discussion of the centrifugal potential adopted in this work is given in

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the following subsection, [Subsection 3.2.1](#). [Subsection 3.2.2](#) is devoted to extending the classical Roche formalism toward a refined version of the potential that better accounts for stellar deformation, and to presenting alternative forms of the Roche potential proposed in the literature.

3.2.1 ON THE CENTRIFUGAL POTENTIAL

We consider a configuration in which each star i in the binary has its own rotation rate Ω_i , which is not necessarily equal to the orbital mean motion n . Assuming that both stellar spin axes are aligned with the orbital angular-momentum vector, the system has two symmetry planes that are exploited by our method: the orbital plane and the meridional plane containing the rotation axes and the line of centres ([Hut, 1981](#); [Sepinsky et al., 2007](#); [Song et al., 2009](#)). A three-dimensional representation of this geometry is shown in [Figure 1.1](#).

Asynchronous binaries can be treated directly by generalising the centrifugal part of the effective potential. In the classical synchronised case, assuming solid-body rotation and a circular orbit, the centrifugal potential in a frame centred on star i can be written as

$$\Psi_c(x, y) = -\frac{n^2}{2} [(x_{\text{CM}} - x)^2 + y^2], \quad (3.6)$$

where n is the orbital angular frequency and x_{CM} is the distance between the centre of the star and the system center of mass ([Hut, 1981](#); [Song et al., 2009](#)). In the more general asynchronous case, still assuming solid-body stellar rotation and a circular orbit, this expression becomes

$$\Psi_c(x, y) = n^2 x_{\text{CM}} x - \frac{\Omega_i^2}{2} (x^2 + y^2), \quad (3.7)$$

In [Equation 3.6](#) and [Equation 3.7](#), we use Cartesian coordinates (x, y) centred on the star under consideration: x is the axis of the binary joining the two stars while y is perpendicular to x but remains in the orbital plane.

As expected, for a synchronised system with solid-body rotation, $\Omega_i = n$, [Equation 3.6](#) and [Equation 3.7](#) are equivalent up to an additive constant. The corresponding effective gravity is obtained from the negative gradient of Ψ_c , as required by the relation between force and potential ([Hut, 1981](#)).

This formalism can be extended further to eccentric systems by considering the instantaneous orbital phase and decomposing the motion of each fluid element into a Keplerian motion about the system center of mass plus a cylindrical rotation about the stellar spin axis ([Sepinsky et al., 2007](#); [Davis et al., 2013](#)). In that case, when the instantaneous separation between the stellar centres is r_{sep} , the factor n^2 in [Equation 3.7](#) is replaced by

$$\tilde{n}^2 = \frac{GM_{\text{tot}}}{r_{\text{sep}}^3}, \quad (3.8)$$

3.2. ROCHE MODEL

so that $\tilde{n}^2 r_{\text{sep}}$ is equal to the norm of the instantaneous relative Keplerian acceleration. Here, M_{tot} denotes the total mass of the system. In practice, this eccentric generalisation should be understood as an instantaneous or quasi-static description of the effective potential at a given orbital phase (Sepinsky et al., 2007; Davis et al., 2013). In that case, the centrifugal potential reduces to

$$\Psi_c(x, y) = \tilde{n}^2 x_{\text{CM}} x - \frac{\Omega_i^2}{2} (x^2 + y^2), \quad (3.9)$$

which, in spherical coordinates (r, θ, ϕ) centred on the star, becomes

$$\Psi_c(r, \theta, \phi) = \tilde{n}^2 x_{\text{CM}} r \sin \theta \cos \phi - \frac{\Omega_i^2}{2} r^2 \sin^2 \theta. \quad (3.10)$$

These expressions form the basis of our non-perturbative treatment of centrifugal effects in synchronised and asynchronous close binaries.

3.2.2 EXTENSION OF THE ROCHE FORMALISM FOR NON-POINT-LIKE BODIES AND NON-TRADITIONAL ROCHE MODEL

The classical Roche potential, written in Equation 3.5, is well suited to describing the exterior gravitational field of a point-mass companion, but it is not a solution of Poisson's equation for the actual density distribution inside a star. For that reason, using the classical Roche potential to modify a stellar model in hydrostatic equilibrium, is not fully self-consistent. To improve upon the classical prescription while keeping the simplicity of the Roche formalism, we constructed a refined Roche potential. The idea is to require that the gravitational potential inside the star be a solution of Poisson's equation for the underlying one-dimensional stellar model, while remaining identical to the classical Roche potential outside the star, where Equation 3.5 remains valid. Assuming that the input stellar model is one-dimensional and spherically symmetric, the density and potential expansions simplify considerably because only the $(\ell, m) = (0, 0)$ term is non-null. In that case,

$$\rho_{i,1D}(r) = \rho_{i,\ell=0}^{m=0}(r), \quad \Psi_i(r) = \Psi_{i,\ell=0}^{m=0}(r),$$

where $\rho_{i,1D}$ is the one dimensional profile of the input stellar model. The solution of Poisson's equation may be written as

$$\Psi_i(r) = 4\pi G \int_{R_i}^{r_i} r_i'^{-2} \left[\int_0^{r_i'} r_i''^2 \rho_{i,1D}(r_i'') dr'' \right] dr' - \frac{4\pi G}{R_i} \int_0^{R_i} r_i'^2 \rho_{i,1D}(r_i') dr'. \quad (3.11)$$

Defining the enclosed mass

$$m_i(r) = 4\pi \int_0^{r_i} r_i'^2 \rho_{i,1D}(r_i') dr', \quad (3.12)$$

the potential can be rewritten more compactly as

$$\Psi_i(r) = \int_{R_i}^{r_i} \frac{Gm_i(r')}{r'^2} dr' - \frac{GM_i}{R_i}. \quad (3.13)$$

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This expression is now consistent with the one-dimensional density profile of the stellar model and therefore provides a much more physically meaningful interior potential than the classical Roche approximation.

The refined Roche prescription nevertheless remains approximate. By construction, all non-spherical terms of the multipole expansion are neglected, which means that the feedback of tidal and rotational deformation on the internal structure of the star, and on the gravitational field felt by its companion, is still ignored. The advantage of the method is that it is already substantially more consistent than the classical Roche model while remaining relatively simple to implement. Its main drawback is that it requires the full internal density profile of the star, whereas the classical Roche model depends only on the total masses or mass ratio and orbital configuration. This simplicity explains why the classical Roche formalism remains so widely used in binary-star modelling despite its simplicity (Eggleton, 1983; Budaj, 2011; Fabry et al., 2022). The non-perturbative method introduced in the following section is designed precisely to recover the missing non-spherical contributions by starting from this spherically symmetric solution and iteratively accounting for the full tidal and rotational distortions.

It should also be noted that the Roche model does not have a unique implementation in the literature. The most classical form assumes circular orbits and synchronous rotation, but the formalism can be generalised to asynchronous and eccentric systems by modifying the centrifugal contribution, as discussed in the previous subsection and in studies of eccentric and non-synchronous binaries (Hut, 1981; Sepinsky et al., 2007; Davis et al., 2013). In addition, some authors adopt an alternative interior potential of the form

$$\Psi_i(r) = -\frac{Gm(r)}{r_i}. \quad (3.14)$$

Although this expression introduces the enclosed mass explicitly, it still does not provide a self-consistent solution of Poisson's equation in the distorted configuration and therefore cannot reproduce the internal response of the star accurately when tidal and rotational deformations become important (Song et al., 2009; Fabry et al., 2022; Fellay et al., 2024). For this reason, it is not commonly used in modern binary-structure calculations. In Subsection 3.7.6, we compare these different Roche-type prescriptions in terms of the resulting internal structure and surface properties, and confront them with the full non-perturbative solution developed in the following section.

3.3 MoBiDICT (MODELLING BINARIES DEFORMATIONS INDUCED BY CENTRIFUGAL AND TIDAL FORCES)

this section is extracted from Fellay & Dupret (2023)

To study the limits of classical models and get an accurate computation of the stellar surface deformations even in the most distorted cases, we developed a new non-perturbative approach in three dimensions (3D) based on a method presented in Roxburgh (2004, 2006) and recently used by Manchon (2021); Houdayer & Reese (2023)

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Figure 3.2: Logo of MoBiDICT.

to model rotating stars in two dimensions. This tool called Modelling Binary Deformation Induced by Centrifugal and Tidal forces (MoBiDICT) is developed in Fortran 95 for the computational efficiency. The principle of MoBiDICT is to iteratively solve Poisson's equation including the centrifugal and tidal forces in a non-perturbative way. More than obtaining accurate surface quantities, such non-perturbative models are giving access to the entire, precise, 3D structure of each binary system component, allowing accurate computation of gravity darkening or apsidal motion constant even in the most distorted cases. Finally, the impact of deformations on the time evolution of the stellar internal structure will be accessible with such detailed models.

3.3.1 GENERAL MODELLING PROCEDURE

The basic idea behind MoBiDICT is to start from two 1D stellar structure given by any 1D stellar evolution code and specifically their density profiles. Then using the barotropic structure (the density is constant on the equipotentials) of our models and assuming that this density profile corresponds to the structure of star in a given direction we can reconstruct the full 3D structure by integrating the Poisson equation. The interactions are then accounted for by iteratively solving the Poisson equation with the newly distorted structures of each star in the system, therefore accounting for the redistribution of mass within the bodies. In this section we are going to detail the modelling procedure in MoBiDICT.

To take advantage of the barotropic properties of the models, we assume that the density of each star is given by a one dimensional, spherically symmetric input model along a given arbitrary direction ($\mu_{\text{crit}}, \phi_{\text{crit}}$). Therefore, in this direction, we can associate a total potential to a density distribution, and by computing the total potential on the whole star, we deduce the 3D density profile of each component of the system.

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The choice of the direction $(\mu_{\text{crit}}, \phi_{\text{crit}})$ is detailed in [Section 3.5](#).

The principle of our modelling method is to solve Poisson's equation iteratively on a spherical-harmonic basis for each stellar component, in order to compute in a non-perturbative way the effect of tidal and centrifugal forces on the hydrostatic structure of the stars. In practice, the method generalises earlier two-dimensional approaches ([Roxburgh, 2004, 2006](#); [Manchon, 2021](#); [Houdayer & Reese, 2023](#)) to a fully three-dimensional treatment adapted to binary systems. Each iteration of the modelling procedure proceeds as follows:

1. We start from a spherically symmetric one-dimensional stellar model for each component i , obtained from a stellar evolution code. The initial density distribution is therefore taken to be

$$\rho_i(r, \mu, \phi) = \rho_{i,1D}(r),$$

where $\rho_{i,1D}(r)$ is the 1D input profile of each star.

2. Using this density distribution, we solve Poisson's equation and compute the total effective potential:

$$\Psi_{\text{tot}} = \Psi_1 + \Psi_2 + \Psi_c,$$

for each star.

3. We then update the density field from the total potential by exploiting the barotropic nature of the problem. More precisely, along a chosen reference direction $(\mu_{\text{crit}}, \phi_{\text{crit}})$, we compute $\Psi_{\text{tot}}(r, \mu_{\text{crit}}, \phi_{\text{crit}})$ and impose

$$\rho_i(r, \mu_{\text{crit}}, \phi_{\text{crit}}) = \rho_{i,1D}(r).$$

Since both ρ_i and Ψ_{tot} are monotonic functions of r along this reference direction, this allows us to construct the functional relation

$$\rho_i = \rho_i(\Psi_{\text{tot}}).$$

The full three-dimensional density field is then obtained by applying this relation throughout the star:

$$\rho_i(r, \mu, \phi) = \rho_i(\Psi_{\text{tot}}(r, \mu, \phi)).$$

4. Finally, we compute the quantities $\delta\rho_i(r, \mu, \phi)$ and $\delta\Psi_i(r, \mu, \phi)$, which serve as convergence indicators and are also used to readjust the model. If convergence has not yet been reached, the updated density field obtained in step 3 is used as the input for the next iteration, and the procedure is repeated from step 2.

The modelling strategy is illustrated in the operating diagram presented in [Figure 3.3](#). This iterative strategy ensures that the density, gravitational potential, and total potential remain mutually consistent at each stage of the computation. In this sense, the method provides a self-consistent route from an initially spherical one-dimensional stellar model to a fully deformed three-dimensional hydrostatic configuration. In [Figure 3.4](#), we illustrate the general structure of a $1.0 M_{\odot}$ MS primary star in a binary system with the equipotentials and the critical direction.

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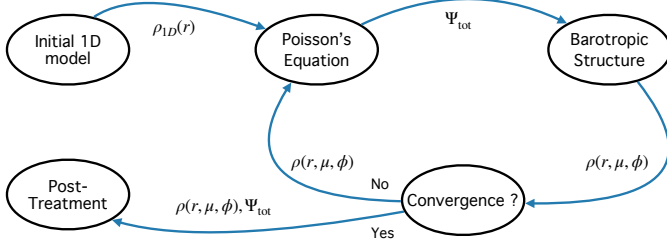


Figure 3.3: Operating diagram of MoBiDICT.

3.3.2 SOLUTION FOR Ψ_{tot} GIVEN ρ_i

Given $\rho_i(r, \mu, \phi)$ on a grid (r, μ, ϕ) for each star i , we want to solve Poisson's equation (Equation 2.72) in order to obtain the potential $\Psi_i(r, \mu, \phi)$ and deduce $\Psi_{\text{tot}} = \Psi_1 + \Psi_2 + \Psi_c$. We decided to adopt an individual spherical coordinates grid (r_i, μ, ϕ) centred on each star i with r_i pointing toward the centre of mass of the system when $\phi = 0$ and $\mu = 0$. Taking advantage of the two symmetrical plans of the problem, we can limit the angular domain to $\mu \in [1, 0]$ and $\phi \in [0, \pi]$.

Generalising the method presented by Roxburgh (2006) in 3D and for binaries, we expressed ρ and Ψ as a finite sum of spherical harmonics with

$$\rho_i(r_i, \mu, \phi) = \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \rho_{i,\ell}^m(r_i) d_\ell^m P_\ell^m(\mu) \cos(m\phi), \quad (3.15)$$

and

$$\Psi_i(r_i, \mu, \phi) = \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \Psi_{i,\ell}^m(r_i) d_\ell^m P_\ell^m(\mu) \cos(m\phi), \quad (3.16)$$

where $p = 0$ if ℓ is even, $p = 1$ if otherwise, $m = 2k + p$ and d_ℓ^m is a normalisation constant for the spherical harmonics. Owing to the symmetries of the system, only spherical harmonics with non-negative values of m and with m having the same parity as ℓ have non-zero coefficients. The free parameter of our model defining the maximum degree of the spherical harmonics to account for is as denoted L . In our work, the spherical harmonics $Y_\ell^m(\mu, \phi) = d_\ell^m P_\ell^m(\mu) \cos(m\phi)$ were normalised as

$$d_\ell^m d_{\ell'}^{m'} \int_0^\pi \int_0^1 P_\ell^m(\mu) \cos(m\phi) P_{\ell'}^{m'}(\mu) \cos(m'\phi) d\mu d\phi = \delta_{\ell\ell'} \delta_{mm'}, \quad (3.17)$$

where δ_{ij} is Kronecker's delta, only $\neq 0$ and $= 1$ when $i = j$, which gives a normalisation constant expressed as

$$d_\ell^m = \begin{cases} \sqrt{\frac{2\ell+1}{\pi}} & m = 0 \\ \sqrt{\frac{2(2\ell+1)(\ell-|m|)!}{\pi(\ell+|m|)!}} & m \neq 0 \end{cases}. \quad (3.18)$$

3.3. MOBIDICT (MODELLING BINARIES DEFORMATIONS INDUCED BY CENTRIFUGAL AND TIDAL FORCES)

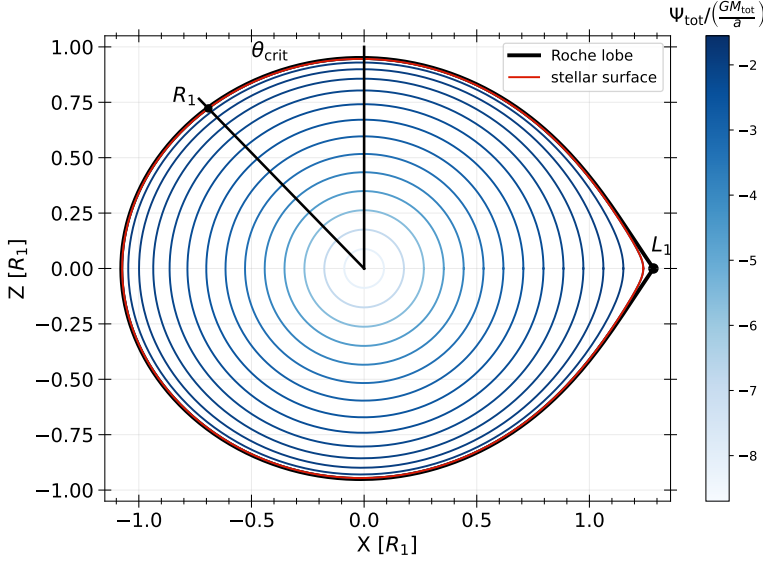


Figure 3.4: View of a $1.0 M_{\odot}$ MS primary star in a binary system, cut in the plane including all the rotations axes of the system. The companion to this star is a $0.2 M_{\odot}$ MS secondary star with an orbital separation of $r = 2R_1$. The black curve corresponds to the Roche lobe of the primary star. The black lines respectively correspond to the reference direction for θ and to the reference direction in which our model is given by the 1D spherical symmetric model ($\theta = \mu_{\text{crit}}$). The blue lines are equipotentials inside the star with the colourbar representing the total potential. On each equipotential the pressure, density, and temperature, are constant. L_1 is the Lagrangian point and R_1 is the radius of the initial 1D model in the reference direction.

To obtain $\Psi_{i,\ell}^m$, we solve Poisson's equation projected onto the spherical-harmonic basis, as given in Equation 2.82. The densities are projected onto the same basis using the integral relation introduced above in Equation 2.68, adapted here to the new integration domain:

$$\rho_{i,\ell}^m(r_i) = d_{\ell}^m \int_0^1 \int_0^{\pi} \rho_i(r_i, \mu, \phi) P_{\ell}^m(\mu) \cos(m\phi) d\phi d\mu. \quad (3.19)$$

As mentioned by Roxburgh (2006), the computation of $\rho_{i,\ell}^m(r_i)$ through Equation 3.19 using a detailed mesh can become a real problem in terms of computation time and the precision required. To reduce this problem, we decided to use a mesh of μ and ϕ following the Gauss-Legendre quadrature method. Detailed informations about the numerical treatments done in MoBiDICT can be found in Section 3.6.

As shown in Section 2.A an integral representation of the solution of Equation 2.82

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with the boundary conditions presented in Equation 2.144 can be written as

$$\Psi_{i,\ell}^m(r_i) = r_i^\ell 4\pi G \int_{R_i}^{r_i} r_i'^{-(2\ell+2)} \left[\int_0^{r_i'} r_i''^{\ell+2} \rho_{i,\ell}^m(r_i'') dr_i'' \right] dr_i' - \frac{r_i^\ell}{R_i^{2\ell+1}} \frac{4\pi G}{2\ell+1} \int_0^{R_i} r_i'^{\ell+2} \rho_{i,\ell}^m(r_i') dr_i'. \quad (3.20)$$

An equivalent representation of Equation 3.20 used by Kopal (1959, 1978); Fitzpatrick (2012), for example, and presented in Section 2.2 is expressed as

$$\Psi_{i,\ell}^m(r_i) = -\frac{4\pi G}{(2\ell+1)r_i^{\ell+1}} \int_0^{r_i'} r_i''^{\ell+2} \rho_{i,\ell}^m(r_i'') dr_i'' - \frac{4\pi G r_i^\ell}{2\ell+1} \int_r^{R_i} r_i'^{1-\ell} \rho_{i,\ell}^m(r_i') dr_i'. \quad (3.21)$$

Both equivalent solutions were implemented in our code, and the only difference between the two is related to the numerical treatment of the integrals in the limit $r_i \rightarrow 0$. To avoid any complications in this region, we exploit the linearity of $\rho_{i,\ell}^m(r_i)/r_i^\ell$ close to the singularity with our trapezoidal integration method to increase the integration precision. With both solutions, the same results were obtained. For all the work presented in the rest of this article, we used the solution given by Equation 3.20.

Inside MoBiDICT, all the equations solved and all the quantities used are written in dimensionless form in order to facilitate the computations. To nondimensionalize the equations and variables, we adopted characteristic constants of the binary system. The corresponding dimensionless variables are defined as

$$x_i = r_i/a; \quad m_i' = m_i/M_{\text{tot}}; \quad \rho_i' = \rho_i / \left(\frac{M_{\text{tot}}}{4\pi a^3} \right); \quad \Psi_i' = \Psi_i / \left(\frac{GM_{\text{tot}}}{a} \right); \quad \Omega_\star'^2 = \Omega_\star^2 / \left(\frac{GM_{\text{tot}}}{a^3} \right), \quad (3.22)$$

where a is the separation of the binaries, M_{tot} is the total mass of the binary system, $\Omega_\star$ is the orbital rotation rate, and G is the gravitational constant. In our case, the quantities used to non-dimensionalise were chosen to simplify the $4\pi G$ constant in Poisson's equation and the integral representation of its solution (Equation 3.20 and Equation 3.21). In the rest of this article, the equations and quantities presented are with dimensions to maintain the physical sense of the problem.

Using the presented method, the following steps are performed to compute the total potential of each star:

1. With the density provided for each star $\rho_i(r_i, \mu, \phi)$, we determine their spectral expansion $\rho_{i,\ell}^m(r_i)$ through Equation 3.19.
2. We numerically compute Poisson's equation (Equation 2.82) solution given by Equation 3.20 for each star in order to obtain their gravitational potential projected on the spherical harmonics basis: $\Psi_{1,\ell}^m(r_1)$ and $\Psi_{2,\ell}^m(r_2)$.
3. Using the proper coordinates conversion and the previously determined spectral quantities, namely $\Psi_{1,\ell}^m(r_1)$ and $\Psi_{2,\ell}^m(r_2)$, we determined each gravitational potential on the grid of each star, obtaining $\Psi_1(r_1, \mu, \phi)$, $\Psi_1(r_2, \mu, \phi)$, $\Psi_2(r_1, \mu, \phi)$, $\Psi_2(r_2, \mu, \phi)$. Finally, we computed the total potential $\Psi_{\text{tot}} = \Psi_1 + \Psi_2 + \Psi_c$ on the grid of each star.

3.3.3 SOLUTION FOR ρ GIVEN Ψ_{tot}

Given Ψ_{tot} , which we determined in the Subsection 3.3.2, we wanted to estimate the corresponding density for of each star. We exploited the fact that the structure of each

3.3. MOBIDICT (MODELLING BINARIES DEFORMATIONS INDUCED BY CENTRIFUGAL AND TIDAL FORCES)

star is barotropic, meaning that the density is constant on the equipotentials.

Assuming that the density of a star i along a given direction μ_{crit} and ϕ_{crit} is the density of the corresponding averaged 1D input model, we could determine the functions $\rho_i(\Psi_{i,\text{tot}}(r_i, \mu_{\text{crit}}, \phi_{\text{crit}})) = \rho_{i,1D}(r_i)$ for each star. Then, by simple function composition and interpolation, we could estimate the density of the entire star with the total potential on each point of the grid. More informations can be found in [Section 3.5](#).

3.3.4 CRITERION FOR THE CONVERGENCE OF THE METHOD

The convergence of our method is guided by the variations of the density and gravitational potential between two successive iterations. These variations quantified the contribution of each iteration to the deformation of each point of a star and thus gave a direct control of the convergence of our modelling. The point at which deformation was expected to be the most important is at the point closest to the Lagrangian point L1 in each star i : $(R_{s,i}, \mu = 0, \phi = 0)$. Thus, if the variation of potential and density at this point was negligible, compared to the previous iteration, then the entire model had converged. In the rest of our work, $R_{s,i}$ denotes the radial coordinate of the stellar surface in a given direction (μ, ϕ) . Our normalised convergence criterion for a model at the iteration j is

$$\delta\rho_{i,j,\ell}^m = \frac{\rho_{i,j}(R_{s,i,j}, 0, 0) - \rho_{i,j-1}(R_{s,i,j-1}, 0, 0)}{\rho_{i,j}(R_{s,i,j}, 0, 0)}, \quad (3.23)$$

or

$$\delta\Psi_{i,j,\ell}^m = \frac{\Psi_{i,j}(R_{s,i,j}, 0, 0) - \Psi_{i,j-1}(R_{s,i,j-1}, 0, 0)}{\Psi_{i,j}(R_{s,i,j}, 0, 0)}. \quad (3.24)$$

Given the normalised precision of our 1D average input models of about 10^{-8} on the density, we used $\delta\rho_{i,j,\ell}^m < 10^{-8}$ and $\delta\Psi_{i,j,\ell}^m < 10^{-8}$ as the criterion for convergence for each star. This value can be adjusted in order to reduce the computation time of our models if high-precision modelling is not required.

3.3.5 VERIFICATION OF KEPLER'S THIRD LAW

At this modelling step, several readjustments were made in order to verify different basic laws impacted by the non-sphericity of the models. In particular, the Kepler's third law has to be verified by looking at the balance of forces at the centre of each star in order to properly recompute the centre of mass and the orbital period of the system.

In our models, the tidal and rotational deformation induces a redistribution of mass and modifies the gravitational field. As a result, the classical Keplerian description may no longer be exact, especially if the centre of mass of each deformed star is displaced relative to that of the corresponding spherically symmetric model. Such an effect is expected in strongly deformed systems, where the mass distribution can differ significantly between the hemisphere facing the companion and the opposite hemisphere. This front-back asymmetry is already apparent at the level of the Roche geometry, therefore, such correction is expected. We consider an arbitrary binary system with an orbital separation r_{sep} , a local orbital rotation rate $\tilde{n}$ (see [Subsection 3.2.1](#)),

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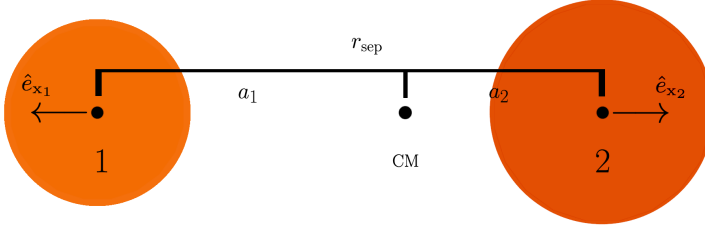


Figure 3.5: Schema of an arbitrary chosen binary system composed of two stars with different masses. a_1 and a_2 are the orbital separation of respectively the primary and secondary star from the centre of mass. $\hat{e}_{x_1}$ and $\hat{e}_{x_2}$ are the unit vectors of the x_1 and x_2 direction considered in this problem.

and a centre of mass noted CM. This system can be illustrated as in [Figure 3.5](#). The orbital separation of the system r_{sep} is a constant of the model. Thus, the centre of each star is at the dynamical equilibrium (i.e. we want the total acceleration to be null in the center), meaning that the sum of the gravitational, tidal, and centrifugal forces is null in their centre. Mathematically this condition is expressed as

$$\begin{cases} \lim_{x_1 \rightarrow 0} \frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_1} = \tilde{n}^2 x_{\text{CM}} r_{\text{sep}} \\ \lim_{x_2 \rightarrow 0} \frac{\partial \Psi_1}{\partial x_2} + \frac{\partial \Psi_2}{\partial x_2} = \tilde{n}^2 (1 - x_{\text{CM}}) r_{\text{sep}} \end{cases}, \quad (3.25)$$

where x_{CM} is defined as a_1/r_{sep} .

The system given in [Equation 3.25](#) can be summed and rewritten as

$$\begin{cases} \lim_{x_1 \rightarrow 0} \frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_1} = \tilde{n}^2 x_{\text{CM}} r_{\text{sep}} \\ \lim_{x_1 \rightarrow 0} \left(\frac{\partial \Psi_1}{\partial x_1} + \frac{\partial \Psi_2}{\partial x_1} \right) + \lim_{x_2 \rightarrow 0} \left(\frac{\partial \Psi_1}{\partial x_2} + \frac{\partial \Psi_2}{\partial x_2} \right) = \tilde{n}^2 r_{\text{sep}} \end{cases}. \quad (3.26)$$

In these equations, all the partial derivatives vanish except those associated with the dipolar component of the tidal potential, $\ell = 1$, because near the stellar centre the spherical-harmonic components of the potential scale as r^ℓ . In the absence of mass redistribution, these relations show that the dipolar part of the tidal potential is normally cancelled by the corresponding inertial or orbital term of the centrifugal potential when the orbital angular frequency satisfies Kepler's third law. Physically, in the center of stars, the $\ell = 1$ contribution describe the uniform acceleration of the stellar centre of mass associated with the orbital motion.

In the MoBiDICT framework, the second equation of this system can be solved to determine the corrected value of $\tilde{n}^2$ required to preserve the orbital separation of the binary once the redistribution of mass inside the stars is taken into account. Once $\tilde{n}^2$ is known, the first equation can then be solved to infer the new position of the system

3.4. POST-TREATMENT OF THE MODELS

center of mass in the presence of this internal mass redistribution. This readjustment implies that, for a binary observed with a given orbital period, non-linear structural effects can slightly modify the actual orbital separation relative to the value predicted by a simpler point-mass treatment and Kepler's third law. The correction to Kepler's third law are explored numerically in [Subsection 3.7.4](#).

3.4 POST-TREATMENT OF THE MODELS

this section is extracted from [Fellay & Dupret \(2023\)](#)

3.4.1 INCLUDING 2D-3D DEFORMATIONS IN 1D STELLAR EVOLUTION CODES

Once we obtained a final model with the desired precision, the models go into a post-processing phase. During this phase, the different properties of the models are extracted and used to potentially include our new 3D models in stellar evolutionary models using the method of [Kippenhahn & Thomas \(1970\)](#). Later revisited by [Meynet & Maeder \(1997\)](#), this method has been extensively used in the literature ([Roxburgh, 2004](#); [Landin et al., 2009](#); [Siess et al., 2013](#); [Fabry et al., 2022](#)) to couple deformations to 1D stellar evolutionary models. Following this method, the stellar structural equations were averaged over isobars of our 3D models to give a one dimensional description of the stellar-distorted structure usable by stellar evolution codes. For a more detailed derivation of the equations presented in this section, we refer the reader to [Manchon \(2021\)](#) in the context of two-dimensional models of rapidly rotating stars.

MASS CONSERVATION EQUATION

We start by introducing m_p , denoting the mass encompassed under an isobar of pressure p and the averaged radius r_p , such as $V_p = \frac{4}{3}\pi r_p^3$, is the volume under this isobar. With this notation, the mass conservation equation is unmodified and given by

$$\frac{dm_p}{dr_p} = 4\pi\rho r_p^2. \quad (3.27)$$

By construction, the critical direction $(\mu_{\text{crit}}, \phi_{\text{crit}})$ was chosen to verify [Equation 3.27](#) at each layer of the star, implying that $r_{p,i} = r_i$ and $m_{p,i} = m_i$ in the whole star.

HYDROSTATIC EQUILIBRIUM

Following [Equation 3.1](#), the infinitesimal distance dn between the equipotentials Ψ_{tot} and $\Psi_{\text{tot}} + d\Psi_{\text{tot}}$ is given by

$$dn = |\mathbf{g}_{\text{eff}}|^{-1} d\Psi_{\text{tot}}. \quad (3.28)$$

With this formalism, the volume between two isobars is obtained with

$$dV_p = \int_{\Psi_{\text{tot}}} dn d\sigma, \quad (3.29)$$

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where the integrals on Ψ_{tot} denote integrals taken over isobars of constant Ψ_{tot} and $d\sigma$ is an isobar surface element expressed as

$$d\sigma = -r(\mu, \phi)^2 \frac{|\mathbf{g}_{\text{eff}}| \cdot |\mathbf{r}|}{\mathbf{g}_{\text{eff}} \cdot \mathbf{r}} d\mu d\phi. \quad (3.30)$$

In Equation 3.30, $|\mathbf{g}_{\text{eff}}|$ denotes the norm of the effective gravity presented in Equation 3.1, and $r(\mu, \phi)$ is the radial position of the chosen isobar in the direction (μ, ϕ) . The analytical expression of the effective gravity in our new models is developed in Subsection 3.4.2. Equation 3.29 can be integrated and combined to Equation 3.28 and Equation 3.30 to obtain the volume under an isobar:

$$V_p = \int_{\Psi_{\text{tot},c}}^{\Psi_{\text{tot}}} d\Psi_{\text{tot}} \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma = \int_{\mu,\phi} d\mu d\phi \int_{\Psi_{\text{tot},c}}^{\Psi_{\text{tot}}} r(\mu, \phi)^2 \frac{\partial r}{\partial \Psi_{\text{tot}}} d\Psi_{\text{tot}} = \frac{1}{3} \int_{\mu,\phi} r(\mu, \phi)^3 d\mu d\phi,$$

where $\Psi_{\text{tot},c}$ is the total potential at the centre of the star. Finally, the mass between two isobars is expressed as

$$dm_p = \rho \int_{\Psi_{\text{tot}}} dnd\sigma = \rho d\Psi_{\text{tot}} \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma. \quad (3.31)$$

Therefore, the equation of hydrostatic equilibrium can be expressed as

$$\frac{dP}{dm_p} = \frac{dP}{d\Psi_{\text{tot}}} \frac{d\Psi_{\text{tot}}}{dm_p} = - \left(\int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \right)^{-1}, \quad (3.32)$$

which can be written as

$$\frac{dP}{dm_p} = -f_p \frac{Gm_p}{4\pi r_p^4}, \quad (3.33)$$

where,

$$f_p = \frac{4\pi r_p^4}{Gm_p} \left(\int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \right)^{-1}. \quad (3.34)$$

TRANSPORT OF ENERGY

In this section, we assume that the chemical composition of each star is constant on each isobar. Therefore, the isobars become isotherms. In the radiative zone, the norm of the local flux is given by

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dn}, \quad (3.35)$$

where T is the temperature of a given isobar, κ is its opacity, a is the radiation constant, and c is the speed of light in the vacuum. Equation 3.35 can be rewritten using Equation 3.28, and the mass conservation equation (Equation 3.27) can be rewritten as

$$F = -\frac{4acT^3}{3\kappa} \frac{dT}{dm_p} \left(\int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \right) |\mathbf{g}_{\text{eff}}|, \quad (3.36)$$

which is the classical formulation of the gravity darkening.

By integrating on an entire equipotential, we obtained the luminosity:

$$L_p = -\frac{4acT^3}{3\kappa} \frac{dT}{dm_p} \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}| d\sigma. \quad (3.37)$$

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The temperature gradient can thus be expressed as

$$\frac{dT}{dm_p} = -\frac{3\kappa L_p}{4acT^3} \left(\int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}| d\sigma \right)^{-1} = -f_T \frac{3\kappa L_p}{64\pi^2 r_p^4 acT^3}, \quad (3.38)$$

where we introduced the corrective factor

$$f_T = 16\pi^2 r_p^4 \left(\int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}|^{-1} d\sigma \int_{\Psi_{\text{tot}}} |\mathbf{g}_{\text{eff}}| d\sigma \right)^{-1}. \quad (3.39)$$

Finally, the correction of the radiative gradient is

$$\nabla_{\text{rad},p} = \frac{f_T}{f_p} \nabla_{\text{rad}}. \quad (3.40)$$

To find the boundaries of convective zones, this corrective factor had to be applied to the radiative gradient. We note that the computed 3D deformations are mainly significant in the upper regions of stars, which means that $f_T/f_p \approx 1$ close to the convective core boundaries for most of the stars.

3.4.2 EFFECTIVE GRAVITY

As shown above, the effective gravity is a key quantity for including the effects of tidal and rotational deformation into stellar-evolution calculations. It is also of direct observational interest, because the local emitted flux depends on it through gravity darkening (Budaj, 2011; Espinosa Lara & Rieutord, 2011; Claret, 2016; Fabry et al., 2022). Consequently, if one wishes to assess the impact of non-perturbative models on the observable properties of close binaries, the computation of the effective gravity is an essential step. The effective gravity is defined as the total acceleration acting on a fluid element at the stellar surface, that is,

$$\mathbf{g}_{\text{eff}} = -\nabla\Psi_{\text{tot}} = -\nabla(\Psi_1 + \Psi_2 + \Psi_c) = -g_{\text{eff},r}\mathbf{e}_r - g_{\text{eff},\mu}\mathbf{e}_\mu - g_{\text{eff},\phi}\mathbf{e}_\phi, \quad (3.41)$$

with each component of a given primary star expressed as

$$g_{\text{eff},r,1} = \frac{\partial\Psi_{\text{tot},1}}{\partial r_1} = \frac{\partial\Psi_1}{\partial r_1} + \frac{\partial\Psi_2}{\partial r_1} + \frac{\partial\Psi_c}{\partial r_1}, \quad (3.42)$$

$$g_{\text{eff},\mu,1} = \frac{1}{r_1} \frac{\partial\Psi_{\text{tot},1}}{\partial\theta_1} = \frac{1}{r_1} \left(\frac{\partial\Psi_1}{\partial\theta_1} + \frac{\partial\Psi_2}{\partial\theta_1} + \frac{\partial\Psi_c}{\partial\theta_1} \right), \quad (3.43)$$

$$g_{\text{eff},\phi,1} = \frac{1}{r_1 \sin(\theta_1)} \frac{\partial\Psi_{\text{tot},1}}{\partial\phi_1} = \frac{1}{r_1 \sin(\theta_1)} \left(\frac{\partial\Psi_1}{\partial\phi_1} + \frac{\partial\Psi_2}{\partial\phi_1} + \frac{\partial\Psi_c}{\partial\phi_1} \right). \quad (3.44)$$

In this case, all the necessary quantities are partial derivatives of solutions to differential equations that have integral representations. Consequently, numerical differentiation methods are not required to perform the derivatives of the potentials and integral formula can be found. The contribution to effective gravity from gravitational potential of the primary star is given by

$$\frac{\partial\Psi_1}{\partial r_1} = \sum_{\ell,m} P_\ell^m(\mu_1) \cos(m\phi_1) \frac{\partial\Psi_{1,\ell}^m}{\partial r_1} = \sum_{\ell,m} P_\ell^m(\mu_1) \cos(m\phi_1) \left(\Psi_{1,\ell}^m \frac{\ell}{r_1} + r_1^{-(\ell+2)} \int_0^{r_1} r_1'^{\ell+2} \rho_{1,\ell}^m(r_1') dr_1' \right), \quad (3.45)$$

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$$\frac{\partial \Psi_1}{\partial \theta_1} = - \sum_{\ell, m} \Psi_{1, \ell}^m(r_1) \cos(m\phi_1) \frac{\partial P_\ell^m(\mu_1)}{\partial \mu_1} \sin(\theta_1), \quad (3.46)$$

$$\frac{\partial \Psi_1}{\partial \phi_1} = - \sum_{\ell, m} \Psi_{1, \ell}^m(r_1) m \sin(m\phi_1) P_\ell^m(\mu_1). \quad (3.47)$$

The contribution from centrifugal force is expressed as

$$\nabla \Psi_c = \tilde{n}^2 x_{\text{CM}} \begin{bmatrix} \sin(\theta) \cos(\phi) \\ r \cos(\theta) \cos(\phi) \\ -r \sin(\theta) \sin(\phi) \end{bmatrix} - \Omega_\star^2 r \sin(\theta) \begin{bmatrix} \sin(\theta) \\ r \cos(\theta) \\ 0 \end{bmatrix}.$$

Finally, the gravitational contribution originating from the companion is written as

$$\begin{bmatrix} \frac{\partial \Psi_2}{\partial r_1} \\ \frac{\partial \Psi_2}{\partial \theta_1} \\ \frac{\partial \Psi_2}{\partial \phi_1} \end{bmatrix} = -\mathbf{J}^T \sum_{\ell, m} \Psi_{2, \ell}^m(R_2) \left(\frac{R_2}{r_2} \right)^{\ell+1} \begin{bmatrix} \frac{\ell+1}{r_2} P_\ell^m(\mu_2) \cos(m\phi_2) \\ \frac{\partial P_\ell^m(\mu_2)}{\partial \mu_2} \sin(\theta_2) \cos(m\phi_2) \\ P_\ell^m(\mu_2) m \sin(m\phi_2) \end{bmatrix}, \quad (3.48)$$

where $\mathbf{J}^T$ is the transpose of the Jacobian matrix taken between the system of coordinates of our primary and secondary star, and expressed as

$$\mathbf{J}^T = \begin{bmatrix} \frac{\partial r_2}{\partial r_1} & \frac{\partial \theta_2}{\partial r_1} & \frac{\partial \phi_2}{\partial r_1} \\ \frac{\partial r_2}{\partial \theta_1} & \frac{\partial \theta_2}{\partial \theta_1} & \frac{\partial \phi_2}{\partial \theta_1} \\ \frac{\partial r_2}{\partial \phi_1} & \frac{\partial \theta_2}{\partial \phi_1} & \frac{\partial \phi_2}{\partial \phi_1} \end{bmatrix}. \quad (3.49)$$

The norm of the effective gravity is simply given by

$$|\mathbf{g}_{\text{eff}}| = (g_{\text{eff}, r, 1}^2 + g_{\text{eff}, \mu, 1}^2 + g_{\text{eff}, \phi, 1}^2)^{1/2}. \quad (3.50)$$

We note that the equations in this section have been expressed in matrix form, as this is the representation used in MoBiDICT for computational efficiency. The main numerical complexity arises from the construction of the matrix $\mathbf{J}^T$, which must be evaluated for each radial layer and angular direction inside the star. The explicit expressions for the derivatives entering this matrix are therefore deferred to the section dedicated to the numerical implementation of MoBiDICT (Section 3.6), since they are closely tied to the grid choices adopted in the code.

3.5 ON THE CRITICAL DIRECTION AND ITS CHOICE

3.5.1 CRITICAL DIRECTION

In the initial design of MoBiDICT we worked with a critical direction to take advantage of the barotropic property of our models. More specifically, along this direction $(\mu_{\text{crit}}, \phi_{\text{crit}})$ we assume that the density of each star and layer is given by a one-dimensional, spherically symmetric input model. Along $(\mu_{\text{crit}}, \phi_{\text{crit}})$ we compute $\Psi_{\text{tot}}(r, \mu_{\text{crit}}, \phi_{\text{crit}})$ and impose that $\rho_i(r, \mu_{\text{crit}}, \phi_{\text{crit}}) \equiv \rho_{i,1D}(r)$. Since ρ_i and Ψ_{tot} are monotonic functions of r , we obtain the function $\rho_i(\Psi_{\text{tot}})$. Using this function, we reconstruct the entire 3D density structure, knowing the total potential at each mesh point and using $\rho_i(r, \mu, \phi) = \rho_i(\Psi_{\text{tot}}(r, \mu, \phi))$.

The choice of critical direction is the main weakness and limitation of this method, owing to the high sensitivity of the results to the chosen direction. This can be understood easily: the profile of $\Psi_{\text{tot}}(r)$ is highly direction dependent. At the equator, for example, the star is distorted and therefore the surface $\Psi_{\text{tot}}(R)$, as given by the single-star model, lies inside the distorted star. Using the equipotential $\Psi_{\text{tot}}(R)$ as the new surface would thus underestimate the deformation. The opposite occurs if the poles are chosen as the critical direction.

Consequently, the critical direction should be defined as the direction most similar to the 1D structure, ideally one with no deformation. Expanding the 3D density profile in a spherical-harmonics basis gives:

$$\rho_i(r_i, \mu, \phi) = \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \rho_{i,\ell}^m(r_i) Y_{\ell,m}(\mu, \phi). \quad (3.51)$$

In the critical direction, we want $\rho_i(r_i, \mu_{\text{crit}}, \phi_{\text{crit}}) = \rho_{1D}$. To satisfy this condition, the $\ell > 0$ terms should cancel, leaving only the $\ell = 0$ term such that $\rho_{i,0}^0(r_i) Y_{0,0}(\mu, \phi) = \rho_{1D,i}$. The dominant terms in the expansion are the $\ell = 2$ contributions. Therefore, by choosing the direction corresponding to the roots of both the $\ell = 2, m = 0$ and $\ell = 2, m = 2$ spherical harmonics, the condition is approximately satisfied. In the initial design of MoBiDICT, we adopted a constant critical direction over all stellar layers, namely the common roots $\mu_{\text{crit}} = 1/\sqrt{3}$ and $\phi_{\text{crit}} = 3\pi/\sqrt{4}$. Using these critical angles yields a precision of about 10^{-2} – 10^{-5} in mass conservation, depending on the deformation and type of star. However, this choice is not ideal, as mass conservation is not always well enforced and depends on the degree of stellar distortion. This leads to systematic over- or underestimation of deformation and errors in the surface quantities by up to a few per cent.

Referring back to [Equation 3.51](#), when the deformation increases the number of non-negligible spherical harmonics terms grows rapidly, and the choice of critical direction worsens, since cancelling only the $\ell = 2$ terms is insufficient and higher-order terms would also need to be cancelled. In our initial article on MoBiDICT we were not fully aware of the severity of this problem, and adopted a critical direction that slightly underestimated the deformation, namely $\mu_{\text{crit}} = 1/\sqrt{3}$, $\phi_{\text{crit}} = 3\pi/\sqrt{4}$.

3.5. ON THE CRITICAL DIRECTION AND ITS CHOICE

THE CRITICAL DIRECTION IN OTHER WORKS

As noted above, the precision achieved by MoBiDICT using a constant critical direction ranges from about 10^{-2} to 10^{-5} . For 2D single rotating stars, [Manchon \(2021\)](#) proposed a method based on finding the roots of the sum of the density spherical-harmonic expansions. Translated to 3D, their principle would be to construct a function:

$$Q(r) = (\rho_{i,0}^0(r_i)Y_{0,0}(\mu, \phi) - \rho_{1D}(r)) + \sum_{\ell>1}^L \sum_{k=0}^{(\ell-p)/2} \rho_{i,\ell}^m(r_i)Y_{\ell,m}(\mu, \phi), \quad (3.52)$$

whose roots correspond to possible critical directions. They then expressed the equation in matrix form, exploiting the recursive properties of Legendre polynomials to solve for the roots. However, this method is hardly applicable in our case, as it introduces even greater degeneracies than in [Manchon \(2021\)](#), especially near the centre.

We implemented alternative solutions. In particular, we varied the critical angle in each stellar layer, guided by the average radius of each equipotential. Expecting the solution to be near $\mu_{\text{crit}} = 1/\sqrt{3}$, $\phi_{\text{crit}} = 3\pi/\sqrt{4}$, we tested different algorithms to explore the parameter space around these angles. The results were unsatisfactory in terms of both computing time and precision. We also found the required development effort disproportionate, given the numerical instabilities and lack of guaranteed improvement in accuracy. We therefore sought a different approach better suited to our problem.

After some developments we devised a method exploiting another property of the models: the 1D solutions of the structure equations at a given radius provide both the average value of each quantity over an equipotential and its value along the critical direction. Mathematically, this can be expressed for a quantity X as:

$$\langle X \rangle_p = X_{1D} = X_{1D}(r_i, \mu_{\text{crit}}, \phi_{\text{crit}}). \quad (3.53)$$

Instead of working directly with the density, for which we have a large number of components and no real criterion to impose except the cancellation of the high order terms, we applied this principle to mass and radius through the mass-conservation equation.

MASS CONSERVATION AS THE NATURAL CHOICE FOR THE CRITICAL DIRECTION

A natural criterion for selecting the critical direction is that the average of each quantity over each equipotential (or isobar) should match the 1D input structure. By imposing this, the averaged properties of the equipotential passing through $(r, \mu_{\text{crit}}, \phi_{\text{crit}})$ are ensured to follow the 1D model. In particular, the mass enclosed within an isobar m_p of pressure p and averaged radius r_p (with volume $V_p = \frac{4}{3}\pi r_p^3$) is given by the 1D input model if the critical direction is defined this way. Expressing mass conservation under an isobar gives:

$$\frac{1}{4\pi r_p^2} \frac{dm_p}{dr_p} = \frac{dm_p}{dV_p} = \rho. \quad (3.54)$$

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In MoBiDICT, the density is assumed to be constant on each equipotential surface and equal to the density of the corresponding layer in the 1D stellar model, that is,

$$\rho \equiv \rho_{1D}. \quad (3.55)$$

By imposing this condition, we ensure that the left-hand side of Equation 3.54, evaluated along the reference direction, is described by the input stellar model. In other words, both the mass and the volume enclosed by each layer are conserved with respect to the 1D model, so that, for each layer,

$$r_p \equiv r, \quad m_p \equiv m. \quad (3.56)$$

The new idea is therefore not to search for a particular direction in which the condition $\rho = \rho_{1D}$ is satisfied and then use this direction to reconstruct the three-dimensional structure, indirectly enforcing mass. Instead, we impose mass conservation locally, at each grid point of the 3D model, by associating each value of r_p with the density of the corresponding layer in the input 1D stellar model such as mass is conserved. In this way, the density assigned to each equipotential surface is chosen so that the enclosed mass is conserved with respect to the original 1D structure. This reformulation removes the need to introduce a critical direction. The construction is instead based entirely on a condition imposed on the equipotential surfaces themselves: each equipotential is associated with a density and an enclosed mass inherited from the 1D model, while its three-dimensional shape is determined self-consistently by the total effective potential. This approach therefore provides a more natural way to reconstruct the deformed stellar structure, as described in the following section.

It should be noted that in the presence of non-conservative forces, such as differential rotation (as explored in 2D by Manchon 2021), a modified version of the mass-conservation equation must be satisfied, and averages must be taken over isobars. We nevertheless strongly believe that our method can be adapted to the 2D case.

3.5.2 IMPOSING MASS CONSERVATION

As outlined above, rather than working with a critical direction, imposing conservation of mass and volume allows us to work directly with equipotentials. The basic principle is to impose that the equipotential-averaged radius of each layer equals the radius of the 1D model, and that the density is constant along equipotentials, thereby satisfying mass conservation.

For each point of the 3D mesh, we first compute the total potential. Referring back to Subsection 3.3.1, this corresponds to the second step of the algorithm. Then, for each value of the total potential sampled on the grid, we compute the volume V_p enclosed by the associated equipotential surface, or equivalently the corresponding volume-equivalent radius r_p . This is done by interpolating in each angular direction to determine the radius $r(\mu, \phi)$ of the equipotential surface, following Equation 3.31 to then obtain r_p associated with each value of the potential. Instead of proceeding with step 3 as described in Subsection 3.3.1, where a critical direction is used to reconstruct the density structure, we adopt the new method described here. Once the

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3D mesh of r_p has been obtained, we impose mass conservation by assigning to each equipotential surface the density of the corresponding layer in the input 1D stellar model. Mathematically, this consists to imposing

$$\frac{dm_p}{dV_p} \equiv \rho_{1D}(r_p), \quad (3.57)$$

so that the left-hand side is fully prescribed by the original 1D model. Once V_p , or equivalently r_p , has been determined, imposing this criterion is straightforward. The density at each mesh point is simply given by

$$\rho(\mathbf{r}) = \rho_{1D}\left[r_p(\Psi_{\text{tot}}(\mathbf{r}))\right], \quad (3.58)$$

because r_p is a function only of the value of the total potential, Ψ_{tot} . In practice, the density is therefore obtained by interpolating the 1D density profile $\rho_{1D}(r)$ at the value of r_p associated with the total potential of each point in the 3D mesh. This procedure directly enforces mass conservation on the equipotential surfaces while avoiding the need to define a critical direction.

3.5.3 REFERENCE DIRECTION FOR THE MASS CONSERVATION

To further improve both precision and computational efficiency, we combine the previous approach with the new equipotential-based reconstruction. We first select an arbitrary reference direction. This direction is not a critical direction of the model and does not affect the final result; it is used only to build an interpolation table. Along this direction, we compute the volume-equivalent radius r_p associated with each equipotential layer, which allows us to construct the monotonic function

$$r_p = r_p(\Psi_{\text{tot}}). \quad (3.59)$$

Once this function is tabulated, the value of r_p at any point of the 3D mesh can be obtained directly from the local value of the total potential Ψ_{tot} , without recomputing the full equipotential geometry. Using the 1D density profile and function composition, we then build the corresponding monotonic relation

$$\rho_i = \rho_i(r_p(\Psi_{\text{tot}})), \quad (3.60)$$

along the same reference direction. The local density at each mesh point can therefore be obtained by a direct interpolation in $\rho_i(\Psi_{\text{tot}})$. This avoids the need to determine $r(\mu, \phi)$ in every angular direction for every value of the potential at every mesh point. As a result, the number of expensive 3D interpolations is drastically reduced, leading to a substantial improvement in execution time while also improving the numerical precision of the reconstruction.

We implemented both approaches (the full 3D grid r_p computations and the reference direction based method) in the iterative process of MoBiDICT and found that the second one is both faster and slightly more accurate. In both cases, the mass-conservation error remains of the order of 10^{-7} . For the remainder of this thesis, we therefore adopt

3.6. NUMERICAL IMPLEMENTATION OF THE METHOD

only the second implementation. Although this method requires the choice of an arbitrary reference direction to construct the function $r_p(\Psi_{\text{tot}})$, this direction does not play the same physical role as the critical direction used in the previous formulation. In particular, it does not affect the resulting stellar structure or the global model outputs. It can, however, introduce small numerical differences through the interpolation procedure. Tests performed with different reference directions showed variations of only $\sim 10^{-7}$, confirming that the method is effectively independent of this choice. We nevertheless found that choosing a direction close to the former critical direction improves the numerical precision by about one additional order of magnitude. Along such directions, the volume-equivalent radii remain close to the radii of the input 1D model, so the interpolation corrections are smaller and the associated numerical noise is reduced. With this choice, the mass-conservation precision reaches $\sim 10^{-8}$. Another important advantage of this formulation is its numerical stability, which makes it well suited for use inside an iterative procedure. In the present implementation, we use simple linear interpolations. More advanced interpolation schemes could be explored in the future and may further improve the precision of the reconstruction.

3.6 NUMERICAL IMPLEMENTATION OF THE METHOD

As with any 3D code, a key challenge lies in the numerical implementation and the computational resources required to execute such calculations. In our case, however, the code is not particularly demanding. This is primarily because the problem does not depend on time, no equations are solved directly on the 3D grid, and we have adopted efficient numerical methods that minimise the number of grid points required in the spatial directions without compromising accuracy. In this section, we describe the numerical implementation of the method, focusing on the main techniques and variables employed in MoBiDICT.

The first point to consider is the choice of spatial grid, which is often computationally expensive in 3D codes. In MoBiDICT, the 3D grid serves only as a support for projecting all quantities into 3D space, which can then be projected onto spherical harmonics or averaged over equipotentials. It is not the grid on which the Poisson equation is solved. Accordingly, the number of angular grid points is determined solely by the required accuracy of the projections and averaged quantities. Rather than employing equidistant mesh points, we use grids tailored to perform the relevant integrals accurately, based on the Gauss–Legendre quadrature method.

Without entering into the full derivation, Gauss–Legendre quadrature is a numerical integration method that exactly integrates polynomial functions of degree $2n - 1$ or less, where n is the number of mesh points. The mesh points are not equidistant, but are instead chosen as the roots of the degree- n Legendre polynomial. Each point x_i is associated with a weight w_i , such that the integral of a function $f(x)$ can be approximated as

$$\int_{-1}^1 f(x)dx \approx \sum_{i=0}^n f(x_i)w_i, \quad (3.61)$$

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which is exact provided that $f(x)$ can be approximated by a polynomial of degree $2n - 1$ or lower. The weights for the Gauss–Legendre method are given by

$$w_i = \frac{2}{(1 - x_i^2)P_n'(x_i)^2}, \quad (3.62)$$

where x_i are the roots of the Legendre polynomials. In practice, Gauss–Legendre quadrature is highly efficient for integrating spherical harmonics, since these functions are polynomials of degree ℓ . Such integrals are central to our method, as they allow the projection of 3D quantities onto the spherical harmonic basis. Hence, only $n = L/2 + 1/2$ mesh points are required in latitude to obtain exact projections onto spherical harmonics up to degree L . In practice, we use more points to compute other grid-based quantities such as m_p or r_p .

For the longitudinal coordinate, we also adopt Gauss–Legendre quadrature. The projections include terms of the form $\cos(m\phi)$, which can be expanded as polynomial series near $\phi = 0$ and are thus well suited to this approach, though higher values of n may be needed to achieve sufficient accuracy. Thanks to the symmetries of the problem (about both the equatorial and polar planes), it is sufficient to model only half a hemisphere, from which the full stellar structure can be reconstructed. The computational domain is therefore restricted to $\mu \in [1, 0]$ and $\phi \in [0, \pi]$.

For the radial direction, the choice is more straightforward. Inside the non-deformed star, the radial grid is set by the one-dimensional input model. Outside the star, an additional grid is introduced and extended up to the inner Lagrangian point L_1 . In this outer region, we adopt an equidistant mesh, with a typical number of points of order 5 000 to 10 000, in order to ensure an accurate determination of the surface and a reliable interpolation of m_p and r_p in highly distorted stars. In practice, two grids are employed, one centred on each star. The axes $\mu, \phi = 0$ correspond to the line connecting the stellar centres to L_1 . [Figure 3.6](#) illustrates a polar cut of such grids for a binary with 50 latitudinal and longitudinal mesh point and 5,000 exterior radial points. As seen in this figure, the grid we use is not evenly spaced, an higher density of points is found close to the equator due to the Gauss-Legendre quadrature method while no mesh points are chosen near the poles.

Once the grids are specified, we turn to code optimisation. Given the many free parameters in binary-star systems (stellar and orbital), it was essential to design a method that runs efficiently. Working in the spectral domain, and using optimised angular quadrature, made the code fast overall. However, two parts remained computationally demanding.

The first computational bottleneck in the code is the interpolation of the density onto the 3D grids. We employ simple linear interpolation, which is adequate given the high density of grid points. Nevertheless, the total number of interpolations remains large, of order $2n_\mu \times n_\phi \times n_r$, where n_μ , n_ϕ , and n_r are the number of mesh points in the μ , ϕ , and r directions, respectively. Even with full parallelisation and optimisation of the index selection based on previous interpolations, this step remains costly, as the interpolation ranges must still be evaluated. In the current version of the code,

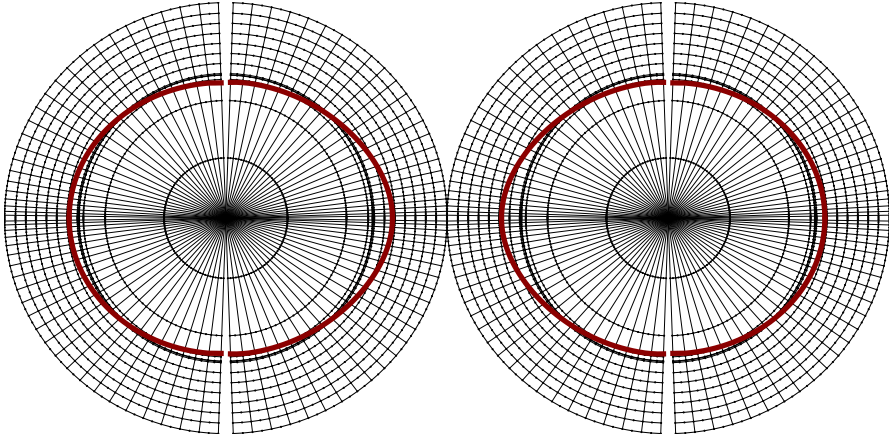


Figure 3.6: Polar cut of the grids used in MoBiDICT using 50 latitudinal and longitudinal mesh point and 5,000 exterior radial points. Each circle-like line represents one in 700 radial points used in the radial mesh. Each solid line is an angular mesh point, we only represented half of them for illustration purposes. Each dot corresponds to the intersection between the radial and angular mesh which are effectively the mesh point of the model. The black solid line region of this plot corresponds to the initial surface of the non-deformed stars. Close to the surface a denser radial grid is present in the initial 1D model hence more mesh points can be found in this region. The red curves correspond to the surface of deformed stars within those grids.

this procedure dominates the runtime, since the other parts of the code have been significantly optimised. The second bottleneck becomes apparent when considering the computations carried out within MoBiDICT. The numerical implementation can be summarised as follows:

1. The input density profile ρ_{1D} is projected onto the spherical harmonics,

$$\rho_{0,i}^0 = \rho_{1D,i} Y_0^0, \quad (3.63)$$

which, owing to spherical symmetry, requires only the $\ell = 0, m = 0$ term (corresponding to the Roche model). This step is low in cost.

2. The Poisson equation is solved for the $\ell = 0$ term. This requires only an integral evaluation, performed using the trapezoidal rule, and is computationally inexpensive.

3. The total potential is then computed on the 3D grid:

$$\begin{aligned} \Psi_{\text{tot}}(r_1, \mu_1, \phi_1) = & \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \Psi_{1,\ell}^m(r_1) P_{\ell}^m(\mu_1) \cos(m\phi_1) \\ & + \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \Psi_{2,\ell}^m(R_2) \left(\frac{1}{r_{1|2}} \right)^{\ell+1} P_{\ell}^m(\mu_{1|2}) \cos(m\phi_{1|2}) \\ & + \tilde{n}^2 x_{\text{CM}} r \sin \theta \cos \phi - \frac{\Omega_{\star,i}^2}{2} r^2 \sin^2 \theta \end{aligned} \quad (3.64)$$

The first term is not problematic, as it consists only of sums of pre-computed quantities. These can be parallelised over any of the spatial coordinates or over (ℓ, m) , and the factors $P_{\ell}^m(\mu_1) \cos(m\phi_1)$ can be tabulated across the entire (ℓ, m, μ_1, ϕ_1) grid at low cost, given the fixed chosen angular resolution. The third term is also straightforward: it is a simple expression independent of the spherical harmonics, which can be directly parallelised over the entire 3D grid. The real computational challenge lies in the second term, in particular the quantities denoted $|_2$. These represent the projection of the coordinates of an element from the primary grid onto the grid of the secondary. With some geometry, these projections can be written in our dimensionless formalism as

$$r_{1|2} = \sqrt{1 + r_1^2 - 2r_1 \sin(\theta_1) \cos(\phi_1)}, \quad (3.65)$$

$$\mu_{1|2} = \cos \left[\frac{\pi}{2} - \text{atan} \left(\frac{r_1 \mu_1}{1 - r_1 \sin(\theta_1) \cos(\phi_1)} \right) \right], \quad (3.66)$$

$$\phi_{1|2} = \text{atan} \left(\frac{r_1 \sin(\theta_1) \sin(\phi_1)}{1 - r_1 \sin(\theta_1) \cos(\phi_1)} \right). \quad (3.67)$$

Each of these quantities must be evaluated at every grid point and then used in the spherical harmonics and potential calculations for all (ℓ, m) . As they appear inside the main iterative loop, these projections are by far the most computationally demanding part of the algorithm.

4. The potential is then used to project the density onto the 3D grid, which constitutes the second most expensive step (or the first in the new versions of the code).
5. The density is re-projected onto the spherical harmonics. The Poisson equation is then solved for each harmonic component.
6. If convergence has not been achieved, the procedure returns to step 3; otherwise, the model proceeds to the post-processing stage.

To mitigate the cost of step 3, we adopted a precomputation strategy, trading memory for speed. Since the spatial grids are fixed, we tabulate the values of

$\left(\frac{1}{r_{1|2}} \right)^{\ell+1} P_{\ell}^m(\mu_{1|2}) \cos(m\phi_{1|2})$ for all $r, \mu_1, \phi_1, \ell, m$. This approach avoids repeated evaluation of the costly coordinate transformations within the iterative loop, with precomputation being easily parallelised. The drawback is the high memory requirement as

3.6. NUMERICAL IMPLEMENTATION OF THE METHOD

Table 3.1: Summary of the numerical resources required by MoBiDICT for the computation of a highly deformed model. The deformation was computed for a twin binary composed of two $0.2 M_{\odot}$ stars (whose model is presented in [Subsection 3.7.1](#)) with an orbital separation of $a/R = 3$. From the numerical point of view, the calculation required 19 iterations. The computations were performed on a 64-core processor running at 7000 MHz (AMD Ryzen Threadripper PRO 5995WX, 128 threads) with 256 GB of RAM. For the most expensive run, about half of the computation time (CT) was spent in the post-processing stage, although this cost can be reduced depending on the output quantities required. In all runs, (n_{μ}, n_{ϕ}) was kept larger than L in order to guarantee the accuracy of the projections. The quantity $R/R_{L=35}$ at $(\mu = 0, \phi = 0)$ indicates the difference in radius at the most distorted point of the star relative to the reference model computed with $L = 35$. Calculations performed at even higher resolution revealed numerical instabilities that will need to be addressed in future work.

L	(n_{μ}, n_{ϕ})	RAM usage [GB]	Total CT [s]	Initialisation CT [s]	CT per iteration [s]	$R/R_{L=35} - 1$ ($\mu = 0, \phi = 0$)
5	(15,15)	0.782	7.854	0.291	0.320	4.66×10^{-4}
10	(15,15)	1.575	11.32	0.876	0.369	1.46×10^{-4}
15	(20,20)	4.861	25.97	2.996	0.729	4.29×10^{-5}
20	(25,25)	11.99	55.68	8.156	1.350	1.59×10^{-5}
25	(30,30)	25.11	107.1	16.96	2.324	6.31×10^{-6}
30	(35,35)	47.15	203.3	32.73	4.122	2.04×10^{-6}
35	(40,40)	81.25	332.0	58.95	5.994	0
40	(45,45)	131.5	526.8	99.02	7.717	/
45	(50,50)	201.9	791.3	149.8	10.22	/

we now have to compute two 5D tables, that we call $c_{r,\mu,\phi,\ell,m}$. The memory cost of those table in bits is given by

$$\text{RAM} = 8n_r n_{\mu} n_{\phi} \frac{L^2}{4}. \quad (3.68)$$

The main limitation of the code lies then in its memory consumption. The precomputation strategy, while efficient, requires storing the five-dimensional tables $c_{r,\mu,\phi,\ell,m}$, whose size scales with L^2 (or L^4 if we account for the required n_{μ} and n_{ϕ}). As a result, increasing the number of spherical harmonics significantly raises the RAM usage, which becomes the dominant cost of high-precision runs and limit the amount of spherical harmonics that can be considered. To quantify a bit more the time and memory consumption of MoBiDICT, in [Table 3.1](#), we summarise the evolution of the resources necessary by MoBiDICT when increasing the amount of spherical harmonics considered in the developments.

As shown in this table, the computational limit reached with the available hardware corresponds to an expansion in spherical harmonics up to $L = 45$. The system considered here is highly deformed and therefore represents a demanding test case for the non-perturbative treatment. Convergence is reached after 19 iterations, using a stopping criterion based on a relative correction to the surface potential below 10^{-8} between two successive iterations. With the lowest acceptable numerical resolution, the total computation time is then 7.854 s. Even at this relatively low resolution, the radial position of the most distorted point on the stellar surface differs relatively by only 4.66×10^{-4} from the value obtained with the most accurate computation performed for this model. Differences of the same order of magnitude are also found,

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for example, in the value of the surface gravitational potential. This shows that, for most applications, low-resolution calculations provide a satisfactory compromise between accuracy and computational cost. They are therefore well suited for producing deformation grids or for computing apsidal-motion properties, as discussed in [Subsection 4.1.1](#). Finally, it should be noted that the timings reported here correspond to a particularly demanding case, since the model is strongly deformed and especially sensitive to the non-perturbative treatment. In practice, for less distorted systems, low-resolution calculations are typically faster by about a factor of two, mainly because the number of iterations required for convergence is also reduced by roughly a factor of two.

3.7 COMPARISON WITH THE ROCHE MODEL

this section is extracted from [Fellay & Dupret \(2023\)](#)

One of the classical approaches used to describe the total potential in a binary system is the Roche model. While this model is based on strong assumptions regarding the internal structure of the stars, it has the advantage of providing simple analytical expressions for the total potential at any point in the system. In [Section 3.2](#), we already presented the theoretical framework of the Roche model together with its underlying assumptions. In this section, we examine the differences between this approximation and our non-perturbative models. In [Subsection 3.7.1](#), we describe the different stellar models and binary systems that are considered and compared in the following sections. In [Subsection 3.7.3](#), we analyse the differences in surface deformation between the Roche model and our models for different types of stars. [Subsection 3.7.5](#) is devoted to the differences in surface gravity, which play a key role in the computation of gravity darkening and thus in the interpretation of binary-star observations. Finally, in [Subsection 3.7.6](#), we investigate the differences in structural modelling between the Roche model and our approach for the different types of stars composing the binaries.

3.7.1 TYPES OF STARS STUDIED

The surface deformation predicted by our models and by the Roche model is studied here for several types of stars. To sample a representative range of stellar structures, we computed four one-dimensional stellar models with the Code Liégeois d'Évolution Stellaire (CLES; [Scuflaire et al., 2008b](#)). The properties of these models are listed in [Table 3.2](#). Our method is not restricted to these particular stellar or binary configurations: as long as a one-dimensional density profile is available for each component of the system, its deformation can be computed. In the most extreme cases, the method can even be applied to systems containing compact objects such as white dwarfs as long as relativistic corrections do not play a strong role.

In this work, we restrict ourselves to twin systems composed of two identical stars, so as to provide a well-defined reference case for comparing the distortions of different stellar models. This is useful because the deformation of a given binary component depends on the internal structure of its companion. Nevertheless, MoBiDICT is equally

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Table 3.2: Summary of the stellar properties of the 1D input models used in this work.

Stellar parameters	Model 1	Model 2	Model 3	Model 4
Mass [$M_{\odot}$]	0.2	1.0	1.5	20.0
Radius [$R_{\odot}$]	0.22	1.03	11.1	6.01
Age [Gyr]	2.0	2.0	2.0	0.002
Evolutionary stage	MS	MS	RGB	MS
Effective temperature [K]	3342	6080	4621	35 859
Luminosity [$L_{\odot}$]	0.005	1.31	51.1	53 921
Metallicity Z_0/X_0	0.014	0.014	0.014	0.014
Core Hydrogen X_c	0.693	0.469	0	0.582

able to treat systems composed of two non-identical stars. In the code, the stellar type is specified solely through the input density profile of each component. In Figure 3.7, we illustrate the one-dimensional density profiles, normalized by the central density, together with the mass profiles as functions of the normalized radius for each stellar model considered in this work.

Two groups of models can be distinguished in Figure 3.7: those with high envelope mass fractions and those with low envelope mass fractions. While the massive and solar-like stars have only a negligible fraction of their mass in the envelope, the low-mass and red-giant-branch (RGB) stars have most of their mass located there. In particular, about 80% of the mass of our RGB model is contained in its convective envelope, with the transition from the radiative core to the convective envelope occurring at $r/R_{1D} \sim 0.05$.

In the following sections, we compare the deformations obtained for the different types of stars presented above with those predicted by the Roche model and by our new class of models. We then relate these differences directly to the one-dimensional density and mass profiles shown in Figure 3.7.

3.7.2 GRAVITATIONAL POTENTIAL

A major approximation of the Roche model is that it keeps only the leading-order contribution, $\ell = 0$ and $\ell = 2$, in the spherical-harmonic expansion of the density and gravitational potential. This assumption can be tested directly with MoBiDICT, since the code provides the spherical-harmonic expansion of the gravitational potential of each deformed star as one of its outputs. Figure 3.8 shows this decomposition for the twin binary system composed of two $0.2 M_{\odot}$ stars at an orbital separation of $a = 2.8R_{1D}$, corresponding to the same system configuration as that presented in Figure 3.9.

Figure 3.8 illustrates the hierarchy of amplitudes of the spherical-harmonic components of the gravitational potential. As assumed in the Roche model, the dominant contributions is the $\ell = 0$ and $\ell = 2$ terms. The dipolar term ($\ell = 1$) is the next most important contribution, especially outside the star, where each harmonic component

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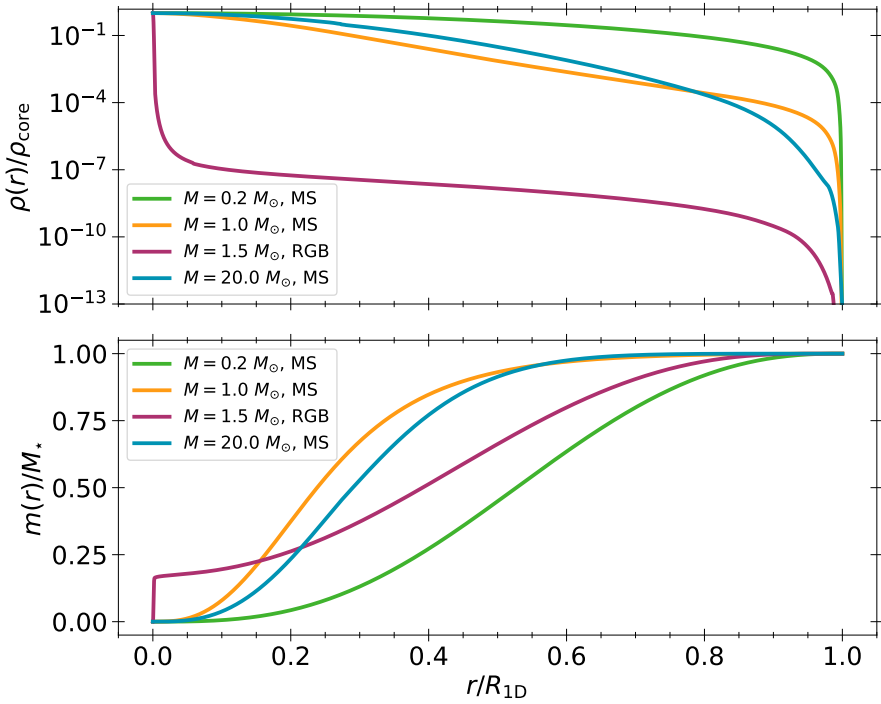


Figure 3.7: 1D input density and mass profiles of the four reference stars. The density profiles of the stellar components in the twin binary systems considered in this work are normalised by the central density of each star. The upper panel shows the stellar density profiles, while the lower panel shows the enclosed mass profiles, normalised by the total mass of each star.

decreases with radius. In these regions, the dipolar term contributes to the tidal potential experienced by the companion star. Neglected in the Roche model, this $\ell = 1$ term is less than one order of magnitude smaller than the $\ell = 2$ term, showing that the tidal force is significantly larger in our models and that neglecting the $\ell = 1$ contribution is not justified in the most distorted cases. The next-order terms ($\ell = 3$ and $\ell = 4$) have amplitudes comparable to that of the $\ell = 1$ term inside the star, but become about one order of magnitude smaller than the dipolar term outside the star. Finally, it should be noted that this non-negligible dipolar term is also responsible for the correction to Kepler’s third law derived in [Subsection 3.3.5](#) and explored numerically in [Subsection 3.7.4](#).

3.7.3 SURFACE DEFORMATION

The surface deformation of binary stars is directly determined by the total potential of the system. In our models, the surface of each stellar component is defined by the equipotential corresponding to the photosphere of the input one-dimensional stellar

3.7. COMPARISON WITH THE ROCHE MODEL

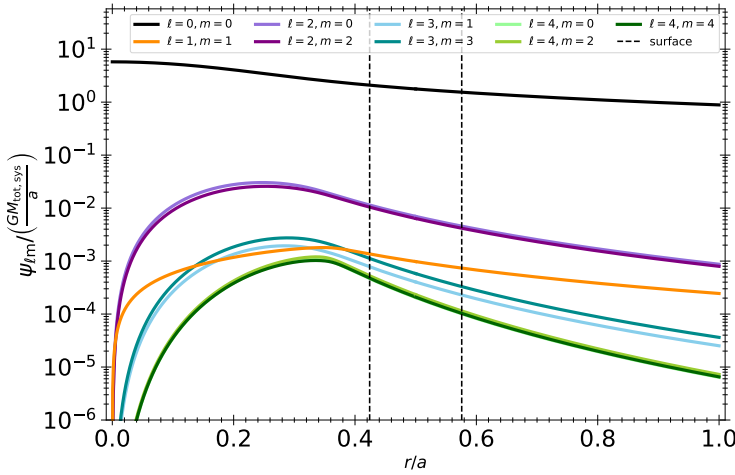


Figure 3.8: Normalised gravitational potential projected onto spherical harmonics in MoBiDICT. The system considered is a twin binary composed of two $0.2 M_{\odot}$ stars with an orbital separation of $a = 2.8 R_{1D}$. The dotted lines indicate the surface of each star along the direction containing both the most distorted point of the stellar surface and the inner Lagrangian point L_1 , that is, for $(\mu, \phi) = (0, 0)$.

model, as described in [Section 3.5](#).

Both the classical Roche model and our refined version predict the same surface deformation, since outside the star the total potential has the same expression in both approaches. To study the deformation predicted by the Roche model, we assume, as in our models, that the volume-averaged radius coincides with the radius of the input one-dimensional model, so that both the volume and the mass are conserved. Under this assumption, the stellar surface in the Roche model can be defined using the same criterion as in our models.

The surface positions predicted by the Roche model and by our model are compared in [Figure 3.9](#) for a $0.2 M_{\odot}$ star in a binary system with an orbital separation of $2.8 R_{1D}$. Here, R_{1D} denotes the radius of the underlying one-dimensional spherically symmetric model. [Figure 3.9](#) shows a clear difference in surface deformation between our models and the Roche model. To quantify this difference, we introduce the quantity

$$\Delta R = \frac{R_{s, \text{MoBiDICT}}(\mu = 0, \phi = 0) - R_{s, \text{Roche}}(\mu = 0, \phi = 0)}{R_{s, \text{Roche}}(\mu = 0, \phi = 0) - R_{1D}}, \quad (3.69)$$

which measures the difference in deformation at the closest surface point, normalised by the deformation predicted by the Roche model with respect to the input one-dimensional model. For the system shown in [Figure 3.9](#), this quantity is $\Delta R = 0.368$, which means that MoBiDICT predicts 36.8% more deformation than the Roche model at the surface point closest to the inner Lagrangian point L_1 .

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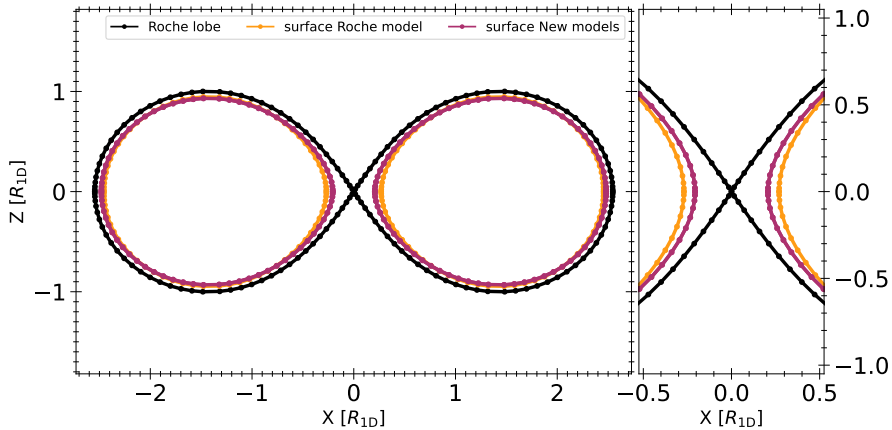


Figure 3.9: Surface deformation of the twin binary system composed of two $0.2 M_{\odot}$ stars with a separation of $2.8 R_{1D}$, viewed from the side. This separation corresponds to an orbital period of 2 h 11 min. The black curve represents the Roche lobe of the system, the orange curve the surface of each star predicted by the Roche model, and the purple curve the surface of each star predicted by MoBiDICT. In this configuration, both models correspond to a Roche-lobe filling factor of 81.7%.

The difference in surface deformation depends on the orbital separation between the two components of the binary system. To compare different types of stars, we use the ratio of the orbital separation to the initial one-dimensional stellar radius, a/R_{1D} , as an indicator of the relative separation. Although the rest of this deformation study focuses on the most distorted region, [Figure 3.9](#) also shows that discrepancies in surface deformation appear near the poles, where our model is more contracted than the Roche model.

To assess the significance of the modifications to the surface deformation predicted by MoBiDICT for the different stellar types, we computed a grid of deformed models for twin binary systems composed of each of the stellar models introduced above. For each system, we varied the orbital separation in order to explore how the discrepancy in surface deformation depends on the separation between the two components. The results obtained from this grid of models are presented in [Figure 3.10](#). [Figure 3.10](#) shows that the deformation differences depend strongly on the type of star considered. For massive and solar-like main-sequence stars, [Figure 3.10](#) shows only minor discrepancies in surface deformation, even when the stars are close to filling their Roche lobes. At most, the deformation difference reaches about 7% before the onset of mass transfer or a common-envelope phase. As shown in [Figure 3.10](#), RGB stars are significantly more deformed in our modelling than in the Roche model, by up to $\sim 35\%$ when $a/R_{1D} \lesssim 3$. Moreover, this discrepancy remains significant even at large a/R_{1D} , with an asymptotic value of $\Delta R \simeq 0.10$. Finally, the largest corrections are found for low-mass stars. As can be seen in [Figure 3.10](#), when $a/R_{1D} \lesssim 3.5$, the Roche model strongly underestimates the deformation, by up to $\sim 80\%$, just before the onset of

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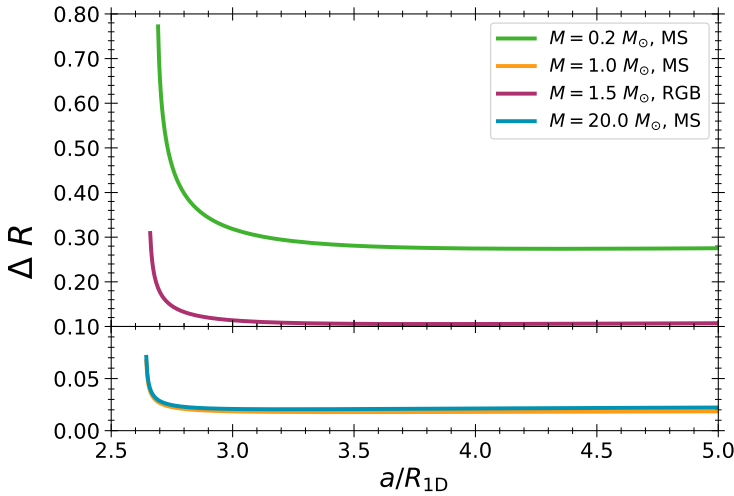


Figure 3.10: Difference in deformation at the closest point, ΔR , between the Roche model and our models as a function of orbital separation. The separation is normalized by the radius of the input one-dimensional stellar model, a/R_{1D} .

mass transfer or a common-envelope phase.

One might expect that, for RGB stars, our new modelling would not significantly modify the surface deformation predicted by the Roche model, since RGB stars are highly evolved and display a strong density contrast between the radiative core and the convective envelope, which seems to justify the point-mass assumption underlying the Roche model. In practice, however, our comparisons between [Figure 3.7](#) and [Figure 3.10](#) show that the deformation discrepancies are more directly related to the one-dimensional mass profiles of the stars. The largest differences are found for stars with the highest fraction of their mass located in the outer layers. In our RGB model, the low density of the envelope is more than compensated by its large volume, so that the envelope contains about $0.8 M_{\text{tot}}$ (see [Figure 3.7](#)). MoBiDiCT is therefore required to model accurately the deformation of close binaries composed of stars with a high envelope mass fraction, such as RGB or low-mass stars.

All the differences presented above can have a significant impact on the theoretical evolution of binary stars, since mass transfer may begin at larger orbital separations than predicted by the Roche model. As a result, phases of mass transfer or common-envelope evolution can occur at earlier stages of the system’s evolution, thereby modifying the entire subsequent evolutionary path of the binary. In this respect, the significant deformation discrepancy found for RGB stars is particularly important, because many mass-transfer episodes are expected to occur during late evolutionary stages, when stellar evolution drives a rapid expansion of the radius. For example, in the system shown in [Figure 3.9](#), a common-envelope phase is expected to occur when the orbital period reaches 2 hours 05 minutes with our modelling, whereas the Roche

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model predicts that an orbital period of 2 hours is required.

3.7.4 CORRECTION OF KEPLER’S THIRD LAW

In [Subsection 3.3.5](#) we mentioned and presented the different readjustments made at the end of each modelling iteration. One of these readjustments, is the computation of the system orbital period taking into account the non-spherical shapes of deformed binaries. For twin twin system composed of two $0.2 M_{\odot}$ stars, whose properties are described in [Subsection 3.7.1](#), [Table 3.3](#) presents the correction to the orbital frequency induced by the non-perturbative treatment as a function of orbital separation in the circular, synchronized case.

Table 3.3: Corrections to Kepler’s third law induced by stellar deformation.

The system considered is a twin binary composed of two $0.2 M_{\odot}$ stars at different orbital separations with circular orbits. For each configuration, the corresponding relative difference between the Keplerian orbital frequency, $\omega_{\text{orb,Kepler}}$, and the non-perturbative orbital frequency, ω_{orb} , obtained through the procedure described in this section, is shown.

a/R_1	2.70	2.80	2.90	3.00	3.10	3.20	3.30	3.40	3.50	4.00	4.5	5.00
$\omega_{\text{orb}}/\omega_{\text{orb,kepler}} - 1$	1.43 %	1.09 %	0.86 %	0.69 %	0.56 %	0.46 %	0.38 %	0.32 %	0.27 %	0.13%	0.07 %	0.04 %

As shown in the table, for $0.2 M_{\odot}$ stars, for which the non-perturbative approach is the most relevant, the correction to the Keplerian period at fixed orbital separation can reach $\sim 1.5\%$ when the stars are very close to contact. Therefore, when observed binary periods are converted into orbital separations using Kepler’s third law, or when orbital separations are used to predict periods, a non-negligible systematic uncertainty is introduced. For close systems such as those considered in this part of the thesis, this systematic effect is orders of magnitude larger than the observational uncertainty on the orbital period and is of the same order as the typical uncertainties affecting the measurements of masses and radii in eclipsing binary systems.

3.7.5 EFFECTIVE SURFACE GRAVITY

On the observational side, MoBiDICT not only predicts significantly larger deformations than the Roche model, but also directly modifies the surface effective gravity through changes in the topology of the total potential. Since the local surface brightness of a star is, for example, proportional to the local effective gravity in the gravity-darkening formalism introduced in [Equation 3.36](#), any modification of the total-potential topology can directly affect the light curves of eclipsing binary systems, and therefore the derivation of their orbital parameters. Because we focus here on the surface effective gravity, and because our refined Roche model gives the same result as the classical Roche model for this quantity, only one Roche model is considered in the remainder of this section.

3.7. COMPARISON WITH THE ROCHE MODEL

In our models, the effective gravity is given by the integral expression presented in [Subsection 3.4.2](#). In the Roche model, it is given by

$$|g_{\text{eff}}| = |\nabla \Psi_{\text{tot,Roche}}|,$$

where $\Psi_{\text{tot,Roche}}$ is defined in [Section 3.2](#). [Figure 3.11](#) compares the surface effective gravity predicted by the Roche model and by our models along the meridian passing through both the most distorted point and the poles for a twin binary system composed of two $0.2 M_{\odot}$ stars with an orbital separation of $2.8 R_{1D}$. [Figure 3.11](#) shows

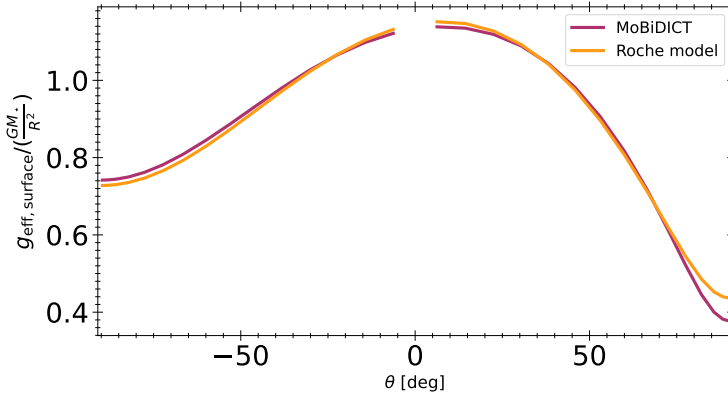


Figure 3.11: Surface effective gravity along the meridian passing through the most distorted point of the star, its opposite point, and the pole. The orange curve corresponds to the surface effective gravity predicted by the Roche model, while the purple curve shows the effective gravity obtained with our modelling. The system considered here is a twin binary composed of two $0.2 M_{\odot}$ stars with an orbital separation of $2.8 R_{1D}$. The absence of points near $\theta = 0$ is due to the Gauss–Legendre quadrature, which does not generate grid points in this region. Positive values of θ correspond to the direction in which the two stars face each other, whereas negative values correspond to the back side of the system.

that a difference in surface effective gravity exists between the Roche model and our modelling. The largest discrepancies in g_{eff} are found in the region closest to the inner Lagrangian point L_1 , where we obtain a relative difference of 12% with respect to the Roche model. These discrepancies can be explained by several factors. First, because our models are noticeably more deformed than the Roche model, the surface effective gravity is correspondingly lower. To a lesser extent, the absence in the Roche model of any correction for the redistribution of mass within the star also affects the surface value of g_{eff} . To summarize these differences, we computed the effective gravity for the grid of models introduced in [Subsection 3.7.3](#) and quantified the discrepancy between our models and the Roche model through the quantity Δg_{eff} , defined as

$$\Delta g_{\text{eff}} = \frac{g_{\text{eff,MobiDiCT}}(\mu = 0, \phi = 0) - g_{\text{eff,Roche}}(\mu = 0, \phi = 0)}{GM_{\star}/R_{1D}^2}. \quad (3.70)$$

3.7. COMPARISON WITH THE ROCHE MODEL

In Figure 3.12, we summarize the differences in surface effective gravity for the twin binary systems presented in Subsection 3.7.3 over a wide range of orbital separations.

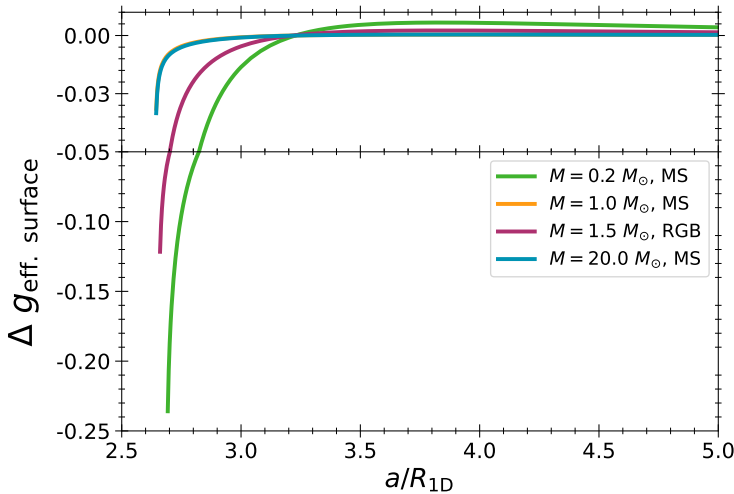


Figure 3.12: Difference of surface effective gravity of the most distorted point as a function of orbital separation for the four types of twin binaries studied.

Figure 3.12 shows the same overall trends as Figure 3.10. The models that display the largest discrepancies in surface deformation relative to the Roche model are also those that exhibit the strongest differences in surface effective gravity. In particular, Figure 3.12 shows that the largest discrepancies in g_{eff} are found for RGB and low-mass stars when $a/R_{1D} \lesssim 3.2$, whereas the other main-sequence stars show only limited corrections in our modelling.

From the observational point of view, the changes in g_{eff} found here may strongly affect the light curves of close binary systems, since the local effective temperature of a star depends directly on the surface effective gravity. With our modelling, the equatorial regions should appear noticeably cooler, and thus redder. Confirming these expectations through binary light-curve simulations based on MoBiDiCT models would be a natural next step.

3.7.6 STRUCTURAL DEFORMATION

The last point of comparison between the Roche model and our models concerns the structural deformation of the stars composing the binary system. For this analysis, we compare the classical Roche model, our refined Roche model, and the model obtained with MoBiDiCT. Structural differences can first be identified in the total potential, Ψ_{tot} , and in the effective gravity, g_{eff} , since these quantities determine the structural properties of our models. Figure 3.13 illustrates the behaviour of Ψ_{tot} and g_{eff} along a specific

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direction, namely the axis joining the centres of the two stars and the centre of mass of the system, for a twin binary composed of two $0.2 M_{\odot}$ stars with an orbital separation of $a = 2.8R_{1D}$.

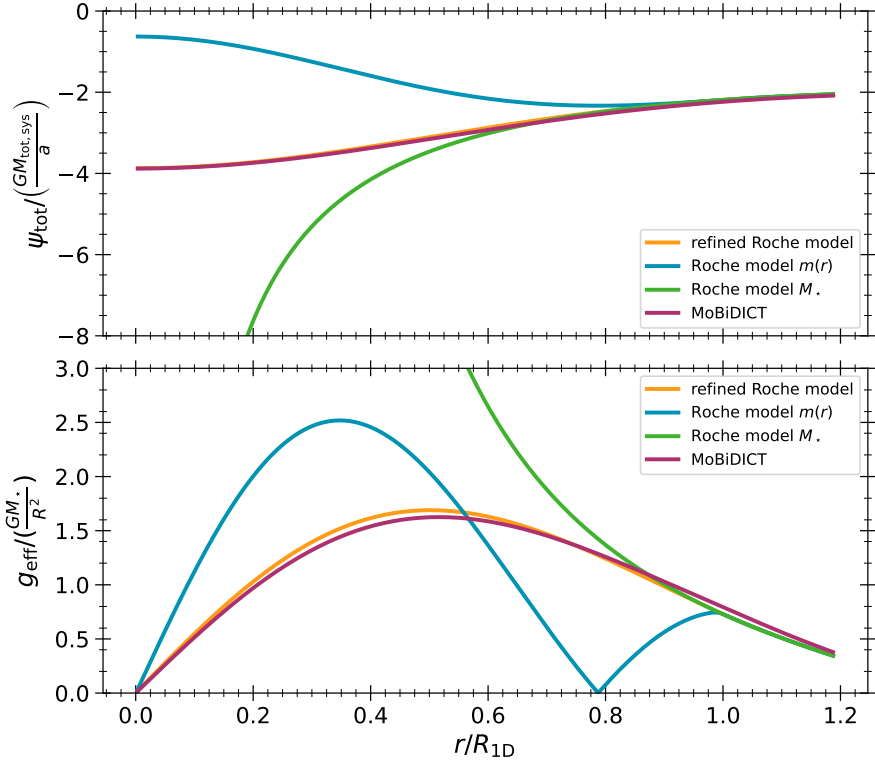


Figure 3.13: Normalized total potential, Ψ_{tot} , and effective gravity, g_{eff} , inside the stars along the axis joining the centres of the two components. The system considered here is a twin binary composed of two $0.2 M_{\odot}$ stars with an orbital separation of $a = 2.8R_{1D}$. The blue and green curves correspond to the classical Roche model, while the orange and violet curves correspond respectively to our refined Roche model and to the model computed with MoBiDICT.

Figure 3.13 illustrates the limitations of Roche-based models for describing the internal structure of deformed binaries. The classical expressions used for the Roche potential (Equation 3.4 and Equation 3.5) do not provide an accurate description of the stellar interior, even though they reproduce the surface properties correctly. In addition, Figure 3.13 shows that our refined Roche model provides a particularly accurate approximation for both Ψ_{tot} and g_{eff} , although small discrepancies remain throughout the star. In the work of Fabry et al. (2022), the potential used to compute the f_p and f_T factors, defined in Equation 3.34 and Equation 3.39, corresponds to the green curve in Figure 3.13. However, with additional assumptions on its derivative, the correspond-

3.7. COMPARISON WITH THE ROCHE MODEL

ing effective gravity is taken from the expression given by our refined Roche model. Although this effective gravity is appropriate for computing f_p and f_T , the potential used to identify the equipotential surfaces on which the averaged quantities are evaluated is not fully representative of the actual deformed stellar structure. As a result, corrections may be expected for the averaged internal quantities f_p and f_T reported by Fabry et al. (2022), for example.

To quantify the structural differences over the whole stellar interior, we used the coefficients f_p and f_T introduced in Subsection 3.4.1. Since this analysis concerns the full deformed structure as a function of radius, we restricted the study to twin binary systems composed of the four stellar types presented in Subsection 3.7.1, all at an orbital separation of $2.7 R_{1D}$. The resulting f_p and f_T values obtained with the different models are shown in Figure 3.14 and Figure 3.15, respectively. Because the classical expressions of the Roche model lead to unphysical results over most of the stellar interior, f_p and f_T are not considered for these models.

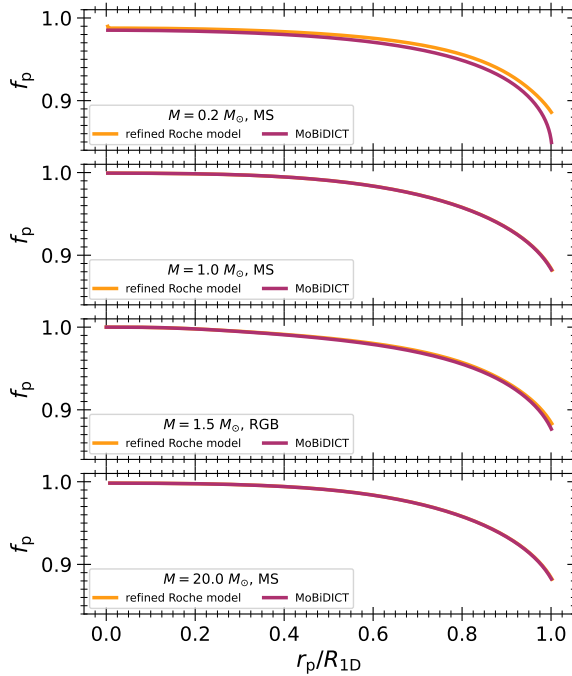


Figure 3.14: Radial variation of f_p in twin binary systems for the different models considered. From top to bottom, the panels show f_p for the 0.2 , 1.0 , 1.5 , and $20.0 M_\odot$ stars, respectively. The orange curves correspond to our refined Roche model, while the purple curves correspond to the results obtained with MoBiDiCT.

Figure 3.14 and Figure 3.15 show that the models exhibiting the largest deformation discrepancies with respect to the refined Roche model are also those displaying

3.8. CONCLUSION

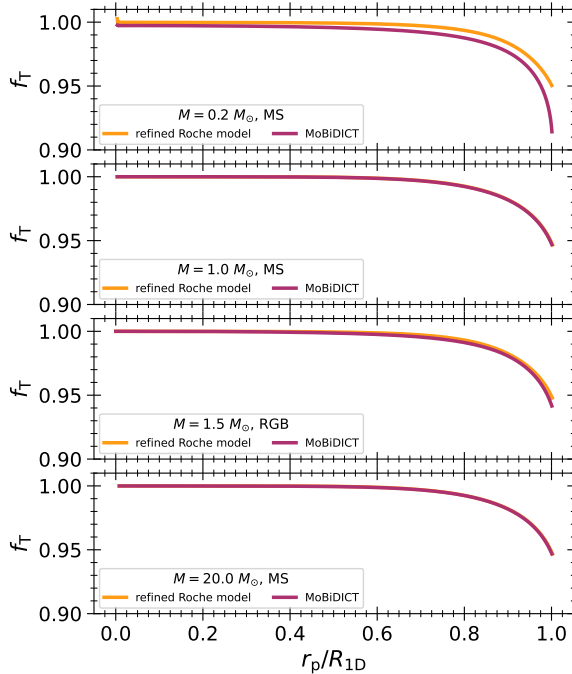


Figure 3.15: Evolution of f_T as a function of radius for the different stellar models in twin binary systems. From top to bottom, the panels correspond respectively to the 0.2 , 1.0 , 1.5 , and $20.0 M_\odot$ stellar models. The orange curves show the results obtained with our refined Roche model, whereas the purple curves show the MoBiDICT results.

the largest differences in f_p and f_T . These structural differences are seen mainly in the outer layers of the 0.2 and $1.5 M_\odot$ stars, which are also the stars for which the deformation discrepancies relative to the Roche model are the largest. Moreover, in the case of the $0.2 M_\odot$ binary system, a discrepancy is visible throughout the entire stellar structure. By predicting lower values of f_p and f_T , our modelling modifies the internal structure of binary stars. Combined with the deformation differences discussed above, these structural discrepancies may significantly alter the evolutionary path of binary systems. Finally, as already seen, our modelling does not significantly modify the structural description of massive and solar-like main-sequence stars.

3.8 CONCLUSION

this section is extracted from [Fellay & Dupret \(2023\)](#)

In this work, we developed MoBiDICT, a new tool designed to compute three-dimensional static models of close, synchronised binary systems in hydrostatic equilibrium using a non-perturbative approach. The technical description of the code is presented in

3.8. CONCLUSION

Section 3.3, while the post-processing required for a future coupling with classical one-dimensional stellar-evolution codes is described in Section 3.4. The numerical implementation of the models and the choice of the critical direction are discussed in Section 3.6 and Section 3.5, respectively. We then compared this new modelling framework with more classical approaches in Section 3.7, starting with the Roche model introduced in Section 3.2.

As shown in Section 3.7, the discrepancies between the Roche model and our non-perturbative models become significant mainly in very close binaries, typically when $a < 5R_{1D}$. The largest differences are found for low-mass and red-giant-branch (RGB) stars in twin systems, reaching up to 80% in our $0.2 M_{\odot}$ models. In contrast, massive and solar-like main-sequence stars are much less affected, with discrepancies not exceeding 7%. These trends indicate that the envelope mass fraction is the key parameter controlling the deformation differences: the larger the envelope mass relative to the total stellar mass, the stronger the structural modification predicted by MoBiDICT. The resulting discrepancies affect all quantities investigated in this work, including the surface effective gravity. This has direct implications for gravity darkening and, consequently, for the interpretation of observations of highly deformed systems. While the Roche model remains a good approximation for surface properties in the weak-deformation regime, our results show that it should not be used to infer the internal structural properties of significantly deformed stars.

A few general conclusions can be drawn from this work. First, in weakly distorted regimes, the classical modelling methods used for binary stars remain valid: the Roche model provides a satisfactory description of the stellar surface, while perturbative models remain adequate for the internal structure. In the most distorted regimes, however, both the Roche and perturbative approaches lose accuracy in reproducing the surface and structural properties of stars with high envelope mass fractions. Significant deformation discrepancies are found in particular for low-mass stars, in agreement with the results of Landin et al. (2009). Our work therefore highlights the need for non-perturbative modelling to study close binary interactions and evolution. Within our framework, the stronger deformations predicted by MoBiDICT imply that mass transfer may begin at larger orbital separations, and potentially earlier during the evolution of the system, than predicted by classical models. In this respect, the deformation discrepancies found for RGB stars are especially important, as they may facilitate the onset of mass transfer during an evolutionary phase already prone to it.

CHAPTER 4

APSIDAL MOTION IN CLOSE BINARY SYSTEMS

this chapter is fully extracted from [Fellay et al. \(2024\)](#).

Tidal forces in binary systems result from the gravitational interaction between non-point-like bodies and can be categorised into two components: the equilibrium tides, corresponding to large-scale circulations resulting from a hydrostatic readjustment (i.e. stellar deformations) and the dynamical tides corresponding to the excitation of eigenmodes of oscillations from the periodic perturbations from a companion ([Zahn, 1975](#)). These tidal forces and the subsequent stellar deformations lead to various phenomena such as orbital synchronisation, mass transfer, and tidal dissipation ([Jeans, 1929](#)), the latter corresponding to the dissipation of orbital energy within the stellar bodies. A direct consequence of the tidal interactions is the apsidal motion in eccentric binaries ([Sterne, 1939](#)). It represents the time variation of the argument of periastron ω or, in other words, the motion of the apsidal line with time. Apsidal motion originates from equilibrium tides, dynamical tides ([Gimenez & Margrave, 1985](#); [Willems & Aerts, 2002](#)), and general relativistic corrections¹ ([Gimenez & Margrave, 1985](#)). It serves as a valuable observational constraint to understand the structure of deformed stars, providing insights into the microphysics of stellar models ([Rosu et al., 2020a, 2022a,b, 2023](#)). Accurate modelling of tidal and centrifugal deformations is essential to determine the precise stellar structure of each component in binary systems and, consequently, to predict the apsidal motion.

When modelling the structure and deformations of binary stars, the perturbative approach is one of the classical and most widely used analytic methods. In this framework, each star is first described by an unperturbed spherically symmetric structure, while tidal and rotational distortions are introduced as small perturbations, usually

¹In the case of a hierarchical triple system, a contribution to the apsidal motion arises from the third body orbiting the inner binary ([Naoz et al., 2013](#)). In the present study, we do not account for such an effect as we focus on binary systems.

keeping only the leading-order terms in the multipolar expansions of the perturbing potentials (Sterne, 1939; Kopal, 1959, 1978). This approach provides a simplified description of rotational and tidal distortions of an initially spherically symmetric stellar structure. In the classical perturbative formalism, the leading quadrupolar response is parameterised by the apsidal-motion/internal-structure constant k_2 , obtained from the Clairaut–Radau equation (Brooker & Olle, 1955; Kopal, 1959; Claret & Gimenez, 1993). In the usual star-centred non-inertial formulation, the $\ell = 1$ component of the companion’s potential is cancelled by the centrifugal term, so the leading structural distortion is treated through $\ell \geq 2$ components (Kopal, 1959; Fitzpatrick, 2012). This cancellation should not be interpreted as the absence of all geometrical asymmetry: full Roche equipotentials contain near–far asymmetries and nonlinear geometric effects that may be absent or underestimated in truncated perturbative treatments. Recent non-perturbative 3D models, such as MoBiDICT, show that perturbative approaches and Roche models can underestimate the deformations in close systems (Fellay & Dupret, 2023). It is therefore important to test the consequences of the assumptions entering the perturbative tidal formalism used in secular tidal-evolution prescriptions, including those based on Zahn and Hut and implemented in widely used stellar-evolution frameworks such as MESA (Zahn, 1977; Hut, 1981; Paxton et al., 2011, 2013, 2015, 2018, 2019).

In this study, we build on the work of Fellay & Dupret (2023) by further investigating the limitations imposed by the assumptions of the perturbative approach, using our non-perturbative method. Our tool, Modelling Binary Deformations Induced by Centrifugal and Tidal forces (MoBiDICT), determines the deformed structure of close binaries in a non-perturbative manner while accounting for the redistribution of mass within the stellar components. Using the deformed structures obtained with MoBiDICT, we can calculate the instantaneous tidal acceleration perturbation and assess its consequences for the apsidal motion of binary systems. Finally, MoBiDICT’s formalism allows us to locate and identify the specific spherical-harmonic order responsible for the observed discrepancies.

In this chapter we start, in Section 4.1, by introducing the physics behind MoBiDICT and the methodology developed to compute the apsidal motion of binaries in a non-perturbative way. In Section 4.2, we explain how we developed the perturbative approach formalism usually used to model the apsidal motion of binaries. Section 4.3 is dedicated to exploring the discrepancies in models for different types of theoretical stars. In Section 4.4 we study binary systems for which the apsidal motion has been measured and provided in the literature. In Section 4.5, we discuss the implication of our findings on the stellar modelling methods used for binary stars. Finally, in the Appendices, we develop all the equations of the perturbative approach starting from a single star’s gravitational potential to arrive at the apsidal motion in binaries.

4.1. NON-PERTUBATIVE MODELLING OF THE APSIDAL MOTION

4.1 NON-PERTUBATIVE MODELLING OF THE APSIDAL MOTION

In classical celestial mechanics theory, the relative acceleration of two bodies in a binary system is entirely given by its radial component expressed as

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = \left[\frac{F_{21}}{M_1} - \frac{F_{12}}{M_2} \right] \hat{\mathbf{e}}_r = \frac{F_{21}}{m_\mu} \hat{\mathbf{e}}_r, \quad (4.1)$$

where the bold symbols denote vectorial quantities, F_{21} is the force exerted by the secondary star on the primary (noting $F_{2 \rightarrow 1} \equiv F_{21}$), F_{12} its opposite, $\hat{\mathbf{e}}_r$ is the radial unit vector centred on the secondary and pointing towards the primary, M_i are the individual masses of the bodies and m_μ is the reduced mass of the system defined as

$$m_\mu = \frac{M_1 M_2}{M_1 + M_2}. \quad (4.2)$$

The last step of Equation 4.1 was deduced from the third Newton Law ($F_{12} = -F_{21}$). The total force per unit of mass exerted by the secondary on its primary can be decomposed as

$$\frac{F_{21}}{m_\mu} = -\frac{GM_{\text{tot}}}{r^2} + F_R, \quad (4.3)$$

In Equation 4.3, F_R is the radial perturbed acceleration resulting from considering stars as non-spherical extended interacting bodies. This perturbation is responsible for all the equilibrium tidal-induced dynamical effects of binary evolution, including orbital migration (i.e. variation of systems semi-major axis), eccentricity dissipation (i.e. reduction of orbital eccentricity), and apsidal motion (i.e. time variation of the argument of the periastron).

In the framework of our modelling, the force exerted by the secondary on the primary can be expressed as

$$F_{21} = - \int \rho_1 F_2 dV_1 = - \int \rho_1 \frac{\partial \Psi_2}{\partial x} dV_1, \quad (4.4)$$

where F_2 is the gravitational force from the secondary inside the primary. By decomposing the densities and the forces in unperturbed spherical symmetric terms, respectively ρ_{10} and F_{20} , and non-spherical perturbations, respectively ρ'_1 and F'_2 , the force exerted by the secondary on the primary can be rewritten as

$$F_{21} = - \int (\rho_{10} + \rho'_1)(F_{20} + F'_2) dV_1. \quad (4.5)$$

The spherically symmetric component of the force can be simply identified as being

$$-\frac{1}{m_\mu} \int \rho_{10} F_{20} dV_1 = -\frac{GM_{\text{tot}}}{r^2}. \quad (4.6)$$

Therefore, the perturbed acceleration is expressed as

$$F_R = -\frac{1}{m_\mu} \int (\rho_{10} F'_2 + \rho'_1 F_{20} + \rho'_1 F'_2) dV_1. \quad (4.7)$$

4.1. NON-PERTUBATIVE MODELLING OF THE APSIDAL MOTION

In MoBiDICT, all quantities are projected on spherical harmonics to facilitate the solving of the Poisson equation (Equation 2.72), for example the densities and potentials are decomposed as

$$\rho_i(r_i, \mu, \phi) = \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \rho_{i,\ell}^m(r_i) Y_{\ell,m}(\mu, \phi), \quad (4.8)$$

and

$$\Psi_i(r_i, \mu, \phi) = \sum_{\ell=0}^L \sum_{k=0}^{(\ell-p)/2} \Psi_{i,\ell}^m(r_i) Y_{\ell,m}(\mu, \phi), \quad (4.9)$$

where L is the free parameter of our modelling defining the maximum degree of the spherical harmonics to be accounted for, $p = 0$ if ℓ is even, $p = 1$ otherwise, and $m = 2k + p$. In this formalism, the decomposition in spherically symmetric and non-symmetric terms is straightforward. The $\ell = 0, m = 0$ term is the spherically symmetric component of each quantity while all other spherical orders are non-symmetric terms. In our formalism the perturbed acceleration can thus be obtained with

$$\begin{aligned} F_R = & -\frac{1}{m_\mu} \int \rho_{1,0}^0 Y_{0,0} \left(\sum_{\ell=1}^L \sum_{k=0}^{(\ell-p)/2} F_{2,\ell}^m \right) dV_1 - \frac{1}{m_\mu} \int \left(\sum_{\ell=1}^L \sum_{k=0}^{(\ell-p)/2} \rho_{1,\ell}^m Y_{\ell,m} \right) F_{2,0}^0 dV_1 \\ & - \frac{1}{m_\mu} \int \left(\sum_{\ell=1}^L \sum_{k=0}^{(\ell-p)/2} \rho_{1,\ell}^m Y_{\ell,m} \right) \left(\sum_{\ell=1}^L \sum_{k=0}^{(\ell-p)/2} F_{2,\ell}^m \right) dV_1. \end{aligned} \quad (4.10)$$

Finally, each force term projected on the spherical harmonic basis $F_{2,\ell}^m$ can be decomposed and re-expressed in a spherical coordinate frame (to be integrated) with

$$F_{2,\ell}^m = \frac{\partial r_1}{\partial x_1} \frac{\partial \Psi_{2,\ell}^m Y_{\ell,m}}{\partial r_1} + \frac{\partial \theta_1}{\partial x_1} \frac{\partial \Psi_{2,\ell}^m Y_{\ell,m}}{\partial \theta_1} + \frac{\partial \phi_1}{\partial x_1} \frac{\partial \Psi_{2,\ell}^m Y_{\ell,m}}{\partial \phi_1}. \quad (4.11)$$

The spherical harmonics $Y_{\ell,m}$ in Equation 4.11 are functions of μ_2 and ϕ_2 . The expressions of the $\partial \Psi_{2,\ell}^m Y_{\ell,m} / \partial r_1$, $\partial \Psi_{2,\ell}^m Y_{\ell,m} / \partial \theta_1$, and $\partial \Psi_{2,\ell}^m Y_{\ell,m} / \partial \phi_1$ terms can be found in Fellay & Dupret (2023). More than having a non-perturbative description of the resulting force, the main advantage of our formalism is the possibility of naturally compare the contributions of the different spherical harmonic orders by selecting the desired orders in the sums.

4.1.1 NON-PERTURBATIVE APSIDAL MOTION COMPUTATION

In this section, we developed the formalism used to model the apsidal motion with our non-perturbative method. Figure 4.1 illustrates the setup of an eccentric binary system, the apsidal motion of a system corresponding to $d\omega/dt$. In eccentric binary systems, apsidal motion is a direct observational constraint on the deformations of distorted bodies. Recent investigations (Rosu et al., 2020a, 2022a,b) highlight its utility in constraining physical processes occurring during stellar evolution, such as internal mixing.

4.1. NON-PERTUBATIVE MODELLING OF THE APSIDAL MOTION

The apsidal motion of an eccentric binary can be decomposed into contributions from the perturbing acceleration associated with the rotational and tidal deformation of the stellar components, from general relativity, and from dynamical tides. The general-relativistic contribution is given by

$$\frac{d\omega_{\text{rel}}}{dt} = \frac{2\pi}{P_{\text{orb}}} \frac{3GM_{\text{tot}}}{c^2 a(1-e^2)}, \quad (4.12)$$

where a is the system's semi-major axis, e its eccentricity, P_{orb} its orbital period, c is the speed of light, and G the gravitational constant. In this chapter, we focus on the deformation-induced component of the apsidal motion and neglect the contribution from dynamical tides, which will be addressed in a forthcoming article. From Gauss'

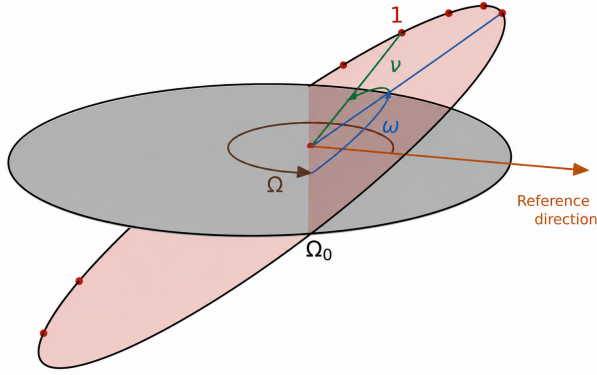


Figure 4.1: Orbital trajectory of a star in a binary system with an eccentricity of 0.2. The orbital plane is in red, the grey plane is a the reference plane (i.e. the plane of the sky), i is the inclination of the system compared to the reference plane, Ω is the longitude of the ascending node, Ω_0 is the ascending node, ω is the argument of the periastron, and φ is the true anomaly (noted ν in the rest of this chapter). The reds dots on the trajectory represent the different points at which the snapshots are taken to compute the apsidal motion.

planetary equations for a perturbed Keplerian orbit, the contribution to the apsidal motion induced by the deformation-related perturbing acceleration can be expressed in terms of the radial and transverse components of this acceleration (see, e.g., [Kopal 1959](#), or [Section 2.4](#)). In the case considered here, the dominant perturbation is radial component, the deformation-induced component of the apsidal motion, denoted $\dot{\omega}_N$, of a binary system is classically given by

$$\dot{\omega}_N \equiv \frac{d\omega_N}{dt} = -\frac{\sqrt{1-e^2}}{e} \sqrt{\frac{a}{GM_{\text{tot}}}} \langle F_R \cos \nu \rangle, \quad (4.13)$$

where ν is the true anomaly and $\langle F_R \cos \nu \rangle$ denotes the time average of the perturbed acceleration expressed as

$$\langle F_R \cos \nu \rangle = \frac{\int F_R \cos \nu dt}{\int dt}. \quad (4.14)$$

4.2. PERTURBATIVE MODEL

To compute the apsidal motion of a system with a non-perturbative approach, the required quantity is the orbit-averaged contribution of the deformation-induced perturbing acceleration entering Gauss’ planetary equation. We evaluate this quantity by exploiting the fast computational capabilities of MoBiDICT, which allows us to compute snapshots of the stellar deformation at several orbital phases, as illustrated in [Figure 4.1](#). The time average is then rewritten as an integral over the true anomaly using Kepler’s second law:

$$\frac{dt}{dv} = \frac{r^2}{na^2 \sqrt{1-e^2}}, \quad (4.15)$$

where r is the separation between the stars expressed, following the first Kepler’s Law, as

$$r = \frac{a(1-e^2)}{1+e \cos(\nu + \omega)}. \quad (4.16)$$

By combining [Equation 4.13](#), [Equation 4.14](#), [Equation 4.15](#) and [Equation 4.16](#), the apsidal motion obtained with our non-perturbative modelling is given by

$$\frac{d\omega_N}{dt} = \frac{1}{\pi} \frac{(1-e^2)^2}{e} \sqrt{\frac{a}{GM_{\text{tot}}}} \int_0^\pi \frac{F_R \cos \nu}{(1+e \cos(\nu))^2} d\nu, \quad (4.17)$$

where the integrals are only taken over half of an orbit by symmetry. In practice, with our method’s implementation, merely fifteen equally spaced values of ν over half of an orbit are sufficient to reach a relative precision of $10^{-3} - 10^{-4}$ on the apsidal motion in the vast majority of close binaries.

4.2 PERTURBATIVE MODEL

4.2.1 THE CLAIRAUT-RADAU EQUATION

In the classical perturbative model, the gravitational interaction within a binary system is described by expanding the gravitational potential around an initially spherically symmetric configuration and retaining only the leading-order terms. In its simplest form, the potential is reduced to the monopolar contribution, $\ell = 0$, associated with the spherically symmetric mass distribution, and the leading non-spherical quadrupolar contribution, $\ell = 2$, which accounts for the dominant tidal and rotational distortions. Higher-order multipolar terms are neglected under the assumption that their contributions remain small in the weakly deformed regime.

Within this framework, the stellar structure is assumed to respond linearly to the perturbing tidal and centrifugal potentials. Although this response is described through internal-structure constants such as k_2 , the method does not solve the fully coupled, nonlinear redistribution of mass and gravitational potential within the deformed stellar components, as done in MoBiDICT. By neglecting these non-linear feedback effects, the perturbative approach provides a simplified and computationally tractable description of tidal and rotational distortions, while remaining valid as long as the induced deformations are sufficiently small. In this section, we present the mathematical development of the perturbative approach used to compute and analyse the apsidal motion,

4.2. PERTURBATIVE MODEL

following the classical treatment usually adopted in the literature (Sterne, 1939; Kopal, 1959, 1978; Fitzpatrick, 2012).

Perturbative models are obtained by assuming that the stellar deformations remain small, more precisely that $|r_\ell^m| \ll 1$ in Equation 4.41, so that only first-order terms are retained in the expansion. As in our modelling approach, the gravitational potential of each star i is obtained by solving the Poisson equation projected onto a spherical-harmonic basis, Equation 3.20. The main distinction is that, in the perturbative framework, the stars are treated as weakly perturbed spherically symmetric bodies. This assumption greatly simplifies the problem and makes it possible to relate the deformation of a body to its unperturbed structure and to the properties of the binary system. In this framework, the response of each body is characterised by the structural coefficients η_ℓ , see Equation 4.41 and Equation 4.58. These coefficients are obtained by solving the first-order non-linear Clairaut–Radau differential equation (Kopal, 1959), derived in Subsection 4.A.1, which can be written as

$$\bar{r} \frac{d\eta_\ell}{d\bar{r}} + 6 \frac{\rho(\bar{r})}{\bar{\rho}} (\eta_\ell + 1) + \eta_\ell (\eta_\ell - 1) = \ell(\ell + 1), \quad (4.18)$$

using the central boundary condition:

$$\eta_\ell(0) = \ell - 2. \quad (4.19)$$

In the Clairaut–Radau equation, $\bar{r}$ is the average radius of a chosen isobar while $\bar{\rho}$ is the average density under the chosen isobar. The relationship between the deformed structure and the η_ℓ coefficients is explained and derived in Subsection 4.A.1.

4.2.2 PERTURBATION OF THE GRAVITATIONAL POTENTIAL AND FORCE BY A COMPANION

We considered the perturbation induced by a point-mass secondary on the primary. As developed in Subsection 4.A.2, the potential perturbation caused by the secondary at the surface of the primary is given by

$$\Psi_{1\ell}^m(R_1) = \frac{1 + \ell - \eta_{1\ell}^m(R_1)}{\eta_{1\ell}^m(R_1) + \ell} \left(\Psi_{2,\ell}^m(R_1) + \Psi_{c,\ell}^m(R_1) \right) = 2k_{\ell,1} \left(\Psi_{2,\ell}^m(R_1) + \Psi_{c,\ell}^m(R_1) \right), \quad (4.20)$$

where $\Psi_{2,\ell}^m(R_1)$ is the (ℓ, m) component of the gravitational potential generated by the secondary evaluated at the surface of the primary, and the centrifugal components of the potential are noted $\Psi_{c,\ell}^m$. The first terms of each of these quantities can be found in Subsection 4.A.3. The coefficients $k_{\ell,1}$ also known as Love numbers, characterise the surface non-spheroidal deformation response of a body subjected to a perturbative potential. Mathematically, these coefficients are defined as:

$$k_{\ell,1} = \frac{1 + \ell - \eta_{1\ell}^m(R_1)}{2(\eta_{1\ell}^m(R_1) + \ell)}. \quad (4.21)$$

Equation 4.20 provides a direct comparison between the models, where the quantity $\Psi_{1,\ell}^m(R_1)$ is equivalent to $\Psi_{1,\ell}^m(R_1)$ of our non-perturbative iterative method. One particular application to Equation 4.20 concerns the dominant contribution to deformations

4.2. PERTURBATIVE MODEL

and perturbations, the quadrupolar terms ($\ell = 2$ terms). For those terms, the sum of the centrifugal and tidal potentials generated at the surface of the primary by the secondary treated as a point-mass body is expressed, in the non-synchronised case, as

$$\Psi_{c,2}^0(R_1) + \Psi_{2,2}^0(R_1) = \left(\frac{R_1}{r}\right)^2 \left(\frac{\Omega_{\star,1}^2}{3} r^2 + \frac{1}{2} \frac{GM_2}{r} \right), \quad (4.22)$$

and

$$\Psi_{c,2}^2(R_1) + \Psi_{2,2}^2(R_1) = -\frac{1}{4} \frac{GM_2}{R_1} \left(\frac{R_1}{r}\right)^3. \quad (4.23)$$

In both synchronised and non-synchronised systems, the dipolar component of the total potential is null ($\Psi_{c,1}^1(R_1) + \Psi_{2,1}^1(R_1)$) when the secondary is treated as a point-mass, as gravitational and centrifugal forces cancel out according to the third Kepler law. As highlighted earlier, when computing the apsidal motion of a system, it is imperative to account for the tidally induced acceleration perturbation F_R . By only considering the spherical order lower than $\ell = 3$, the perturbed acceleration can be decomposed as follows:

$$F_R = -\frac{1}{m_\mu} [F_{2'10} + F_{201'} + \cancel{F_{21'1}}], \quad (4.24)$$

where $F_{2'10}$ is the force exerted by the quadrupolar deformation of the secondary on the unperturbed primary, $F_{201'}$ is the force exerted by the unperturbed secondary on the already perturbed primary, and $F_{2'1'}$ is the force exerted by the perturbed secondary on the already perturbed primary. The last term, $F_{2'1'}$, is a second order term, thus neglected in the perturbative approach. In accordance with Newton's third law, the force exerted by the unperturbed primary on the already perturbed secondary is the opposite of the force exerted by the perturbed secondary on the unperturbed primary, mathematically $F_{201'} = -F_{1'20}$. As shown in [Subsection 4.A.4](#), the force exerted by the perturbed secondary on the unperturbed primary is given by

$$F_{2'10} = 2 \frac{GM_1^2}{r^2} k_{2,2} \left(\frac{R_2}{r}\right)^5 \left[3 + \frac{1}{2} \frac{M_{\text{tot}}}{M_1} \left(\frac{\Omega_{\star,1}}{\tilde{n}}\right)^2 \right], \quad (4.25)$$

similarly, $F_{1'20}$ can be obtained by permuting the primary and secondary in [Equation 4.25](#). Finally, combining Eqs. (4.24) and (4.25) the perturbed acceleration is expressed as

$$F_R = -2 \frac{GM_{\text{tot}}}{r^2} \left[k_{2,2} \frac{1}{q} \left(\frac{R_2}{r}\right)^5 \left(3 + \frac{q+1}{2} \left(\frac{\Omega_{\star,2}}{\tilde{n}}\right)^2 \right) + k_{2,1} q \left(\frac{R_1}{r}\right)^5 \left(3 + \frac{1}{2} \frac{q+1}{q} \left(\frac{\Omega_{\star,1}}{\tilde{n}}\right)^2 \right) \right], \quad (4.26)$$

with $q = M_2/M_1$.

4.2.3 PERTURBATIVE APSIDAL MOTION COMPUTATION

In the case of an eccentric orbit, the mean orbital rotation rate n is distinct from the instantaneous orbital rotation rate $\tilde{n}$ experienced by each individual binary component.

4.2. PERTURBATIVE MODEL

Using this formalism, the perturbed acceleration can be rewritten by grouping the tidal and centrifugal terms as

$$F_R = -2 \frac{GM_{\text{tot}}}{a^2} \left[\left(3k_{2,2} \frac{1}{q} \left(\frac{R_2}{a} \right)^5 + 3k_{2,1} q \left(\frac{R_1}{a} \right)^5 \right) \left(\frac{a}{r} \right)^7 + \left(k_{2,2} \frac{q+1}{2q} \left(\frac{R_2}{a} \right)^5 \left(\frac{P_{\text{orb}}}{P_{\star,2}} \right)^2 + k_{2,1} \frac{q+1}{2} \left(\frac{R_1}{a} \right)^5 \left(\frac{P_{\text{orb}}}{P_{\star,1}} \right)^2 \right) \left(\frac{a}{r} \right)^4 \right] = -2 \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} \left(\frac{a}{r} \right)^7 + R_c \left(\frac{a}{r} \right)^4 \right], \quad (4.27)$$

where $P_{\star,1}$ and $P_{\star,2}$ are respectively the rotation period of the primary and secondary, and R_{tides} and R_c can be identified in Equation 4.27. The apsidal motion of a system is dependent on the time average of the perturbed acceleration over an orbit $\langle F_R \cos \varphi \rangle$. Using Equation 4.14, the perturbed acceleration averaged over an orbit is given by

$$\langle F_R \cos \varphi \rangle = -\frac{2}{P_{\text{orb}}} \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} \int_0^{P_{\text{orb}}} \left(\frac{a}{r} \right)^7 \cos \varphi \, dt + R_c \int_0^{P_{\text{orb}}} \left(\frac{a}{r} \right)^4 \cos \varphi \, dt \right].$$

The integrals in Equation 4.28 can be re-expressed with the zero-order Hansen coefficients $X_0^{n,m}(e)$ defined for every positive m as

$$X_0^{n,m}(e) \equiv \frac{1}{P_{\text{orb}}} \int_0^{P_{\text{orb}}} \left(\frac{r}{a} \right)^n \cos(m\varphi) \, dt. \quad (4.28)$$

Without entering into the detailed expressions of these coefficients, the two orders that are of interest in our case are the $n = -4, m = 1$ and $n = -7, m = 1$ that are given by

$$X_0^{-4,1}(e) = e(1 - e^2)^{-5/2}, \quad (4.29)$$

and,

$$X_0^{-7,1}(e) = \frac{5}{2} e(1 - e^2)^{-11/2} \left(1 + \frac{3}{2} e^2 + \frac{1}{8} e^4 \right). \quad (4.30)$$

The average perturbed acceleration is finally given by

$$\langle F_R \cos \varphi \rangle = -2 \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} X_0^{-4,1}(e) + R_c X_0^{-7,1}(e) \right], \quad (4.31)$$

and can be inserted in Equation 4.13 to obtain the apsidal motion of a binary system with the perturbative approach:

$$\frac{d\omega_N}{dt} = \frac{2\pi}{P_{\text{orb}}} \left[15f(e) \left(\left(\frac{R_1}{a} \right)^5 q k_{2,1} + \left(\frac{R_2}{a} \right)^5 \frac{k_{2,2}}{q} \right) + g(e) \left(k_{2,1} (q+1) \left(\frac{P_{\text{orb}}}{P_{\star,1}} \right)^2 \left(\frac{R_1}{a} \right)^5 + k_{2,2} \frac{q+1}{q} \left(\frac{P_{\text{orb}}}{P_{\star,2}} \right)^2 \left(\frac{R_2}{a} \right)^5 \right) \right], \quad (4.32)$$

with,

$$f(e) = \frac{1 + \frac{3}{2} e^2 + \frac{1}{8} e^4}{(1 - e^2)^5}, \quad (4.33)$$

and,

$$g(e) = \frac{1}{(1 - e^2)^2}. \quad (4.34)$$

4.3. GENERAL MODEL COMPARISONS

4.3 GENERAL MODEL COMPARISONS

This section is dedicated to a theoretical comparison of different types of stars encompassing a range of potential binary systems across different mass regimes. Here, we present the input stellar models used to compare the perturbative and non-perturbative modelling approaches. We chose four models to represent a broad spectrum of stars and structural characteristics. The physical ingredients of the chosen models are detailed in [Table 4.1](#), those models are the same models compared in [Fellay & Dupret \(2023\)](#). Table 1 in [Fellay & Dupret \(2023\)](#) has only been extended to include the k_2

Table 4.1: Summary of the stellar properties of the 1D input models used in this work.

Stellar parameters	$0.2 M_{\odot}$	$1.0 M_{\odot}$	$1.5 M_{\odot}$	$20 M_{\odot}$
Mass [$M_{\odot}$]	0.2	1.0	1.5	20.0
Radius [$R_{\odot}$]	0.22	1.03	11.1	6.01
Age [Gyr]	2.0	2.0	2.0	0.002
Evolutionary stage	MS	MS	RGB	MS
Effective temperature [K]	3342	6080	4621	35 859
Luminosity [$L_{\odot}$]	0.005	1.31	51.1	53 921
Initial Hydrogen X_0	0.72	0.72	0.72	0.72
Initial Metallicity Z_0	0.010	0.010	0.010	0.010
Core Hydrogen X_c	0.693	0.469	0	0.582
k_2	0.151	0.0104	0.0593	0.0127
Ω_i/n	1.0	1.0	1.0	1.0

value of each model. As shown by [Equation 4.20](#), to a higher k_2 corresponds a greater surface response from a body when subjected to a potential perturbation to its surface. Moreover, k_2 can be used as an indicator of a body’s deformability. In addition, our results from [Fellay & Dupret \(2023\)](#), showed that the low-mass and red giant branch (RGB) stars are the most distorted by a companion while the massive and solar type stars are expected to be less deformed and impacted by our non-perturbative method for a given a/R_1 ratio. In the following sections, we compared the surface deformations and apsidal motion obtained with our non-perturbative modelling method and the perturbative modelling for twin binary systems ($q = 1$) composed of the stars presented in [Table 4.1](#). Twin binary systems being systems composed of two identical stars (i.e. stars having the same mass, radius, effective temperature, and rotational period).

4.3.1 VALIDITY OF THE CLAIRAUT-RADAU EQUATION

In this section, we verify the limits of the perturbative approach for close binaries, in particular when the stars are undergoing high deformations. To compare the different models, we directly look at the coefficient η_{ℓ} obtained through [Equation 4.60](#) for per-

4.3. GENERAL MODEL COMPARISONS

turbative models, while with our models these quantities are computed with a projection on the spherical harmonics of the deformations. We also compared these models to our refined Roche model through the same coefficients η_ℓ obtained by the same way that with our models. We focus our work on the $\ell = 2$ term as this coefficient is assumed, by the perturbative model, to be the leading contribution to the apsidal motion of binary system that is used for example in Rosu et al. (2020a, 2022a,b) to constrain the stellar structure.

The evolution of η_2 as a function of the average radius is illustrated in Figure 4.2 for the extensively studied system of this article, the twin binary system composed of $0.2 M_\odot$ stars at an orbital distance of $2.8R_{1D}$.

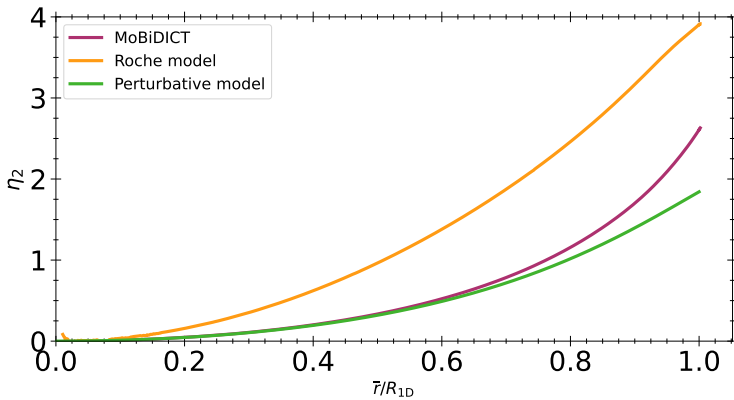


Figure 4.2: Evolution of η_2 as a function of the average radius for the different models studied in this work. The system studied here is a twin binary system composed of two $0.2 M_\odot$ stars with an orbital distance of $2.8R_{1D}$.

Figure 4.2 illustrates different significant results from our modelling. First comparing the Roche model to other models, we see that both perturbative and MoBiDICT models are predicting an important difference of η_2 in all the stars. These results can be interpreted as difficulties of the Roche model to precisely depict the stellar deformations, these limitations originating from the assumptions on the stellar gravitational potential.

The second major result illustrated in Figure 4.2 is the comparison between our new models and the perturbative models. First, in the less distorted regions of the star (when $\bar{r} < 0.5R_{1D}$) the same results are obtained with models, indicating that our modelling is giving a good depiction of stellar deformation in these regions, as we expect to be in the limit of validity of perturbative modelling. Then, after $\bar{r} > 0.5R_{1D}$, non negligible differences between the two models are appearing, with discrepancies increasing with $\bar{r}$, as the upper layers of the star are more deformed. These differences can be explained by the limitations of the perturbative approach in these regimes of deformation, while our model is designed to study cases with high stellar deformations.

To study the apsidal motion of binary system one would require the apsidal motion

4.3. GENERAL MODEL COMPARISONS

constant k_2 that is usually obtained through η_2 evaluated at the surface of the primary star. To study how our new modelling procedure is impacting this apsidal motion constant depending on the twin binary system studied and orbital separation we introduce a new quantity $\Delta\eta_2$ defined as the difference of η_2 at the stellar surface between our model and the perturbative model, $\Delta\eta_2$ is expressed as

$$\Delta\eta_2 = \eta_{2,\text{MoBiDiCT}}(\bar{R}_s) - \eta_{2,\text{pert.}}(\bar{R}_s). \quad (4.35)$$

The comparison of $\Delta\eta_2$ from the grid of twin binary models, presented in [Subsection 3.7.1](#), as a function of their orbital separation, is illustrated in [Figure 4.3](#).

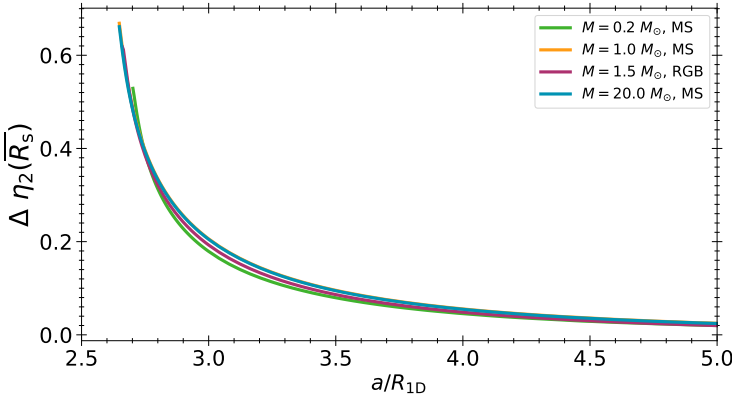


Figure 4.3: Evolution of the surface $\Delta\eta_2$ for different types of stars composing the binaries and with different orbital separation.

[Figure 4.3](#) illustrates that the differences of η_2 between the perturbative and MoBiDiCT models are almost independent from the type of star composing the binary system and only dependent on the orbital separation of the components. Significant differences of surface η_2 can only be seen in the most distorted cases, when $a < 5R_{1D}$, otherwise the perturbative treatment of the tidal and centrifugal forces is justified to compute η_2 . The differences found are expected to directly impact the apsidal motion computation, predicting correction to apsidal motion constant k_2 and adding k_1 constant to account for neglected dipolar terms. The impact of our improved modeling on apsidal motion will be deeper investigated in another forthcoming article.

4.3.2 POTENTIAL AND FORCES DISCREPANCY

The perturbation of the surface potential is a reliable indicator of the deformations undergone by a star. Within either our formalism or the perturbative formalism, the perturbation of the surface potential is given by all spectral terms of the surface potential $\Psi_{i\ell}^m(R_i)$ with $\ell > 0$. In the perturbative method, this quantity is given by [Equation 4.20](#), which is applied to the $\ell = 2$ terms in [Equation 4.85](#) and [Equation 4.84](#). Within our non-perturbative method, $\Psi_{i\ell}^m(R_i)$ are obtained in an iterative process, solving Poisson's equation at each step and for each spherical order. Consequently, there is no

4.3. GENERAL MODEL COMPARISONS

analytical expressions available for this quantity in our method. To compare the surface deformations with different modelling procedures we introduce the discrepancy of surface potential, denoted $\Delta\Psi_\ell^m(R_1)$ and defined as:

$$\Delta\Psi_\ell^m(R_1) = \Psi_{\ell,\text{MoBi}}^m(R_1) - \Psi_{\ell,\text{pert}}^m(R), \quad (4.36)$$

where $\Psi_{\ell,\text{MoBi}}^m$ is the spectral surface potential obtained with our non-perturbative method and $\Psi_{\ell,\text{pert}}^m$ is the same quantity obtained with the perturbative approach. In Figure 4.4, we show this surface potential discrepancy as a function of the orbital separation for twin binary systems composed of the stars presented in Table 4.1. Our focus in this analysis is on the dominant terms, the $\ell = 2$ terms.

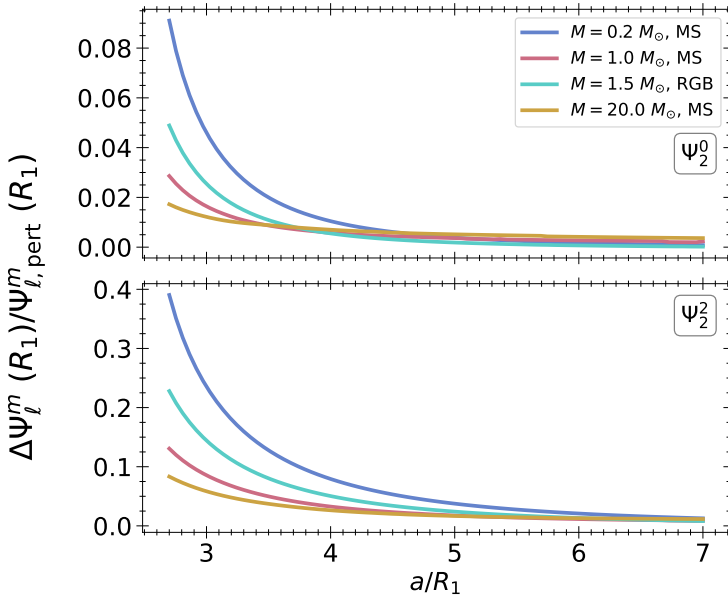


Figure 4.4: Surface potential discrepancy between perturbative and non-perturbative modelling approaches as a function of the orbital separation normalised by the stellar radii for the different binary components presented in Table 4.1. The upper panel corresponds to the $\ell = 2, m = 0$ component of the potential while the lower panel is the $\ell = 2, m = 2$ term.

The surface potential discrepancy, varies significantly depending on the stellar type and the orbital separation. For both $\ell = 2$ components, stars with the higher k_2 exhibit more pronounced effects from our non-perturbative treatment, resulting in differences of up to $\sim 40\%$ in the $\ell = 2, m = 2$ term for low mass stars. This implies that the higher the deformations, the more substantial the underestimation of the surface deformations by the perturbative approach. Furthermore, orbital separation plays a critical role, with closer star configurations yielding stronger deformations and more

4.3. GENERAL MODEL COMPARISONS

notable disparities in the surface potential.

By comparing the two panels in [Figure 4.4](#), we observe a significantly smaller difference in surface potential for the $\ell = 2, m = 0$ term, regardless of the model. The $\ell = 2, m = 0$ component arises from both centrifugal and tidal deformations, while the $\ell = 2, m = 2$ term is solely originating from the tidal deformation. Both our modelling and perturbative modelling are sharing the same assumptions on the centrifugal potential, therefore, the difference originating from tidal contribution in the $\ell = 2, m = 0$ term is diluted by the centrifugal contribution. The major difference between the perturbative and non-perturbative modelling approaches is thus appearing in the treatment of the tidal deformation.

4.3.3 APSIDAL MOTION

In [Subsection 4.1.1](#) we showed that the apsidal motion of an eccentric binary system is proportional to the time average of the perturbed acceleration. As detailed in [Subsection 4.3.2](#), the perturbed acceleration denotes the tidal acceleration originating from a nearby companion. A modification of the surface $\ell = 2$ potential as illustrated in [Figure 4.4](#) introduces variations in the perturbed acceleration. In this section, we investigated the perturbed acceleration discrepancy resulting from the potential differences seen and the consequent impact on the apsidal motion of eccentric binaries.

In the classical perturbative model, the perturbed acceleration F_R is solely originating from the quadrupolar component of the deformation. The detailed expression of F_R is given in [Equation 4.26](#) and derived in [Subsection 4.A.4](#). In opposition, our methodology incorporates a non-perturbative treatment of all spherical orders spanning $\ell = 1, 2, \dots, L$. The exact expression of F_R is given by [Equation 4.10](#). To quantify the disparities between perturbative and non-perturbative perturbed accelerations, we define their difference ΔF_R as follows:

$$\Delta F_R = F_{R,\text{MoBi}} - F_{R,\text{pert}}. \quad (4.37)$$

The difference of perturbed accelerations as a function of the separation between the twin binary systems components presented in [Table 4.1](#) is illustrated in [Figure 4.5](#). The perturbed acceleration discrepancy observed in [Figure 4.5](#) is in agreement with the results showed in [Figure 4.4](#). Models exhibiting the greatest surface potential discrepancies are also exhibiting notable perturbed acceleration differences. Moreover, models with higher k_2 , namely the low mass and RGB stars, are the most impacted by our modelling. Solar-like stars and massive stars are comparatively less affected, even if in close orbit F_R discrepancies can still reach up to, respectively, 20% and 12%.

To assess the discrepancy of apsidal motion resulting from the perturbed acceleration differences, we need to consider binaries with eccentric orbits. To limit the free parameters of the system we compare binary systems with synchronised rotations and a fixed eccentricity, $e = 0.1$. The impact of the eccentricity on the apsidal motion is explored for systems in [Subsection 4.3.4](#). We introduce the apsidal motion difference:

$$\Delta\dot{\omega} = \dot{\omega}_{\text{MoBi}} - \dot{\omega}_{\text{pert}}, \quad (4.38)$$

4.3. GENERAL MODEL COMPARISONS

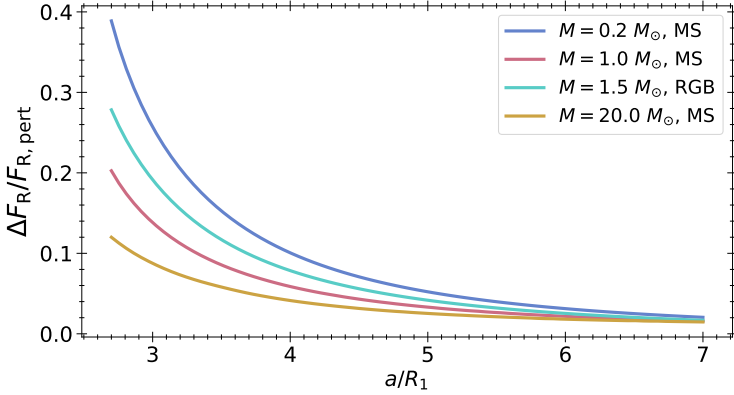


Figure 4.5: Perturbed acceleration discrepancy as a function of the separation normalised by the stellar radii for the twin binary systems composed of the stars presented in Table 4.1. The colour code denotes the different twin binary systems.

to compare the apsidal motion originating from the non-perturbative and perturbative approaches. In the perturbative approach, the apsidal motion is given by Equation 4.106. With our non-perturbative method we required to capture snapshots of the binary system along half of an orbit to estimate the perturbed acceleration time average and hence the apsidal motion (see Subsection 4.1.1). For each binary system composed of the stars presented in Table 4.1, Figure 4.6 illustrates $\Delta\dot{\omega}$ as the function of the binaries orbital separation normalised by the stellar radii. The apsidal motion

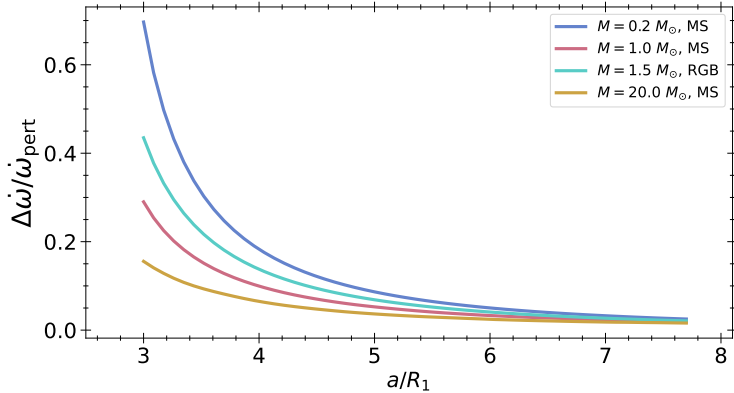


Figure 4.6: Evolution of the apsidal motion relative difference as a function of the orbital separation normalised by the stellar radii. The eccentricity of the orbits were set to $e = 0.1$ and the colour code denotes the different twin binary systems.

discrepancy is a direct result of the perturbed acceleration illustrated in Figure 4.5. Binary systems with greater F_R discrepancies exhibit the highest apsidal motion differences. For the low mass and RGB stars, the apsidal motion discrepancy can reach up

4.3. GENERAL MODEL COMPARISONS

to respectively 70% and 45% when the stars are close to contact, in their periastron. Similarly, for the solar type and massive stars we found an apsidal motion discrepancy up to respectively 30% and 15%. These discrepancies vary depending on the exact structure of the stars or the architecture and the orbital parameters of the system.

4.3.4 DEPENDENCY ON THE ORBITAL ECCENTRICITY

In the previous section, we fixed the eccentricity of the systems to infer the modelled apsidal motion discrepancy. In this section, we study the impact of varying the orbital eccentricity on the apsidal motion discrepancy. We expect that binaries with higher eccentricities are closer in their periastron, consequently more deformed and more impacted by our methodology. We focussed on the twin binary system composed of $20 M_{\odot}$ stars as this system is less impacted by our modelling and will consequently give a lower limit to the modelling discrepancies for a given semi-major axis. We computed a grid of models with different eccentricities and orbital separations, looking at the apsidal motion relative difference. The results from this computation are illustrated in Figure 4.7². The eccentricity is indeed a crucial parameter directly impacting the

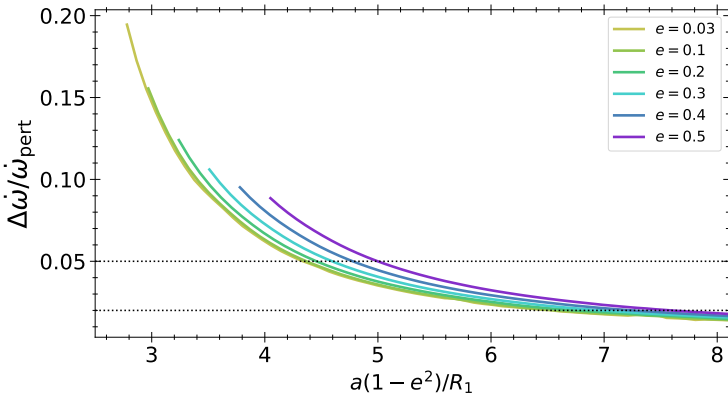


Figure 4.7: Apsidal motion relative different for a twin binary system composed of two $20 M_{\odot}$ stars presented in Table 4.1. Each colour correspond to the variation of discrepancy as a function of the orbital separation scaled by the eccentricity and stellar radii for different values of the orbital eccentricity. The black lines are lines of constant model relative difference, corresponding to 5% and 2% discrepancy.

modelling discrepancies. As expected, binary systems with higher eccentricities are more significantly impacted by our modelling method for a given orbital separation. For our least deformed model, we found that the modelling discrepancy reaches at least 5% when $a(1 - e^2)/R_1 \lesssim 4.5$ and 2% when $a(1 - e^2)/R_1 \lesssim 6.5$. For all our other theoretical models these thresholds are higher, by applying the same methodology to the twin system composed of $0.2 M_{\odot}$ stars, we found that the modelling discrepancy

²In Figure 4.7, we have chosen the empirical parameter $a(1 - e^2)/R_1$ for the x -axis such as the curves are the most superposed at high orbital separation.

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thresholds of 5% and 2% are respectively located at $a(1 - e^2)/R_1 \lesssim 6.5$ and $a(1 - e^2)/R_1 \lesssim 4.5$.

In addition to this work, we verified the precision of our method at higher orbital separation and found that for $a(1 - e^2)/R_1 = 20$ the discrepancies between the models reach 0.1% indicating a slow decrease of the apsidal motion discrepancy at high orbital separation and a sufficient precision to impose the thresholds given in this section.

4.3.5 DEPENDENCY ON k_2 AND THE ORBITAL SEPARATION

In the previous sections, we saw that a hierarchy exists between the perturbed acceleration of different stellar models and the resulting apsidal motion discrepancy. We observed that stellar models with higher k_2 values were more affected by our modelling approach. Another crucial quantity impacting the discrepancies is the separation of the binary components. In this section, we aim to comprehensively assess modelling discrepancies across a diverse range of stellar models and orbital separation.

To explore this, we constructed a grid of models of MS stars with masses ranging from $0.2M_\odot$ to $30M_\odot$ using the Code Liégeois d'Evolution Stellaire (CLES, [Scuflaire et al., 2008b](#)). For each mass in our grid we took several models along the MS to have diverse models and stellar structures with similar k_2 to explore whether discrepancies are solely dependent on k_2 and the orbital separation.

[Figure 4.8](#), illustrates the evolution of the acceleration perturbation relative difference $\Delta F_R/F_{R,\text{pert}}$ between our modelling and the perturbative approach. The top panel of the figure represents a group of stars that are either fully convective or composed of a radiative core and convective envelope, specifically stars with masses below $1.25M_\odot$. The lower panel corresponds to stars with a convective core, which includes stars with masses greater than $1.5M_\odot$.

[Figure 4.8](#) reveals a distinct behaviour between the two groups of stars. While the lower k_2 limit of the low mass stars and the upper limit of the intermediate-high mass stars have similar k_2 , their acceleration perturbation discrepancy is not similar. This result indicates that discrepancies between the models are not only a function of k_2 but also involve a component related to the intrinsic stellar structure. From our model grid, we observed that stars with different structures, particularly those with radiative or convective cores, displayed significant variations in discrepancy despite similar k_2 values. This phenomenon is probably related to the differences in density profiles between stars having or not a convective core. If looking for a possible parametrisation solely dependent on k_2 and orbital separation to correct the modelling discrepancies in the perturbative approach, the non-linear behaviours of the models greatly reduces its feasibility. We attempted an MCMC analysis using linear combinations of power laws of the orbital separation, for both the intermediate-high mass stars and low-mass stars. However, the results were unsatisfactory to propose an empirical parametrisation. As detailed in [Subsection 4.4.2](#), the modelling discrepancies indeed arise from a combination of discrepancies in the $\ell = 1, 2, 3$ terms. Non-perturbative modelling is necessary to recover the discrepancies seen in this section. However, [Figure 4.8](#) can be used to get an order of magnitude of the underestimation of tidal forces introduced by

4.3. GENERAL MODEL COMPARISONS

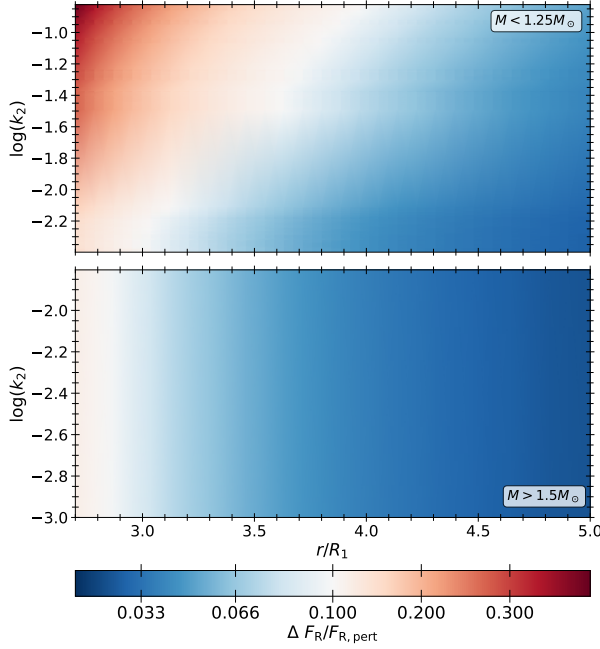


Figure 4.8: Mapping of the acceleration perturbation discrepancy for our grid of low-mass stars and intermediate-high mass stars. The colour code denotes the difference of perturbed acceleration between the perturbative and our non-perturbative approach as a function of the models k_2 and orbital separation.

the perturbative modelling. The updated grid of k_2 values provided by [Claret \(2023\)](#) could for example be used to easily locate observed stellar systems in this diagram.

4.3.6 DEPENDENCY ON THE MASS RATIO q

Our analysis so far focussed on the dependency on R_1/a as this latter parameter has the most important impact on the results. However, the perturbative apsidal motion has a direct dependency on 8 parameters: $(n, (n/\Omega_1)^2, (n/\Omega_2)^2, (R_1/a)^5, (R_2/a)^5, q, e, k_{21}, k_{22})$, see [Equation 4.106](#). With our non-perturbative modelling the dependencies on k_{21}, k_{22} become dependencies on the entire density profile of each star. Characterising the dependency of our result on mass ratio q is useful as this quantity can strongly impact the results and is a direct sub-product of the observations of binaries. To study and modify the q parameter in our modelling, different stellar models with the same age and initial chemical properties have to be used for the secondary star, impacting 4 of the parameters of our modelling for a given orbital separation: $(n, (R_2/a)^5, q, k_{22}$ or $\rho_2(r))$. Consequently the effect of a change of q will be seen indirectly, through, in particular $(R_2/a)^5$. For a given q parameter different than one, the behaviour of the apsidal motion discrepancy as a function of the orbital separation is dependent on the regime of $(R_2/R_1)^5$, as discussed hereafter.

4.3. GENERAL MODEL COMPARISONS

$$R_1^5 \gg R_2^5$$

This regime describes the case of a system composed of either a star plus a compact object or an evolved star plus a MS star. To simulate this situation, we chose to model the primary star as being the $1.5 M_\odot$ RGB star described in Table 4.1. The secondary is a MS star with a lower mass, ranging between 0.2 and $1.25 M_\odot$, the same age (2 Byr) and initial chemical composition ($X_0 = 0.72$, $Z_0 = 0.010$). In Figure 4.9, we show the evolution of the apsidal motion discrepancy for the different systems considered. Based on Equation 4.106, we chose the x -axis of Figure 4.9 to be $q^{-1/5} R_1/a$ to maintain the scale between $(R_1/a)^5$ and q . We can see that all the systems

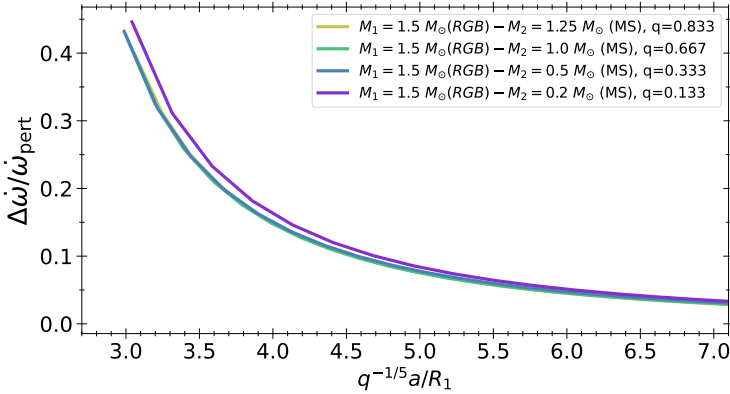


Figure 4.9: Apsidal motion discrepancy as a function of the orbital separation scaled with the q parameter. Each system is composed of a primary $1.5 M_\odot$ RGB star presented in Table 4.1, and a secondary star with a lower mass. The stars composing the systems have the same age (2 Byr) and initial chemical composition ($X_0 = 0.72$, $Z_0 = 0.010$). Each colour corresponds to a secondary with a different mass.

are superposed in Figure 4.9 which indicates a clear proportionality of the apsidal motion discrepancy on q . When $R_1^5 \gg R_2^5$, as seen in Equation 4.106, the terms with $(R_1/a)^5$ are dominating for a given orbital separation, and the contribution from the secondary is negligible. In this situation, only the deformations of the primary impact the apsidal motion. As both the deformations and the apsidal motion are proportional to q for a given orbital separation, the apsidal motion discrepancy is also proportional to q .

$$R_1^5 \simeq R_2^5$$

This regime describes systems composed of stars in similar evolutionary stages and structural properties. To simulate this situation we combined a primary MS star with a mass of $1.5 M_\odot$, an age of 1.5 Byr, an initial chemical composition of $X_0 = 0.72$, $Z_0 = 0.010$, with a secondary star with the same properties but a lower or equal mass. In Figure 4.10, we explore the apsidal motion discrepancy as a function of $q^{-1/5} R_1/a$ for these binary systems. In the case where the primary and secondary have similar

4.4. APPLICATION TO OBSERVED SYSTEMS

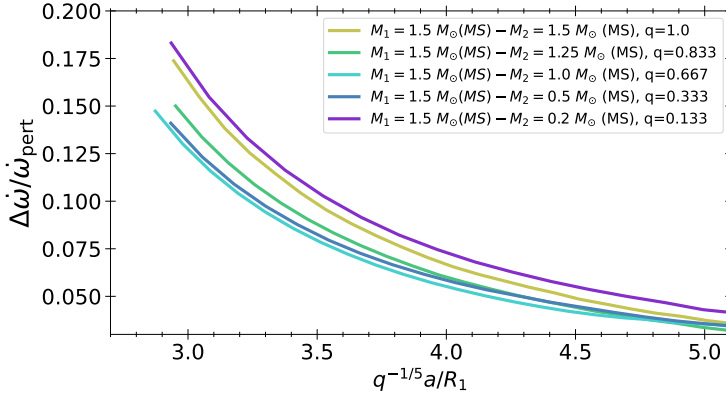


Figure 4.10: Apsidal motion discrepancy as a function of the orbital separation scaled with the q parameter. Each system is composed of a primary $1.5M_{\odot}$ MS star, and a secondary star with a lower or equal mass. The stars composing the systems have the same age (1.5 Byr) and initial chemical composition ($X_0 = 0.72$, $Z_0 = 0.010$). Each colour corresponds to a secondary with a different mass.

radii despite different q , the properties of the secondary have an impact on the apsidal motion discrepancy, as illustrated by comparing Figs. [Figure 4.9](#) and [Figure 4.10](#). The apsidal motion discrepancy is still dominated by the contribution from the primary, consequently a partial proportionality to q can be seen. However, additional contributions from the other parameters of the secondary, in particular its density profile, have to be accounted for.

In our determination of the limit after which the modelling discrepancy reaches 2% and 5% (see [Subsection 4.3.4](#)), we assumed that $q = 1$. In practice, observed systems often have $q < 1$, and, to include the dependency of these limits on q we modified them. As for both cases presented above the dependency on q dominates the apsidal motion discrepancy, we need that $q^{-1/5}a(1 - e^2)/R_1 \lesssim 4.5$ to have at least 5% modelling discrepancy and $q^{-1/5}a(1 - e^2)/R_1 \lesssim 6.5$ to have at least 2% modelling discrepancy. By adopting these modifications we introduced a little bias in the regime where $R_1^5 \simeq R_2^5$ however, as showed before, the dependency on q still dominates.

4.4 APPLICATION TO OBSERVED SYSTEMS

4.4.1 MODELLING OF OBSERVED BINARIES AND QUANTIFICATION OF THE EXTRA MIXING

In this section, we apply our non-perturbative modelling technique to existing twin binary systems, comparing the stellar parameters obtained with our approach to those from the perturbative theory. As we study twin binary systems, modelling one star is sufficient to describe the entire system. We integrated MoBiDICT to a minimisation

4.4. APPLICATION TO OBSERVED SYSTEMS

method following a Levenberg-Marquardt algorithm to compute on the fly the non-perturbative apsidal motion resulting from twin binaries which unperturbed structure are given by our 1D single star stellar evolution code, CLES. By using this methodology we followed the procedure of [Rosu et al. \(2020a, 2022a,b, 2023\)](#), neglecting the interactions of the binaries during their evolution. To avoid such assumptions, our modelling method would have to be directly coupled to binary stellar evolution codes, this work will be conducted in the future with the code BINSTAR ([Siess et al., 2013](#)) and will be presented in a forthcoming article.

We selected four well-known close observed twin binary systems from the literature: PV Cas ([Torres et al., 2010](#); [Claret et al., 2021](#); [Marcussen & Albrecht, 2022](#)), IM Per ([Lacy et al., 2015](#); [Claret et al., 2021](#)), Y Cyg ([Gimenez et al., 1987](#); [Harmanec et al., 2014](#); [Claret et al., 2021](#); [Marcussen & Albrecht, 2022](#)), and HD152248 ([Rosu et al., 2020a,b](#)). The first binary system is composed of MS intermediate mass stars, the second one of sub-giant stars, and the last two ones of massive stars. The properties of these systems are given in the second and fifth line blocks of [Table 4.2](#). For each of these binary systems, an apsidal motion was determined in the literature and is used in this section as a constraint on the stellar structure. The systems were selected to be twin systems with accurate determination of the orbital and stellar parameters, and an apsidal motion primarily driven by tidal forces.

For this modelling we were inspired by the findings of [Rosu et al. \(2020a, 2022a,b, 2023\)](#), who demonstrated that modelling twin binary systems using 1D stellar evolution codes with apsidal motion as a constraint necessitates significant additional mixing during stellar evolution. In our modelling, we used the same constraints for all observed systems: the masses, radii, effective temperatures, and apsidal motions. The constraints for each system are listed in the second line block of [Table 4.2](#). The objective of our modelling technique is to generate an evolutionary track such as the end model is fitting all the observed constraints of the system. As we have four constraints, we used four free parameters for our stars, their initial mass M_0 , initial hydrogen mass fraction X_0 , and age, as well as the overshooting parameter α_{over} . In our modelling, we used the overshooting to control the additional mixing necessary to reproduce the observed constraints as a strong degeneracy is existing between the overshooting and the mixing (see e.g. [Rosu et al. \(2020a, 2022a\)](#)). For all the models, we used the AGSS09 abundances ([Asplund et al., 2009](#)), the FreeEOS equation of state ([Irwin, 2012](#)), the OPAL opacities ([Iglesias & Rogers, 1996](#)), the $T(\tau)$ relation from Model-C of [Vernazza et al. \(1981\)](#) for the atmosphere, the mixing length theory of convection implemented as in [Cox & Giuli \(1968\)](#), and the nuclear reaction rates of [Adelberger et al. \(2011\)](#). For the massive stars, the mass loss rate was computed, accounting for the metallicity of the models, with the prescription of [Vink et al. \(2001\)](#) assuming that the scaling factor for the mass loss rate equals $\xi = 1$. The equivalent averaged mass loss along the entire evolution is given in the fourth line block of [Table 4.2](#).

In our modelling procedure, we started by finding models reproducing the observations with the perturbative approach. We then fixed the initial mass and hydrogen mass fraction, only using the constraints on the radii and apsidal motion to model the systems with MoBiDICT. This approach aims to directly evaluate the impact of our

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non-perturbative modelling on the required additional mixing and age. The objective of our modelling is not to provide an accurate estimation of the stellar properties of each system but rather to compare the stellar properties obtained when modelling the apsidal motion with the perturbative and non-perturbative approaches. The results from our modelling procedure are given both for the non-perturbative and perturbative modelling approaches in the third line block of [Table 4.2](#).

Table 4.2: Observed and model properties of the systems studied. The first line block corresponds to the observational constraints used in our modelling, the second line block gives the values of the free parameters obtained, the third line block gives the assumed parameters of the models, and the fourth, provides the assumed parameters of the system originating from observations. All the models were compared using χ^2 defined as the sum of the squared relatives differences between the observed constraints and the models.

Targets		PV Cas ^{1,2,3}		IM Per ^{2,4}		Y Cyg ^{5,2,3}		HD152248 ^{6,7}	
		Pert.	Non-Pert.	Pert.	Non-Pert.	Pert.	Non-Pert.	Pert.	Non-Pert.
Obs. Const.	$M[M_{\odot}]$	2.78(8)		1.78(1)		17.72(30)		29.5(5)	
	$R[R_{\odot}]$	2.28(4)		2.38(4)		5.8(1)		15.07(12)	
	$T_{\text{eff}}[K]$	10200(250)		7570(160)		33200(500)		34000(1000)	
	$\dot{\omega} [^{\circ}\text{cycle}^{-1}]$	0.0212(2)		0.0146(4)		0.06186(30)		0.0293(13)	
Free Param.	$M_0[M_{\odot}]$	2.780	2.780	1.78	1.78	20.4	20.4	33.16	33.16
	X_0	0.750	0.750	0.730	0.730	0.743	0.743	0.726	0.726
	α_{over}	0.923	0.951	0.306	0.333	1.01	1.05	1.29	1.28
	Age [Gyr]	3.07	3.09	1.33	1.35	0.00347	0.00353	0.00521	0.00520
Models Fixed Param.	Z_0	0.0245		0.012		0.012		0.017	
	α_{MLT}	1.95		1.95		1.80		1.80	
	$\dot{M}[M_{\odot}\text{yr}^{-1}]$	0		0		$0.8 \cdot 10^{-6}$		$1.18 \cdot 10^{-6}$	
Obs. Param.	P_{orb} [days]	1.75047		2.25422		2.996321		5.816498	
	$i [^{\circ}]$	85.8		84.4		86.47		67.6	
	e	0.0325		0.0491		0.145		0.13	
	P/P_{orb}	0.998		0.933		0.688		0.88	
χ^2		0.11	0.18	0.11	0.20	0.04	0.23	0.05	0.03

References. (1) [Torres et al. \(2010\)](#); (2) [Claret et al. \(2021\)](#); (3) [Marcussen & Albrecht \(2022\)](#); (4) [Lacy et al. \(2015\)](#); (5) [Harmanec et al. \(2014\)](#); (6) [Rosu et al. \(2020a\)](#); (7) [Rosu et al. \(2020b\)](#).

For the modelling of each system, the parameters presented in the fourth line block of [Table 4.2](#) were empirically selected to facilitate the convergence of our Levenberg-Marquardt method. For the first system, Z_0 was chosen from a model grid to minimise the χ^2 and be able to reproduce the observations as we could not achieve a coherent fit with a solar metallicity. For the other systems, a solar metallicity was adopted as

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no constraints were given from the observations.

Table 4.2 shows that both the perturbative and non-perturbative approaches can accurately reproduce all observed system constraints. For the four systems, we find that extremely large extra-mixing is required to reproduce the observational properties of stars. Comparing the stellar parameters obtained with MoBiDICT and the perturbative approach, our non-perturbative technique necessitates even more extra mixing for the first three systems. While for the three first systems, the leading contribution to the χ^2 is the reproduction of the apsidal motion and the radius of the stars, for HD152248, the major difficulty was to reproduce the observed effective temperature. This can partly explain the smaller impact of the non-perturbative treatment on the overshooting parameter found in this system. Nonetheless, our approach does not reduce the extremely high extra-mixing required to reproduce the apsidal motion motion inferred by Rosu et al. (2020a) but rather exacerbate it. This additional mixing could stem from various sources, including rotation or dynamical tides. We do not think that it should be concluded from our study that such large overshooting is present in these stars. Other physical processes neglected in our study such as dynamical tides are known to significantly affect the apsidal motion (Willems & Aerts, 2002).

4.4.2 APSIDAL MOTION CONTRIBUTION FROM THE DIFFERENT SOURCES

In this section, we focus on exploring the origin of the apsidal motion discrepancy seen for the observed targets presented in Section 4.4. We reused the structures from the modelling done in Table 4.2 and looked at the different contributions to the apsidal motion for each binary system. Table 4.3 provides a decomposition of the apsidal motion for each observed binary system using the models obtained with our non-perturbative method. In the right part of Table 4.3, we also decomposed the difference between the perturbative and non-perturbative approaches based on the considered spherical orders, revealing the key contributors to the apsidal motion discrepancies. The analysis of Table 4.3 reveals that, for all chosen targets, the contributions from our non-perturbative modelling approach surpass the uncertainties in the observational apsidal motion determinations. Generally, the corrections provided by MoBiDICT are comparable in magnitude to the general relativistic corrections. Then, going into more details, the perturbed acceleration discrepancy between the non-perturbative and perturbative methods can be attributed to several main components:

- $\ell = 1$: Constituting $\sim 23\%$ of the modelling discrepancy, this component cannot be recovered by the perturbative model due to its inherent formalism and assumptions;
- $\ell = 2$: Representing $\sim 40\%$ of the modelling discrepancy, this component cannot either be recovered by the perturbative approach. The differences arise from the first-order assumption of the perturbative approach. We tried taking into account the effect from the quadrupolar deformation of the primary star on the quadrupolar component of its companion and the reverse impact of this modification on the quadrupolar deformation of the primary star. However this effect is negligible as each of such additional effect considered has an increase dependency to $(R/a)^3$ of a power three;

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Table 4.3: Decomposition of the apsidal motion of the modelled observed systems. The relativistic component of the apsidal motion is noted $\dot{\omega}_{\text{rel.}}$, $\dot{\omega}_{\text{pert.}}$ is the contribution from the perturbative approach and $\dot{\omega}_{\text{non-pert.}}$ is the correction induced by the non-perturbative approach. The latest contribution is then decomposed as spherical harmonics in the right region of the table noted $\dot{\omega}_{\text{non-pert.}}$ contributions.

Targets	a/R_1	e	$\dot{\omega}_{\text{obs}}$ [°cycle ⁻¹]	$\dot{\omega}_{\text{rel.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{pert.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{non-pert.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{non-pert.}}$ contributions [%]				
							$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell > 3$	$F'\rho'$
PV Cas ^{1,2,3}	4.80	0.0325	0.0212(2)	0.0012	0.0192	0.0008	23	38	38	2	~ 0
IM Per ^{2,4}	4.60	0.0491	0.0146(4)	0.0007	0.0131	0.0008	25	37	37	2	~ 0
Y Cyg ^{5,2,3}	4.95	0.145	0.0618(3)	0.0029	0.0565	0.0025	22	45	32	2	~ 0
HD152248 ^{6,7}	3.47	0.13	0.0293(13)	0.0026	0.0250	0.0018	23	42	32	2	~ 0

References. (1) [Torres et al. \(2010\)](#); (2) [Claret et al. \(2021\)](#); (3) [Marcussen & Albrecht \(2022\)](#); (4) [Lacy et al. \(2015\)](#); (5) [Harmanec et al. \(2014\)](#); (6) [Rosu et al. \(2020a\)](#); (7) [Rosu et al. \(2020b\)](#).

- $\ell = 3$: Accounting for ~ 35% of the modelling discrepancy, this component can be at least partially recovered by the perturbative model by going at higher spherical orders than usually considered. In [Subsection 4.4.3](#), we expand the computation of the perturbative apsidal motion to the $\ell = 3$ assessing the extent to which these discrepancies can be rectified;
- $\ell > 3$: The contribution of these components only represents 2% of the discrepancies. Our analysis suggests that computing spherical orders higher than three is unnecessary, in particular when comparing these discrepancies with the contributions from other spherical orders;
- $F'\rho'$: This term corresponds to the assumption made to neglect the third contribution to the apsidal motion in [Equation 4.24](#). With our modelling method we see that this assumption is totally valid even in the most distorted cases.

For the case of IM Per, we examine the individual contributions to this discrepancy. [Figure 4.11](#) illustrates the evolution of the components of the perturbed acceleration with orbital separation normalised by the stellar radii for the perturbative and non-perturbative models. For the $\ell = 2$ terms, [Figure 4.11](#) shows that most of the discrepancies are originating from the $\ell = 2, m = 2$ term. This aspect is explained by the fact that in the $\ell = 2, m = 0$ term both tidal force and centrifugal force are included while in $\ell = 2, m = 2$ only tidal force is present. This indicates that the perturbative approach lacks precision when modelling tidal forces at low orbital separations, corroborating findings from [Subsection 4.3.3](#) and [Figure 4.4](#).

4.4.3 HIGHER ORDER PERTURBATIVE METHOD

An advantage of the perturbative approach lies in the possibility to extend the modelling to higher spherical orders. In the previous section we showed that the $\ell = 3$

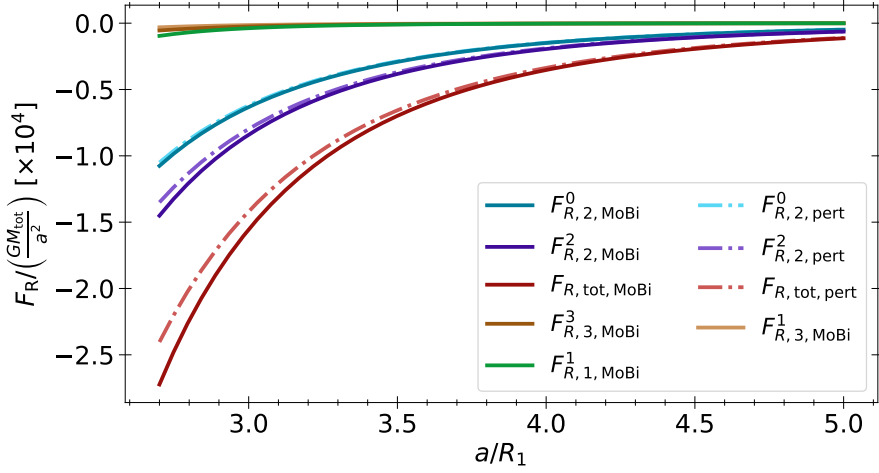


Figure 4.11: Decomposition of the perturbed acceleration for the IM-Per system with the perturbative and non-perturbative models as a function of the system orbital separation normalised by the stellar radii. Each colour represents one spherical order: the lighter curves correspond to the perturbative model while the darker curves correspond to the with non-perturbative approach. The red curves are the total of all the spherical orders.

component of the acceleration perturbation represents $\sim 35\%$ of the discrepancies seen between our non-perturbative and the perturbative models. In [Subsection 4.A.6](#), we developed the equations necessary to extend the perturbative approach to the $\ell = 3$ term. In this section, we verify the relevance of adding the $\ell = 3$ term in the computation of the apsidal motion in the perturbative approach. As shown in [Subsection 4.A.6](#), the apsidal motion arising from the $\ell = 3$ component of the perturbed acceleration is given by

$$\frac{d\omega}{dt} = \frac{56\pi}{P_{\text{orb}}} v(e) \left(k_{2,3} \frac{1}{q} \left(\frac{R_2}{a} \right)^7 + k_{1,3} q \left(\frac{R_1}{a} \right)^7 \right), \quad (4.39)$$

with

$$v(e) = \frac{1 + \frac{15}{4}e^2 + \frac{15}{8}e^4 + \frac{5}{64}e^6}{(1 - e^2)^7}. \quad (4.40)$$

We took back the model of observed stars extensively presented in [Subsection 4.4.1](#) and computed the apsidal motion using our method and the perturbative approach including the $\ell = 3$ term. The results from this modelling are presented in [Table 4.4](#). By comparing [Table 4.3](#) and [Table 4.4](#) we conclude that by including the $\ell = 3$ component of the apsidal motion we corrected $\sim 20\%$ of the discrepancies seen in [Subsection 4.4.2](#). However, the $\ell = 3$ term is only partially corrected when included in the perturbative approach. Indeed, we found that for all the systems modelled this term stills represent $\sim 10\%$ of the observed model discrepancies. Due to the reduction of the contribution from the $\ell = 3$ term, the quadrupolar term now represents

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Table 4.4: Decomposition of the apsidal motion of the modelled observed systems including the $\ell = 3$ term of the perturbative approach. The relativistic component of the apsidal motion is noted $\dot{\omega}_{\text{rel.}}$, $\dot{\omega}_{\text{pert.}}$ is the contribution from the perturbative approach and $\dot{\omega}_{\text{non-pert.}}$ is the correction induced by the non-perturbative approach. The latest contribution is then decomposed as spherical harmonics in the right region of the table noted $\dot{\omega}_{\text{non-pert.}}$ contributions.

Targets	$\dot{\omega}_{\text{obs}}$ [°cycle ⁻¹]	$\dot{\omega}_{\text{rel.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{pert.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{non-pert.}}$ [°cycle ⁻¹]	$\dot{\omega}_{N,\text{non-pert.}}$ contributions [%]			
					$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell > 3$
PV Cas ^{1,2,3}	0.0212(2)	0.0012	0.0194	0.0006	31	53	14	2
IM Per ^{2,4}	0.0146(4)	0.0007	0.0133	0.0005	33	50	14	3
Y Cyg ^{5,2,3}	0.0618(3)	0.0029	0.0570	0.0020	29	59	11	2
HD152248 ^{6,7}	0.0293(13)	0.0026	0.0255	0.0013	30	55	12	3

References. (1) [Torres et al. \(2010\)](#); (2) [Claret et al. \(2021\)](#); (3) [Marcussen & Albrecht \(2022\)](#); (4) [Lacy et al. \(2015\)](#); (5) [Harmanec et al. \(2014\)](#); (6) [Rosu et al. \(2020a\)](#); (7) [Rosu et al. \(2020b\)](#).

$\sim 50\%$ of the observed discrepancies while the dipolar term contributes to $\sim 30\%$ of the differences. The exact repartition of these contributions can vary depending on the orbital separation and the stellar type, in particular, we saw a highly non-linear comportment of the dipolar term. Nonetheless, in the future, the $\ell = 3$ contribution to the apsidal motion have to be included when modelling the apsidal motion of close binaries with the perturbative approach.

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In the previous sections, we showed that non-perturbative modelling is necessary for a precise interpretation of observations in close binary systems. Aside from the orbital evolution considerations, the detection of tight systems with apsidal motion is favoured due to the $(R/a)^5$ dependency of the tidal component. The combination of high-precision observations and strong tidal effects on apsidal motion motivates their investigation as stellar laboratories and, consequently, the need to model them consistently. Our modelling has conclusively shown that, for close binary systems, a non-perturbative treatment of deformations is necessary for accurately characterising stellar global properties and stellar structure.

To assess the impact of our modelling method on observed binary systems with apsidal motion, we constructed a comprehensive catalogue of such systems. In total, our catalogue includes 61 systems, encompassing all systems with constraints on their stellar parameters and observed apsidal motion found in the literature. These systems were obtained by combining two recent catalogues of observed binary systems with apsidal

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motion (Claret et al., 2021; Marcussen & Albrecht, 2022) to systems from various literature sources where stellar parameter determinations were available (Gimenez et al., 1987; Benvenuto et al., 2002; Wolf et al., 2006, 2008, 2010; Torres et al., 2010; Pablo et al., 2015; Rosu et al., 2020b; Baroch et al., 2021, 2022; Rosu et al., 2022a,b, 2023). With this catalogue, Figure 4.12 illustrates the fraction of observed apsidal motion arising from relativistic corrections against the parameter $q^{-1/5}a(1 - e^2)/R_1$ of each observed system.

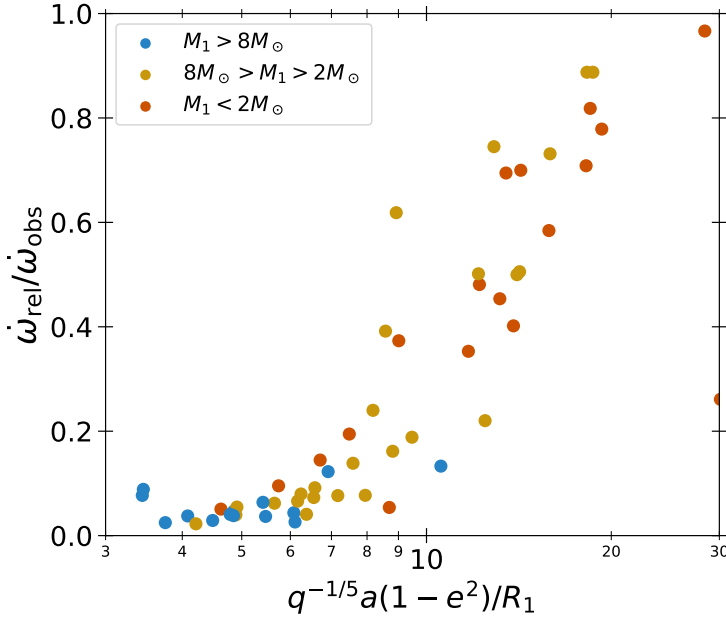


Figure 4.12: Repartition of observed binary systems from our catalogue affected by our methodology. Each point correspond to an observed system with a given orbital separation and a relativistic contribution to the apsidal motion. The blue dots correspond to massive binaries, the yellow points are intermediate mass binary stars, and the red dots are low-mass stars. For this diagram, the primary was chosen as the most massive star in the system and used to determine the color code.

The observed binary systems in our catalogue can be broadly categorised into two equal-sized groups: close binaries with most of the apsidal motion driven by tidal effects and wider binaries where apsidal motion is mostly due to relativistic corrections. The boundary between these two groups is located around $q^{-1/5}a(1 - e^2)/R_1 = 8 - 9$. Because of their low tidal contribution to the apsidal motion, the group of systems with high relativistic contribution is not favoured when wanting to constrain the stellar structure with apsidal motion. For such systems, as shown in Section 4.3, no significant correction is expected from our non-perturbative method. On the other hand, for binary systems where apsidal motion is dominated by tidal effects (close systems), which is the group preferred for drawing constraints on stellar structure

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with apsidal motion, our non-perturbative models show discrepancies compared to perturbative models. In [Subsection 4.3.5](#) we showed that, for a given orbital separation, intermediate-high mass stars are the least impacted by our non-perturbative method, independently from the k_2 value. In [Figure 4.7](#) we also showed that for a theoretical massive star the apsidal motion relative difference reaches 5% when $q^{-1/5}a(1 - e^2)/R_1 \lesssim 4.5$ and 2% when $q^{-1/5}a(1 - e^2)/R_1 \lesssim 6.5$. Therefore, we can estimate that in general when $q^{-1/5}a(1 - e^2)/R_1 \lesssim 4.5$ the relative difference of tidally induced apsidal motion reaches at least 5% which corresponds to about 10% of the stars in the catalogue presented in [Figure 4.12](#). Similarly, for 42% of the catalogue we expect at least a 2% apsidal motion correction from our methodology, corresponding to almost all the systems with the apsidal motion highly dominated by the tidal component. Among the binaries impacted by our methodology, a majority are high-mass or intermediate-mass binaries. In these catalogues and in general, low-mass binaries are under-represented, and not commonly observed with apsidal motion.

The discrepancies observed in the apsidal motion are a consequence of the underlying tidal force discrepancy. However, various aspects and phenomena of binary systems' evolutions are impacted by an underestimation of the tidal force. One key quantity affected by this underestimation is tidal dissipation, which plays a crucial role in the evolution of binary systems through orbital circularisation, transfer of angular momentum between bodies, and orbital migrations. Current models of the tidal dissipation rely on the perturbative approach ([Hut, 1981](#)) combined to a free parameter that can be adjusted to control the dissipation. Even if this parameter can be tuned to reproduce the dissipation obtained by our non-perturbative models with enhanced tidal interactions at a given point of the evolution, it cannot capture the non-linearities that arise during the evolution of binary systems. Consequently, our methodology predicts that tidal dissipation models used by the majority of binary stellar evolution codes underestimate the dissipation at low orbital separation. This leads to imprecise predictions regarding the evolution of global orbital parameters, stellar rotation, and the overall outcomes of these models for systems with close orbits. One specific area impacted by these findings is population synthesis, where the modelled population strongly depends on stellar interactions. Therefore, population models focused on the outcomes of close massive binary systems and the remnants of these systems face significant uncertainties, even before considering mass exchange and processes that rely on imprecise models with free parameters. These findings highlight the need for improved models that better account for the complex interplay of tidal forces in close binary systems.

The discrepancies revealed by our modelling have implications that extend beyond global stellar and orbital parameters, since the full structure of the distorted bodies is affected. The stronger stellar interaction obtained with our modelling leads to larger surface deformations, as already shown in [Fellay & Dupret \(2023\)](#). Moreover, the internal structure of the stars, which is usually treated through one-dimensional spherical models with perturbative corrections, may also be significantly modified in strongly deformed systems. Accurately describing the interior deformation, mass redistribution, and structural properties of such stars therefore requires a transition toward fully three-dimensional modelling. This is particularly important for the study

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of stellar pulsations in highly distorted binary components. Tidally excited oscillations, or dynamical tides when the time-dependent tidal response is considered, can contribute significantly to the orbital evolution of binary systems. However, most current dynamical-tide calculations rely on perturbative tidal forcing applied to one-dimensional, spherically symmetric stellar structures, often combined with linear adiabatic or non-adiabatic oscillation codes (Willems et al., 2003; Valsecchi et al., 2013; Fuller, 2017; Sun et al., 2023; Fellay & Dupret, 2025; Fellay et al., 2026). These approaches are well suited to weakly deformed stars, but they are not expected to capture accurately the oscillations of extremely close binaries, where the stellar structure and oscillation cavities are significantly deformed. A fully self-consistent treatment of this regime would require 3D non-perturbative equilibrium models, together with corresponding 3D non-adiabatic oscillation calculations. In this context, the present work provides a first essential step toward addressing the problem by constructing the non-perturbative 3D stellar structures required for future oscillation modelling.

Our methodology shares with the classical perturbative approach a notable limitation: the assumption of solid-body rotation within the stellar components. This approximation is restrictive for isolated stars, whose interiors may develop significant differential rotation during their evolution beyond the MS (Zahn, 1992; Maeder & Zahn, 1998; Mathis & Zahn, 2004; Mathis et al., 2004). In close binaries, however, tidal interactions, angular-momentum exchange, and tidally induced mixing can reduce differential rotation, making the solid-body approximation more appropriate in strongly interacting systems. As shown in Subsection 4.4.1, the interactions found by our models lead to enhanced mixing within deformed stars. Even so, angular-momentum transport in binary systems remains a complex and open problem. MoBiDICT therefore constitutes an important step in this direction by providing a 3D non-perturbative computation of the stellar interaction and of the associated force field throughout the deformed stellar structure.

Finally, the modelling of mass transfer and phases of common envelopes in binary systems is still associated with significant uncertainties and often relies on the use of multiple free parameters. In theory, MoBiDICT has the potential to be adapted for the static modelling of common envelope phases or, at the very least, to provide initial 3D deformed structures that can be used in conjunction with more sophisticated codes to simulate common envelope evolution. Furthermore, our 3D non-perturbative models can be valuable in modelling mass transfer processes in binaries or providing initial 3D deformed structures for further studies. In the near future, it would be useful to investigate the oscillation properties of binaries before they fill their Roche lobes as determining their configurations and properties before these phases play a crucial role in determining their ultimate fate.

4.6 CONCLUSION

In this work, we investigated the impact of non-perturbative modelling on the theoretical computation of the apsidal motion and tidal force in eccentric binaries. We developed formalisms for apsidal motion in our non-perturbative approach and the

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perturbative approach in respectively [Section 4.1](#) and [Section 4.2](#). Then, we compared the perturbed acceleration and apsidal motion for theoretical models, in [Section 4.3](#), and observed binary systems, in [Section 4.4](#). Finally, we discussed the implications of our results, in [Section 4.5](#), on the observed population of binaries and the general problem of binary modelling.

All theoretical stellar models were significantly impacted at low orbital separation by our methodology. For instance, we observed a maximum increase in tidal force of about 40% for our low-mass stars, 28% for the RGB, 20% for the solar types, and 12% for massive stars. Consequently, the apsidal motion of these systems is impacted. For an orbital eccentricity of $e = 0.1$, these discrepancies led to a respectively increase in apsidal motion of 70%, 45%, 30%, and 15% for the low-mass, RGB, solar type, and massive twin binary systems. The discrepancies seen may vary depending on the exact structure of the stars or the architecture and the orbital parameters of the system. These results are consistent with [Fellay & Dupret \(2023\)](#): the models with higher envelope mass experience greater impacts from our methodology. From this study we noted that models with higher k_2 are more impacted by our methodology. We attempted to establish an empirical relationship between tidal force discrepancy, the orbital separation, and k_2 . However, due to high non-linearities and the numerous free parameters of the problem we were not able to identify such a relationship. We also explored the dependency of the apsidal motion modelling discrepancy on q and found that this difference is proportional to q except when stars have similar radii. In the latter case, the properties of the secondary are modifying this relationship and the exact properties of the system have to be accounted for.

In our analysis of observed systems, we explored in more details the origins of these discrepancies. Specifically, we modelled four observed twin binary systems, PV Cas, IM Per, Y Cyg, and HD152248, composed of respectively, intermediate mass, subgiants, and massive stars for the two latests. We started by modelling these systems using the perturbative model to reproduce the observed values of the constraints parameters. When best fitting models were found, we started from this solution to model stars using our non-perturbative method to obtain the theoretical apsidal motion. In agreement with the results of [Rosu et al. \(2020a, 2022a,b, 2023\)](#), our results highlighted the necessity for significant extra mixing in the 1D stellar models to accurately reproduce the observational properties of these systems. Moreover, our methodology induced an additional increase of extra mixing due to the modification of apsidal motion, reinforcing the results of [Rosu et al. \(2020a, 2022a,b, 2023\)](#).

For the systems we examined, the discrepancies in apsidal motion exceeded the observed uncertainties, indicating their non-negligible impact. Our investigation into the origins of these discrepancies revealed that the $\ell = 1$ term is responsible for $\sim 23\%$ of the discrepancies, $\sim 40\%$ for the $\ell = 2$, $\sim 35\%$ for the $\ell = 3$, and less than 2% for the $\ell > 3$ components. The exact distribution of these discrepancies may vary based on the model. For the $\ell = 1$ and $\ell = 2$ terms the discrepancies cannot be recovered by the perturbative approach due to its formalism and approximations. We were able to correct $\sim 20\%$ of the $\ell = 3$ discrepancies by employing a higher spherical order perturbative approach. In opposition to the results of [Rosu et al. \(2020b\)](#), the $\ell = 3$

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contribution to the apsidal motion cannot be neglected in close binaries.

Finally, we constructed a catalogue of observed binary systems with apsidal motion from the literature to assess the fraction of the observed binaries with apsidal motion effectively impacted by our modelling method. Our analysis indicated that differences in tidal force, and consequently apsidal motion, are non-negligible for systems where $q^{-1/5}a(1 - e^2)/R_1 \lesssim 6.5$. As a result, a significant fraction of the observed binaries (42%) with apsidal motion are indeed affected by our modelling approach.

When modelling the dynamics and deformations of binary stars, the perturbative approach is one of the most widely used methods in the literature. In this article, we demonstrated that this approach loses precision for binaries with $q^{-1/5}a(1 - e^2)/R_1 \lesssim 6.5$. The most immediate observable quantity affected is the apsidal motion. While, in this work, we only considered the contribution from the deformations and the general relativistic contributions, additional phenomenon could impact the determination of the apsidal motion. Furthermore, including the dynamical tides can highly modify the theoretical determination of the apsidal motion. However, a proper non-perturbative non-addiabatic asteroseismic modelling is required to study the propagation of gravito-inertial waves in the stellar interior and their impact on the dynamics of binary systems. Developing such a tool is the theme of the second part of this thesis. In addition, in the future, the inclination of binary systems have to be considered when modelling the tidal contribution to the apsidal motion to account for all the parameters of the orbits. An extension of MoBiDICT for non-aligned systems can be developed; however, all the system's symmetries exploited in this work will disappear, consequently our methodology will require significantly more resources to run.

Importantly, the underestimation identified here is not limited to apsidal motion, but may also affect the modelling of tidal interactions between binary components and, consequently, tidal dissipation processes. Since many binary stellar-evolution codes rely on perturbative tidal prescriptions, these limitations may also impact their predictions for short-period or strongly distorted systems. Given the significant discrepancies in the tidal force predicted by our models at small orbital separations, it is reasonable to expect that standard perturbative prescriptions may underestimate the strength of tidal interactions, and potentially the associated tidal dissipation, in this regime. A non-perturbative treatment such as the one developed here could therefore lead to stronger angular-momentum exchanges in close binary systems and, in turn, to shorter tidal-evolution timescales. To test this expectation quantitatively, the next step will be to couple our method to existing binary stellar-evolution codes. Such an implementation would provide a more self-consistent framework for studying the dynamics and long-term evolution of close binary systems.

4.A PERTURBATIVE MODELLING

4.A.1 THE CLAIRAUT-RADAU EQUATION

Let us express the radial coordinate of a given equipotential in the direction $\tilde{r}(\theta, \phi)$ with its spherical expansion:

$$\tilde{r}(\tilde{r}, \theta, \phi) = \bar{r} \left(1 + \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} r_{\ell}^m(\bar{r}) Y_{\ell m}(\theta, \phi) \right), \quad (4.41)$$

where $\bar{r}$ denotes the averaged radial coordinate of the considered equipotential and $r_{\ell}^m(r)$ the spectral terms of the equipotential radii that can be obtained with

$$r_{\ell}^m(\bar{r}) = \frac{1}{\bar{r}} \int_0^1 \int_0^{\pi} \tilde{r}(\bar{r}, \mu, \phi) Y_{\ell m}(\theta, \phi) d\phi d\mu, \quad (4.42)$$

where the spherical harmonics have been appropriately normalized. Throughout this entire section, [Figure 4.13](#) illustrates the different variables and orbital quantities used to describe the deformations of a binary system. In the perturbative approach, $\bar{r}$ can

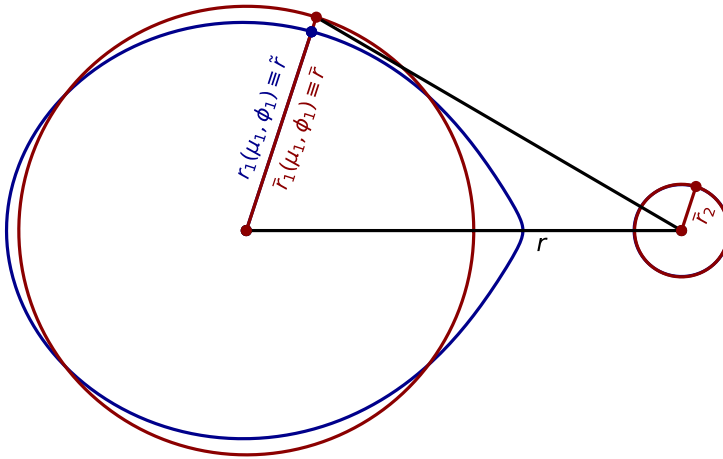


Figure 4.13: Scheme of a binary system separated by a distance a . The different quantities used to model the deformations in the perturbative framework are detailed and illustrated here. The blue curve is the surface of the deformed star while the red curves are the spheres of equivalent averaged radii. As the surface is represented here, $\bar{r}_i(\mu_i, \phi_i) = R_i$, R_i being the radial coordinate of the stellar surface in the spherically symmetric case.

be obtained by expanding [Equation 4.41](#) to the first order ($r_{\ell}^m(\tilde{r}) \ll 1$):

$$\bar{r}(\tilde{r}, \theta, \phi) = \frac{\tilde{r}}{1 + \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} r_{\ell}^m(\tilde{r}) Y_{\ell m}(\theta, \phi)} \simeq \tilde{r} \left(1 - \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} r_{\ell}^m(\tilde{r}) Y_{\ell m}(\theta, \phi) \right) \quad (4.43)$$

4.A. PERTURBATIVE MODELLING

The density of an equipotential passing by $(\tilde{r}, \theta, \phi)$ denoted $\rho(\tilde{r}(\tilde{r}, \theta, \phi))$ can be approximated to the first order using a Taylor expansion of $\tilde{r}$ around $\tilde{r}$:

$$\rho(\tilde{r}(\tilde{r}, \theta, \phi)) = \rho(\tilde{r}) + \frac{d\rho}{d\tilde{r}}(\tilde{r} - \tilde{r}) = \rho(\tilde{r}) - \frac{d\rho}{d\tilde{r}}(\tilde{r}) \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} r_{\ell}^m(\tilde{r}) Y_{\ell m}(\theta, \phi), \quad (4.44)$$

where $\rho(\tilde{r})$ is the density of the equipotential with $\tilde{r} = \tilde{r}$. The densities of a sphere of radius $\tilde{r}$ projected on the spherical harmonics basis is obtained through

$$\rho_{\ell}^m(\tilde{r}) = \int_0^{\pi} \int_0^{2\pi} \rho(\tilde{r}, \theta, \phi) Y_{\ell m}(\theta, \phi) \sin(\theta) d\phi d\theta \quad (4.45)$$

that can be rewritten with

$$\rho_{\ell}^m(\tilde{r}) = \int_0^{\pi} \int_0^{2\pi} \left[\rho(\tilde{r}) - \tilde{r} \frac{d\rho}{d\tilde{r}}(\tilde{r}) \sum_{\ell'=1}^{L'} \sum_{m'=-\ell'}^{\ell'} r_{\ell'}^{m'}(\tilde{r}) Y_{\ell' m'}(\theta, \phi) \right] Y_{\ell m}(\theta, \phi) \sin(\theta) d\phi d\theta. \quad (4.46)$$

Using the orthogonality properties of the spherical harmonics, at the first order the projected densities are

$$\rho_{\ell}^m(\tilde{r}) = \begin{cases} \rho(\tilde{r}) & \ell, m = 0 \\ -\tilde{r} \frac{d\rho}{d\tilde{r}}(\tilde{r}) r_{\ell}^m(\tilde{r}) & \ell > 0 \end{cases}. \quad (4.47)$$

The spectral development of the gravitational potential on a sphere of radius $\tilde{r}$ is given by [Equation 6.24](#). The gravitational potential on a sphere of radius $\tilde{r}$ will be noted in the future $\Psi_r(\tilde{r}, \theta, \phi)$ while the gravitational potential on an equipotential is noted $\Psi_{\tilde{r}}(\tilde{r}, \theta, \phi)$ and can be expressed as

$$\Psi_{\tilde{r}}(\tilde{r}, \theta, \phi) = \Psi'_0(\tilde{r}) + \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} \Psi_{\ell}^m(\tilde{r}) Y_{\ell m}(\theta, \phi) = \Psi_r(\tilde{r}(\tilde{r}, \theta, \phi), \theta, \phi) \quad (4.48)$$

that can be developed to the first order using the Taylor approximation around $\tilde{r} = \tilde{r}$:

$$\begin{aligned} \Psi_r(\tilde{r}(\tilde{r}, \theta, \phi), \theta, \phi) &= \Psi_0(\tilde{r}(\tilde{r}, \theta, \phi)) + \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} \Psi_{\ell}^m(\tilde{r}(\tilde{r})) Y_{\ell m}(\theta, \phi) \\ &= \Psi_0(\tilde{r}) + \frac{d\Psi_0}{d\tilde{r}}(\tilde{r} - \tilde{r}) \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} \Psi_{\ell}^m(\tilde{r}) Y_{\ell m}(\theta, \phi) \\ &= \Psi_0(\tilde{r}) + \sum_{\ell=1}^L \sum_{m=-\ell}^{\ell} \left(\Psi_{\ell}^m(\tilde{r}) + \tilde{r} \frac{d\Psi_0}{d\tilde{r}} r_{\ell}^m(\tilde{r}) \right) Y_{\ell m}(\theta, \phi), \end{aligned} \quad (4.49)$$

therefore

$$\begin{cases} \Psi'_0(\tilde{r}) = \Psi_0(\tilde{r}) \\ \Psi_{\ell}^m(\tilde{r}) = \Psi_{\ell}^m(\tilde{r}) + \tilde{r} \frac{d\Psi_0}{d\tilde{r}} r_{\ell}^m(\tilde{r}) \end{cases}. \quad (4.50)$$

4.A. PERTURBATIVE MODELLING

The spectral components of the gravitational potential (Equation 6.24) on an equipotential can be rewritten using Equation 4.47 before integrating by parts:

$$\begin{aligned}
 \Psi_\ell^m(\bar{r}) &= -\frac{4\pi G}{(2\ell+1)\bar{r}^{\ell+1}} \left(-\int_0^{\bar{r}} \bar{r}'^{\ell+3} \frac{d\rho}{d\bar{r}'}(\bar{r}') r_\ell^m(\bar{r}') d\bar{r}' \right) \\
 &\quad - \frac{4\pi G \bar{r}^\ell}{2\ell+1} \left(-\int_{\bar{r}}^\infty \bar{r}'^{2-\ell} \frac{d\rho}{d\bar{r}'}(\bar{r}') r_\ell^m(\bar{r}') d\bar{r}' \right) \\
 &= -\frac{4\pi G}{(2\ell+1)\bar{r}^{\ell+1}} \left(-\left[\bar{r}'^{\ell+3} r_\ell^m \rho \right]_0^{\bar{r}} + \int_0^{\bar{r}} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{\ell+3}) \right) \\
 &\quad - \frac{4\pi G \bar{r}^\ell}{2\ell+1} \left(-\left[\bar{r}'^{2-\ell} r_\ell^m \rho \right]_{\bar{r}}^{R_s} + \int_{\bar{r}}^{R_s} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{2-\ell}) \right) \\
 &= -\frac{4\pi G}{(2\ell+1)\bar{r}^{\ell+1}} \int_0^{\bar{r}} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{\ell+3}) - \frac{4\pi G \bar{r}^\ell}{2\ell+1} \int_{\bar{r}}^{R_s} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{2-\ell}),
 \end{aligned} \tag{4.51}$$

where at the second step the two crossed terms are cancelling each other out. As the total potential is constant on an equipotential, one can write the that:

$$\Psi_{\text{tot}} = \Psi_1 + \Psi_2 + \Psi_c = \text{const}, \tag{4.52}$$

implying that for all the ℓ and m different from 0:

$$\Psi_{1\ell}^m + \Psi_{2,\ell}^m + \Psi_{c,\ell}^m = 0. \tag{4.53}$$

Equation 4.51 can be developed and multiplied by $\bar{r}^{\ell+1}/4\pi G$:

$$\begin{aligned}
 &\frac{-1}{(2\ell+1)} \int_0^{\bar{r}} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{\ell+3}) - \frac{\bar{r}^{2\ell+1}}{2\ell+1} \int_{\bar{r}}^{R_s} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{2-\ell}) \\
 &\quad + \frac{\bar{r}^\ell m(\bar{r})}{4\pi} r_\ell^m(\bar{r}) + \frac{\bar{r}^{\ell+1}}{4\pi G} (\Psi'_{2\ell m} + \Psi'_{\text{cent},\ell m}) = 0.
 \end{aligned} \tag{4.54}$$

By differentiating Equation 4.54, we obtain

$$-\bar{r}^{2\ell} \int_{\bar{r}}^{R_s} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{2-\ell}) + \frac{m(\bar{r})}{4\pi} \bar{r}^{\ell-1} \left(\bar{r} \frac{dr_\ell^m}{d\bar{r}} + \ell r_\ell^m \right) + \frac{1}{4\pi G} \frac{d}{d\bar{r}} \left((\Psi_{2,\ell}^m + \Psi_{c,\ell}^m) \bar{r}^{\ell+1} \right) = 0, \tag{4.55}$$

this expression can then be divided by $\bar{r}^{2\ell}$ and gain differentiated by $\bar{r}$ to obtain

$$\bar{r}^2 \frac{d^2 r_\ell^m}{d\bar{r}^2} + \frac{6\rho(\bar{r})}{\bar{\rho}(\bar{r})} \left(\bar{r} \frac{dr_\ell^m}{d\bar{r}} + r_\ell^m(\bar{r}) \right) - \ell(\ell+1) r_\ell^m(\bar{r}) + \frac{3\bar{r}^{\ell-1}}{4\pi G \bar{\rho}(\bar{r})} \frac{d}{d\bar{r}} \left(\frac{1}{\bar{r}^{2\ell}} \frac{d}{d\bar{r}} \left((\Psi_{2,\ell}^m + \Psi_{c,\ell}^m) \bar{r}^{\ell+1} \right) \right) = 0, \tag{4.56}$$

where we used the definition of the mean density under an isobar:

$$\bar{\rho} = \frac{m(\bar{r})}{\frac{4}{3}\pi \bar{r}^3}. \tag{4.57}$$

To the first order $(\Psi_{2,\ell}^m + \Psi_{c,\ell}^m) \propto \bar{r}^\ell$ meaning that $\frac{d}{d\bar{r}} \left((\Psi_{2,\ell}^m + \Psi_{c,\ell}^m) \bar{r}^{\ell+1} \right) \propto \bar{r}^{2\ell}$ therefore the last term of the previous equation is cancelling out during the differentiation.

4.A. PERTURBATIVE MODELLING

Finally by defining

$$\eta_\ell^m \equiv \frac{\bar{r}}{r_\ell^m} \frac{dr_\ell^m}{d\bar{r}}, \quad (4.58)$$

and developing the second derivative of r_ℓ^m :

$$\frac{d^2 r_\ell^m}{d\bar{r}^2} = \frac{d^2}{d\bar{r}^2} \left(\frac{r_\ell^m \eta_\ell^m}{\bar{r}} \right) = \eta_\ell^m \left(\frac{1}{\bar{r}} \frac{d\eta_\ell^m}{d\bar{r}} + \frac{\eta_\ell^m (\eta_\ell^m + 1)}{\bar{r}^2} \right), \quad (4.59)$$

we obtain the well known Clairaut-Radau equation expressed as

$$\bar{r} \frac{d\eta_\ell}{d\bar{r}} + 6 \frac{\rho(\bar{r})}{\bar{\rho}(\bar{r})} (\eta_\ell + 1) + \eta_\ell (\eta_\ell - 1) = \ell(\ell + 1), \quad (4.60)$$

with the boundary conditions:

$$\eta_\ell(0) = \ell - 2. \quad (4.61)$$

4.A.2 PERTURBATION OF THE SURFACE TOTAL POTENTIAL

In this section our aim is to look at how the perturbative model is modifying the surface gravitational potential of each star composing the system. Let us go back to [Equation 4.54](#) and divide it by $\bar{r}^{2\ell+1}$ and next differentiating it to make $\int_{\bar{r}}^{R_s} \dots$ fall :

$$\frac{1}{\bar{r}^{2\ell+2}} \int_0^{\bar{r}} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{\ell+3}) + \frac{1}{4\pi G} \frac{d}{d\bar{r}} \left(\frac{\Psi_{2,\ell}^m + \Psi_{c,\ell}^m}{\bar{r}^\ell} \right) + \frac{r_\ell^m(\bar{r})}{\bar{r}^{\ell+2}} \int_0^{\bar{r}} \rho(\bar{r}') \bar{r}'^2 d\bar{r} (\eta_\ell^m - 1 - \ell) = 0. \quad (4.62)$$

The previous equation is then multiplied by $\frac{\bar{r}^{\ell+1} 4\pi G}{2\ell + 1}$ before being injected in [Equation 4.51](#) to eliminate $\int_0^{\bar{r}} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{\ell+3})$ and obtain the spectral projections of the gravitational potentials expressed as

$$\Psi_{1\ell}^m(\bar{r}) = \frac{4\pi G}{2\ell + 1} \left(\frac{r_\ell^m(\bar{r}) m(\bar{r})}{4\pi \bar{r}} (\eta_{1\ell}^m - 1 - \ell) \bar{r}^\ell \int_{\bar{r}}^{R_s} \rho(\bar{r}') d(r_\ell^m \bar{r}'^{2-\ell}) \right) + \frac{\bar{r}^{\ell+1}}{2\ell + 1} \frac{d}{d\bar{r}} \left(\frac{\Psi_{2,\ell}^m + \Psi_{c,\ell}^m}{\bar{r}^\ell} \right), \quad (4.63)$$

the integral term $\int_{\bar{r}}^{R_s} \dots$ is negligible at first order. For an arbitrary chosen primary with a radius R_1 the surface gravitational potential is now expressed as

$$\Psi_{1\ell}^m(R_1) \simeq \frac{r_\ell^m(R_1) GM}{(2\ell + 1) R_1} (\eta_{1\ell}^m(R_1) - 1 - \ell). \quad (4.64)$$

[Equation 4.55](#) is then evaluated at the stellar surface and to the first order to obtain

$$r_\ell^m(R_1) GM = -R_1 (2\ell + 1) \frac{\Psi_{2,\ell}^m(R_1) + \Psi_{c,\ell}^m(R_1)}{\eta_{1\ell}^m(R_1) + \ell}, \quad (4.65)$$

that we inject in [Equation 4.64](#) to express the perturbation of the spectral surface gravitational potential

$$\Psi_{1\ell}^m(R_1) = \frac{1 + \ell - \eta_{1\ell}^m(R_1)}{\eta_{1\ell}^m(R_1) + \ell} (\Psi_{2,\ell}^m(R_1) + \Psi_{c,\ell}^m(R_1)). \quad (4.66)$$

4.A. PERTURBATIVE MODELLING

With the definition of the Love numbers given by:

$$k_{\ell,1} = \frac{1 + \ell - \eta_{1\ell}^m(R_1)}{2(\eta_{1\ell}^m(R_1) + \ell)}, \quad (4.67)$$

we have finally:

$$\Psi_{1\ell}^m(R_1) = 2k_{\ell,1} \left(\Psi_{2,\ell}^m(R_1) + \Psi_{c,\ell}^m(R_1) \right). \quad (4.68)$$

4.A.3 FIRST TERMS OF THE CENTRIFUGAL, GRAVITATIONAL AND TIDAL POTENTIALS

CENTRIFUGAL POTENTIAL IN UNSYNCHRONISED SYSTEMS

In a non-synchronized system, the centrifugal potential expressed with the Cartesian coordinates centred on the primary is given by

$$\Psi_c(x, y, z) = -\tilde{n}^2 x_{\text{CM}} x - \frac{\Omega^2}{2} (x^2 + y^2), \quad (4.69)$$

where (x, y, z) are the Cartesian coordinates centered on the star studied, x pointing toward the center of mass of the system, $\tilde{n}$ is the orbital rotation rate of the binary system with a separation r and Ω is the rotation rate of this star. The quantity $x^2 + y^2$ can be written when passing in spherical coordinates as $\tilde{r}^2 - z^2$ with $z = \tilde{r} \cos \theta$ that can be expressed with the spherical harmonics

$$x^2 + y^2 = \frac{2}{3} \tilde{r}^2 - Y_2^0(\cos \theta). \quad (4.70)$$

The centrifugal potential can therefore be expressed as

$$\Psi_c(x, y, z) = -\frac{\Omega^2}{3} \tilde{r}^2 + \tilde{n}^2 x_{\text{CM}} \tilde{r} Y_1^1(\mu) + \frac{\Omega^2}{3} \tilde{r}^2 x_{\text{CM}}^2 Y_2^0(\mu), \quad (4.71)$$

and decomposed as

$$\Psi_{c,0}^0 = -\frac{\Omega^2}{3} \tilde{r}^2, \quad (4.72)$$

$$\Psi_{c,1}^1 = -\tilde{n}^2 x_{\text{CM}} \tilde{r}, \quad (4.73)$$

$$\Psi_{c,2}^0 = \frac{\Omega^2}{3} \tilde{r}^2, \quad (4.74)$$

$$\Psi_{c,\ell}^m = 0 \text{ for all other combinations of } \ell \text{ and } m.$$

GRAVITATIONAL AND TIDAL POTENTIAL

Treating the secondary as a point source, the spectral component of the gravitational potential exerted by the secondary star at the surface of the primary can be trivially expressed as

$$\Psi_{2,\ell}(R_1) = -\frac{GM_2}{R_1} \left(\frac{R_1}{r} \right)^{\ell+1} P_\ell(\sin \theta_1 \cos \phi_1), \quad (4.75)$$

4.A. PERTURBATIVE MODELLING

where r is the instantaneous orbital separation of the binaries and $P_\ell(\sin \theta_1 \cos \phi_1)$ is the classical Legendre polynomial that can be decomposed using spherical harmonics as

$$P_\ell(\sin \theta_1 \cos \phi_1) = \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta_1, \phi_1) d_\ell^m, \quad (4.76)$$

where d_ℓ^m are constant coefficients linked to the normalisation of the spherical harmonics. In the following we will define and use

$$\lambda_1 \equiv \sin \theta_1 \cos \phi_1. \quad (4.77)$$

Using this notation:

$$P_1(\lambda_1) = \lambda_1 = \sin \theta_1 \cos \phi_1 = \frac{x}{r} = -Y_1^1(\theta_1, \phi_1), \quad (4.78)$$

yield that $d_1^0 = 0$, $d_1^1 = -1$. Similarly,

$$P_2(\lambda_1) = \frac{1}{4} Y_2^2(\theta_1, \phi_1) - \frac{1}{2} Y_2^0(\theta_1, \phi_1), \quad (4.79)$$

yield that $d_2^0 = -\frac{1}{2}$ and $d_2^2 = \frac{1}{4}$. Therefore,

$$\Psi_{2,1}^1(R_1) = \frac{GM_2}{R_1} \left(\frac{R_1}{r} \right)^2 \quad (4.80)$$

$$\Psi_{2,2}^0(R_1) = \frac{1}{2} \frac{GM_2}{R_1} \left(\frac{R_1}{r} \right)^3 \quad (4.81)$$

$$\Psi_{2,2}^2(R_1) = -\frac{1}{4} \frac{GM_2}{R_1} \left(\frac{R_1}{r} \right)^3 \quad (4.82)$$

Summing [Equation 4.71](#), [Equation 4.80](#) and using the definition of $\tilde{n}$ ([Equation 3.8](#)), we have:

$$\Psi_{c,1}^1(R_1) + \Psi_{2,1}^1(R_1) = \frac{GM_2}{R_1} \left(\frac{R_1}{a} \right)^2 - \frac{GM_2}{R_1} \left(\frac{R_1}{a} \right)^2 = 0. \quad (4.83)$$

Substituting this result in the right hand side of [Equation 4.68](#), we see that the perturbative theory predicts that the dipolar component of the gravitational potential generated by a star is equal to zero when its companion is treated as a point source. This is an important result explaining why the dipolar component is not expected to contribute to deformations.

The quadrupolar terms are finally given by

$$\Psi_{c,2}^0(R_1) + \Psi_{2,2}^0(R_1) = \left(\frac{R_1}{r} \right)^2 \left(\frac{\Omega^2}{3} r^2 + \frac{1}{2} \frac{GM_2}{r} \right), \quad (4.84)$$

and,

$$\Psi_{c,2}^2(R_1) + \Psi_{2,2}^2(R_1) = -\frac{1}{4} \frac{GM_2}{R_1} \left(\frac{R_1}{r} \right)^3. \quad (4.85)$$

4.A.4 PERTURBATION OF THE FORCE

In this section, we develop the expression of the tidal force exerted by a deformed body on its companion, specifically $F_{2,10}$ as introduced in [Section 4.1](#). We limit ourselves to the dominant term, which arises from the $\ell = 2$ deformation on the unperturbed companion. The system of coordinates used in this Section is described in [Figure 4.13](#). Let us consider the secondary, which is deformed due to the presence of an unperturbed primary. In the section, r_i denotes the radial coordinate of the spherical coordinate frame centered on each star (in previous sections r_1 were noted $\tilde{r}$). With this formalism the potential perturbation at the surface of the secondary is given by [Equation 4.66](#) applied to the $\ell = 2$ terms:

$$\Psi_{2,2}(R_2) = \Psi_{2,2}^0(R_2)Y_0^2(\theta_2, \phi_2) + \Psi_{2,2}^2(R_2)Y_2^2(\theta_2, \phi_2) \quad (4.86)$$

that can be rewritten, using [Equation 4.75](#) as

$$\Psi_{2,2}(R_2, \theta_2, \phi_2) = 2k_{2,2} \left(\frac{R_2}{r} \right)^2 \left[-\frac{GM_1}{r} P_2(\lambda_2) + \frac{\Omega_2^2}{3} r^2 P_2(\mu_2) \right], \quad (4.87)$$

where P_2 is the Legendre polynomial of order two and $\lambda_i \equiv \sin \theta_i \cos \phi_i$. To study the impact of this potential perturbation on the primary, we can express the $\ell = 2$ component of the potential outside of the secondary as

$$\Psi_{2,2}(r_2, \lambda_2, \mu_2) = \Psi_{2,2}(R_2, \lambda_2, \mu_2) \left(\frac{R_2}{r_2} \right)^3 = 2k_{2,2} \left(\frac{R_2}{r_2} \right)^5 \left(\frac{r}{r_2} \right)^3 \left[-\frac{GM_1}{r} P_2(\lambda_2) + \frac{\Omega_2^2}{3} r^2 P_2(\mu_2) \right]. \quad (4.88)$$

Following [Kopal \(1959\)](#), different coordinates related quantities from the secondary star can be approximated with series developments in the coordinates of the primary star:

$$\left(\frac{r}{r_2} \right)^3 = 1 + 3 \frac{r_1}{r} P_1(\lambda_1) + \left(\frac{r_1}{r} \right)^2 (5P_2(\lambda_1) + 1) + \dots \quad (4.89)$$

$$\begin{aligned} P_2(\lambda_2) \left(\frac{r}{r_2} \right)^3 &\simeq \left(1 + \left(\frac{r_1}{r} \right)^2 (P_2(\lambda_1) - 1) \right) \times \left(1 + 3 \frac{r_1}{r} P_1(\lambda_1) + \dots \right) \\ &\simeq 1 + 3 \frac{r_1}{r} P_1(\lambda_1) + 6 \left(\frac{r_1}{r} \right)^2 P_2(\lambda_1), \end{aligned} \quad (4.90)$$

and,

$$\begin{aligned} P_2(\mu_2) &= \frac{1}{2} \left(3 \frac{z_1^2}{r_2^2} - 1 \right) = \frac{1}{2} \left(3 \frac{z_1^2}{r^2} - 1 \right) + \mathcal{O} \left(\left(\frac{r_1}{r} \right)^3 \right) \simeq \frac{3}{2} \left(\frac{r_1}{r} \right)^2 \mu_1^2 - \frac{1}{2} + \mathcal{O} \left(\left(\frac{r_1}{r} \right)^3 \right) \\ &= \left(\frac{r_1}{r} \right)^2 P_2(\mu_1) + \frac{1}{2} \left(\frac{r_1}{r} \right)^2 - \frac{1}{2} + \mathcal{O} \left(\left(\frac{r_1}{r} \right)^3 \right), \end{aligned} \quad (4.91)$$

finally,

$$P_2(\mu_2) \left(\frac{r}{r_2} \right)^3 = -\frac{1}{2} - \frac{3}{2} \frac{r_1}{r} P_1(\lambda_1) + \left(\frac{r_1}{r} \right)^2 \left[P_2(\mu_1) - \frac{5}{2} P_2(\lambda_1) \right] + \mathcal{O} \left(\left(\frac{r_1}{r} \right)^3 \right). \quad (4.92)$$

4.A. PERTURBATIVE MODELLING

All those approximations can be inserted back in Equation 4.88 to obtain the perturbation of the secondary gravitational potential in the coordinates of the primary:

$$\begin{aligned} \Psi_{2,2}(r_1, \lambda_1, \mu_1) = & \Psi_{2,0}(r_1^2) - 2k_{2,2} \left(\frac{R_2}{r} \right)^5 \frac{r_1}{r} \left(3 \frac{GM_1}{r} + \frac{\Omega_2^2}{2} r^2 \right) P_1(\lambda_1) \\ & - 2k_{2,2} \left(\frac{R_2}{r} \right)^5 \left(\frac{r_1}{r} \right)^2 \left[\left(6 \frac{GM_1}{r} + \frac{5}{6} \Omega_2^2 r^2 \right) P_2(\lambda_1) - \frac{\Omega_2^2}{3} r^2 P_2(\mu_1) \right], \end{aligned} \quad (4.93)$$

that can be rewritten using the third Kepler Law including the periods of the system as follow

$$\begin{aligned} \Psi_{2,2}(r_1, \lambda_1, \mu_1) = & \Psi_{2,0}(r_1^2) - 2k_{2,2} \frac{GM_1}{r} \left(\frac{R_2}{r} \right)^5 \left[\frac{r_1}{r} \left(3 + \frac{1}{2} \frac{M_{\text{tot}}}{M_1} \left(\frac{\Omega_2}{\tilde{n}} \right)^2 \right) P_1(\lambda) \right. \\ & \left. + \left(\frac{r_1}{r} \right)^2 \left(6 + \frac{5}{6} \frac{M_{\text{tot}}}{M_1} \left(\frac{\Omega_2}{\tilde{n}} \right)^2 \right) P_2(\lambda) \frac{1}{3} \frac{M_{\text{tot}}}{M_1} \left(\frac{\Omega_2}{\tilde{n}} \right)^2 P_2(\mu) \right]. \end{aligned} \quad (4.94)$$

We can note that all the constant appearing in the expressions have been included in the $\Psi_{2,0}$ as they are not depending on r_1/r .

As highlighted earlier, when computing the apsidal motion of a system, it is imperative to account for the tidally induced acceleration perturbation F_R . By only considering the spherical order lower than $\ell = 3$, the perturbed acceleration can be decomposed as follows:

$$F_R = -\frac{1}{m_\mu} [F_{2'10} + F_{201'} + \cancel{F_{2'1'}}], \quad (4.95)$$

where $F_{2'10}$ is the force exerted by the quadrupolar deformation of the secondary on the unperturbed primary, $F_{201'}$ is the force exerted by the unperturbed secondary on the already perturbed primary, and $F_{2'1'}$ is the force exerted by the perturbed secondary on the already perturbed primary. The last term, $F_{2'1'}$, is a second order term, thus neglected in the perturbative approach. In accordance with Newton's third law, the force exerted by the unperturbed primary on the already perturbed secondary is the opposite of the force exerted by the perturbed secondary on the unperturbed primary, mathematically $F_{201'} = -F_{1'20}$. Let us now compute the force exerted by the perturbation of the secondary on the unperturbed primary, $F_{2'10}$, that is given by

$$F_{2'10} = - \int \rho_{10} \frac{\partial \Psi_{2,2}}{\partial x_1} dV_1 \simeq -M_1 \frac{\partial \Psi_{2,2}}{\partial x_1}(r_1 = 0), \quad (4.96)$$

when treating the primary as point like $r_1 = 0$. In this case, only the gradient of $\ell = 1$ component is non-null at the center of the star ($r_1 = 0$) as

$$\frac{r_1}{r} P_1(\lambda) = \frac{x_1}{r}. \quad (4.97)$$

Therefore only taking the $\ell = 1$ component, the force can be expressed as

$$F_{2'10} = -M_1 \frac{d\Psi_{2,2}}{dx_1}(r_1 = 0) = 2k_{2,2} \frac{GM_1^2}{r^2} \left(\frac{R_2}{r} \right)^5 \left(3 + \frac{1}{2} \frac{M_{\text{tot}}}{M_1} \left(\frac{\Omega_2}{\tilde{n}} \right)^2 \right). \quad (4.98)$$

4.A. PERTURBATIVE MODELLING

Similarly, $F_{1'20}$ can be obtained by permuting the primary and secondary in Equation 4.98. Finally, combining Equation 4.95 and Equation 4.98 the perturbed acceleration is expressed as

$$F_R = -2 \frac{GM_{\text{tot}}}{r^2} \left[k_{2,2} \frac{1}{q} \left(\frac{R_2}{r} \right)^5 \left(3 + \frac{q+1}{2} \left(\frac{\Omega_2}{\tilde{n}} \right)^2 \right) + k_{2,1} q \left(\frac{R_1}{r} \right)^5 \left(3 + \frac{1}{2} \frac{q+1}{q} \left(\frac{\Omega_1}{\tilde{n}} \right)^2 \right) \right], \quad (4.99)$$

with $q = M_2/M_1$.

4.A.5 PERTURBATIVE APSIDAL MOTION COMPUTATION

In the case of an eccentric orbit, the mean orbital rotation rate n is distinct from the instantaneous orbital rotation rate $\tilde{n}$ experienced by each individual binary component. Using this formalism, the perturbed acceleration can be rewritten by grouping the tidal and centrifugal terms as

$$\begin{aligned} F_R = -2 \frac{GM_{\text{tot}}}{a^2} & \left[\left(3k_{2,2} \frac{1}{q} \left(\frac{R_2}{a} \right)^5 + 3k_{2,1} q \left(\frac{R_1}{a} \right)^5 \right) \left(\frac{a}{r} \right)^7 + \left(k_{2,2} \frac{q+1}{2q} \left(\frac{R_2}{a} \right)^5 \left(\frac{P_{\text{orb}}}{P_2} \right)^2 \right. \right. \\ & \left. \left. + k_{2,1} \frac{q+1}{2} \left(\frac{R_1}{a} \right)^5 \left(\frac{P_{\text{orb}}}{P_1} \right)^2 \right) \left(\frac{a}{r} \right)^4 \right] = -2 \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} \left(\frac{a}{r} \right)^7 + R_c \left(\frac{a}{r} \right)^4 \right], \end{aligned} \quad (4.100)$$

where P_1 and P_2 are respectively the rotation period of the primary and secondary, and R_{tides} and R_c can be identified in Equation 4.100. The apsidal motion of a system is dependent on the time average of the perturbed acceleration over an orbit $\langle F_R \cos \nu \rangle$. Using Equation 4.14, the perturbed acceleration averaged over an orbit is given by

$$\langle F_R \cos \nu \rangle = -\frac{2}{P_{\text{orb}}} \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} \int_0^{P_{\text{orb}}} \left(\frac{a}{r} \right)^7 \cos \nu \, dt + R_c \int_0^{P_{\text{orb}}} \left(\frac{a}{r} \right)^4 \cos \nu \, dt \right]. \quad (4.101)$$

The integrals in Equation 4.101 can be re-expressed with the zero-order Hansen coefficients $X_0^{n,m}(e)$ defined for every positive m as

$$X_0^{n,m}(e) \equiv \frac{1}{P_{\text{orb}}} \int_0^{P_{\text{orb}}} \left(\frac{r}{a} \right)^n \cos(m\nu) \, dt. \quad (4.102)$$

Without entering into the detailed expressions of these coefficients, the two orders that are of interest in our case are the $n = -4, m = 1$ and $n = -7, m = 1$ that are given by

$$X_0^{-4,1}(e) = e(1 - e^2)^{-5/2}, \quad (4.103)$$

and,

$$X_0^{-7,1}(e) = \frac{5}{2} e(1 - e^2)^{-11/2} \left(1 + \frac{3}{2} e^2 + \frac{1}{8} e^4 \right). \quad (4.104)$$

The average perturbed acceleration is finally given by

$$\langle F_R \cos \nu \rangle = -2 \frac{GM_{\text{tot}}}{a^2} \left[R_{\text{tides}} X_0^{-4,1}(e) + R_c X_0^{-7,1}(e) \right], \quad (4.105)$$

4.A. PERTURBATIVE MODELLING

and can be inserted in [Equation 4.13](#) to obtain the apsidal motion of a binary system with the perturbative approach:

$$\frac{d\omega_N}{dt} = n \left[15f(e) \left(\left(\frac{R_1}{a} \right)^5 q k_{2,1} + \left(\frac{R_2}{a} \right)^5 \frac{k_{2,2}}{q} \right) g(e) (k_{2,1}(q+1) \right. \\ \left. \left(\frac{P_{\text{orb}}}{P_1} \right)^2 \left(\frac{R_1}{a} \right)^5 + k_{2,2} \frac{q+1}{q} \left(\frac{P_{\text{orb}}}{P_2} \right)^2 \left(\frac{R_2}{a} \right)^5 \right) \right], \quad (4.106)$$

with,

$$f(e) = \frac{1 + \frac{3}{2}e^2 + \frac{1}{8}e^4}{(1 - e^2)^5}, \quad (4.107)$$

and,

$$g(e) = \frac{1}{(1 - e^2)^2}. \quad (4.108)$$

4.A.6 HIGHER ORDER PERTURBATIONS

In this section, we apply the developments done in [Subsection 4.A.4](#) and [Subsection 4.A.5](#) to obtain the apsidal motion resulting from the $\ell = 3$ perturbation of the potential. By combining [Equation 4.68](#) and [Equation 4.75](#), the surface potential perturbation ℓ of the secondary originating from presence of the primary is given by

$$\Psi_{2,3}(R_2, \theta_2, \phi_2) = -2k_{2,3} \frac{GM_1}{R_2} \left(\frac{R_2}{r} \right)^4 P_3(\lambda_2). \quad (4.109)$$

At a distance r_2 from the surface of the secondary, the potential perturbation from the $\ell = 3$ perturbed potential of the secondary is expressed as

$$\Psi_{2,3}(r_2, \theta_2, \phi_2) = -2k_{2,3} \frac{GM_1}{R_2} \left(\frac{R_2}{r} \right)^4 P_3(\lambda_2) \left(\frac{R_2}{r_2} \right)^3 = -2k_{2,3} \frac{GM_1}{R_2} \left(\frac{R_2}{r} \right)^7 P_3(\lambda_2) \left(\frac{r}{r_2} \right)^3. \quad (4.110)$$

Based on [Kopal \(1959\)](#), λ_2 can be expanded as

$$\lambda_2 = 1 + \frac{1}{3}(P_2(\lambda_1) + 1) \left(\frac{r_1}{r} \right)^2 + \dots, \quad (4.111)$$

meaning that

$$P_3(\lambda_2) = \frac{1}{2}(5\lambda_2^2 - 3\lambda_2) = 1 - \frac{1}{2}(P_2(\lambda_1) + 1) \left(\frac{r_1}{r} \right)^2 + \dots, \quad (4.112)$$

and

$$P_3(\lambda_2) \left(\frac{r}{r_2} \right)^3 \simeq 1 + 3 \frac{r_1}{r} P_1(\lambda_1) + \dots, \quad (4.113)$$

which can be inserted back in [Equation 4.110](#) to obtain the perturbation originating from the $\ell = 3$ perturbation of secondary in the coordinate frame of the primary:

$$\Psi_{2,3}(r_1, \lambda_1) = \Psi_{2,0}(r_1) - 6k_{2,3} \frac{GM_1}{r} \left(\frac{R_2}{r} \right)^7 \frac{r_1}{r} P_1(\lambda_1) + \dots \quad (4.114)$$

4.A. PERTURBATIVE MODELLING

By using the same methodology than in the late [Subsection 4.A.4](#), the force perturbation originating from the $\ell = 3$ deformation of the secondary on the primary is expressed as

$$F_{2',10} = -6k_{2,3} \frac{GM_1^2}{r^2} \left(\frac{R_2}{r} \right)^7 \quad (4.115)$$

Following the late [Subsection 4.A.5](#), the acceleration perturbation undergone by the system to the tidal perturbation is expressed as

$$F_R = -6 \frac{GM_{\text{tot}}}{a^2} \left(k_{2,3} \frac{1}{q} \left(\frac{R_2}{a} \right)^7 + k_{1,3} q \left(\frac{R_1}{a} \right)^7 \right) \left(\frac{a}{r} \right)^9 \equiv R_{\text{tides}, \ell=3} \left(\frac{a}{r} \right)^9, \quad (4.116)$$

when defining a as the semi-major axis of the system. Following the same approach as in [Subsection 4.A.5](#), the time orbital average of the perturbation is formally expressed as

$$\langle F_R \cos \nu \rangle = \frac{1}{P_o} R_{\text{tides}, \ell=3} \int_0^{P_o} \left(\frac{a}{r} \right)^9 \cos \nu \, dt, \quad (4.117)$$

which can be developed into polynomial functions of the eccentricity using Hansen's functions:

$$\langle F_R \cos \nu \rangle = R_{\text{tides}, \ell=3} X_0^{-9,1}(e), \quad (4.118)$$

with,

$$X_0^{-9,1}(e) = \frac{7}{2} e (1 - e^2)^{-15/2} \left(1 + \frac{15}{4} e^2 + \frac{15}{8} e^4 + \frac{5}{64} e^6 \right). \quad (4.119)$$

The apsidal motion caused by the $\ell = 3$ acceleration perturbation can finally be obtained by inserting [Equation 4.118](#) in [Equation 4.13](#) and simplifying:

$$\frac{d\omega}{dt} = 21 \frac{2\pi}{P_o} v(e) \left(k_{2,3} \frac{1}{q} \left(\frac{R_2}{a} \right)^7 + k_{1,3} q \left(\frac{R_1}{a} \right)^7 \right), \quad (4.120)$$

with,

$$v(e) = \frac{1 + \frac{15}{4} e^2 + \frac{15}{8} e^4 + \frac{5}{64} e^6}{(1 - e^2)^7}. \quad (4.121)$$

PART II

ASTEROSEISMOLOGY AND TIDALLY EXCITED OSCILLATIONS IN INTERMEDIATE MASS STARS

The observational determination of stellar parameters has a long tradition and is fundamental not only for stellar physics, but also for the characterisation of exoplanet host stars and for Galactic archaeology, where reliable stellar ages are required to reconstruct the formation and assembly history of Galactic stellar populations (Andersen, 1991; Freeman & Bland-Hawthorn, 2002; Sozzetti et al., 2007; Soderblom, 2010; Torres et al., 2010; Chaplin & Miglio, 2013; Miglio et al., 2013). Classical observations, in particular spectroscopy and broad-band photometry, provide constraints on atmospheric properties such as effective temperature, surface gravity, and photospheric chemical abundances (Gray, 2005). When combined with trigonometric parallaxes and bolometric corrections, these measurements also yield intrinsic luminosities (Gaia Collaboration et al., 2016, 2018, 2020). Additional constraints can be obtained from long-baseline interferometry, which provides angular diameters and therefore stellar radii when distances are known (Boyajian et al., 2012; Ligi et al., 2016). Detached eclipsing binaries remain the primary source of precise, model-independent masses and radii, and therefore provide essential benchmarks for stellar-evolution models (Andersen, 1991; Torres et al., 2010; Heiter et al., 2015; Southworth, 2021). Projected surface rotation velocities can furthermore be estimated from the rotational broadening of spectral lines, yielding $v \sin i$ rather than the true equatorial rotation velocity (Gray, 2005; Reiners et al., 2012). The determination of accurate stellar masses and ages is especially important in exoplanetary science, because the inferred planetary radii, masses, and bulk densities are often limited by the precision with which the properties of the host star are known (Seager & Mallén-Ornelas, 2003; Winn, 2010). It is equally essential for Galactic archaeology, which relies on robust stellar ages to trace the formation and evolution of Galactic stellar populations (Freeman & Bland-Hawthorn, 2002; Soderblom, 2010). However, classical observables do not provide direct measurements of stellar mass (except in eclipsing binaries) or age. These quantities must instead be inferred by comparing the observed effective temperatures, luminosities, surface gravities, and chemical compositions with stellar-evolution calculations and isochrone grids, such as BaSTI and PARSEC (Pietrinferni et al., 2004; Bressan et al., 2012; Marigo et al., 2017; Hidalgo et al., 2018). This procedure is, however, strongly degenerate, particularly for main-sequence and subgiant stars, so that different combinations of mass, age, and composition may reproduce nearly identical classical observables (Pont & Eyer, 2004; Jørgensen & Lindegren, 2005; Soderblom, 2010). As a consequence, stellar parameters derived from classical constraints alone often remain affected by substantial uncertainties.

In this context, asteroseismology, the study of stellar oscillation modes, has become a key tool for probing stellar internal structure and dynamics and for tightly constraining global properties such as mass, radius, and age (Aerts et al., 2010; Chaplin & Miglio, 2013; Aerts, 2021). This progress builds on the foundations laid by helioseismology, and on more recent advances driven by high-precision space photometry (Unno et al., 1989; Christensen-Dalsgaard, 2002; Borucki et al., 2010; Ricker et al., 2015). For intermediate-mass main-sequence stars, roughly $1.3 \leq M/M_{\odot} \leq 8.0$, gravity-mode asteroseismology has proved particularly powerful because high-order g modes probe the near-core radiative layers surrounding the convective core (Miglio et al., 2008; Aerts, 2021). Forward seismic modelling of individual Slowly Pulsating B (SPB) ($3 \leq M/M_{\odot} \leq 8.0$) stars and γ -Doradus ($1.3 \leq M/M_{\odot} \leq 1.8$) stars has

provided tight constraints on convective-core overshooting, near-core mixing, and additional diffusive mixing in the radiative envelope (Mombarg et al., 2019; Pedersen et al., 2021; Mombarg et al., 2021; Pedersen, 2022). In parallel, ensemble analyses of gravity-mode period-spacing patterns in γ -Doradus and related g-mode pulsators have made it possible to infer near-core rotation rates for large samples of intermediate-mass stars (Van Reeth et al., 2016; Aerts et al., 2019; Li et al., 2020b). These studies indicate that many apparently single intermediate-mass stars have radiative interiors with relatively low radial differential rotation, pointing to efficient internal angular-momentum transport (Li et al., 2020b). Moreover, sharp structural and compositional features at the edge of the convective core and in the associated chemical-gradient region leave characteristic dips or quasi-periodic modulations in gravity-mode period-spacing patterns (Miglio et al., 2008; Moravveji et al., 2015). These signatures arise from mode trapping caused by the buoyancy glitch associated with the μ -gradient region, and can therefore be exploited to constrain near-core chemical stratification and convective-boundary mixing (Michielsen et al., 2019, 2021). In the case of δ -Scuti stars ($1.6 \leq M/M_{\odot} \leq 2.2$), the detection of regular frequency spacings in high-precision space photometry has opened the way to using their acoustic-mode spectra as seismic diagnostics. In particular, the observed large-frequency-separation-like spacings scale with the square root of the mean stellar density and can therefore provide precise constraints on the mean density when interpreted with appropriate stellar models (García Hernández et al., 2013; Suárez et al., 2014; García Hernández et al., 2015, 2017). By combining individual oscillation frequencies with global seismic diagnostics such as the asymptotic gravity-mode period spacing, forward seismic modelling of intermediate-mass stars has therefore become feasible and now allows for determinations of their masses, ages, near-core rotation rates, and mixing properties (Mombarg et al., 2019; Pedersen et al., 2021; Mombarg et al., 2021; Bowman & Michielsen, 2021).

The advent of space-based photometric missions has transformed asteroseismology. Nearly continuous, high-precision light curves obtained with satellites such as *MOST*, *CoRoT*, *Kepler*, and, more recently, *TESS* have provided the photometric stability, cadence, high duty cycle, and temporal baselines required to detect low-amplitude oscillations across the HR diagram (Walker et al., 2003; Baglin et al., 2006; Auvergne et al., 2009; Borucki et al., 2010; Koch et al., 2010; Ricker et al., 2015; Aerts, 2021). These facilities have complemented and, for many applications, surpassed earlier ground-based multisite campaigns by allowing for homogeneous analyses of large stellar samples and by revealing oscillation spectra that were previously inaccessible from the ground (Aerts et al., 2010; Chaplin & Miglio, 2013; Aerts, 2021). Continuous space photometry has thus uncovered rich mode spectra in low-mass main-sequence stars, subgiants, and red giants, while also opening the way to detailed seismic studies of compact pulsators such as hot subdwarf B/O stars (Fontaine & Brassard, 2008; Gilliland et al., 2010; Huber et al., 2010; Mosser et al., 2011; Chaplin et al., 2011; Fontaine et al., 2012; Chaplin & Miglio, 2013; Lund et al., 2017; Charpinet et al., 2019; Aerts, 2021). For solar-like oscillators and red-giant stars, the measurement of global seismic observables, including the large frequency separation and the frequency of maximum oscillation power, has allowed to estimate stellar masses and radii through seismic scaling relations (Kjeldsen & Bedding, 1995; Chaplin & Miglio,

2013). The subsequent detection and interpretation of mixed modes in evolved stars have extended these diagnostics to stellar cores, making it possible to distinguish evolutionary states, constrain ages, and infer internal rotation for large stellar ensembles (Bedding et al., 2011; Beck et al., 2011; Mosser et al., 2011, 2012; Aerts, 2021). More specifically, rotational splittings of mixed modes in subgiants and red giants have shown that stellar cores generally rotate faster than their envelopes, while rotating much more slowly than predicted by models lacking efficient angular-momentum transport (Beck et al., 2012; Mosser et al., 2012; Marques et al., 2013; Deheuvels et al., 2014; Triana et al., 2017; Aerts et al., 2019). Ensemble asteroseismology of main-sequence and red-giant stars therefore continues to refine the treatment of convective-boundary mixing, diffusive mixing, rotational mixing, and angular-momentum transport in stellar-evolution models (Chaplin & Miglio, 2013; Aerts et al., 2019; Pedersen et al., 2021; Aerts, 2021). In parallel, space-based observations of gravity and gravito-inertial modes in intermediate- and high-mass main-sequence stars have revealed characteristic period-spacing patterns and mode-trapping signatures that probe near-core mixing, internal rotation, and angular-momentum transport in radiative interiors (Miglio et al., 2008; Degroote et al., 2010; Pápics et al., 2014; Van Reeth et al., 2015, 2016; Aerts et al., 2018; Li et al., 2020b; Aerts, 2021; Pedersen et al., 2021; Pedersen, 2022). Likewise, detailed seismic modelling of pulsating white dwarfs has constrained their internal chemical stratification, crystallisation, and cooling histories (Winget et al., 1991; Metcalfe et al., 2004; Fontaine & Brassard, 2008; Córscico et al., 2019; Córscico, 2020). These advances have established asteroseismology as a central tool of modern stellar astrophysics and of Galactic stellar-population studies (Miglio et al., 2013; Chaplin & Miglio, 2013; Miglio et al., 2017; Aerts, 2021).

Beyond what can be probed by adiabatic frequency modelling alone, major uncertainties still affect the microphysics adopted in stellar models. A prominent example is the radiative opacity of iron-group elements, whose Rosseland mean depends on a very large number of atomic transitions and still differs significantly among independent calculations such as OPAL, OP, and OPLIB (Iglesias & Rogers, 1993, 1996; Seaton, 2005; Badnell et al., 2005; Colgan et al., 2016). In intermediate- and high-mass B-type pulsators, the iron-group opacity bump near $\log T \simeq 5.2$ plays a central role in non-adiabatic mode excitation through the κ -mechanism. It is responsible for the excitation of β Cephei low-order p and g modes and SPB-star high-order g modes (Dziembowski & Pamyatnykh, 1993; Pamyatnykh, 1999; Aerts et al., 2010). Changes in the adopted opacity tables, metal mixture, or local iron-group opacity profile can substantially modify the predicted instability domains and the occurrence of hybrid β Cephei/SPB behaviour (Zdravkov & Pamyatnykh, 2008; Salmon et al., 2012; Moravveji et al., 2015). In addition, element-transport processes such as atomic diffusion and radiative levitation can locally modify chemical abundances and opacity profiles, thereby affecting mode excitation in intermediate-mass stars (Mombarg et al., 2020; Rehm et al., 2024). These issues make pulsating B stars particularly valuable astrophysical laboratories for testing opacity physics and the non-adiabatic ingredients of stellar models (Bailey et al., 2015; Lynas-Gray et al., 2018). Another major source of uncertainty concerns convection. In most stellar-evolution calculations, convection is still described using local mixing-length prescriptions (Cox & Giuli, 1968), which constitute a strong approximation to a much more complex phenomenon. Convection also

plays an important role in the excitation and damping of oscillation modes in δ -Scuti and γ -Doradus stars, where the interaction between convection and pulsation can determine whether modes are driven or damped. A more complete treatment is therefore required, such as time-dependent convection (TDC), in which perturbations to the convective flux are treated consistently with the pulsation equations (Gough, 1977; Xiong, 1989; Balmforth, 1992a; Houdek et al., 1995; Xiong et al., 1997; Houdek et al., 1999; Houdek, 2000; Grigahcène et al., 2005; Houdek & Dupret, 2015; Xiong et al., 2015, 2016). Such formalisms have significantly improved the non-adiabatic modelling of intermediate-mass pulsators, especially δ -Scuti and γ -Doradus stars, by yielding more realistic mode-stability predictions and instability-strip boundaries (Grigahcène et al., 2005; Houdek & Dupret, 2015; Xiong et al., 2016). This is precisely where non-adiabatic asteroseismology becomes indispensable. By explicitly modelling the exchange of heat between oscillations and the surrounding stellar layers, non-adiabatic calculations predict mode growth rates, damping rates, and the boundaries of instability domains, thereby complementing the structural constraints obtained from adiabatic seismic modelling (Unno et al., 1989; Aerts et al., 2010; Houdek & Dupret, 2015). Consequently, the observed distribution of excited and damped modes in β -Cephei, SPB, δ -Scuti, and γ -Doradus stars places strong constraints on radiative opacities, convective transport, and related microphysics that cannot be obtained from adiabatic frequency modelling alone (Houdek & Dupret, 2015; Aerts, 2021).

In binary systems, stars can be viewed as forced oscillators. An isolated star possesses a discrete spectrum of eigenmodes, with eigenfrequencies corresponding to the oscillation modes studied in standard asteroseismology. In a close binary, however, the perturbation is externally imposed by the companion through the time-dependent tidal potential. For eccentric systems, this potential can be expanded into a series of orbital harmonics, so that the forcing frequencies in the frame corotating with the star are combinations of the orbital mean motion and the stellar spin frequency (Zahn, 1975; Smeyers & Willems, 2001; Willems, 2003; Ogilvie, 2014). When one of these forcing frequencies approaches a stellar eigenfrequency in the corotating frame, the dynamical tide can resonantly excite the corresponding mode and produce an observable tidally excited oscillation (Willems et al., 2003; Fuller & Lai, 2012; Burkart et al., 2012; Guo, 2021). Unlike the self-excited or stochastically excited modes of single stars, which are driven internally by mechanisms such as the κ -mechanism, convective flux blocking, or turbulent convection, tidally excited modes are maintained by the external tidal forcing of the companion. They therefore connect the pulsation spectrum directly to the orbital dynamics (Fuller & Lai, 2012; Burkart et al., 2012; Ogilvie, 2014; Guo, 2021). Their amplitudes, phases, and frequency patterns depend not only on the orbital period, eccentricity, and viewing geometry, but also on the internal stellar structure, rotation, tidal coupling coefficients, and dissipation of the excited tide (Fuller & Lai, 2012; Fuller, 2017; Guo, 2020; Guo et al., 2020). Tidally excited oscillations thus provide a powerful framework in which seismic diagnostics can be combined with orbital modelling to probe dynamical tides and to investigate the coupled evolution of stellar interiors and binary orbits (Hambleton et al., 2013; Cheng et al., 2020; Fuller, 2021; Guo, 2021).

From an observational perspective, space-based photometry has now established that

tidal forcing can excite rich oscillation spectra in close binaries over a wide range of stellar masses (Welsh et al., 2011; Fuller & Lai, 2012; Fuller, 2017; Guo, 2021). Early ground-based evidence already came from the SPB system HD 177863, whose dominant variability occurs at an integer multiple of the orbital frequency and has been interpreted as being consistent with a resonantly excited high-order g mode produced by the dynamical tide (De Cat & Aerts, 2002; Willems & Aerts, 2002). Another important early example is the peculiar A/F-type eccentric binary HD 209295, in which several g-mode frequencies are exact orbital harmonics while a δ -Scuti-like pressure-mode (p-mode) is also present (Handler & Shobbrook, 2002). In the A/F-star regime, *CoRoT* and *Kepler* subsequently showed that tidally excited oscillations can coexist with self-excited p- and g-mode pulsations in eccentric binaries, providing direct observational access to the interaction between classical pulsation physics and tidal forcing (Southworth et al., 2011; Maceroni et al., 2013; Hambleton et al., 2013; Maceroni et al., 2014). At higher masses and in more eccentric systems, *Kepler* and *TESS* have shown that tides can excite modes at very high orbital harmonics. The prototype KOI-54, also known as HD 187091, displays strong periastron brightening and a rich set of pulsations locked to the orbital frequency, with the two dominant modes occurring at the 90th and 91st orbital harmonics (Welsh et al., 2011). Detailed modelling of this system showed that these oscillations can be interpreted as tidally excited gravity modes driven close to resonance (Fuller & Lai, 2012; Burkart et al., 2012). The broader *Kepler* sample analysed by Thompson et al. (2012) helped establish heartbeat stars as a distinct observational class of eccentric binaries. In these systems, the close periastron passage produces a characteristic, phase-dependent brightening caused by the combined effects of tidal distortion, irradiation, and viewing geometry. In many cases, the same time-dependent tidal forcing also excites stellar oscillation modes, producing tidally induced pulsations at harmonics of the orbital frequency. Thus, heartbeat stars provide direct observational evidence of dynamical tidal interactions in eccentric binaries. Subsequent spectroscopic follow-up provided orbital solutions and mass constraints for several systems (Smullen & Kobulnicky, 2015; Shporer et al., 2016). More recently, *TESS* has broadened the sample toward massive heartbeat stars (Kołaczek-Szymański et al., 2021), while long-baseline survey photometry has revealed extreme systems such as MACHO 80.7443.1718, where tidally excited oscillations are detected up to the 230th orbital harmonic (Kołaczek-Szymański et al., 2022). *TESS* has also uncovered qualitatively new tidal-pulsation geometries, most notably HD 74423, in which the pulsation axis is aligned with the tidal axis and the oscillation amplitude is concentrated predominantly in one stellar hemisphere (Handler et al., 2020; Fuller et al., 2020). The current literature therefore shows that tidally excited or tidally modified oscillations can appear in many different forms and in a wide variety of systems. Each case may involve distinct stellar, orbital, and tidal properties, as well as different physical mechanisms. Understanding this diversity, and explaining the detailed origin of the observed pulsation behaviour, remains an important challenge for future theoretical modelling.

In binaries containing intermediate- and high-mass main-sequence stars, tidally excited oscillations can have a particularly important impact on orbital evolution because these stars typically combine convective cores with extended radiative envelopes. In such stars, equilibrium-tide dissipation by turbulent viscosity is relatively inefficient as

there is no substantial convective envelope, whereas the dominant tidal dissipation is often the dynamical tide, carried by internal gravity modes (g-modes) in the radiative envelope (Zahn, 1975, 1977; Witte & Savonije, 1999a; Ogilvie, 2014). Through the coupling of the tidal potential to these waves and modes, orbital energy and angular momentum can be transferred to the stellar interior and dissipated through radiative damping, non-linear wave breaking, or other damping mechanisms (Witte & Savonije, 1999a; Fuller, 2017; Ahuir et al., 2021). Dynamical tides are therefore a key agent of orbital circularisation, spin synchronisation, and, more generally, secular spin-orbit evolution in radiative-envelope stars (Zahn, 1977; Hut, 1981; Ogilvie, 2014). Because the tidal potential in an eccentric binary can be decomposed into a spectrum of orbital harmonics, the coupling to stellar oscillations becomes especially efficient when one of the forcing frequencies in the stellar corotating frame approaches an eigenfrequency (Willems, 2003; Fuller & Lai, 2012; Ogilvie, 2014). In such cases, repeated periastron passages can maintain enhanced tidal dissipation over long intervals, and in favourable circumstances the coupled evolution of the orbit and stellar eigenfrequency can lead to resonance locking (Fuller, 2017; Hambleton et al., 2018). At the same time, many intermediate-mass stars naturally support self-excited high-order g-modes and, in more evolved objects, mixed modes. In SPB stars these oscillations are driven by the iron-bump κ -mechanism, whereas in γ -Doradus stars they are excited by convective flux blocking near the base of the envelope convection zone (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Pamyatnykh, 1999; Guzik et al., 2000; Dupret et al., 2005; Aerts et al., 2010). If the tidal forcing couples resonantly to an intrinsically unstable mode, in principle, tides do not act as purely dissipative, the phase lag can change sign and energy and angular momentum can be transferred from the stellar pulsation back into the orbit, a process referred to as inverse tides (Fuller, 2021). The competition between ordinary dissipative tides and inverse tidal effects therefore depends sensitively on the resonant mode spectrum, the stellar rotation profile, the tidal coupling coefficients, and the damping or growth rates of the participating modes (Fuller, 2017, 2021; Guo, 2021). By combining asteroseismic constraints from tidally excited oscillations with detailed orbital modelling, one can directly connect internal structure, near-core rotation, and mode damping to the secular evolution of binary systems.

Being able to model the rotational and orbital evolution of such systems is essential for constraining the properties of the binary components and for predicting their future evolution. It is also important for understanding the broader evolution of intermediate-mass binary populations (Zahn, 1977; Hut, 1981; Ogilvie, 2014). Depending on the properties of the excited modes, especially their detuning, damping or growth rate, and tidal coupling coefficients, these resonant interactions may either enhance dissipation or temporarily return energy and angular momentum to the orbit or stellar spin (Witte & Savonije, 1999a, 2001; Fuller, 2017, 2021). The net tidal torque, and thus the secular evolution of the system, is therefore governed by the time-dependent sum of many resonant and non-resonant contributions, whose relative importance changes as the orbit, stellar rotation, and stellar eigenfrequency spectrum evolve. Following this evolution over long timescales requires a framework able, at each evolutionary step, to compute the stellar response over the relevant forcing-frequency spectrum, to update that response as the orbital and rotational parameters evolve, and to inte-

grate the cumulative effect of the resulting torques self-consistently (Witte & Savonije, 1999a, 2001; Fuller, 2017). Such calculations remain challenging over evolutionary timescales comparable to the 10^8 – 10^9 yr main-sequence lifetimes of intermediate-mass stars, especially when the oscillation spectrum is dense and the tidal potential contains many significant harmonics. The main limitation is numerical rather than conceptual: even with an efficient algorithm, one must evaluate millions, and often tens of millions, of tidal responses while repeatedly updating the stellar structure, rotation, and mode spectrum as the system evolves. This difficulty explains why most long-term binary-evolution calculations still rely on simplified tidal prescriptions, while detailed forced-oscillation calculations have generally been applied to shorter integrations, or with individual forcing frequencies (Hurley et al., 2002; Valsecchi et al., 2013; Fuller & Lai, 2012; Fuller, 2017). Today, however, improved numerical resources, renewed theoretical interest in resonance locking and long-term tidal evolution, and the observational breakthrough provided by *Kepler* heartbeat stars make the problem particularly timely (Thompson et al., 2012; Welsh et al., 2011; Fuller, 2017; Hambleton et al., 2018; Guo, 2021). The goal of this part of the thesis is therefore to study tidally excited oscillations and dynamical tides in sufficient detail to predict their long-term impact on the orbital and rotational evolution of close binary systems.

In the first chapter of this part, Chapter 5, I introduce the general formalism of non-adiabatic asteroseismology used to describe stellar oscillations, focusing in particular on intermediate-mass main-sequence stars and on the excitation and damping mechanisms that operate in such stars. In Chapter 6, I then apply this framework to develop a stellar oscillation code capable of computing the response of a star to tidal forcing induced by a companion. By including non-adiabatic effects, this approach makes it possible to study the driving and damping of tidally excited oscillations, which regulate the exchange of angular momentum and energy between the stars and their orbit, and therefore shape the orbital and rotational evolution of the system. Finally, in Chapter 7, I develop the numerical framework and integrator required to follow the effect of dynamical tides throughout the full main-sequence evolution of intermediate-mass stars, by coupling the new stellar oscillation code to an orbital evolution integrator.

CHAPTER 5

INTRODUCTION TO ASTEROSEISMOLOGY IN SINGLE STARS

Before presenting the mathematical formalism of stellar oscillations, it is useful to consider the relevant physical timescales, as these govern the dynamical and thermal processes within stars. In both stellar evolution and asteroseismology, timescales play a crucial role in defining the physical regime, motivating approximations (e.g. hydrostatic or thermal equilibrium), and determining whether oscillations can be treated as adiabatic or require a fully non-adiabatic description.

The first of these is the dynamical timescale, which characterises rapid dynamical processes such as free-fall collapse and the global response of a star to perturbations of hydrostatic equilibrium. It can be approximated as the free-fall time,

$$\tau_{\text{dyn}} = \sqrt{\frac{R^3}{GM}} = \sqrt{\frac{3}{4\pi} \frac{1}{G\rho}}, \quad (5.1)$$

where ρ is the mean density of the star, M its mass, R its radius and G is the gravitational constant. It sets the natural scale for acoustic (p-mode) periods: for the Sun, p-mode oscillations have periods of a few minutes, which are only a bit shorter than τ_{dyn} , while in more compact stars the dynamical timescale is correspondingly shorter (Christensen-Dalsgaard, 2002; Chaplin & Miglio, 2013). Other phenomena governed by this timescale include dynamical stellar instabilities and the rapid phases of gravitational collapse.

A second important global timescale is the Kelvin–Helmholtz timescale, which represents the time it would take a star to radiate away its internal (gravitational and thermal) energy at the current luminosity if nuclear reactions were suddenly turned off. It can be written as

$$\tau_{\text{KH}} = \frac{GM^2}{LR}, \quad (5.2)$$

where M is the stellar mass, and R the radius. This timescale, which describe the time taken by global thermal processes take place, is for the Sun approximately $\tau_{\text{KH}} \simeq 3 \times$

10^7 yr, while the corresponding nuclear timescale, the time over which nuclear burning can power the observed luminosity, is of order 10^{10} yr (Christensen-Dalsgaard, 2002; Chaplin & Miglio, 2013). The strong hierarchy

$$\tau_{\text{dyn}} \ll \tau_{\text{KH}} \ll \tau_{\text{nuc}}$$

implies that main-sequence stars are close to thermal equilibrium, so that the energy produced by nuclear reactions approximately balances the luminosity radiated at the surface. This separation of timescales also implies that stellar structures adjust quasi-hydrostatically on evolutionary timescales, while oscillations occur on timescales that are extremely short compared to global thermal changes. It therefore underpins the standard approximation that stellar oscillations can often be treated as adiabatic in the stellar interior, with the background structure considered fixed during an oscillation cycle. However, this argument is based on global timescales. The relevant quantity for pulsation driving and damping is instead the local thermal relaxation timescale, τ_{th} , defined as,

$$\tau_{\text{th}}(r) = \int_{\Delta m} \frac{T c_v}{L} dm, \quad (5.3)$$

which is not uniform throughout the star. In the previous equation c_v is heat capacity, T is the local temperature, L the global luminosity, and Δm the mass encompassed between two layers. For intermediate-mass stars, the local thermal relaxation timescale is usually much longer than the mode periods in the deep interior, supporting the use of the adiabatic approximation in these regions. In the outermost layers, however, and particularly in partial-ionisation zones and near the base of thin convective envelopes, the local thermal timescale can become comparable to the oscillation periods (Gautschy & Saio, 1995, 1996; Chaplin & Miglio, 2013; Aerts et al., 2019). Consequently, in these regions, radiative and convective heat exchange strongly affects the oscillations, and a non-adiabatic treatment is required to describe mode driving and damping, for example via the κ -mechanism or convective flux blocking. The competition between driving and damping in these zones plays a central role in setting the instability strips of classical pulsators and in determining which modes are excited or damped.

In the following section, Section 5.1, I provide a concise mathematical description of the oscillation equations. I outline the standard assumptions used in the theory of stellar oscillations, discuss their physical meaning in the light of the hierarchy of timescales introduced above, and clarify the limits of their applicability in both adiabatic and non-adiabatic asteroseismic modelling. Then, in Section 5.2, I extend this discussion by presenting the non-adiabatic formalism used to study the excitation and damping of oscillation modes in intermediate-mass stars, including time-dependent treatments of the convection-oscillation interactions. In Section 5.3, I then present the different tools commonly used in asteroseismology, together with the basic numerical framework that will be modified in the following chapters. Next, in Section 5.4, I describe the classical instability strips of intermediate-mass stars, that is, the regions of the HR diagram in which oscillation modes are excited, and I review the main excitation mechanisms responsible for them. I then explore in greater depth, in Section 5.5, the γ -Doradus instability strip and the impact of various physical quantities on its location and extent. Finally, in Section 5.6, I develop the formalism required to include

rotation in non-adiabatic computations using the traditional approximation of rotation, which is then applied to the case of tidally excited oscillations in the following chapters.

5.1 ADIABATIC ASTEROSEISMOLOGY

5.1.1 EQUATIONS OF HYDRODYNAMICS

To fully understand the behaviour of stellar oscillations and the limitations of our theory, we begin with the basic equations of hydrodynamics in a fluid. In this section, we neglect the thermal component of these equations in order to focus on the physical nature of oscillation modes. The thermal aspects and non-adiabatic effects will be addressed in the following section. The dynamical part of the hydrodynamic equations consists of the conservation equations for mass and momentum, supplemented by Poisson's equation for the gravitational potential. These are given as follows:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (5.4)$$

Conservation of momentum:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Psi, \quad (5.5)$$

Poisson's equation:

$$\Delta \Psi = 4\pi G \rho, \quad (5.6)$$

These equations describe the evolution of a fluid element inside the star with density ρ and velocity vector $\mathbf{v}$, subject to pressure forces P and gravity via the gravitational potential Ψ . Bold quantities denote vectors, and ∇ is the nabla operator. These equations are general and do not assume equilibrium. However, in the context of stellar structure, we apply them under the assumption of hydrostatic equilibrium. The Kelvin–Helmholtz timescale of a star (ranging from minutes to days) is orders of magnitude shorter than the nuclear timescale (millions to billions of years). Hence, it is reasonable to assume that a star is in hydrostatic equilibrium at each stage of its evolution. For non-rotating or slowly rotating stars, the effects of rotation on the structure (e.g. Coriolis and centrifugal forces) can be neglected, allowing us to treat the star as spherically symmetric. In this case, the equations reduce to the classical form of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad (5.7)$$

and the expression for the mass contained within a spherical shell:

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (5.8)$$

where r is the radial coordinate within the star.

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5.1.2 PERTURBATION METHOD

In oscillating stars, local physical quantities oscillate around their equilibrium values, which are described by the hydrostatic equilibrium equations in [Subsection 5.1.1](#). To describe these small perturbations, one can adopt either an Eulerian or a Lagrangian framework. Eulerian perturbations describe variations at a fixed point in space, whereas Lagrangian perturbations describe variations as experienced by a fluid element moving with the flow. Let $f_0(\mathbf{r})$ be a physical quantity in equilibrium. Its Eulerian perturbation is denoted $f'(\mathbf{r}, t)$, so that the perturbed quantity becomes:

$$f(\mathbf{r}_0, t) = f_0(\mathbf{r}_0) + f'(\mathbf{r}_0, t). \quad (5.9)$$

The corresponding Lagrangian perturbation is defined as:

$$f(\mathbf{r}, t) = f_0(\mathbf{r}_0) + \delta f(\mathbf{r}, t), \quad (5.10)$$

where δf is the Lagrangian perturbation. Under the linear approximation, where second-order terms in the perturbations are neglected, the Eulerian and Lagrangian perturbations are related by:

$$\delta f(\mathbf{r}, t) = f'(\mathbf{r}_0, t) + \boldsymbol{\xi} \cdot \nabla f_0(\mathbf{r}_0), \quad (5.11)$$

where $\boldsymbol{\xi} = \mathbf{r} - \mathbf{r}_0$ is the displacement vector, with $\mathbf{r}$ denoting the perturbed position and $\mathbf{r}_0$ the equilibrium position. It should be emphasised that the validity of the linear approximation relies on the assumption that the perturbations remain small throughout the domain.

5.1.3 EQUATIONS OF LINEAR ADIABATIC NON-RADIAL OSCILLATIONS

In this section, we apply small perturbations to the fluid conservation equations, which, after some mathematical development, allow us to derive the governing conditions for various types of stellar oscillations. We begin with the continuity equation:

$$\rho' + \nabla \cdot (\rho_0 \boldsymbol{\xi}) = 0. \quad (5.12)$$

The conservation of momentum:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla P' - \rho_0 \nabla \Psi' - \rho' \nabla \Psi_0, \quad (5.13)$$

and the Poisson equation:

$$\Delta \Psi' = 4\pi G \rho', \quad (5.14)$$

where ρ_0 , P_0 , and Ψ_0 denote the equilibrium density, pressure, and gravitational potential, respectively, while ρ' , P' , and Ψ' are the corresponding perturbed quantities. To close this system of differential equations, the energy conservation equation must typically be included. However, under the adiabatic approximation, energy and heat exchange between the oscillations and the surrounding medium are neglected, so the energy equation is not required. Instead, we use a thermodynamic relation connecting the perturbations of pressure and density, where the entropy dependence is neglected.

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Further details on this relation can be found in [Unno et al. \(1989\)](#). The perturbed thermodynamic relation is given by:

$$P' + \xi \cdot \nabla P_0 = c^2 (\rho' + \xi \cdot \nabla \rho_0), \quad (5.15)$$

with $c^2 = \frac{\Gamma_1 P_0}{\rho_0}$ the adiabatic sound speed, and $\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho} \right)_S$ the first adiabatic exponent. In solar-like stars, the adiabatic approximation is valid within the stellar interior but fails in the outer layers, where oscillation excitation and damping occur. These so-called "surface effects" arise due to inadequate modelling of the upper layers and the neglect of non-adiabatic processes. A detailed discussion of the implications of the adiabatic approximation can be found in [Section 5.2](#) and [Section 5.4](#), the latter focusing on the non-adiabatic theoretical modelling of intermediate-mass stellar oscillations. The resulting set of differential equations is linear in time, allowing solutions to be expressed as infinite sums with time dependence of the form $e^{i\sigma_n t}$. Thus, a general solution may be written as:

$$X(\mathbf{r}, t) = \text{Re} \left(\sum_{n=0}^{\infty} X_n(\mathbf{r}) e^{i\sigma_n t} \right), \quad (5.16)$$

where $X_n(\mathbf{r})$ represents the spatial component of the eigenfunction, and σ_n the associated oscillation frequency. Under the adiabatic approximation, both X_n and σ_n are real. With this formalism, the perturbed momentum conservation becomes:

$$\sigma^2 \rho_0 \xi = -\nabla P' - \rho_0 \nabla \Psi' - \rho' \nabla \Psi_0. \quad (5.17)$$

We now adopt a spherical coordinate system (r, θ, ϕ) centred on the star, with an orthonormal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. The displacement vector can be expressed as:

$$\xi = \xi_r \mathbf{e}_r + \xi_\theta \mathbf{e}_\theta + \xi_\phi \mathbf{e}_\phi. \quad (5.18)$$

For brevity, we now omit the subscript '0' on equilibrium quantities. In this coordinate system, the nabla operator is:

$$\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \mathbf{e}_\phi. \quad (5.19)$$

The three components of the perturbed equation of motion are:

$$\sigma^2 \xi_r = \frac{\partial \Psi'}{\partial r} + \frac{\rho'}{\rho} g + \frac{1}{\rho} \frac{\partial P'}{\partial r}, \quad (5.20)$$

$$\sigma^2 \xi_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \left(\Psi' + \frac{P'}{\rho} \right), \quad (5.21)$$

$$\sigma^2 \xi_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\Psi' + \frac{P'}{\rho} \right), \quad (5.22)$$

where $g = Gm/r^2$. The mass conservation equation becomes:

$$\delta \rho + \rho \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \xi_\theta) + \frac{1}{r \sin \theta} \frac{\partial \xi_\phi}{\partial \phi} \right) = 0. \quad (5.23)$$

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Substituting the expressions for ξ_θ and ξ_ϕ yields:

$$\frac{\delta\rho}{\rho} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi_r) - \frac{1}{r^2 \sigma^2} \mathcal{L}^2 \left(\Psi' + \frac{P'}{\rho} \right) = 0, \quad (5.24)$$

where $\mathcal{L}^2$ is the angular (Legendre) operator defined by:

$$\mathcal{L}^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (5.25)$$

The perturbed thermodynamic relation then becomes:

$$\rho' = \frac{P'}{c^2} + \xi_r \rho \left(\frac{1}{\Gamma_1} \frac{d \ln P}{dr} - \frac{d \ln \rho}{dr} \right). \quad (5.26)$$

The perturbed Poisson equation reads:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi'}{dr} \right) - \frac{1}{r^2} \mathcal{L}^2 \Psi' = 4\pi G \rho'. \quad (5.27)$$

We now make use of spherical harmonics $Y_\ell^m(\theta, \phi)$, as defined and used in the previous part. A key property is that they are eigenfunctions of the operator $\mathcal{L}^2$:

$$\mathcal{L}^2 Y_\ell^m = \ell(\ell + 1) Y_\ell^m. \quad (5.28)$$

With this, the full solution becomes separable and takes the form:

$$X(r, \theta, \phi, t) = \text{Re} \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=0}^{\infty} A_{\ell,m,k} X_k(r) Y_\ell^m(\theta, \phi) e^{i\sigma_k t} \right). \quad (5.29)$$

After substitution and simplification, the system reduces to:

$$\frac{\delta\rho}{\rho} + \frac{1}{r^2} \frac{d}{dr} (r^2 \xi_r) - \frac{\ell(\ell + 1)}{r^2 \sigma^2} \left(\frac{P'}{\rho} + \Psi' \right) = 0, \quad (5.30)$$

$$\sigma^2 \rho \xi_r = \frac{dP'}{dr} + \rho \frac{d\Psi'}{dr} + g\rho', \quad (5.31)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi'}{dr} \right) - \frac{\ell(\ell + 1)}{r^2} \Psi' = 4\pi G \rho'. \quad (5.32)$$

The Lagrangian perturbation of density in the adiabatic limit follows:

$$\frac{\delta\rho}{\rho} = \frac{1}{\Gamma_1} \frac{\delta P}{P} = \frac{P'}{\rho c^2} - \frac{\xi_r g}{c^2}, \quad (5.33)$$

where this expression results from combining the Eulerian and Lagrangian formalisms with the hydrostatic equilibrium equation. We introduce two characteristic frequencies:

$$S_\ell^2 = \frac{\ell(\ell + 1)c^2}{r^2}, \quad (5.34)$$

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$$N^2 = g \left(\frac{1}{\Gamma_1} \frac{d \ln P}{dr} - \frac{d \ln \rho}{dr} \right), \quad (5.35)$$

corresponding to the squared Lamb and squared Brunt–Väisälä frequencies, respectively. The final system becomes:

$$\frac{dP'}{dr} + \rho \frac{d\Psi'}{dr} + g \frac{P'}{c^2} = (\sigma^2 - N^2) \rho \xi_r, \quad (5.36)$$

$$\frac{P'}{\rho c^2} \left(1 - \frac{S_\ell^2}{\sigma^2} \right) - \frac{\ell(\ell+1)}{r^2 \sigma^2} \Psi' - \frac{g \xi_r}{c^2} + \frac{1}{r^2} \frac{d}{dr} (r^2 \xi_r) = 0, \quad (5.37)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi'}{dr} \right) - \frac{\ell(\ell+1)}{r^2} \Psi' = 4\pi G \left(\frac{P'}{c^2} + \frac{\rho N^2}{g} \xi_r \right). \quad (5.38)$$

This constitutes a system of three differential equations in the unknowns ξ_r , P' , and Ψ' . To solve it, three boundary conditions are required. These vary depending on the author, the oscillation code, and the adopted set of variables. We will discuss our choice of boundary conditions in more detail in [Section 5.2](#), where we present the full set of non-adiabatic equations.

5.1.4 THE PHYSICAL MEANING AND ASYMPTOTIC PROPERTIES OF MODES

The solutions of the pulsation equations are typically obtained through numerical solving. However, in order to better understand the physical nature of stellar oscillations, it is useful to apply several approximations, following the approach outlined in [Gabriel & Scuflaire \(1979\)](#). The first approximation is to neglect the perturbation of the gravitational potential, known as the Cowling approximation ([Cowling, 1941](#)), which is only valid when both ℓ and n are large. Under this approximation, the system can be rewritten as:

$$\frac{d\xi_r}{dr} = - \left(\frac{2}{r} + \frac{1}{\Gamma_1 H_p} \right) \xi_r + \frac{1}{\rho c^2} \left(\frac{S_\ell^2}{N^2} - 1 \right) P', \quad (5.39)$$

$$\frac{dP'}{dr} = \rho(\omega^2 - N^2) \xi_r + \frac{P'}{\Gamma_1 H_p}, \quad (5.40)$$

with $H_p = \frac{dr}{d \ln P}$ the pressure scale height. The system is further simplified by introducing new variables:

$$v = f_1 r^2 \xi_r, \quad (5.41)$$

$$w = f_2 \frac{P'}{\rho}, \quad (5.42)$$

where f_1 and f_2 are defined as:

$$f_1 = \exp \left(\int_0^r \frac{1}{\Gamma_1} \frac{d \ln P}{dr} dr \right), \quad (5.43)$$

$$f_2 = \exp \left(- \int_0^r \frac{N^2}{g} dr \right). \quad (5.44)$$

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With this substitution, the system becomes:

$$\frac{dw}{dr} = (\omega^2 - N^2) \frac{f_2}{r^2 f_1} v, \quad (5.45)$$

$$\frac{dv}{dr} = \left(\frac{S_\ell^2}{\omega^2} - 1 \right) \frac{r^2 f_1}{c^2 f_2} w. \quad (5.46)$$

This system can be combined into a second-order differential equation:

$$\frac{d}{dr} \left(\frac{1}{1 - \frac{S_\ell^2}{\omega^2}} \frac{c^2 f_2}{r^2 f_1} \frac{dv}{dr} \right) + (\omega^2 - N^2) \frac{f_2}{r^2 f_1} v = 0, \quad (5.47)$$

or

$$\frac{d}{dr} \left(\frac{1}{N^2 - \omega^2} \frac{r^2 f_1}{f_2} \frac{dw}{dr} \right) + \left(\frac{S_\ell^2}{\omega^2} - 1 \right) \frac{c^2 f_2}{r^2 f_1} w = 0. \quad (5.48)$$

From these equations, it follows that oscillatory solutions exist only when either $\omega^2 < N^2$ and $\omega^2 < S_\ell^2$, or $\omega^2 > N^2$ and $\omega^2 > S_\ell^2$. If one of these conditions is met, the mode is trapped in a so-called oscillation cavity and decays exponentially outside of it. Two classes of modes can be identified: low-frequency gravity modes (g-modes) and high-frequency pressure modes (p-modes).

PRESSURE MODES.

Pressure modes propagate in regions where

$$\omega^2 > N^2 \quad (5.49)$$

and

$$\omega^2 > S_\ell^2. \quad (5.50)$$

They are high-frequency acoustic waves, with pressure acting as the restoring force. Their propagation is mainly controlled by the adiabatic sound speed c . In the acoustic cavity, and away from the turning points, the terms N^2/ω^2 and S_ℓ^2/ω^2 are small in Eqs. 5.47 and 5.48. For non-radial modes, the inner turning point occurs where $\omega = S_\ell$. Since $S_\ell^2 = \ell(\ell+1)c^2/r^2$, low-degree modes penetrate more deeply into the stellar interior than high-degree modes, while radial modes can reach the centre.

GRAVITY MODES.

The propagation conditions for gravity modes are

$$\omega^2 < N^2, \quad (5.51)$$

and

$$\omega^2 < S_\ell^2. \quad (5.52)$$

These non-radial modes propagate in stably stratified radiative regions, where buoyancy provides the restoring force. In the deep interior, the high sound speed generally

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leads to a large Lamb frequency, so that $S_\ell^2 \gg \omega^2$. For high-order gravity modes, $N^2 \gg \omega^2$ also typically holds, and therefore $N^2 - \omega^2 \simeq N^2$. Gravity modes are evanescent in convective regions, where $N^2 \lesssim 0$, which makes them difficult to observe in stars with extended convective envelopes unless they are coupled to an outer acoustic cavity as mixed modes.

MIXED MODES.

In evolved stars, pressure- and gravity-mode cavities can be coupled through the intermediate evanescent region. Acoustic modes trapped in the outer envelope may then tunnel into the deep interior and couple to gravity modes trapped in the core, giving rise to mixed modes. Such modes are particularly important in stars with dense, contracted cores and extended envelopes, such as subgiants and red giants. In these stars, the Brunt–Väisälä frequency increases in the core because of the strong stratification and steep chemical and density gradients, while the Lamb frequency decreases in the expanded envelope. This produces separated acoustic and gravity-mode cavities that can nevertheless remain coupled.

Depending on the coupling strength, mixed modes can behave either predominantly as pressure modes or predominantly as gravity modes. Kepler-56 provides an example of strong coupling between the two cavities, as studied in detail by [Takata \(2016b,a\)](#). Earlier work on mixed modes in red giants includes [Mosser et al. \(2012\)](#); [Goupil et al. \(2013\)](#). In the strong-coupling regime, a single acoustic resonance can interact with many gravity-dominated modes over a finite frequency range.

Mixed modes have become powerful probes of the deep interiors of evolved stars. Their period spacings allow red-giant-branch stars to be distinguished from core-helium-burning red-clump stars, while their rotational splittings provide constraints on internal rotation profiles. These measurements have shown that standard stellar-evolution models generally predict cores rotating much faster than observed, revealing missing or underestimated angular-momentum transport processes.

5.1.5 ASYMPTOTIC PROPERTIES OF PRESSURE AND GRAVITY MODES.

With additional assumptions, it is possible to derive asymptotic expressions for high-order stellar oscillation modes ([Shibahashi, 1979](#); [Tassoul, 1980](#)). These expressions are particularly useful because they reveal the propagation cavities of acoustic and gravity modes and provide simple frequency-spacing relations. A complete mathematical derivation can be found, for example, in [Gough \(1993\)](#); here, we only present the main results needed for the following discussion. Starting from [Equation 5.39](#) and [Equation 5.40](#), and still adopting the Cowling approximation, the system can be reduced to a second-order wave equation by introducing the modified displacement variable

$$X = c^2 \sqrt{\rho} \nabla \cdot \xi, \quad (5.53)$$

where c is the adiabatic sound speed, ρ the equilibrium density, and ξ the displacement vector. After neglecting rapidly varying gravitational and density-perturbation terms

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with respect to the dominant thermodynamic terms, one obtains

$$\frac{d^2 X}{dr^2} = -K(r)X, \quad (5.54)$$

where the radial wavenumber squared is given by

$$K(r) = \frac{1}{c^2} \left[S_\ell^2 \left(\frac{N^2}{\omega^2} - 1 \right) + \omega^2 - \omega_c^2 \right]. \quad (5.55)$$

Here, ω_c is the acoustic cut-off frequency. In terms of the density scale height

$$H_\rho = - \left(\frac{d \ln \rho}{dr} \right)^{-1}, \quad (5.56)$$

the acoustic cut-off frequency can be written as

$$\omega_c^2 = \frac{c^2}{4H_\rho^2} \left(1 - 2 \frac{dH_\rho}{dr} \right). \quad (5.57)$$

The sign of $K(r)$ determines the local behaviour of the solution. In regions where $K(r) > 0$, the solutions are oscillatory and waves can propagate. Conversely, where $K(r) < 0$, the solutions are evanescent and decay exponentially. The points where $K(r) = 0$ are turning points that delimit the propagation cavity. Near the stellar surface, the Lamb-frequency contribution generally becomes small, so the sign of $K(r)$ is mainly controlled by the comparison between ω and the acoustic cut-off frequency ω_c . Modes with $\omega < \omega_c$ are reflected below the surface, whereas modes with $\omega > \omega_c$ may leak into the atmosphere.

The JWKB approximation applies when the eigenfunction varies on a much shorter length scale than the equilibrium stellar structure. In that case, the solution may be written as

$$X(r) = a(r)e^{i\phi(r)}, \quad (5.58)$$

where $a(r)$ is a slowly varying amplitude and

$$k_r = \frac{d\phi}{dr} \quad (5.59)$$

is the local radial wavenumber. Applying this approximation to [Equation 5.54](#) gives, to leading order,

$$k_r^2 \simeq K(r). \quad (5.60)$$

Thus, inside a propagation cavity bounded by two turning points r_1 and r_2 , the phase accumulated by the wave must satisfy the quantisation condition

$$\int_{r_1}^{r_2} K^{1/2}(r) dr = \left(n - \frac{1}{2} \right) \pi, \quad (5.61)$$

where n is the radial order of the mode. This condition is the stellar-oscillation analogue of a Bohr–Sommerfeld quantisation rule: only those frequencies for which an integer number of half-wavelengths fits inside the propagation cavity correspond to eigenmodes.

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ASYMPTOTIC EXPRESSION FOR PRESSURE MODES.

For p modes, for which $\omega^2 \gg N^2$, Equation 5.61 simplifies to

$$\int_{r_i}^R \left(1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}\right)^{1/2} \frac{dr}{c} = \frac{(n + \alpha(\omega))\pi}{\omega}, \quad (5.62)$$

which is known as Duvall's law (Duvall, 1982). Here, r_i is the lower turning point, defined by $\omega = S_\ell$, and R denotes the stellar surface, with the near-surface phase behaviour absorbed into the function $\alpha(\omega)$. For low-degree modes, the lower turning point lies close to the stellar centre, and the high-order p-mode frequencies approximately follow

$$\nu_{n,\ell} = \frac{\omega_{n,\ell}}{2\pi} \approx \left(n + \frac{\ell}{2} + \epsilon\right) \Delta\nu, \quad (5.63)$$

where ϵ is a phase term that depends mainly on the near-surface layers. The large frequency separation is given by

$$\Delta\nu = \left(2 \int_0^R \frac{dr}{c}\right)^{-1} \approx \nu_{n,\ell} - \nu_{n-1,\ell}. \quad (5.64)$$

This asymptotic relation predicts an approximately uniform spacing between consecutive high-order p modes of the same angular degree. A more precise second-order asymptotic formulation (Shibahashi, 1979; Tassoul, 1980) gives

$$\nu_{n,\ell} \simeq \left(n + \frac{\ell}{2} + \epsilon\right) \Delta\nu - [A\ell(\ell+1) - \delta] \frac{\Delta\nu^2}{\nu_{n,\ell}}, \quad (5.65)$$

where ϵ and δ are phase terms that depend mainly on the near-surface layers. The coefficient A is given by

$$A = \frac{1}{4\pi^2 \Delta\nu} \left[\frac{c(R)}{R} - \int_0^R \frac{1}{r} \frac{dc}{dr} dr \right]. \quad (5.66)$$

This term contains information on the sound-speed gradient and is therefore sensitive to the deeper stellar structure, especially the core regions through the small frequency separations. Modern seismic modelling generally relies on the direct numerical computation of oscillation frequencies from stellar models, rather than on the asymptotic formulae alone. Nevertheless, these relations remain essential for interpreting the frequency pattern. In particular, frequency-separation ratios such as those introduced by Roxburgh & Vorontsov (2003) combine large and small separations in a way that reduces sensitivity to near-surface modelling errors, thereby providing more robust constraints on the stellar interior.

ASYMPTOTIC EXPRESSION FOR GRAVITY MODES.

Since gravity modes propagate only in stably stratified regions, their propagation cavity is determined by the regions where

$$\omega^2 < N^2 \quad \text{and} \quad \omega^2 < S_\ell^2.$$

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In convective regions, where $N^2 \leq 0$, gravity modes are evanescent. The boundaries of the g-mode cavity are therefore associated with the turning points where the mode frequency becomes comparable to either the Brunt–Väisälä or Lamb frequency, as well as with convective boundaries and sharp variations in N . In stars with a radiative core and a convective envelope, such as solar-like stars, high-order gravity modes are trapped mainly in the radiative interior. In main-sequence intermediate-mass stars, such as γ Doradus and SPB stars, the situation is reversed: gravity modes propagate mainly in the radiative envelope surrounding the convective core. In the asymptotic limit of high radial order, the periods of gravity modes can be written as (Shibahashi, 1979; Tassoul, 1980)

$$P_{n,\ell} \simeq \frac{\Pi_0}{\sqrt{\ell(\ell+1)}} (n + \epsilon_g), \quad (5.67)$$

where n is the radial order, and ϵ_g is a phase term that depends on the properties of the turning points. The buoyancy radius Π_0 is defined as

$$\Pi_0 = 2\pi^2 \left(\int_{r_1}^{r_2} \frac{N}{r} dr \right)^{-1}, \quad (5.68)$$

where r_1 and r_2 are the inner and outer boundaries of the g-mode propagation cavity. A direct consequence of this expression is that high-order g modes of a given degree ℓ are approximately equally spaced in period:

$$\Delta P_\ell = P_{n+1,\ell} - P_{n,\ell} \simeq \frac{\Pi_0}{\sqrt{\ell(\ell+1)}}. \quad (5.69)$$

Thus, while high-order p modes are approximately equally spaced in frequency, high-order g modes are approximately equally spaced in period. In real stars, however, the period spacing is not perfectly constant. Sharp chemical-composition gradients, especially those produced near the boundary of a receding convective core during main-sequence evolution, introduce periodic deviations from the uniform asymptotic spacing. These deviations provide a powerful seismic diagnostic of the near-core structure, the extent of extra mixing, and the evolutionary state of the star (Miglio et al., 2008).

5.2 NON-ADIABATIC TREATMENT OF THE OSCILLATIONS

In the previous chapter, we derived the linear adiabatic oscillation equations in order to understand the physical behaviour of stellar oscillation modes and their asymptotic properties. In this chapter, we go one step further by including non-adiabatic effects, that is, the exchange of heat and energy between the oscillations and the medium in which they propagate. These effects are essential for determining whether a mode is damped or excited, and therefore whether it can be observed (Dupret, 2001; Grahcène et al., 2005; Dupret et al., 2005; Houdek & Dupret, 2015). Let us return to Subsection 5.1.1. In that section, the oscillations were assumed to be adiabatic, and only the dynamical conservation equations and their perturbations were considered. To this system of equations, we must now add the thermal description of the star, namely the conservation of energy:

$$T \frac{dS}{dt} = \epsilon - \frac{\nabla \cdot \mathbf{F}}{\rho}, \quad (5.70)$$

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where T is the temperature of the fluid element, S its entropy, ϵ the nuclear energy-generation rate per unit mass, and $\mathbf{F}$ the energy flux. In stellar interiors, energy is transported mainly by radiation and convection, so that the total flux can be decomposed as

$$\mathbf{F} = \mathbf{F}_R + \mathbf{F}_C, \quad (5.71)$$

where $\mathbf{F}_R$ is the radiative flux and $\mathbf{F}_C$ is the convective flux. In the diffusion approximation, the radiative flux is written as

$$\mathbf{F}_R = -\frac{4ac_{\text{light}}T^3}{3\kappa\rho}\nabla T, \quad (5.72)$$

where a is the radiation-density constant, c_{light} is the speed of light, and κ is the Rosseland mean opacity. Convection is one of the main sources of uncertainty in stellar modelling, affecting both stellar evolution and stellar oscillations. This difficulty arises because convection is a turbulent, multi-scale, and intrinsically three-dimensional process. A complete treatment would require solving the hydrodynamical equations in 3D over evolutionary timescales, which remains beyond current numerical capabilities for most stellar-evolution applications. In one-dimensional stellar models, convection must therefore be represented by averaged prescriptions. Most commonly, this is done using mixing-length-type formalisms (Cox & Giuli, 1968), which introduce free parameters that set the characteristic length scale of convective motions and thereby control the efficiency of convective heat transport. The associated mixing of chemical elements is then treated through additional assumptions, such as instantaneous homogenisation inside convective regions or diffusive mixing prescriptions. In this work, the stellar models used in this part were computed with the mixing-length theory, in which the mixing-length parameter α_{MLT} controls the characteristic transport length (Böhm-Vitense, 1958; Cox & Giuli, 1968). The treatment of the interaction between convection and oscillations is discussed in Subsection 5.2.1 and Subsection 5.2.2. By perturbing the energy-conservation equation and the radiative-diffusion equation, we obtain

$$i\sigma T\delta S = \epsilon' + \frac{\rho'}{\rho^2}\nabla \cdot \mathbf{F} - \frac{\nabla \cdot \mathbf{F}'}{\rho},$$

and

$$\mathbf{F}'_R = \left(3\frac{T'}{T} - \frac{\kappa'}{\kappa} - \frac{\rho'}{\rho}\right)\mathbf{F}_R - \frac{4acT^3}{3\kappa\rho}\nabla T'. \quad (5.73)$$

At this stage, it is useful to emphasize that, in contrast to the adiabatic case, non-adiabatic oscillation calculations generally yield complex eigenfrequencies and complex eigenfunctions. The real part of the eigenfrequency gives the oscillation frequency, whereas the imaginary part measures the net growth or damping rate of the mode. The sign convention for stability depends on the adopted temporal dependence of the perturbations, such as $\exp(i\omega t)$ or $\exp(-i\omega t)$. Before proceeding further, we introduce several quantities that will be used throughout the following discussion.

$$\delta L = \delta(4\pi r^2 F_r), \quad \delta L_R = \delta(4\pi r^2 F_{R,r}), \quad \delta L_C = \delta(4\pi r^2 F_{C,r}), \quad (5.74)$$

where the subscript r denotes the radial component of the flux. The quantities L , L_R , and L_C denote the total, radiative, and convective luminosities, respectively, while F_r ,

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$F_{R,r}$, and $F_{C,r}$ are the corresponding radial fluxes. Since the total energy flux contains both radiative and convective contributions, one has $L = L_R + L_C$, and therefore $\delta L = \delta L_R + \delta L_C$. Closing the non-adiabatic pulsation equations requires an additional prescription for the response of convection to the oscillation. Two main treatments are commonly adopted: the frozen-convection approximation and time-dependent convection. In the former, the perturbation of the convective flux is neglected or simplified, whereas in the latter it is included consistently in the pulsation equations. These two approaches are described in the following sections.

5.2.1 FROZEN-CONVECTION APPROXIMATION FOR CONVECTION–OSCILLATION INTERACTIONS

The frozen-convection approximation consists in neglecting the response of the convective flux to the pulsation. In the formalism adopted in our code, this is done by neglecting the Lagrangian perturbation of the convective luminosity, δL_C , as well as the Lagrangian perturbation of the transverse components of the convective flux. Different implementations of the frozen-convection approximation are possible; the one described here corresponds to the prescription used in the present work. With these assumptions, the equation of energy conservation takes the form

$$i\sigma T \delta S = -\frac{1}{4\pi r^2 \rho} \frac{\partial \delta L}{\partial r} + \epsilon \left(\frac{\delta \epsilon}{\epsilon} + \frac{\delta \rho}{\rho} \frac{1}{r^2} \frac{\partial(r^2 \xi_r)}{\partial r} \right) + \frac{1}{4\pi \rho r^3} \mathcal{L}^2 \left[L_R \left(\frac{\delta T}{r \, dT/dr} - \frac{\xi_r}{r} \right) - \frac{L_C}{r^2 \sigma^2} \left(\Psi' + \frac{P'}{\rho} \right) \right]. \quad (5.75)$$

And the diffusion equation can be written as

$$\frac{\delta L}{L} = \frac{L_R}{L} \frac{\delta L_R}{L_R} = \frac{L_R}{L} \left(2 \frac{\xi_r}{r} + 3 \frac{\delta T}{T} - \frac{\delta \kappa}{\kappa} - \frac{\delta \rho}{\rho} + \frac{\partial \delta T / \partial r}{dT/dr} - \frac{\partial \xi_r}{\partial r} \right). \quad (5.76)$$

5.2.2 TIME-DEPENDENT TREATMENT OF CONVECTION–OSCILLATION INTERACTIONS

For a time-dependent treatment of the interaction between oscillation and convection, and following the standard Reynolds decomposition used in turbulence studies, each physical quantity is split into a mean component and a fluctuating component. We denote mean quantities with an overline, for example $\bar{\rho}$ for the density, and convective fluctuations with Δ , so that $\rho = \bar{\rho} + \Delta\rho$. The hydrodynamical equations are first averaged to define the equilibrium structure, and the resulting mean equations are then linearly perturbed to derive the oscillation equations including time-dependent convection (Gautschy & Saio, 1995; Grigahcène et al., 2005; Houdek & Dupret, 2015).

Assuming, for simplicity, an isotropic Reynolds-stress tensor, the radial component of the momentum equation may be written as

$$\sigma^2 \xi_r = \frac{d\delta\psi}{dr} + \frac{1}{\rho} \frac{d}{dr} (\delta p_{\text{th}} + \delta p_t), \quad (5.77)$$

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where p_{th} is the sum of the gas and radiation pressures, and p_t is the turbulent pressure. Its equilibrium value is defined by

$$\bar{p}_t = \overline{\rho V_r^2}, \quad (5.78)$$

where V_r is the radial convective velocity. The corresponding perturbation can then be expressed as

$$\frac{\delta p_t}{p_t} = \frac{\delta \rho}{\rho} + 2 \frac{\overline{V_r \delta V_r}}{\overline{V_r^2}}. \quad (5.79)$$

Under the same isotropic approximation, the horizontal component of the momentum equation becomes

$$\sigma^2 r \xi_h = \delta \psi + \frac{1}{\rho} (\delta p_{\text{th}} + \delta p_t). \quad (5.80)$$

The perturbed energy-conservation equation is

$$\begin{aligned} i\sigma T \delta s = & - \frac{d\delta(L_R + L_c)}{dm} + \left(\frac{\delta \epsilon}{\epsilon} + \ell(\ell+1) \frac{\xi_h}{r} \right) \epsilon + \frac{\ell(\ell+1)}{4\pi r^3 \rho} \left[L_R \left(\frac{\delta T}{r dT/dr} - \frac{\xi_r}{r} \right) - L_c \frac{\xi_h}{r} \right] \\ & + \frac{\ell(\ell+1)}{r\rho} \delta F_{c,h} + \delta \epsilon_t, \end{aligned} \quad (5.81)$$

where ϵ_t is the rate at which turbulent kinetic energy is dissipated into heat, and $F_{c,h}$ is the horizontal component of the convective flux. Finally, the perturbation of the convective flux $\mathbf{F}_c$ can be written as

$$\delta \mathbf{F}_c = \mathbf{F}_c \left(\frac{\delta \rho}{\rho} + \frac{\delta T}{T} \right) + \bar{\rho} \bar{T} (\delta \Delta s \bar{\mathbf{V}} + \overline{\Delta s \delta \mathbf{V}}), \quad (5.82)$$

where $\mathbf{V}$ is the convective velocity. A full derivation of the equations used in the anisotropic case is given in the appendices of [Grigahcène et al. \(2005\)](#).

In practice, the main difficulty in time-dependent mixing-length treatments of the convection–oscillation interaction lies in evaluating the various perturbation terms, for example, $\delta \Delta s \bar{\mathbf{V}}$, $\overline{\Delta s \delta \mathbf{V}}$, and $\overline{V_r \delta V_r / V_r^2}$, that enter the perturbed convective-flux and momentum equations, and that ultimately determine the driving or damping of the modes. Different time-dependent convection formalisms mainly differ in the way these terms are modelled ([Houdek & Dupret, 2015](#)). In one-dimensional stellar pulsation studies, three main families of models are commonly discussed. A first family descends from the formalism introduced by [Unno \(1967, 1977\)](#); [Unno et al. \(1989\)](#) and later generalized by [Gabriel \(1996\)](#), and is the basis of the treatment presented by [Grigahcène et al. \(2005\)](#). A second family is based on the formulation introduced by [Gough \(1977\)](#) and extended to include a non-local treatment of convection consistent with stellar models ([Balmforth, 1992a](#); [Houdek et al., 1995, 1999](#); [Houdek, 2000](#)). This formalism has been used extensively in the study of both solar-like and classical pulsators ([Balmforth, 1992a](#); [Houdek et al., 1995, 1999](#); [Houdek, 2000](#)). A third family is based on more complex non-local and anisotropic convection formalisms, such as those developed by [Xiong \(1989\)](#); [Xiong et al. \(1997, 2015, 2016\)](#), based on [Spiegel \(1963\)](#), which have also been applied successfully to the instability-strip boundaries of

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δ -Scuti and related pulsators. The present section is limited to the basic principles of these models, a more detailed comparison and discussion of the different formalisms can be found in [Houdek & Dupret \(2015\)](#).

From a practical point of view, the non-adiabatic oscillation calculations presented in this work were carried out with the MAD code ([Dupret, 2001](#); [Dupret et al., 2002](#)), which implements the time-dependent convection formalism of [Grigahcène et al. \(2005\)](#). Other non-adiabatic pulsation tools exist, including codes based on the formalism developed by [Houdek et al. \(1995, 1999\)](#); [Houdek \(2000\)](#), but these can only be applied to radial modes. In addition, the formalism developed by [Xiong \(1989\)](#); [Xiong et al. \(1997, 2015, 2016\)](#) is associated with codes that are not publicly available. In practice, for the present study of gravity-mode pulsators with a time-dependent convection treatment, MAD is therefore the most appropriate tool as it also includes the effect of the rotation through the tradition approximation ([Bouabid et al., 2013](#)), see [Section 5.6](#). This makes γ -Doradus stars a natural target for this part of the thesis, not only because their excitation requires such a treatment, but also because they benefit from the extensive observational constraints provided by *Kepler*.

5.2.3 BOUNDARY CONDITIONS

At this stage, the system contains more unknowns than independent equations. The closure is provided by the equation of state and its perturbation, which relate the various perturbed thermodynamic quantities. By perturbing the equation of state, one can express some of these quantities in terms of the others and reintroduce them into the system, so that the problem ultimately reduces to a set of five differential equations for five unknowns.

These differential equations must be supplemented by boundary conditions. At the centre, the boundary conditions are obtained by imposing the regularity of the solution, that is, by requiring the perturbed quantities and their derivatives to remain finite. In practice, these conditions are derived by performing a Taylor expansion about the centre and retaining only the leading-order terms. Since I did not work specifically on this aspect, I do not discuss the derivation in more detail here, I recommend reading the thesis of Marc-Antoine for the detailed central boundary conditions used.

For non-radial modes, regularity at the centre implies the following asymptotic behaviours:

$$\xi_r \propto r^{\ell-1}, \quad \delta\psi \propto r^{\ell}, \quad \frac{\delta P}{P} \propto r^{\ell}, \quad \frac{\delta S}{c_v} \propto r^{\ell}, \quad \frac{\delta L}{L} \propto r^{\ell}.$$

At the stellar surface, three additional boundary conditions must be imposed. The first one considered here concerns the Poisson equation.

POISSON EQUATION

The first boundary condition to impose concern the perturbation of the potential, with this surface boundary condition, we want to impose a first order continuous match between the inner solution of the Poisson equation and the outer solution of the Laplace

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equation at the surface of the star. I will detail the derivation of this particular boundary condition as it will be necessary later in the manuscript, this demonstration is extracted from the lessons of Richard Scufflaire. In the vacuum the solution of the Laplace equation tending towards 0 at infinity takes the form

$$\Psi_{\ell,\text{out}}^m(r) = \Psi_{\ell}^m(R) \left(\frac{R}{r} \right)^{\ell+1}, \quad (5.83)$$

where $\Psi_{\ell}^m(R)$ is the component of the gravitational potential at the surface of stars. For what follows the dependency on ℓ and m is implicit. The first derivative with respect to the radial coordinate of the solution of the Laplace equation in the vacuum is

$$\frac{d\Psi_{\text{out}}}{dr} = -\frac{\ell+1}{r} \Psi_{\text{out}}. \quad (5.84)$$

The continuity of the Lagrangian perturbation of the gravitational potential and its derivative can be expressed as

$$\delta\Psi = \delta\Psi_{\text{out}}, \quad (5.85)$$

and,

$$\delta \frac{d\Psi}{dr} = \delta \frac{d\Psi_{\text{out}}}{dr}. \quad (5.86)$$

Those equations can be rewritten in the Eulerian formalism as

$$\Psi' + \delta r \frac{d\Psi}{dr} = \Psi'_{\text{out}} + \delta r \frac{d\Psi_{\text{out}}}{dr}, \quad (5.87)$$

and,

$$\frac{d\Psi'}{dr} + \delta r \frac{d^2\Psi}{dr^2} = \frac{d\Psi'_{\text{out}}}{dr} + \delta r \frac{d^2\Psi_{\text{out}}}{dr^2}. \quad (5.88)$$

The first derivative of the potentials are continuous, mathematically:

$$\frac{d\Psi}{dr} = \frac{d\Psi_{\text{out}}}{dr}. \quad (5.89)$$

Concerning the second derivative of the gravitational potential, we only have the continuity if the density is null, at the exterior of the star we have,

$$\frac{d^2\Psi_{\text{out}}}{dr^2} + \frac{2}{r} \frac{d\Psi_{\text{out}}}{dr} = 0, \quad (5.90)$$

and in the interior we have:

$$\frac{d^2\Psi}{dr^2} + \frac{2}{r} \frac{d\Psi}{dr} = 4\pi G\rho. \quad (5.91)$$

By subtracting the last two equations, we obtain:

$$\frac{d^2\Psi_{\text{out}}}{dr^2} - \frac{d^2\Psi}{dr^2} = -4\pi G\rho. \quad (5.92)$$

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By inserting this equality in [Equation 5.88](#), and using the expression of the radial derivative of the outer potential we obtain:

$$\frac{d\Psi'}{dr} + \frac{\ell + 1}{r}\Psi' = -4\pi G\rho\delta r. \quad (5.93)$$

which is the boundary condition for the Poisson equation. Usually the right hand side of [Equation 5.93](#) is neglected as the surface density is significantly smaller than in the rest of the star.

MECHANICAL BOUNDARY CONDITION

The mechanical boundary condition is usually obtained by imposing the total reflection of pressure modes at the stellar surface. Assuming that

$$\frac{P}{\rho g r} \rightarrow 0 \quad (5.94)$$

at the surface, or that

$$\frac{d}{dr}\left(\frac{\delta P}{P}\right) = 0, \quad (5.95)$$

one obtains the following mechanical boundary condition:

$$\sigma^2 \xi_r = \frac{d\Psi'}{dr} + \frac{d(g\xi_r)}{dr} + g\left(\frac{\delta\rho}{\rho} - \frac{\delta P}{P}\right). \quad (5.96)$$

THERMAL BOUNDARY CONDITION

The thermal boundary condition is usually written as

$$\frac{\delta L}{L} = 2\frac{\xi_r}{r} + 4\frac{\delta T}{T}. \quad (5.97)$$

This expression corresponds to the perturbation of the Stefan–Boltzmann relation, assuming that the photospheric luminosity perturbation is mainly determined by the variations of the stellar radius and effective temperature. It should be noted, however, that these simple boundary conditions are not used exactly in our non-adiabatic oscillation calculations. This is due to the specific treatment of the atmosphere in the oscillation code, as well as to the limitations and simplifying assumptions involved in the boundary conditions written above. During this thesis, I did not specifically work on the non-adiabatic treatment of the atmosphere. I therefore do not discuss these aspects in further detail here. For a complete description of the physical assumptions and numerical implementation, I refer the reader to the thesis of M.-A. Dupret and to the non-adiabatic treatment described in [Dupret \(2001\)](#); [Dupret et al. \(2002\)](#).

5.3 ADIABATIC AND NON-ADIABATIC OSCILLATION CODES

Throughout this thesis, I worked with the tools developed within the ASTA group. For stellar oscillations in the adiabatic approximation, I relied on LOSC ([Scuflaire et al.](#),

5.3. ADIABATIC AND NON-ADIABATIC OSCILLATION CODES

2008a), whereas for the non-adiabatic treatment I used MAD Dupret (2001); Dupret et al. (2002). My own work did not focus primarily on LOSC itself. Although I implemented the effect of rotation through the traditional approximation of rotation in the most recent version of LOSC in order to obtain a reliable version of the code including this approximation, the calculations presented in this thesis were mostly carried out with older versions, which were computationally more efficient and already included the traditional approximation. Accordingly, the present section is mainly devoted to MAD and to the specific non-adiabatic treatment implemented in this code. It should be viewed as an updated description of the current state of MAD following the extensive revisions of its numerical treatment over the past two decades. The improvements I made to the treatment of rotation, as well as the new version of MAD developed for tidally excited oscillations, are presented in Section 5.6 and Chapter 6. In this section, however, I restrict the discussion to the non-rotating case.

In practice, MAD is composed of three separate components: `intnad`, `LOSC`, and `nonad`. The role of `intnad` is to take a stellar model computed with CLES and improve the precision of its structure through interpolation. In addition, `intnad` computes several quantities that are required exclusively for the non-adiabatic treatment. Finally, `intnad` is responsible for constructing the staggered grid used in MAD. For the non-adiabatic computations, the code relies on two interlaced meshes following the approaches of Castor (1971) and Pesnell (1990), which provide both high numerical accuracy and good stability. I return to the staggered-grid formalism below when discussing the discretisation currently used in the code. At this stage, the role of `intnad` is to interpolate each relevant quantity accurately onto the points of the two interlaced meshes.

A subset of these refined structures is then passed to `LOSC`, whose role is to compute the adiabatic frequencies of the oscillation modes within a prescribed frequency range. Using the resulting list of adiabatic frequencies and the associated mode identifications, the final component of the code, `nonad`, is then used. The purpose of this program is to combine the outputs of the two previous components in order to compute the non-adiabatic corrections for each mode identified by `LOSC`, and thus determine the complex eigenfrequencies together with quantities such as the work integral for each oscillation mode.

At the beginning of my work on MAD, a substantial part of the effort was devoted to modernizing the code so that it could be compiled with modern compilers and made portable enough to ensure its long-term usability by different users. I first worked on the `intnad` component. I adopted a version developed by Mélanie Godart around 2012–2013, in which `intnad` had already been modified to handle more modern stellar structures and to interpolate directly the relevant quantities from the stellar models. Starting from that version, I still had to adapt it to newer versions of the stellar structure code and to the type of stars considered here, and perform a large-scale verification of the interpolated structures.

Once this work had been completed, I turned to the `nonad` component of the code. The version of `nonad` used as a starting point was the version developed by Mehdi

5.3. ADIABATIC AND NON-ADIABATIC OSCILLATION CODES

Bouabid in 2014, in which the traditional approximation had already been implemented to include the effects of rotation. My work focused mainly on the internal structure of the code, with the goal of making it easier to modify, understand, and extend toward the computation of tidally excited oscillations. In this section, I discuss in more detail two of the main technical aspects involved in these developments. The first is the principle of the staggered grid, which must be introduced before presenting the central part of this section, namely the equations that are solved, their discretisation, and the associated numerical algorithm. These elements are also necessary for understanding the modifications introduced later for the computation of tidally excited oscillations in [Chapter 6](#).

5.3.1 STAGGER GRID

As presented previously, MAD uses two interlaced meshes, following the approaches of [Castor \(1971\)](#) and [Pesnell \(1990\)](#), for all numerical computations. The arrangement of the mesh points on the two grids used in MAD is illustrated in [Figure 5.1](#).

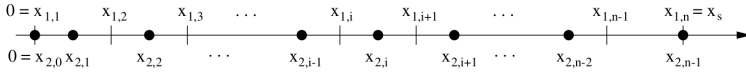


Figure 5.1: Illustration of the discretisation based on the two interlaced meshes used in MAD, as a function of x ($x = r/R$), for a model with n grid points. This figure is extracted from the thesis of Marc-Antoine Dupret.

The main advantage of using these two interlaced grids is the improved numerical stability when estimating derivatives with respect to x . Let us define a function f discretised on grid 1, which contains n points, as

$$(f_{1,1}, f_{1,2}, \dots, f_{1,n-1}, f_{1,n}) = (f(x_{1,1}), f(x_{1,2}), \dots, f(x_{1,n-1}), f(x_{1,n})). \quad (5.98)$$

Similarly, we define a function g discretised on grid 2, also with n points, such that

$$(g_{2,0}, g_{2,1}, \dots, g_{2,n-2}, g_{2,n-1}) = (g(x_{2,0}), g(x_{2,1}), \dots, g(x_{2,n-2}), g(x_{2,n-1})). \quad (5.99)$$

With this discretisation scheme, the derivative of a quantity defined on one grid is naturally evaluated at the points of the other grid using centred finite differences. For a function defined on grid 1, the derivative evaluated at a point of grid 2 is given by

$$\left. \frac{df}{dx} \right|_{x_{2,i}} = \frac{f(x_{1,i+1}) - f(x_{1,i})}{x_{1,i+1} - x_{1,i}} = \frac{f(x_{1,i+1}) - f(x_{1,i})}{d_{1,i}}. \quad (5.100)$$

Similarly, for a function defined on grid 2, the derivative evaluated at a point of grid 1 is written as

$$\left. \frac{dg}{dx} \right|_{x_{1,i}} = \frac{g(x_{2,i}) - g(x_{2,i-1})}{x_{2,i} - x_{2,i-1}} = \frac{g(x_{2,i}) - g(x_{2,i-1})}{d_{2,i}}. \quad (5.101)$$

This staggered-grid formulation makes it possible to compute derivatives with centred differences while maintaining good numerical stability.

5.3.2 EQUATIONS USED IN MAD

When solving the oscillation equations numerically, it is convenient to introduce a set of dimensionless variables. In MAD, the dimensionless radial coordinate, enclosed mass, angular frequency, and density are defined as

$$x = \frac{r}{R}, \quad q = \frac{m}{M}, \quad \omega^2 = \frac{R^3}{GM} \sigma^2, \quad \bar{\rho} = \frac{4\pi R^3}{M} \rho. \quad (5.102)$$

The dimensionless perturbation variables are then written as

$$\zeta = \frac{1}{x^{\ell-1}} \frac{\xi_r}{R}, \quad \phi = \frac{R}{GMx^\ell} \Psi', \quad \beta = \frac{4\pi R^4}{GM^2 x^\ell} \delta P, \quad \gamma = \frac{1}{x^\ell} \frac{\delta \rho}{\rho}, \quad (5.103)$$

and

$$\eta = \frac{1}{x^\ell} \frac{\delta S}{c_v}, \quad \vartheta = \frac{1}{x^\ell} \frac{\delta T}{T}, \quad \zeta_h = \frac{1}{x^{\ell-1}} \frac{\xi_h}{R}, \quad \delta \epsilon_x = \frac{1}{x^\ell} \frac{\delta \epsilon}{\epsilon}. \quad (5.104)$$

These variables are chosen so that the central behaviour of the eigenfunctions is regular and so that the discretised equations remain numerically well conditioned (Dupret, 2001). In MAD, the stellar model is divided into three distinct regions: the central region, the main stellar interior, and the atmosphere. The equations and discretisation schemes used in these regions are not identical; a detailed description is given in the thesis of M.-A. Dupret and in Dupret (2001). In this section, we focus only on the discretisation used in the main part of the star, since this is the region where the modifications introduced in this work are applied. Moreover, in the most recent version of MAD, the central region is reduced as much as possible, so that the main-interior discretisation plays the dominant role in the numerical treatment.

With the dimensionless formalism, the radial motion equation takes the form:

$$\omega^2 \zeta_i = \frac{x_{2,i}}{\bar{\rho}_{1,i} d_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\ell-1} \beta_i - \frac{x_{2,i-1}}{\bar{\rho}_{1,i} d_{2,i}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\ell-1} \beta_{i-1} + \frac{x_{2,i}}{d_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\ell-1} \varphi_i \quad (5.105)$$

$$- \frac{x_{2,i-1}}{d_{2,i}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\ell-1} \varphi_{i-1} + \frac{d_{1,i-1} \bar{\rho}_{2,i-1} q_{2,i-1}}{2d_{2,i} \bar{\rho}_{1,i} x_{2,i-1}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\ell-1} \gamma_{i-1} + \frac{d_{1,i} \bar{\rho}_{2,i} q_{2,i}}{2d_{2,i} \bar{\rho}_{1,i} x_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\ell-1} \gamma_i \quad (5.106)$$

$$+ \frac{q_{2,i} x_{1,i+1}^2}{2d_{2,i} x_{2,i}^4} \left(\frac{x_{1,i+1}}{x_{1,i}} \right)^{\ell-1} \zeta_{i+1} + \frac{x_{1,i+1}^2}{2d_{2,i}} \left(\frac{q_{2,i}}{x_{2,i}^4} - \frac{q_{2,i-1}}{x_{2,i-1}^4} \right) \zeta_i - \frac{q_{2,i-1} x_{1,i-1}^2}{2d_{2,i} x_{2,i-1}^4} \left(\frac{x_{1,i-1}}{x_{1,i}} \right)^{\ell-1} \zeta_{i-1},$$

the horizontal motion + mass conservation equation is written as

$$\omega^2 \left[x_{2,i}^2 \gamma_i + x_{1,i+1} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^\ell \frac{\zeta_i}{d_{1,i}} - x_{1,i} \left(\frac{x_{1,i}}{x_{2,i}} \right)^\ell \frac{\zeta_i}{d_{1,i}} \right] \quad (5.107)$$

$$= \ell(\ell+1) \left[\varphi_i + \frac{\beta_i}{\rho_{2,i}} + \frac{q_{2,i}}{2x_{2,i}^3} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{\ell+1} \zeta_{i+1} + \frac{q_{2,i}}{2x_{2,i}^3} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{\ell+1} \zeta_i \right], \quad (5.108)$$

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and the Poisson expression is expressed as

$$\begin{aligned} & \frac{1}{d_{2,i+1}d_{1,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{2\ell+2} \varphi_{i+1} - \left[\frac{1}{d_{2,i+1}d_{1,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{2\ell+2} + \frac{1}{d_{2,i}d_{1,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{2\ell+2} \right] \varphi_i \\ & + \frac{1}{d_{2,i}d_{1,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{2\ell+2} \varphi_{i-1} = \bar{\rho}_{2,i} \gamma_i - \frac{\bar{\rho}_{1,i+1} - \bar{\rho}_{1,i}}{2d_{1,i}x_{2,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{\ell+1} \zeta_{i+1} - \frac{\bar{\rho}_{1,i+1} - \bar{\rho}_{1,i}}{2d_{1,i}x_{2,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{\ell+1} \zeta_i. \end{aligned} \quad (5.109)$$

5.3.3 SOLVING THE DISCRETISED SET OF DIFFERENTIAL EQUATIONS

To solve the system of differential equations, we need to cast it into the canonical matrix form:

$$(\mathbf{A} - \omega \mathbf{B})\mathbf{x} = 0, \quad (5.110)$$

where $\mathbf{A}$ and $\mathbf{B}$ are complex square matrices. To achieve this, we rewrite the system by introducing auxiliary variables in order to explicitly isolate the variables that are multiplied by ω . For the radial displacement equation, we define the auxiliary variable

$$\xi_i = \omega \zeta_i, \quad (5.111)$$

such that the radial displacement equation can be rewritten in the interior of the star as:

$$\omega \xi_i = G_{11,i} \gamma_{i-1} + G_{12,i} \gamma_i + G_{21,i} \beta_{i-1} + G_{22,i} \beta_i + G_{31,i} \varphi_{i-1} + G_{32,i} \varphi_i + G_{41,i} \zeta_{i-1} + G_{42,i} \zeta_i + G_{43,i} \zeta_{i+1}, \quad (5.112)$$

where the coefficients $G_{jk,i}$ can be identified from [Equation 5.105](#). Similarly, for the mass conservation and horizontal motion equations, we introduce the auxiliary variable χ_i :

$$\chi_i = \omega (M_{1,i} \gamma_i + M_{41,i} \zeta_i + M_{42,i} \zeta_{i+1}), \quad (5.113)$$

such that:

$$\omega \chi_i = \omega C_{12,i} \gamma_i + \omega C_{22,i} \theta_i + \omega C_{32,i} \varphi_i + \omega C_{41,i} \varphi_i + \omega C_{42,i} \varphi_{i+1}. \quad (5.114)$$

For the Poisson equation, it is not necessary to introduce an auxiliary variable, and it can simply be written as:

$$R_{1,i} \gamma_i + R_{41,i} \zeta_i + R_{42,i} \zeta_{i+1} + R_{31,i} \varphi_{i-1} + R_{32,i} \varphi_i + R_{33,i} \varphi_{i+1} = 0. \quad (5.115)$$

In each of these equations, the expressions for the various coefficients can be obtained by comparison with the discretised equations presented in the previous section. Although we do not detail the discretisation of the thermal equations here, the same procedure applies to those as well. Now that the auxiliary variables have been introduced, the system of differential equations can be rewritten in its canonical form:

$$(\mathbf{A} - \omega \mathbf{B})\mathbf{x} = 0, \quad (5.116)$$

where $\mathbf{A}$ and $\mathbf{B}$ are complex, square band matrices (except for the last column), of dimension $7n + 4$. The vectors $\mathbf{x}$ and ω represent, respectively, the eigenvector and the eigenvalue (or the complex frequency) associated with each oscillation mode.

5.4. DRIVING AND DAMPING MECHANISMS IN INTERMEDIATE MASS STARS

In this thesis, we do not describe the structure of the complex matrices $\mathbf{A}$ and $\mathbf{B}$ in detail; their full description can be found in the thesis of Marc-Antoine Dupret. The matrix form in Equation 5.116 resembles a classical eigenvalue problem, with the only distinction being that $\mathbf{B}$ is not the identity matrix.

To solve such eigenvalue problems, the inverse iteration method proves to be particularly efficient. This method has been adapted by Marc-Antoine Dupret to handle cases where $\mathbf{B}$ is not the identity. For a detailed explanation of the algorithm, see Dupret (2001). The only remark we wish to add here is that each iteration of the inverse method requires solving the linear system in Equation 5.116. To this end, MAD uses LU factorisation with a custom pivoting strategy, as described in Dupret (2001), in order to avoid introducing significant numerical inaccuracies.

5.4 DRIVING AND DAMPING MECHANISMS IN INTERMEDIATE MASS STARS

One of the main applications of non-adiabatic oscillation codes is to quantify the exchange between the stellar thermal reservoir and the energy of the oscillation modes, and thus to identify the regions that drive or damp the modes (Cox, 1974; Gautschy & Saio, 1995). In the convention adopted here, a mode that is globally damped in the stellar interior is said to be vibrationally stable, or simply stable, and is characterized by a positive imaginary part of the eigenfrequency, $\text{Im}(\sigma) > 0$; in that case, $\text{Im}(\sigma)$ is the damping rate. Conversely, a mode that is globally driven is said to be vibrationally unstable, or simply unstable, and satisfies $\text{Im}(\sigma) < 0$; the quantity $-\text{Im}(\sigma)$ is then the growth rate (Gautschy & Saio, 1995; Dupret et al., 2005).

From a theoretical point of view, several driving mechanisms operate in different classes of pulsating stars. For many self-excited pulsators, the instability can be interpreted as a heat-engine process, in which part of the stellar heat is converted into oscillation work through a thermodynamic cycle (Cox, 1974; Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Gautschy & Saio, 1995). This framework includes, for example, opacity-driven pulsations and mechanisms associated with the modulation of convective flux. For the discussion below, it is useful to introduce the total energy of an oscillation mode,

$$\mathcal{E} = \frac{1}{2} \sigma_{\text{R}}^2 I, \quad (5.117)$$

where σ_{R} denotes the real part of the eigenfrequency and I is the mode inertia, defined as

$$I = 4\pi \int_0^R \left(|\xi_r|^2 + \ell(\ell+1) |\xi_h|^2 \right) \rho r^2 dr. \quad (5.118)$$

Here, ξ_r and ξ_h are the radial and horizontal components of the displacement eigenfunction, respectively. To localize the regions that contribute to mode driving and damping, one also introduces the work integral,

$$W(r) = -\frac{4\pi}{\text{Re}(\sigma)I} \int_0^x \text{Im} \left[\frac{\delta\rho}{\rho} T \delta S \right] (\Gamma_3 - 1) r^2 \rho dr, \quad (5.119)$$

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where the chosen normalization implies $W(R) = \text{Im}(\sigma)$, and Γ_3 is the third adiabatic exponent. With this convention, the work integral measures the cumulative exchange of energy between the oscillation mode and the surrounding stellar medium: a decreasing $W(x)$ indicates local damping, whereas an increasing $W(x)$ indicates local driving (Balmforth, 1992b; Gautschy & Saio, 1995; Grigahcène et al., 2005). When non-adiabatic effects are included, the star can be divided into three broad regions according to the relation between the local thermal timescale and the pulsation period (Cox, 1974; Balmforth, 1992b; Gautschy & Saio, 1995). Deep layers are quasi-adiabatic, with $\delta S/c_v \simeq 0$, and therefore contribute little to the net driving or damping. In the outermost layers, the thermal content is too small for these regions to store or release a significant amount of heat over one pulsation cycle, so their contribution to the total work is also weak (Balmforth, 1992b; Goldreich & Wu, 1999). The dominant contribution generally arises in the transition region, where the local thermal relaxation time is of the same order as the pulsation period. This is where classical heat-engine driving mechanisms are expected to operate (Guzik et al., 2000; Dupret et al., 2004, 2005). For the stars considered in this work, we therefore focus on two mechanisms: the κ mechanism, often denoted the κ - γ mechanism when the role of the adiabatic exponent is emphasized, and convective-flux blocking (Cox, 1974; Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Guzik et al., 2000; Dupret et al., 2004, 2005).

5.4.1 OBSERVED AND THEORETICAL INSTABILITY STRIPS

Recent advances in space-based photometry have considerably expanded our view of stellar pulsations and have revealed the rich variety of pulsators observed in nature (Gautschy & Saio, 1995; Kaye et al., 1999; Aerts, 2021). In many stars, pulsations are self-excited: internal non-adiabatic processes inject energy into a subset of eigenmodes and allow them to reach observable amplitudes, provided that the driving overcomes the various damping processes acting in the stellar interior and envelope (Cox, 1974; Gautschy & Saio, 1995; Dupret et al., 2005). In binary systems, oscillations may also be forced by the tidal potential of the companion, but this will be discussed in a later chapter. In the present section, we focus on the mechanisms responsible for the self-excitation of stellar oscillations, because the same non-adiabatic properties that govern intrinsic mode stability also influence the response of stars to tidal forcing. Figure 5.2 illustrates the zoo of observed stellar pulsators across the H-R diagram. Different classes of pulsating stars exhibit distinct oscillation properties and are associated with different driving mechanisms. In this thesis, we focus on intermediate-mass main-sequence stars, both because they are the primary targets of our non-adiabatic calculations and because our treatment of rotation is best suited to stars without deep convective envelopes. In Figure 5.2, the main sequence is indicated by the dashed curve; the remainder of this section therefore concentrates on the main classes of pulsators found along this evolutionary phase.

Intermediate-mass main-sequence stars, roughly in the range $1.3 \lesssim M/M_\odot \lesssim 8$, correspond to F-, A-, and B-type stars in the H-R diagram. They are characterized by convective cores surrounded by extended radiative envelopes (Gautschy & Saio, 1995; Aerts, 2021; Bowman, 2020). In the cooler part of this mass range, namely among A-

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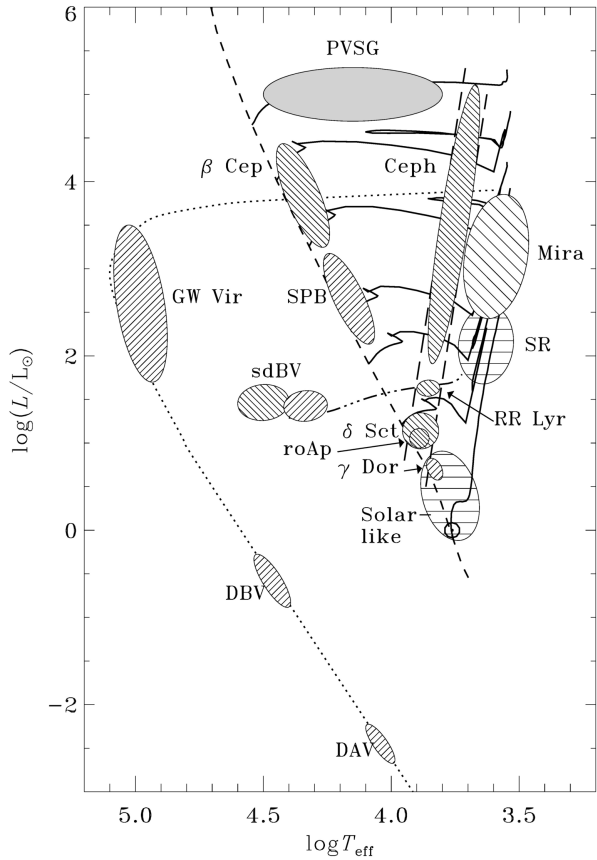


Figure 5.2: Schematic overview of the main classes of pulsating stars across the H–R diagram. The dashed curve indicates the main sequence. In this work, we focus on the main-sequence intermediate-mass regime. This figure is extracted from [Aerts et al. \(2010\)](#).

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and F-type stars, shallow surface convection zones are also present. This global structure makes intermediate-mass stars prime targets for asteroseismology, because their self-excited oscillations provide direct constraints on near-core mixing, internal rotation, and envelope structure (Kaye et al., 1999; Guzik et al., 2000; Aerts, 2021).

The instability regions of intermediate-mass main-sequence stars are traditionally organized into several classical pulsator classes, each associated with a specific type of excited mode and with one or more driving mechanisms (Gautschy & Saio, 1995; Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Guzik et al., 2000; Miglio et al., 2007). In the lower-mass part of this regime, γ -Doradus stars pulsate predominantly in high-order gravity modes with periods of the order of one day (Kaye et al., 1999; Guzik et al., 2000; Dupret et al., 2005). These modes probe the deep radiative interior and are especially sensitive to the structure of the near-core region. Their excitation is usually attributed to the convective flux-blocking mechanism operating near the base of the shallow convective envelope (Guzik et al., 2000; Grigahcène et al., 2005; Dupret et al., 2005). Efficient driving requires the base of the convective envelope to lie within a restricted temperature range, which naturally confines the γ -Doradus instability strip to a limited interval in effective temperature, near early-F spectral types (Guzik et al., 2000; Dupret et al., 2005). The corresponding theoretical instability strip is illustrated in Figure 5.3.

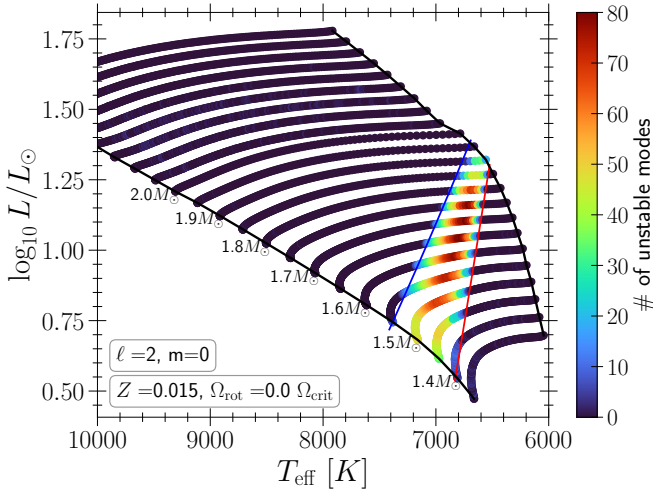


Figure 5.3: Theoretical instability strip for $\ell = 2$ - γ Doradus stars. The color scale indicates the number of unstable modes predicted by our models. The blue and red curves correspond to the theoretical blue and red edges of the instability strip obtained in this work.

At slightly higher effective temperatures, the classical instability strip contains the δ -Scuti stars, which are typically late-A to early-F stars on or close to the main sequence. Their pulsation spectra are dominated by low-order pressure modes and, in

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some cases, by mixed p/g modes (Gautschy & Saio, 1995; Dupret et al., 2004; Aerts, 2021). In these stars, the dominant driving mechanism is the κ -mechanism acting in the second helium ionization zone, while the interaction between pulsation and convection becomes important close to the cool edge of the instability strip (Dupret et al., 2004; Grigahcène et al., 2005; Dupret et al., 2005). A theoretical description of the κ -mechanism is given in Subsection 5.4.2. The δ -Scuti instability strip is shown in Figure 5.4.

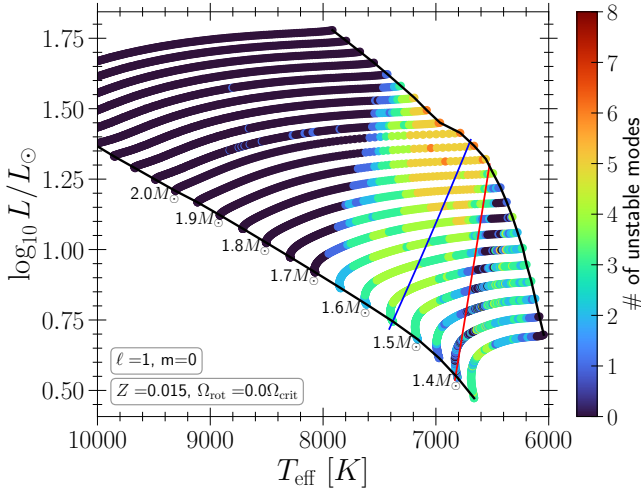


Figure 5.4: Theoretical instability strip for $\ell = 1$ δ -Scuti stars. The color scale indicates the number of unstable modes predicted by our models. The blue and red curves show, for comparison, the theoretical blue and red edges of the γ -Doradus instability strip.

Because the γ -Doradus and δ -Scuti instability domains partially overlap, some stars simultaneously exhibit high-order g modes and low-order p modes. These hybrid pulsators are particularly valuable because they probe both the near-core layers and the stellar envelope within the same object (Kaye et al., 1999; Dupret et al., 2004; Grigahcène et al., 2010; Aerts, 2021). Space photometry has shown that such hybrid behaviour is common in the overlap region and is not limited to a few exceptional objects (Grigahcène et al., 2010; Uytterhoeven et al., 2011).

At higher effective temperatures, mid- and early-B main-sequence stars in the upper part of the intermediate-mass range develop pulsational instabilities associated with the iron-group opacity bump, usually called the Z-bump, located at temperatures of order 2×10^5 K in their radiative envelopes (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007). This opacity enhancement enables κ -mechanism driving and gives rise to two neighbouring instability domains. Slowly Pulsating B (SPB) stars preferentially excite high-order g modes, whereas the hotter β -Cephei stars preferentially excite low-order p modes and

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mixed modes (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007; Bowman, 2020). Although SPB and β -Cephei stars are often discussed as distinct observational classes, they are physically related, since both owe their instability to the same opacity-driven mechanism associated with the iron-group bump (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007). The SPB instability strip, computed here for a solar-like metallicity, is shown in Figure 5.5.

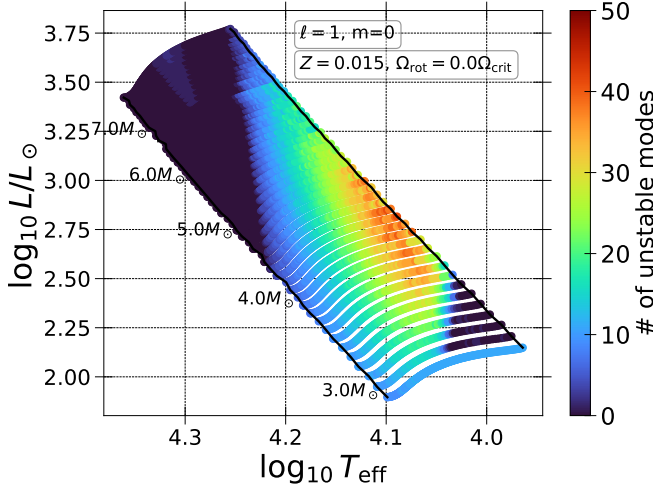


Figure 5.5: Theoretical instability strip for $\ell = 1$ SPB stars. The color scale indicates the number of unstable modes predicted by our models. This instability strip was computed for a solar-like metallicity.

5.4.2 κ -MECHANISM IN SPB AND δ -SCUTI STARS

The κ -mechanism is a thermodynamic driving mechanism associated with opacity bumps produced by different elements at different temperatures. It is responsible for the excitation of oscillation modes in many classes of classical pulsators (Cox, 1974; Gautschy & Saio, 1995; Aerts, 2021). Its basic principle is that, in layers where the opacity increases during compression, the radiative flux is temporarily blocked, so that part of the thermal energy stored in the layer is converted into mechanical work over a pulsation cycle (Cox, 1974; Gautschy & Saio, 1995). In partial-ionisation zones, variations of the adiabatic exponents may further reinforce this effect; in that case one often refers to the κ - γ mechanism rather than to the pure κ -mechanism (Cox, 1974; Gautschy & Saio, 1995).

The pulsation cycle may be idealised as the succession of a hot, compressed phase and a cool, expanded phase. The thermodynamic cycle, illustrated in Figure 5.6, proceeds as follows. During the hot phase, compression increases the local temperature, which in turn increases the opacity around the top of the driving layer. The radiative

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flux that enters from bottom cannot go out, heat is trapped, causing heat to accumulate and favouring subsequent expansion. During the cool phase, the opacity decreases around the top of the driving layer and increases at the bottom. The radiative flux no longer enters from bottom but can go out, heat is released from this zone, and the layer contracts again. If the gain in mechanical work during compression exceeds the losses over the full cycle, the mode is driven (Cox, 1974; Gautschy & Saio, 1995; Aerts, 2021).

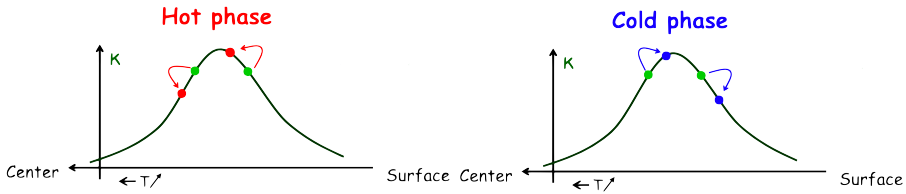


Figure 5.6: Schematic representation of the thermodynamic cycle associated with the κ -mechanism. This figure is extracted from the course of Stellar stability and asteroseismology of Marc-Antoine Dupret

From a mathematical point of view, the origin of the driving can be understood from the perturbed radiative diffusion equation (see Equation 5.76). In practice, the κ -mechanism is efficient when the opacity perturbation varies strongly enough across the transition region to dominate the other contributions to the luminosity perturbation. The opacity perturbation may be written as

$$\frac{\delta\kappa}{\kappa} = \kappa_{\text{PS}} \frac{\delta P}{P} = \frac{(\Gamma_3 - 1)\kappa_T + \kappa_\rho}{\Gamma_1} \frac{\delta P}{P}, \quad (5.120)$$

where $\kappa_T = (\partial \ln \kappa / \partial \ln T)_\rho$ and $\kappa_\rho = (\partial \ln \kappa / \partial \ln \rho)_T$. In most of the stellar envelope, P'/P increases smoothly outward while κ_{PS} remains negative or only weakly varying. In that case, the luminosity perturbation $\delta L/L$ tends to increase outward, which contributes to damping. By contrast, near a sufficiently strong opacity bump, κ_{PS} may increase steeply and become positive. The luminosity perturbation then decreases outward, which produces driving (Cox, 1974; Gautschy & Saio, 1995; Dupret et al., 2005).

In Figure 5.7, we show the run of κ , κ_{PS} , and the thermal relaxation time τ_{th} in the envelope of a $5 M_\odot$ SPB model, together with the work integral of an unstable mode excited by the κ -mechanism.

Figure 5.7 shows that, in SPB stars, two major opacity bumps are present in the envelope. The first, located around $\log_{10} T \simeq 5.2$, is associated with iron-group elements and is responsible for the excitation of SPB and β Cephei pulsations through the κ -mechanism (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007). The second, located near $\log_{10} T \simeq 4.7$,

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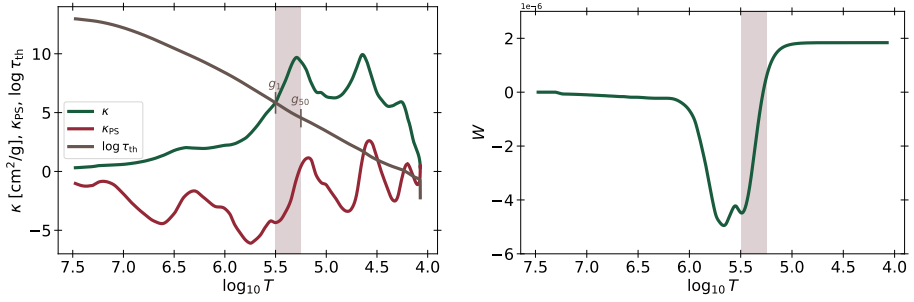


Figure 5.7: Illustration of mode driving by the κ -mechanism. The left panel shows the opacity κ (green), the quantity κ_{PS} (red), and the thermal relaxation time τ_{th} (brown) as functions of $\log_{10} T$. The grey region indicates the transition region, where the pulsation period and the local thermal relaxation time are of the same order. The right panel shows the work integral as a function of $\log_{10} T$ for an unstable g mode with radial order $n = 20$; the grey region again marks the transition region.

is associated with the second ionisation of helium and is the main opacity bump involved in the excitation of δ -Scuti stars (Dupret et al., 2004; Grigahcène et al., 2005; Dupret et al., 2005).

For SPB stars, the driving process follows the standard picture described above. In the transition region, the iron-group opacity bump produces a sharp increase in κ_{PS} , which may become positive. This causes a local decrease in $\delta L/L$ and leads to a positive contribution to the work integral, showing that thermal energy is being converted into pulsational work (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007). Depending on the stellar class, one may also speak of the κ - γ mechanism, because the adiabatic exponents Γ_1 and Γ_3 are modified in partial-ionisation regions and can therefore contribute to the driving. This additional γ -effect is especially important in lower-mass stars, whereas in B-type stars the driving is dominated by the opacity bump itself (Cox, 1974; Gautschy & Saio, 1995; Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993).

Because the SPB instability strip is highly sensitive to the iron-group opacity bump, it can be used to constrain both the metal content of the driving region and the transport processes that shape the local chemical composition (Dziembowski & Pamyatnykh, 1993; Dziembowski & Moskalik, 1993; Iglesias & Rogers, 1996; Miglio et al., 2007). It also provides a way to test the opacity calculations themselves, which remain challenging because they involve a very large number of atomic transitions under the thermodynamic conditions found in stellar envelopes (Badnell et al., 2005; Miglio et al., 2007; Moravveji et al., 2016). This issue has gained further importance with laboratory measurements of iron opacity at conditions close to those of stellar interiors, which suggest that current opacity tables may still underestimate the true opacity in some regimes (Bailey et al., 2015).

5.4. DRIVING AND DAMPING MECHANISMS IN INTERMEDIATE MASS STARS

In δ -Scuti stars, the dominant driving mechanism is the κ -mechanism acting in the He II partial-ionisation zone, with additional contributions from the γ -mechanism and from pulsation–convection coupling near the cool edge of the instability strip (Dupret et al., 2004; Grigahcène et al., 2005; Dupret et al., 2005). Because helium, rather than iron-group elements, plays the central role in that case, the excitation of δ -Scuti pulsations provides a useful probe of the helium abundance in the driving region and therefore of the transport processes that may modify it.

5.4.3 RADIATIVE DAMPING

Radiative damping is a non-adiabatic damping process that operates mainly in the deep radiative layers of the star, that is, in the quasi-adiabatic region. Physically, it corresponds to the leakage of heat from the oscillation to the surrounding equilibrium medium through radiative diffusion. This effect arises because compressed fluid elements maintain temperature differences with respect to their environment, allowing thermal energy to be exchanged with the background stellar structure (Cox, 1974; Gautschy & Saio, 1995). Radiative damping is especially important for high-order gravity modes, whose eigenfunctions vary rapidly with radius in the radiative cavity (Dupret et al., 2005).

Mathematically, in the quasi-adiabatic region the eigenfunctions oscillate on a much shorter spatial scale than the equilibrium quantities, which may therefore be treated as slowly varying. In the perturbed diffusion equation, the dominant contribution in the radiative zone is then the term involving $(\partial\delta T/\partial r)/(dT/dr)$, so that the work integral can be approximated as

$$W(r) = -\frac{1}{\text{Re}(\sigma)} \int_0^r \frac{L}{d \ln T/dr} \frac{d^2}{dr^2} \left(\frac{\delta T}{T} \right) dr. \quad (5.121)$$

Because $\delta T/T$ oscillates with radius, its second derivative tends to have the opposite sign locally; since $d \ln T/dx < 0$ in the radiative interior, the net contribution to the work integral is negative. Radiative diffusion therefore extracts energy from the mode and damps the oscillation (Cox, 1974; Gautschy & Saio, 1995; Dupret et al., 2005).

In Figure 5.8, we illustrate the effect of radiative damping on both the work integral and the corresponding luminosity perturbation. As illustrated in this figure, radiative damping becomes very strong for high-radial-order modes. In that case, the damping occurring below the outer convective envelope can dominate the net work balance, so that even if convective blocking provides some driving, it is not sufficient to destabilize the mode (Dupret et al., 2005; Bouabid et al., 2013). Near the red edge of the γ -Doradus instability strip, this radiative damping becomes even more efficient. In particular, Dupret et al. (2005) showed that, for cooler models, strong damping already develops in the deep radiative region, close to the convective core (see their Figs. 13 and 14).

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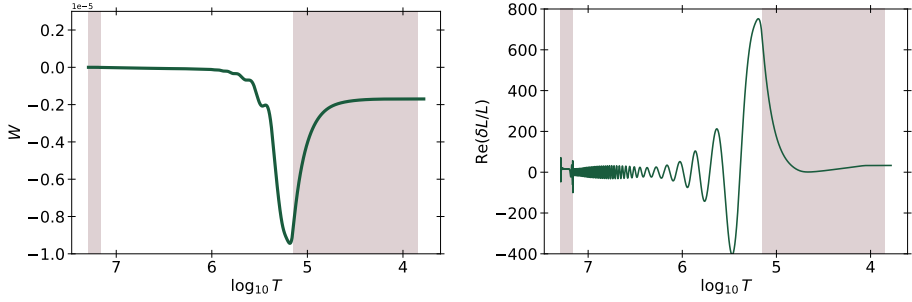


Figure 5.8: Illustration of radiative damping in a $1.5 M_{\odot}$ γ -Doradus model. The left panel shows the work integral as a function of $\log_{10} T$ for a stable $\ell = 2$, $m = 0$, $n = 85$ gravity mode. The grey regions indicate the convective zones of the model. The right panel shows the real part of the luminosity perturbation.

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this section is fully extracted from Fellay & Dupret (2026), submitted.

The instability strips (IS) of γ -Doradus stars were extensively investigated in the late 1990s and early 2000s to identify the driving mechanisms responsible for the observed pulsations and to reconcile theory with the observational constraints available at that time (Kaye et al., 1999; Handler, 1999; Handler & Shobbrook, 2002). γ -Doradus stars, typically spanning spectral types A7 to F5, exhibit high-order g -mode instabilities driven by convective flux blocking at the base of their thin convective envelopes. While the theoretical instability strips of γ -Doradus stars were studied in detail in the early 2000s, only a few works have revisited their theoretical description in light of the vastly improved observational constraints provided by past (*Kepler*; (Borucki et al., 2010)), current (*TESS*; (Ricker et al., 2015)), and future (*PLATO*; (Rauer et al., 2014)) space missions. Most detailed non-adiabatic computations including time-dependent convection (TDC) have so far been based on relatively small grids of models (Dupret et al., 2005; Grigahcène et al., 2005), and later studies mainly focused on improved convection treatments and updated input physics (Xiong et al., 2016). On the observational side, space-based photometry has revealed large samples of γ -Doradus and hybrid δ -Scuti/ γ -Doradus pulsators with rich gravity-mode spectra. In particular, *TESS* has already delivered ensemble catalogues, extensive period-spacing samples, and detailed case studies of individual stars and clusters (Antoci et al., 2019; Van Reeth et al., 2022; Garcia et al., 2022; Zhou, 2025; Kliapets et al., 2025; Zhou, 2025). However, the effect of rotation on the γ -Doradus instability strip has not yet been explored systematically on large model grids. The influence of rotation has previously been addressed only for a limited number of models (Bouabid et al., 2013) and without an exhaustive parameter study, and the excitation of Rossby (r) modes (Saio et al., 2018) has not been investigated in the γ -Doradus context. Regarding metallicity, pre-

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vious work has considered its impact on the global IS boundaries without studying in detail the excitation mechanisms or making the corresponding model grids publicly available (Grigahcène et al., 2006).

The revolution brought by the Kepler space telescope led to the discovery of hundreds to thousands of additional γ -Doradus stars. Notably, Li et al. (2020b) was able to determine the near-core rotation rates of 611 γ -Doradus stars, 32 of which were found to be in binary systems (an aspect that will be of interest in the latter parts of this thesis). Figure 5.9 illustrates the HR diagram of the γ -Doradus population observed by Kepler and analysed by Li et al. (2020b).

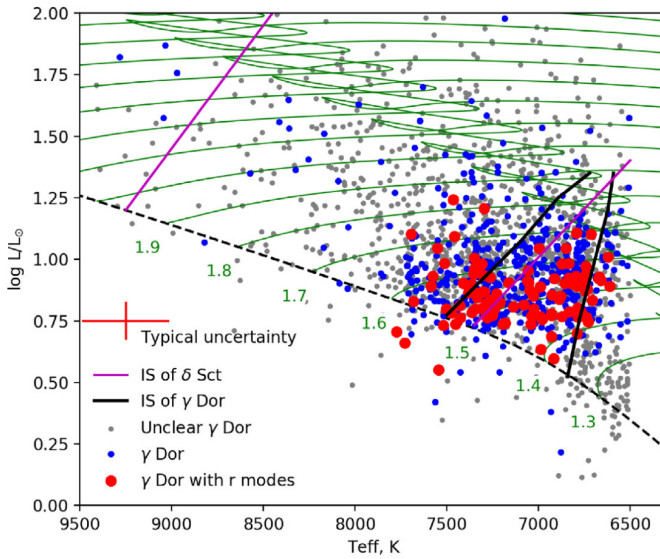


Figure 5.9: Stars on the HR diagram. The blue points are the γ -Doradus stars with clear g-mode patterns. The red points are the stars with both g- and r-mode patterns. The grey points are the stars we inspected but do not show clear period-spacing patterns. The black lines show the theoretical instability strip (IS) of γ -Doradus star while the purple lines depict the observed IS of δ -Scuti star by Murphy et al. (2019). The figure and description were taken from Li et al. (2020b).

From a theoretical perspective, the early 2000s saw significant progress in the development of non-adiabatic pulsation codes, aimed at explaining the observed instability strip of γ -Doradus stars. The first successful explanation was proposed by Guzik et al. (2000), who identified the convective flux blocking mechanism (described in Subsection 5.5.1) as the likely driving mechanism for these pulsations. However, their modelling relied on the frozen convection approximation, which limited the reliability of the predicted instability strip.

Accurate modelling of the instability strips of γ -Doradus stars (in particular the theoretical red edge) requires non-adiabatic stellar oscillation codes that includes a time-

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dependent treatment of convection–oscillation interactions (Dupret et al., 2005; Xiong et al., 2016). Guzik et al. (2000) demonstrated that modes in γ -Doradus stars can be driven by convective flux blocking at the base of their thin convective envelopes using the frozen-convection approximation. However, subsequent work showed that only TDC models yield reliable physical interpretations of instability strips (Dupret et al., 2005; Grigahcène et al., 2005). Modern stellar oscillation codes such as GYRE (Townsend & Teitler, 2013; Townsend et al., 2018; Goldstein & Townsend, 2020) do not yet include a full TDC treatment, making it impossible to compute realistic instability strips or detailed mode damping/growth rates for γ -Doradus stars, since the frozen-convection approximation breaks down when the lifetime of convective eddies is shorter than the pulsation period.

Comparisons between existing theoretical instability strips and observations show broad qualitative agreement, but several persistent discrepancies remain. Using the complete *Kepler* sample of 611 γ -Doradus stars with detected period-spacing patterns, Li et al. (2019, 2020b) demonstrated that more oscillation modes are excited in observations than predicted by current theory, even when rotation is included. A further discrepancy, is the presence of a sub-population of γ -Doradus stars located at the extreme blue edge or even outside the theoretical instability strip. Similar tensions between theory and observations are emerging from large *Kepler* or *TESS*-based catalogues of γ -Doradus and hybrid pulsators, which reveal very rich and diverse mode spectra over a wide range of stellar parameters (Antoci et al., 2019; Li et al., 2019, 2020b; Garcia et al., 2022; Zhou, 2025; Kliapets et al., 2025; Zhou, 2025). Ongoing and upcoming observations with *TESS* and, in the longer term, *PLATO* will provide additional constraints on these discrepancies and on the physical processes shaping the γ -Doradus instability strip.

In this context, one of the major components of the work carried out during this thesis was a systematic exploration of the γ -Doradus instability strip. The resulting grids have been made publicly available for the benefit of the observational and asteroseismic communities. This work is described in this section.

5.5.1 CONVECTIVE LUMINOSITY BLOCKING

In the case of γ -Doradus stars, the driving mechanism responsible for the observed g -mode instabilities is the convective flux blocking that operates at the base of their thin, near-surface convective envelopes (Guzik et al., 2000; Dupret et al., 2005; Grigahcène et al., 2005; Xiong et al., 2016). Physically, this mechanism arises because the convective envelope cannot adjust instantaneously to the periodic perturbations induced by the pulsations. As a result, part of the radiative flux that would normally be transported outward by convection is temporarily trapped at the base of the convective zone at the hot phase of the cycle. During the pulsation cycle, this trapped thermal energy is periodically converted into work, providing net driving as part of a thermodynamic heat engine. The classical picture of convective flux blocking, following Dupret et al. (2005), is that the efficiency of this mechanism, and thus the excitation of unstable modes, is mainly controlled by the location of the base of the convective envelope relative to the so-called transition region, where the pulsation period is of the same order

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as the local thermal relaxation timescale. When the base of the convective zone lies within this transition region, the thermodynamic heat engine has a period close to that of the g-modes, the thermodynamic heat engine is the most efficient, leading to the driving. If, on the other hand, the base of the convective zone is situated well above or below the transition region, radiative damping dominates and the modes remain stable.

In this section, we explore the driving and damping processes behind the range of oscillations modes driven and damped. For a $1.55 M_{\odot}$ evolutionary track (see [Subsection 5.5.2](#) for a detail description of the physical parameters used in the stellar models), we illustrate in [Figure 5.10](#) the range of radial orders corresponding to stable and unstable modes as a function of the effective temperature.

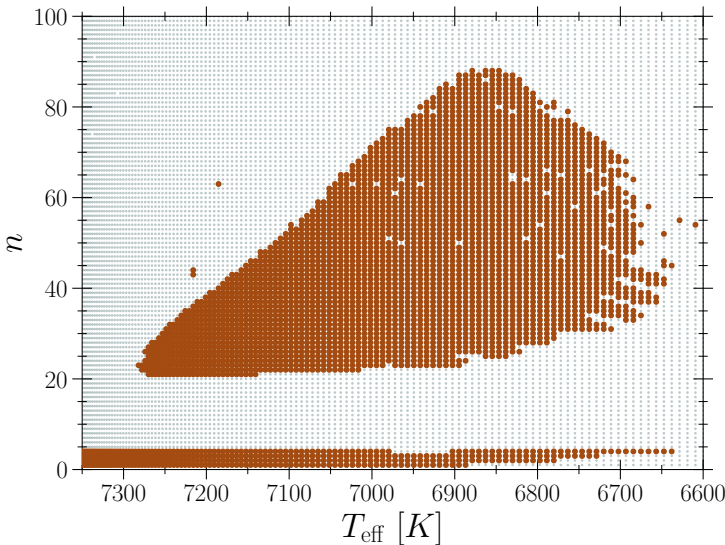


Figure 5.10: Radial orders of the stable and unstable $\ell = 1$, $m = 0$ modes for a $1.55 M_{\odot}$ track. Each grey dot corresponds to a stable mode, while orange dots denote unstable modes. The detail description of the physical parameters used in the stellar models is given in [Subsection 5.5.2](#).

This figure shows the range of radial orders at which unstable modes are found in classical γ -Doradus stars. In general, unstable modes are found with radial orders between $n \simeq 20$ and $n \simeq 90$, with lower radial orders ($n \sim 20$) favoured near the blue edge of the instability strip and higher radial orders favoured near the red edge. This theoretical range can be compared with the observed radial orders reported by [Li et al. \(2019\)](#). Overall, the agreement between theory and observations is satisfactory, although the observed instability strip appears more extended in radial order, with a non-negligible number of observed $\ell = 1$ modes at $n < 20$ and up to $n \sim 100$. Low-radial-order unstable modes ($n < 10$) are also visible in this diagram; these correspond to the δ -Scuti instability strip, which will be discussed in a separate article. Compared

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to the work of [Bouabid et al. \(2013\)](#), we find a larger number of excited modes in the central region of the instability strip. The models explored by [Bouabid et al. \(2013\)](#) were likely concentrated towards the blue edge, which naturally leads to the excitation of predominantly low-radial-order modes.

ON THE LOW RADIAL ORDER/PERIOD LIMIT

First, a region of stable modes exists between the δ -Scuti instability strip (low radial orders) and the γ -Doradus instability strip (higher radial orders). Two arguments can account for this stabilisation region. The first concerns the evanescent zone. For a g mode to propagate, its angular frequency must be lower than both the Brunt-Väisälä frequency and the Lamb frequency. When inspecting the propagation diagram of a typical γ -Doradus model (see [Figure 5.11](#)), we find that low-radial-order g modes must traverse a wide evanescent region before reaching the potential excitation layer at the base of the convective envelope. In this case, the modes are strongly attenuated while travelling through the evanescent zone, so that radiative damping dominates and they remain stable. By contrast, higher-radial-order g modes have propagation cavities that are better connected to the excitation region and can therefore overcome the damping and become unstable ([Dupret et al., 2005](#); [Xiong et al., 2016](#)). The sec-

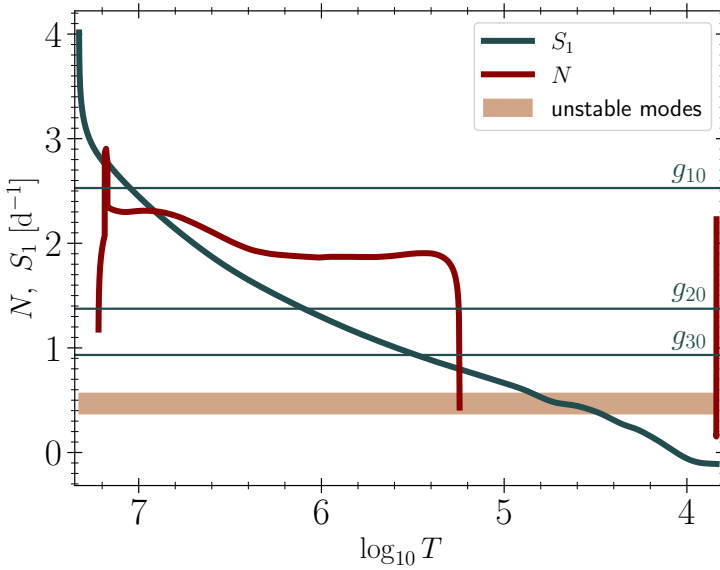


Figure 5.11: Propagation diagram for a $1.55 M_{\odot}$ γ -Doradus star. The red and blue lines corresponds to respectively the Brunt-Väisälä frequency and the Lamb frequency of the $\ell = 1$ modes. The orange region corresponds to the frequency a which unstables modes are observed in this stars, the blue horizontal lines corresponds to the frequency of the $n = 10, 20, 30$ g-modes which are stable.

ond explanation arises from the behaviour of the eigenfunctions themselves. Near the

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surface, the pressure perturbation δP , following [Buta & Smith \(1979\)](#), can be approximated by:

$$\frac{\delta P}{P} = \left[\ell(\ell + 1)K - 4 - K^{-1} \right] \frac{\xi_r}{r}, \quad (5.122)$$

where K is the square of the inverse dimensionless frequency of the mode:

$$K = \frac{GM}{\sigma^2 R^3}. \quad (5.123)$$

At the surface, the range of dimensionless frequencies corresponding to the intermediate stable region coincides with the point where the right-hand side of [Equation 5.122](#) approaches zero. As a result, the amplitude of the modes near the surface remains small and cannot be efficiently excited by convective blocking. Both explanations are plausible, and it is not currently possible to distinguish between them; in both cases, the affected modes remain stable.

ON THE HIGH RADIAL ORDER/PERIOD LIMIT

A second important aspect of mode excitation concerns the range of radial orders and periods that are effectively driven by convective luminosity blocking. As mentioned previously, this range depends strongly on the stellar effective temperature. This dependence is particularly striking near the blue edge of the instability strip, where only a few tens of modes (~ 30) are excited, compared with 60–70 modes in the middle of the IS. The detailed explanation for these ranges is far from trivial, because several effects shape the final stability of the modes: changes in the efficiency of convective blocking, the strength of radiative damping, and the impact of stellar evolution on the internal structure and on the excitation/damping mechanisms.

Several possibilities were considered to explain this behaviour, and we find that the depth of the base of the convective envelope (BCE) largely controls the excitation efficiency. First, it sets the region where convective blocking becomes possible: the blue edge of the instability strip is systematically defined by models in which the BCE lies within the transition zone, where the local thermal timescale is comparable to the mode period. However, this location does not correspond to the region where convective blocking is most efficient (more than 75 % of the IS lies outside this regime), but rather to the point where it becomes sufficiently efficient to overcome radiative damping for low-radial-order modes. Following the results of [Godart et al. \(2009\)](#), radiative damping scales approximately as σ^{-4} and therefore becomes increasingly efficient for higher-radial-order modes. As a consequence, low-radial-order modes are more easily driven when convective blocking is still relatively weak. To illustrate this phenomenon, [Figure 5.12](#) shows the work integrals of modes with different periods (and radial orders) for a $1.55 M_{\odot}$ model with $T_{\text{eff}} = 7000$ K.

As shown in this figure, several effects contribute to the driving and damping of the modes. As expected, the radiative damping strengthens with increasing period, as indicated by the decreasing work integral, which stabilises long-period modes. Most work integrals also exhibit a non-negligible driving contribution around $\log_{10} T = 5.2$, which was initially unexpected and corresponds to the κ -mechanism associated with

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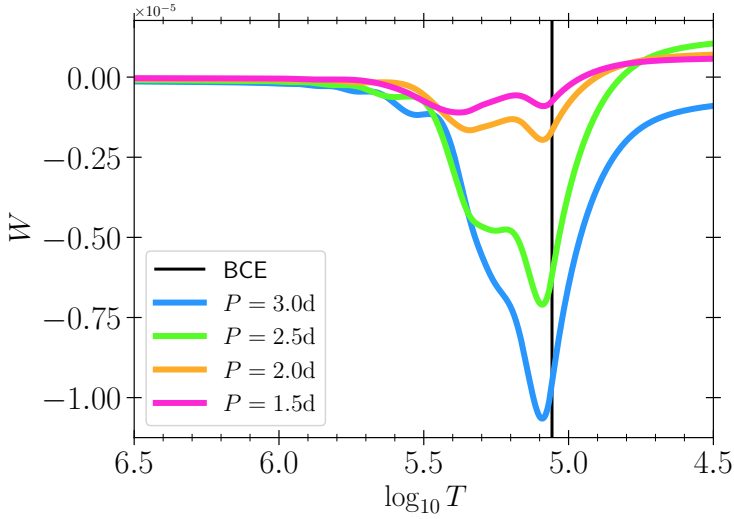


Figure 5.12: Work integral for selected stable and unstable γ -Doradus g modes as a function of $\log T$ within a $1.55, M_{\odot}$ model with $T_{\text{eff}} = 7000$, K. The black vertical line marks the location of the base of the convective envelope (BCE). Each colour corresponds to a different mode (i.e. a different period). The net damping or driving is given by the surface value of the work integral: positive values indicate unstable (excited) modes, while negative values correspond to damped modes. A decreasing work integral indicates damping, whereas an increasing work integral indicates driving.

the iron-group opacity bump (Cox et al., 1966; Pamyatnykh, 1999). This contribution is stronger at shorter periods, again favouring the excitation of low-radial-order modes. The dominant driving, however, arises close to the BCE, and the efficiency of this convective blocking contribution clearly depends on the mode period. To clarify this dependence and the enhanced efficiency of the κ -mechanism at low periods, Figure 5.13 displays the eigenfunctions of the same modes.

A comparison of the eigenfunctions shows that increasing the mode period shifts the last radial node of the eigenfunction closer to the BCE. As a result, the location of the maximum amplitude in $\delta P/P$ varies with the period. In particular, for low-frequency modes the amplitude maximum occurs far below the $\log_{10} T = 5.2$ region where the κ -mechanism acts, whereas for high-period modes the maximum lies close to or beyond this region. The efficiency of the κ -mechanism is therefore strongly linked to the behaviour of the eigenfunctions near the opacity bump. Since the κ -mechanism contributes to the driving of the modes, variations in opacity and metallicity will influence both the number of excited modes and their periods; this impact is investigated in the next section. In addition, shifting the last node closer to the BCE with increasing period also affects the efficiency of convective blocking, as the behaviour of the eigenfunctions near the BCE plays a crucial role in the driving. To confirm this conjecture, we examined the eigenfunctions of the highest-period unstable modes across several

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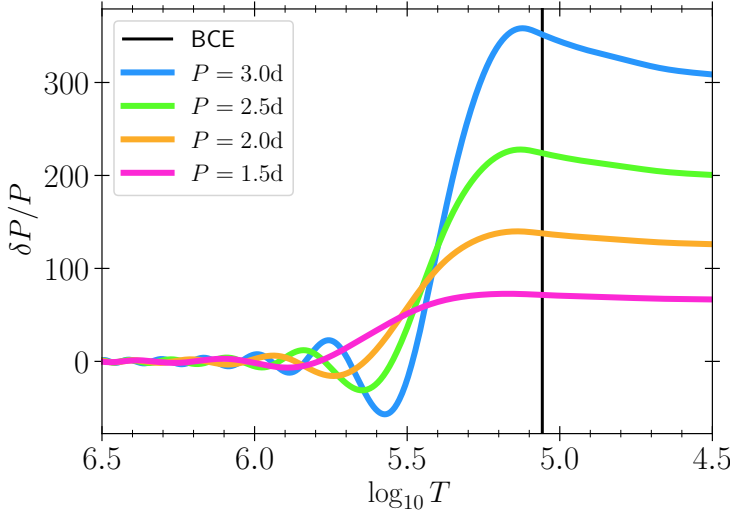


Figure 5.13: $\delta P/P$ eigenfunctions for the same modes presented in Figure 5.12 as a function of the temperature of the different layers of a $1.55 M_{\odot}$ model with $T_{\text{eff}} = 7000 K$. Both the near core region and the surface were cut in this figure to focus on the region close to the BCE where the driving and damping of the oscillations takes place.

models along the $1.55 M_{\odot}$ evolutionary track, shown in Figure 5.14. Despite clear differences between the stellar models and the radial orders of their highest-period unstable modes, the eigenfunctions display remarkably similar behaviour below the BCE. In all cases, the minimum preceding the last node lies near $\log_{10} T \sim 5.6$, and the maximum amplitude lies just below $\log_{10} T \sim 5.2$, immediately before the BCE. Since the eigenfunctions behave similarly close to the BCE, the dimensionless buoyancy radius in this region must also vary in a similar way, leading to

$$\frac{1}{\sigma} \int_{r_{\log_{10} T=5.5}}^{r^{BCE}} \frac{N}{r'} dr' \simeq \text{const}, \quad (5.124)$$

where σ is the frequency of the highest-period unstable mode. This evolutionary trend is visible in Figure 5.15, which shows the periods of stable and unstable modes as a function of effective temperature for the $1.55 M_{\odot}$ sequence. In this figure we compare the modelled unstable periods with the various theoretical limits derived analytically (Dupret et al., 2005; Dupret et al., 2009; Xiong et al., 2016). The blue edge of the instability strip is clearly associated with the region where the thermal relaxation timescale at the BCE matches the mode period. As the star evolves, the period of the longest unstable modes increases, and the slope of this transition is governed by the evolution of the buoyancy radius in the vicinity of the BCE, as discussed previously. Finally, a second transition occurs, in which the maximum unstable period decreases significantly. This happens when the BCE migrates clearly outside the transition region and the efficiency of convective blocking declines. This regime is difficult to

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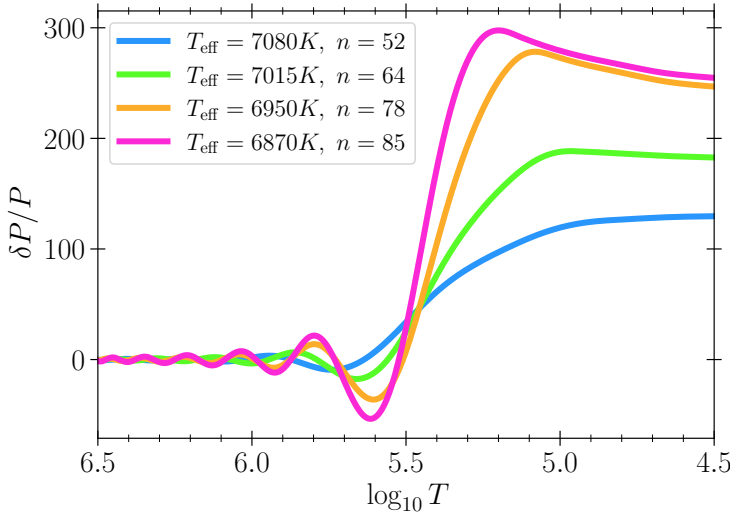


Figure 5.14: $\delta P/P$ eigenfunctions of the unstable mode with the highest period across a few models in the $1.55M_{\odot}$ IS. Both the near core region and the surface were cut in this figure to focus on the region close to the BCE where the driving and damping of the oscillations takes place.

characterise precisely, because it depends sensitively on the adopted time-dependent convection–oscillation interaction formalism (Dupret et al., 2005; Houdek & Dupret, 2015). From an observational point of view, the longest observed unstable periods can place constraints on the buoyancy radius in the BCE region for stars located in the blue part of the instability strip (Van Reeth et al., 2015; Li et al., 2019).

5.5.2 IMPACT OF DIFFERENT PARAMETERS ON THE THEORETICAL γ -DOR IS

Using the classical view of the problem, the position of the instability strip should be particularly dependent on the mixing-length parameter chosen in the stellar models and all the parameters related to the convection. Dupret et al. (2005) pointed out that, to reproduce the observations available at the time, the best-suited mixing-length theory (MLT, Cox & Giuli, 1968) convective parameter α_{MLT} was $\alpha_{\text{MLT}} = 2.0$ in CLES. At that time, the standard chemical composition used for all such computations in the literature was a metallicity of $Z = 0.02$, adopting the GN98 solar abundances (Grevesse & Sauval, 1998). Even though metallicity is not the dominant factor affecting the instability strip, it directly affects stellar evolution and the efficiency of the κ -mechanism responsible for part of the driving. In addition, rotation also impacts the instability strips of these stars, stabilising modes with particular azimuthal orders m and shifting their frequencies in the observer frame, the effect of the rotation is discussed in Subsection 5.6.3.

In this section we are exploring the effect of different modifications of the stellar struc-

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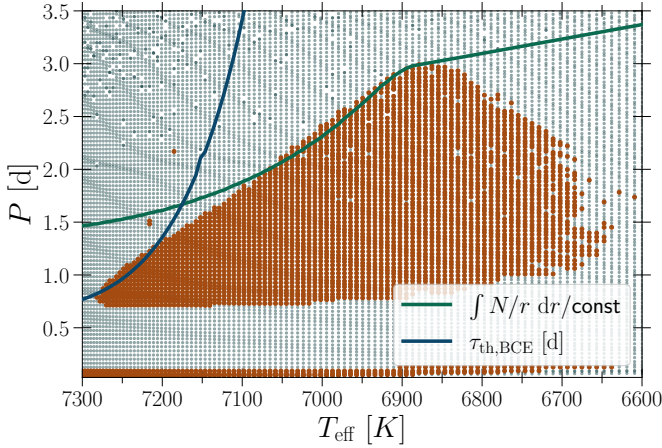


Figure 5.15: Periods of the stable and unstable $\ell = 1, m = 0$ modes for the reference $1.55 M_{\odot}$ track. Each black dot corresponds to a stable mode, while red dots denote unstable modes.

ture and physical ingredients starting from a 'reference' set of stellar ingredients kept constant for the computations of a reference instability trip across the HR diagram. Unless precised in the main text, the physical ingredients used to compute the instability strip remain the same as our reference grid. All the evolutionary sequences were computed with the Code Liégeois d'Evolution Stellaire (CLES, [Scuflaire et al., 2008b](#)). For the reference grid of models we used an initial hydrogen mass fraction of $X = 0.72$, a solar metallicity $Z = 0.015$, a MLT convective parameter of $\alpha = 2.0$, a step-like overshooting of $\alpha_{\text{over}} = 0.2$ (only acting in the core convective zone). The models were stopped when the central hydrogen mass fraction reached $X_C = 0.05$. For all the stellar modelling we used the AGSS09 abundances ([Asplund et al., 2009](#)), the FreeEOS equation of state ([Irwin, 2012](#)), the OPAL opacities ([Iglesias & Rogers, 1996](#)), the $T(\tau)$ relation from Model-C of [Vernazza et al. \(1981\)](#) for the atmosphere, and the nuclear reaction rates of [Adelberger et al. \(2011\)](#).

For the reference grid of models, [Figure 5.16](#) illustrates the associated HR diagram of the γ -Doradus IS as computed with MAD for $\ell = 1$ gravity modes. For this reference grid, the γ -Doradus IS is located for models with an effective temperate between 7300K and 6600K and luminosities $\log(L/L_{\odot})$ between 0.6 and 1.3. When comparing this IS to the one obtained by [Dupret et al. \(2005\)](#), the results seems compatible and the IS is roughly in the same position on the HR-diagram, which was expected as we did not modified any treatment of the convection/oscillations interactions.

IMPACT OF THE METALLICITY

To explore the instability strip of γ -Doradus stars in more detail, we computed a grid of models with masses ranging from $1.35 M_{\odot}$ to $2.5 M_{\odot}$ in steps of $0.05 M_{\odot}$, and metallicities from $Z = 0.010$ to $Z = 0.025$ in steps of 0.005. In [Figure 5.17](#), we illustrate the

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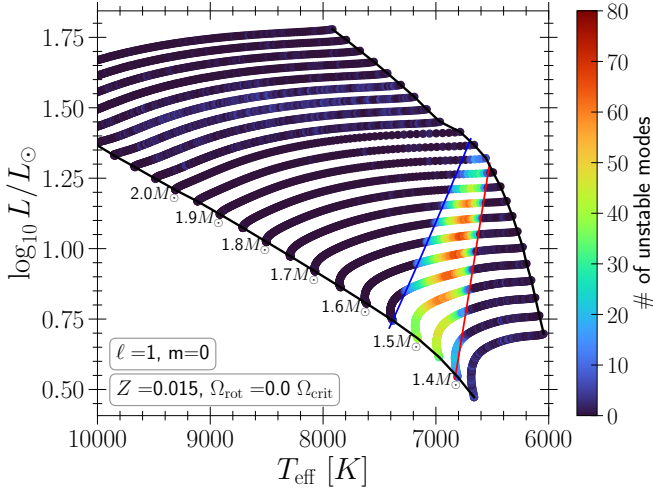


Figure 5.16: HR-diagram of the γ -Doradus IS for our reference grid. The color code corresponds to the number of unstable $\ell = 1$ g modes for each model in each evolutionary sequence.

γ -Doradus IS for a metallicity of $Z = 0.025$, which can be compared to the reference case at $Z = 0.015$. Figure 5.17 shows that, as expected (Grigahcène et al., 2006), the position of the instability strip in the HR diagram is not strongly affected by metallicity. For comparison, the red-edge and blue-edge used in this figure are the same as for the reference grid. We only see a small increase of the size of the IS of about 50K in each direction, which is expected as the amount of metals increases and so is the strength of the κ mechanism. In terms of stellar evolutionary impact models with higher metallicity reach the instability strip at higher masses and younger ages compared to low-metallicity models. Similarly, decreasing metallicity shifts the instability strip to lower stellar masses. Exploring cases of extremely low metallicities, such as those that could potentially in globular clusters, could therefore be of particular interest, as γ -Doradus stars in these environments may correspond to low-mass stars still on the main sequence despite the advanced age of the clusters.

In addition to the effect of the metallicity on the stellar structure, the driving of the modes should also be affected as the efficiency of the κ -mechanism should be impacted by the metallicity. On a global scale we do not see a particularly significant impact on the amount of unstable modes found. However, their period range can be modified, in Figure 5.18 we illustrate the range of period at which unstable modes are found in a $1.75 M_{\odot}$ track with $Z = 0.025$. By comparing the period range of unstable modes in Figure 5.18 and for the solar metallicity case in Figure 5.15 we see an effect of the metallicity. The modes driven in the high metallicity case are shifted toward higher periods, which is logical as the efficiency of the κ -mechanism is increased favouring the driving of higher period modes. We also note that the stellar evolution has a similar effect on the modes periods, it is therefore difficult to distinguish the two

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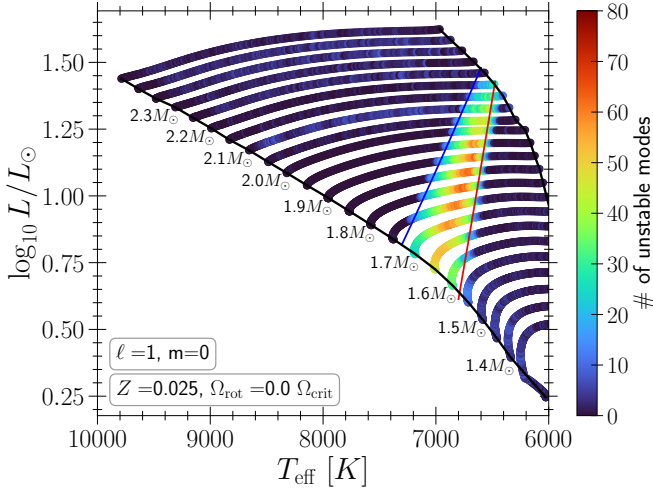


Figure 5.17: HR diagram of the γ -Doradus IS for a grid with metallicity $Z = 0.025$. The colour scale indicates the number of unstable $\ell = 1$ g modes for each model in each evolutionary sequence.

effects as metallicity also modify the evolution and the masses of the stars considered.

IMPACT OF THE MIXING LENGTH PARAMETER

As mentioned previously, the mixing-length parameter has a strong effect on the temperature at which the instability strip appears, as it controls the size of the convective zone. Without considering the effect of rotation, we explored the instability strips for two values of the mixing-length parameter: $\alpha_{\text{MLT}} = 2.0$, and 2.2 . As shown in [Figure 5.19](#), the mixing-length parameter has a strong impact on the instability strip, as already reported by [Dupret et al. \(2005\)](#). In particular, increasing α_{MLT} shifts the IS towards hotter models. While the observed population of γ -Doradus stars might appear to provide constraints on α_{MLT} . However, the treatment of convection–pulsation interactions remains uncertain, and even with TDC models, only an estimate of α_{MLT} that yields instability strips in approximate agreement with observations for our stellar structure code can be provided. Our purpose here is to illustrate the sensitivity of the instability strip to α_{MLT} , which remains one of the dominant parameters affecting its position in the HR diagram. We also notice that the

IMPACT OF THE OVERSHOOTING

One additional parameters can affect the size of convective zones, the overshooting, which controls the extension of the convective core. As overshooting only affects the size of the convective core and thus the stellar evolution, overshooting does not directly impact the effective temperature at which γ -Doradus stars are found ([Gri-gahcène et al., 2006](#)). To verify this, we extended our grid with models including

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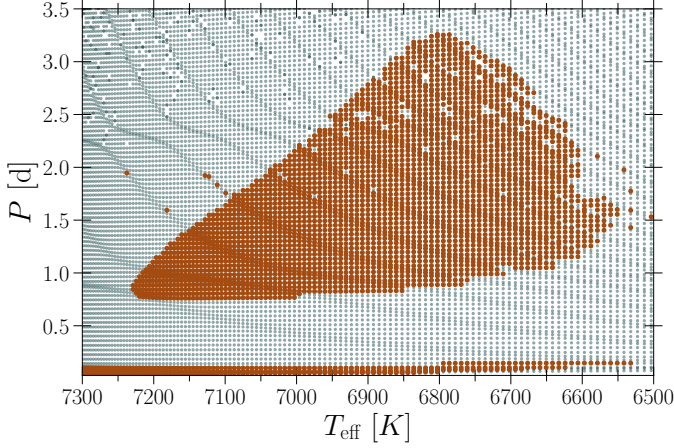


Figure 5.18: Periods of the stable and unstable $\ell = 1$, $m = 0$ modes for the $1.55 M_{\odot}$ track with $Z = 0.025$. Each black dot corresponds to a stable mode, while red dots denote unstable modes.

overshooting, adopting $\alpha_{\text{over}} = 0.4$. In Figure 5.20 we illustrate the γ -Doradus IS for the case $\alpha_{\text{over}} = 0.4$. As expected, we did not find any effect of overshooting on the instability strip, except for the shift in evolutionary tracks caused by the modification of the convective core size.

5.6 INCLUSION OF THE ROTATION THROUGH THE TRADITIONAL APPROXIMATION

In this section, we include the effect of rotation on non-radial oscillations by accounting for the Coriolis force, while neglecting the centrifugal force and the associated stellar deformation. We therefore focus on cases where the centrifugal distortion remains sufficiently small. In practice, this approximation is usually adopted for moderately rotating stars, although its precise domain of validity depends on the stellar model and on the oscillation modes considered (Townsend, 2003; Bouabid et al., 2013; Dhouib et al., 2021). We define the critical angular velocity as

$$\Omega_{\text{crit}} = \sqrt{\frac{GM}{R^3}}, \quad (5.125)$$

where M and R are the stellar mass and radius. The Coriolis force is included through the so-called traditional approximation of rotation, which is described below. When the Coriolis force is included, the only modified equation is the equation of motion, which becomes

$$\frac{\partial^2 \xi}{\partial t^2} = -\nabla \Psi' - \frac{\nabla P'}{\rho} + \frac{\rho'}{\rho^2} \nabla P - 2i\sigma \Omega_{\text{rot}} \times \xi, \quad (5.126)$$

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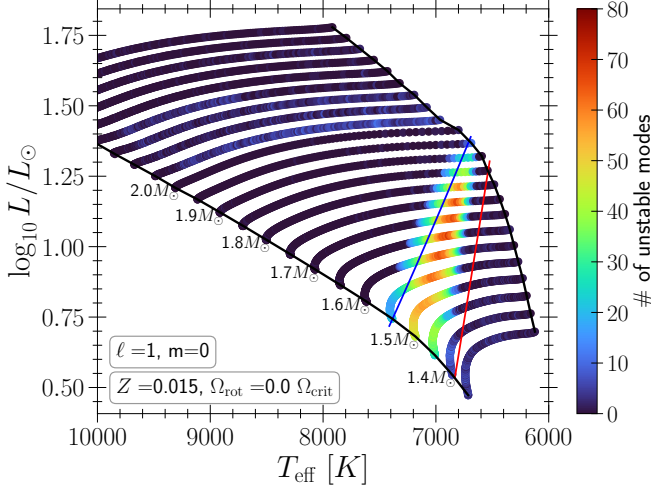


Figure 5.19: HR diagram of the γ -Doradus instability strip for a grid with $\alpha_{\text{MLT}} = 2.2$. The colour code corresponds to the number of unstable $\ell = 1$ g modes for each model in each evolutionary sequence. The blue and red edges are the same than in Figure 5.16.

where ξ is the displacement vector, σ is the oscillation frequency in the corotating frame, and $\times$ denotes the vector cross product. The radial, latitudinal, and azimuthal components of this equation are then written as

$$\sigma^2 \xi_r = \frac{\partial \Psi'}{\partial r} + \frac{\rho'}{\rho} \frac{Gm}{r^2} + \frac{1}{\rho} \frac{\partial P'}{\partial r} - 2i\sigma\Omega_{\text{rot}} \sin \theta \xi_\phi, \quad (5.127)$$

$$\sigma^2 \xi_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \left(\Psi' + \frac{P'}{\rho} \right) - 2i\sigma\Omega_{\text{rot}} \cos \theta \xi_\phi, \quad (5.128)$$

and

$$\sigma^2 \xi_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\Psi' + \frac{P'}{\rho} \right) + 2i\sigma\Omega_{\text{rot}} (\cos \theta \xi_\theta + \sin \theta \xi_r). \quad (5.129)$$

Up to this point, no approximation has been made apart from neglecting centrifugal deformation. The traditional approximation is introduced at this stage. It is particularly suited to gravito-inertial modes, for which the horizontal displacement dominates over the radial displacement. Under this assumption, the radial component of the Coriolis force is neglected, as is the contribution of the radial displacement to the horizontal components of the Coriolis acceleration. Such an approximation is justified only when the radial component of the Coriolis force is negligible compared with buoyancy. This is usually fulfilled in radiative zones, where the oscillation and rotation frequencies are both smaller than the Brunt–Väisälä frequency (Townsend, 2003; Dhouib et al., 2021). It is, however, not valid in stars with in cold stars which have deep convective envelopes, and where the restoring effect of buoyancy is weak. With

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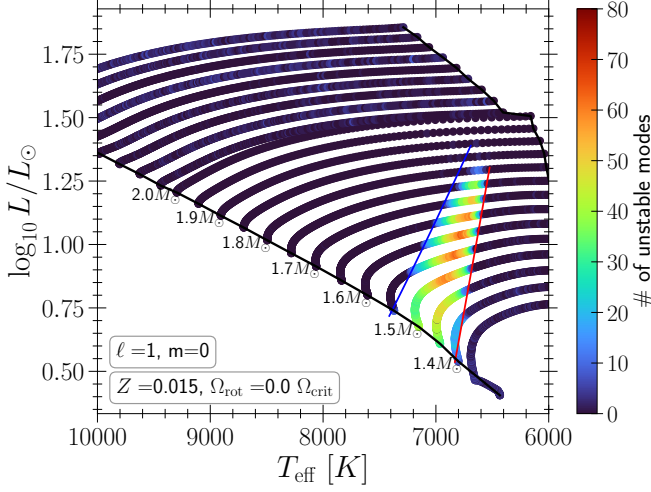


Figure 5.20: HR diagram of the γ -Doradus instability strip for a grid with $\alpha_{\text{MLT}} = 2.0$ and overshooting parameter $\alpha_{\text{over}} = 0.4$. The colour code corresponds to the number of unstable $\ell = 1$ g modes for each model in each evolutionary sequence.

this assumption, the equation of motion becomes

$$\sigma^2 \xi_r = \frac{\partial \Psi'}{\partial r} + \frac{\rho'}{\rho} \frac{Gm}{r^2} + \frac{1}{\rho} \frac{\partial P'}{\partial r}, \quad (5.130)$$

$$\sigma^2 \xi_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \left(\Psi' + \frac{P'}{\rho} \right) - 2i\sigma\Omega_{\text{rot}} \cos \theta \xi_\phi, \quad (5.131)$$

and

$$\sigma^2 \xi_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\Psi' + \frac{P'}{\rho} \right) + 2i\sigma\Omega_{\text{rot}} \cos \theta \xi_\theta. \quad (5.132)$$

It should be noted that the traditional approximation is often used together with the Cowling approximation, in which the perturbation of the gravitational potential is neglected. This is mainly because separability issues appear when the Poisson equation is retained. These difficulties are discussed below. For the moment, however, we do not impose the Cowling approximation. Starting from the horizontal components of the equation of motion, the horizontal displacements can be isolated and written as

$$\xi_\theta = \frac{1}{r\sigma^2} \frac{1}{1 - \nu^2 \cos^2 \theta} \left[\frac{\partial}{\partial \theta} \left(\Psi' + \frac{P'}{\rho} \right) - i\nu \cot \theta \frac{\partial}{\partial \phi} \left(\Psi' + \frac{P'}{\rho} \right) \right], \quad (5.133)$$

and

$$\xi_\phi = \frac{1}{r\sigma^2} \frac{1}{1 - \nu^2 \cos^2 \theta} \left[i\nu \cos \theta \frac{\partial}{\partial \theta} \left(\Psi' + \frac{P'}{\rho} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left(\Psi' + \frac{P'}{\rho} \right) \right], \quad (5.134)$$

where

$$\nu = \frac{2\Omega_{\text{rot}}}{\sigma} \quad (5.135)$$

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is the spin parameter. These expressions can then be substituted into the continuity equation, leading to the horizontal-motion plus continuity equation

$$\begin{aligned} \frac{\delta\rho}{\rho} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r^2 \sigma^2 \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\frac{\sin \theta}{1 - \nu^2 \mu^2} \left[\frac{\partial}{\partial \theta} - i\nu \cot \theta \frac{\partial}{\partial \phi} \right] \left[\Psi' + \frac{P'}{\rho} \right] \right) \right. \\ \left. + \frac{\partial}{\partial \phi} \left(\frac{\sin \theta}{1 - \nu^2 \mu^2} \left[i\nu \mu \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right] \left[\Psi' + \frac{P'}{\rho} \right] \right) \right] = 0, \end{aligned} \quad (5.136)$$

where $\mu = \cos \theta$. At this stage, we search for separable solutions of the form

$$X' = X'(r)f(\mu)e^{im\phi}, \quad (5.137)$$

where $f(\mu)$ describes the latitudinal dependence and will be specified below. Substituting this expression into the continuity equation gives

$$\begin{aligned} \frac{\delta\rho}{\rho} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r^2 \sigma^2} \left[\frac{d}{d\mu} \left(\frac{1 - \mu^2}{1 - \nu^2 \mu^2} \frac{df}{d\mu} \right) \right. \\ \left. - \frac{1}{1 - \nu^2 \mu^2} \left(\frac{m^2}{1 - \mu^2} + m\nu \frac{1 + \nu^2 \mu^2}{1 - \nu^2 \mu^2} \right) f \right] \left[\Psi' + \frac{P'}{\rho} \right] = 0. \end{aligned} \quad (5.138)$$

We therefore define the angular operator $\mathcal{L}_\nu$ as

$$\mathcal{L}_\nu[f(\mu)] \equiv \frac{d}{d\mu} \left(\frac{1 - \mu^2}{1 - \nu^2 \mu^2} \frac{df}{d\mu} \right) - \frac{1}{1 - \nu^2 \mu^2} \left(\frac{m^2}{1 - \mu^2} + m\nu \frac{1 + \nu^2 \mu^2}{1 - \nu^2 \mu^2} \right) f(\mu). \quad (5.139)$$

This operator corresponds to the latitudinal part of the horizontal-motion plus continuity equation. The functions $f(\mu)$ must therefore be eigenfunctions of $\mathcal{L}_\nu$. These eigenfunctions are the Hough functions, denoted here by $\Theta_{\nu,k}^m(\mu)$, and they satisfy the Laplace tidal equation

$$\mathcal{L}_\nu [\Theta_{\nu,k}^m(\mu)] = -\lambda_{\nu,k}^m \Theta_{\nu,k}^m(\mu), \quad (5.140)$$

where $\lambda_{\nu,k}^m$ is the corresponding eigenvalue, which depends on m , ν , and the mode index k (Townsend, 2003, 2005; Bouabid et al., 2013). The separable solution can then be written as

$$X' = X'(r)\Theta_{\nu,k}^m(\mu)e^{im\phi}. \quad (5.141)$$

It is important to emphasise that, within the traditional approximation, the Coriolis force modifies only the horizontal-motion plus continuity equation. If the frozen-convection approximation is also adopted, the full system of differential equations to be solved can be written as

$$\sigma^2 \rho \xi_r = \frac{dP'}{dr} + g\rho' + \rho \frac{d\Psi'}{dr}, \quad (5.142)$$

$$\frac{\delta\rho}{\rho} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r^2 \sigma^2} \mathcal{L}_\nu \left[\Psi' + \frac{P'}{\rho} \right] = 0, \quad (5.143)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi'}{dr} \right) - \frac{1}{r^2} \mathcal{L} [\Psi'] = 4\pi G\rho', \quad (5.144)$$

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$$i\sigma T\delta S = -\frac{1}{4\pi r^2\rho}\frac{\partial\delta L}{\partial r} + \epsilon\left(\frac{\delta\epsilon}{\epsilon} + \frac{\delta\rho}{\rho}\frac{1}{r^2}\frac{\partial(r^2\xi_r)}{\partial r}\right) + \frac{1}{4\pi\rho r^3}\mathcal{L}^2\left[L_R\left(\frac{\delta T}{r(dT/dr)} - \frac{\xi_r}{r}\right) - \frac{L_C}{r^2\sigma^2}\left(\Psi' + \frac{P'}{\rho}\right)\right], \quad (5.145)$$

and

$$\frac{\delta L}{L} = \frac{L_R}{L}\frac{\delta L_R}{L_R} = \frac{L_R}{L}\left(2\frac{\xi_r}{r} + \frac{T'}{T} - \frac{\kappa'}{\kappa} - \frac{\rho'}{\rho} + \frac{\partial\delta T/\partial r}{dT/dr} - \frac{\partial\xi_r}{\partial r}\right). \quad (5.146)$$

This system highlights the origin of the separability problem. In the continuity equation, the latitudinal dependence is naturally described by Hough functions, which are eigenfunctions of $\mathcal{L}_v$. By contrast, in the Poisson equation and in the angular part of the energy-conservation equation, the angular dependence is governed by the usual spherical Laplacian $\mathcal{L}$, whose eigenfunctions are associated Legendre polynomials. Thus, once the perturbation of the gravitational potential is retained, the latitudinal dependence cannot be represented by a single Hough function in all equations simultaneously. Instead, different spherical degrees become coupled, and the problem turns into a coupled system of radial differential equations. This difficulty is one of the main reasons why the traditional approximation is often combined with the Cowling approximation. By neglecting the perturbation of the gravitational potential, the Poisson equation no longer needs to be solved, and the remaining separability issue is avoided. The angular contribution in the energy equation, where the operator $\mathcal{L}$ appears, generally has only a weak influence on the solutions and can also be neglected in many practical applications. Another approach sometimes adopted in the literature is to replace the spherical Laplacian operator $\mathcal{L}$ by the traditional-approximation operator $-\mathcal{L}_v$. This procedure avoids the Cowling approximation and allows the perturbation of the gravitational potential to be included in an approximate way. However, it does not correspond to the exact Poisson equation, since the gravitational potential is naturally expanded on associated Legendre polynomials rather than on Hough functions. This substitution should therefore be regarded as a practical approximation rather than a rigorous treatment. In the next section, we investigate the validity of the Cowling approximation and quantify the impact of this alternative treatment on the computed mode frequencies.

From a numerical point of view, including rotation within the traditional approximation is relatively straightforward. The main modification consists in replacing the eigenvalue of the spherical Laplacian, $-\ell(\ell+1)$, by the eigenvalue of Laplace's tidal equation, $\lambda_{v,k}^m$, whenever the angular operator acts on a perturbed quantity (Townsend, 2003; Bouabid et al., 2013). In practice, it is convenient to define an effective degree $\tilde{\ell}$ such that

$$\tilde{\ell}(\tilde{\ell}+1) = \lambda_{v,k}^m. \quad (5.147)$$

The radial pulsation equations then keep essentially the same form as in the non-rotating case, provided that the terms involving $\ell(\ell+1)$ are replaced by $\lambda_{v,k}^m$, or equivalently by $\tilde{\ell}(\tilde{\ell}+1)$.

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5.6.1 NUMERICAL IMPLEMENTATION OF THE TRADITIONAL APPROXIMATION

The inclusion of rotation through the traditional approximation is straightforward to implement in MAD, and more generally in one-dimensional stellar oscillation codes. As shown by the previous equations, the main modification to the system of differential equations consists in replacing ℓ by the effective angular degree $\tilde{\ell}$. The discretisation is therefore modified accordingly, with all occurrences of ℓ replaced by $\tilde{\ell}$. This treatment had already been implemented in MAD and LOSC by M.-A. Dupret and M.-P. Bouabid in [Bouabid et al. \(2013\)](#). With the traditional approximation, the discretised radial momentum equation becomes

$$\omega^2 \zeta_i = \frac{x_{2,i}}{\bar{\rho}_{1,i} d_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\tilde{\ell}-1} \beta_i - \frac{x_{2,i-1}}{\bar{\rho}_{1,i} d_{2,i}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\tilde{\ell}-1} \beta_{i-1} + \frac{x_{2,i}}{d_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\tilde{\ell}-1} \varphi_i \quad (5.148)$$

$$- \frac{x_{2,i-1}}{d_{2,i}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\tilde{\ell}-1} \varphi_{i-1} + \frac{d_{1,i-1} \bar{\rho}_{2,i-1} q_{2,i-1}}{2 d_{2,i} \bar{\rho}_{1,i} x_{2,i-1}} \left(\frac{x_{2,i-1}}{x_{1,i}} \right)^{\tilde{\ell}-1} \gamma_{i-1} + \frac{d_{1,i} \bar{\rho}_{2,i} q_{2,i}}{2 d_{2,i} \bar{\rho}_{1,i} x_{2,i}} \left(\frac{x_{2,i}}{x_{1,i}} \right)^{\tilde{\ell}-1} \gamma_i \quad (5.149)$$

$$+ \frac{q_{2,i} x_{1,i+1}^2}{2 d_{2,i} x_{2,i}^4} \left(\frac{x_{1,i+1}}{x_{1,i}} \right)^{\tilde{\ell}-1} \zeta_{i+1} + \frac{x_{1,i+1}^2}{2 d_{2,i}} \left(\frac{q_{2,i}}{x_{2,i}^4} - \frac{q_{2,i-1}}{x_{2,i-1}^4} \right) \zeta_i - \frac{q_{2,i-1} x_{1,i-1}^2}{2 d_{2,i} x_{2,i-1}^4} \left(\frac{x_{1,i-1}}{x_{1,i}} \right)^{\tilde{\ell}-1} \zeta_{i-1}. \quad (5.150)$$

The horizontal momentum equation, combined with mass conservation, is then written as

$$\omega^2 \left[x_{2,i}^2 \gamma_i + x_{1,i+1} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{\tilde{\ell}} \frac{\zeta_i}{d_{1,i}} - x_{1,i} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{\tilde{\ell}} \frac{\zeta_i}{d_{1,i}} \right] \quad (5.151)$$

$$= \tilde{\ell}(\tilde{\ell} + 1) \left[\varphi_i + \frac{\beta_i}{\rho_{2,i}} + \frac{q_{2,i}}{2 x_{2,i}^3} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{\tilde{\ell}+1} \zeta_{i+1} - \frac{q_{2,i}}{2 x_{2,i}^3} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{\tilde{\ell}+1} \zeta_i \right].$$

If the Cowling approximation is adopted, the Poisson equation does not need to be solved. However, when the perturbation of the gravitational potential is retained, a common practical approximation consists in replacing ℓ by $\tilde{\ell}$ in the discretised Poisson equation as well. With this approximation, the Poisson equation becomes

$$\frac{1}{d_{2,i+1} d_{1,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{2\tilde{\ell}+2} \varphi_{i+1} - \left[\frac{1}{d_{2,i+1} d_{1,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{2\tilde{\ell}+2} + \frac{1}{d_{2,i} d_{1,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{2\tilde{\ell}+2} \right] \varphi_i$$

$$+ \frac{1}{d_{2,i} d_{1,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{2\tilde{\ell}+2} \varphi_{i-1} = \bar{\rho}_{2,i} \gamma_i - \frac{\bar{\rho}_{1,i+1} - \bar{\rho}_{1,i}}{2 d_{1,i} x_{2,i}} \left(\frac{x_{1,i+1}}{x_{2,i}} \right)^{\tilde{\ell}+1} \zeta_{i+1} - \frac{\bar{\rho}_{1,i+1} - \bar{\rho}_{1,i}}{2 d_{1,i} x_{2,i}} \left(\frac{x_{1,i}}{x_{2,i}} \right)^{\tilde{\ell}+1} \zeta_i. \quad (5.152)$$

In MAD, we adopt this approximation and use the discretisation of the Poisson equation given above. Within the traditional approximation, the central behaviour of the eigenfunctions is no longer governed by the usual powers r^ℓ or $r^{\ell+1}$, but instead by $r^{\tilde{\ell}}$ or $r^{\tilde{\ell}+1}$. Consequently, the dimensionless formalism introduced in [Section 5.3](#) must be modified consistently by replacing ℓ with $\tilde{\ell}$.

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5.6.2 TRADITIONAL APPROXIMATION WITHOUT THE COWLING APPROXIMATION: THE UNCOUPLED APPROXIMATION

In this section, we present the new methodology we developed to include the Coriolis force through the traditional approximation, using what we refer to as the ‘uncoupled approximation’ to separate the Poisson equation.

In the framework of the traditional approximation, as shown in [Section 5.6](#), we require all the equations and quantities to be projected and decomposed on the Hough basis to solve the system of differential equations presented in [Equation 5.142](#), [Equation 5.143](#), and [Equation 2.72](#). In the case of the Poisson equation, as written in [Equation 2.72](#) the projection on the Hough basis can be obtained using

$$\langle \Theta_{\ell,v}^m | \nabla^2 \Psi' \rangle = 4\pi G \langle \Theta_{\ell,v}^m | \rho' \rangle, \quad (5.153)$$

where $\langle \dots | \dots \rangle$ denotes the scalar product on a sphere. The right-hand side of [Equation 5.153](#) is not the major problem here, as it can be separated by decomposing ρ' on the Hough basis:

$$\rho' = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{m=\ell} \rho_{\ell}^{\prime m} \Theta_{\ell,v}^m. \quad (5.154)$$

The problem of separability comes from the left-hand side of the equation. By decomposing Ψ' on the Hough basis for a given m , it can be rewritten as

$$\langle \Theta_{\ell,v}^m | \nabla^2 \Psi' \rangle = \left\langle \Theta_{\ell,v}^m \left| \sum_{\ell'} \nabla^2 \Psi_{\ell'}^m(r) \Theta_{\ell',v}^m \right. \right\rangle, \quad (5.155)$$

which can be rewritten to emphasise the dominant term of decomposition as follows:

$$\langle \Theta_{\ell,v}^m | \nabla^2 \Psi' \rangle = \langle \Theta_{\ell,v}^m | \nabla^2 \Psi_{\ell}^m(r) \Theta_{\ell,v}^m \rangle + \left\langle \Theta_{\ell,v}^m \left| \sum_{\ell' \neq \ell} \nabla^2 \Psi_{\ell'}^m(r) \Theta_{\ell',v}^m \right. \right\rangle. \quad (5.156)$$

If the system of equations is uncoupled and one mode dominates the decomposition, then the last term of the previous equation can be neglected; we call this assumption the ‘uncoupled approximation’. This approximation is valid when there is one dominant mode in the decomposition, which typically occurs at low spin parameters or near resonance, as we show below. By using this approximation, the Poisson equation can be rewritten as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi'}{dr} \right) - \frac{z_{\ell}^m}{r^2} \Psi' = 4\pi G \left(\delta\rho - \frac{d\rho}{dr} \xi_r \right), \quad (5.157)$$

where

$$z_{\ell}^m = \langle \Theta_{v,\ell}^m | \nabla^2 \Theta_{v,\ell}^m \rangle \quad (5.158)$$

is the coupling term as given by [Fuller \(2017\)](#) in the case of modal decomposition. To estimate z_{ℓ}^m , [Fuller \(2017\)](#) uses an integral representation of the scalar product. However, in our case we are using the properties of the Hough functions to obtain this parameter; these properties are described in [Equation 5.6.2](#). To maintain consistency with our stellar oscillation code, we introduce a modified spherical order $\hat{\ell}$, such as

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$\hat{\ell}(\hat{\ell} + 1) = z_\ell^m$. On the Hough basis and under the uncoupled approximation, the potential close to the centre of the star is proportional to $r^{\hat{\ell}}$, while close to the surface and outside of the star it is proportional to $r^{-(\hat{\ell}+1)}$. The surface boundary conditions of the Poisson equation are also modified by the inclusion of the Coriolis force. Those conditions are obtained by imposing the continuity of the potential perturbation and its derivative between the inner solution of the Poisson equation and the outer solution of the Laplace equation. Using the same formalism as in [Equation 5.157](#) the boundary condition of the Poisson equation can be expressed as

$$\frac{\partial \Psi'_1}{\partial r} + \frac{\hat{\ell} + 1}{r} \Psi'_1 = 0, \quad (5.159)$$

where the density has been neglected at the surface. With the uncoupled approximation, [Equation 5.157](#) can be solved with [Equation 5.142](#) and [Equation 5.143](#) on the Hough basis, as is traditionally done when computing tidally excited oscillations without including the effect of rotation. The main uncertainty of this approximation is the degree of validity of the uncoupled approximation; in [Subsection 6.1.4](#) we present a new modelling method that fully includes the mode coupling.

PROJECTIONS FROM AND ON THE HOUGH BASIS

Mathematically, each Hough function can be defined as a combination of associated Legendre polynomials:

$$\Theta_{\nu, \ell, k}^m(\mu) = \sum_{\ell'=m}^{\infty} c_{\ell', \ell, k}^m P_{\ell'}^m(\mu), \quad (5.160)$$

where $c_{\ell', \ell, k}^m$ can be obtained by solving the Laplace tidal equation, we used a matrix representation of this equation as described in [Unno et al. \(1989\)](#) to obtain simultaneously the eigenvalue of each Hough function λ and the spectral decomposition on the associated Legendre polynomials. Under the uncoupled approximation, these properties can be used to compute the coefficient z_ℓ^m defined in [Equation 5.158](#). Starting from [Equation 5.158](#), the Hough functions appearing in the scalar product are first decomposed onto associated Legendre polynomials using [Equation 5.160](#). The angular Laplace operator can then be applied term by term, since the associated Legendre polynomials are its eigenfunctions. Finally, using the orthogonality properties of the associated Legendre polynomials, one obtains

$$z_\ell^m = \sum_{\ell'=m}^{\infty} \ell' (\ell' + 1) \left(c_{\ell', \ell, k}^m \right)^2. \quad (5.161)$$

Throughout this thesis, this sum is truncated at $\ell' = 50$. Owing to parity selection rules, only terms with the same parity as ℓ contribute to the sum.

NUMERICAL IMPLEMENTATION OF THE TRADITIONAL APPROXIMATION WITHOUT THE COWLING APPROXIMATION

The numerical implementation of this treatment of the Poisson equation proved to be difficult, in particular because of the central discretisation adopted in MAD. Ideally, the

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oscillation equations would retain the discretisation used in the traditional approximation, and only the Poisson equation would be modified. In practice, however, the other equations also depend on the perturbation of the gravitational potential. Once this perturbation is expanded with a different normalisation, additional factors of the form $x^{\tilde{\ell}-\ell}$ appear. These factors diverge near the centre when $\tilde{\ell} > \ell$, which occurs for prograde and Rossby modes. This makes a direct implementation of the fully coupled treatment in MAD numerically delicate. For this reason, we did not develop a version of MAD capable of solving the complete coupled problem in this form. The difficulty is reduced in the case of tidally forced oscillations, because the tidal perturbation enters the system as an inhomogeneous forcing term. This makes it possible to treat the Poisson equation outside the main system of differential equations, providing a simpler and more stable way to include the effect of the gravitational-potential perturbation on the forced oscillations. In what follows, we therefore investigate the impact of the Cowling approximation using the tidally excited oscillation code developed in Chapter 6. This approach allows us to quantify how the inclusion of the perturbed gravitational potential modifies the amplitudes and spatial properties of the tidally forced modes, while avoiding the numerical difficulties associated with a fully coupled implementation in MAD.

5.6.3 IMPACT OF THE ROTATION ON THE γ -DORADUS IS.

this subsection is fully extracted from Fellay & Dupret (2026, submitted).

As mentioned previously, rotation is expected to impact the instability of oscillation modes in the γ -Doradus instability strip, in particular by modifying the number of unstable modes and shifting their frequencies depending on the azimuthal order m . In addition, rotation is expected to create an instability strip for Rossby modes within the γ -Doradus domain. Rossby modes are discussed in more detail in Figure 5.6.3. In this work we include the impact of the rotation through the traditional approximation for rotation, which is valid for high radial orders gravity modes as in γ -Doradus stars. A key consequence of adopting the traditional approximation is that modes are no longer decomposed onto spherical harmonics but onto Hough functions, which depend on rotation, frequency, and azimuthal order (Unno et al., 1989). Their associated eigenvalue λ plays a central role in determining the physical properties and observability of oscillation modes. The larger λ is for a given mode, the more confined the mode is to the equatorial regions, reducing its visibility. Thus, we generally expect to favor the observation of modes with low values of λ . The eigenvalue λ depends on the spin parameter ν , defined as

$$\nu = \frac{2\Omega_{\text{rot}}}{\omega}, \quad (5.162)$$

where ω is the oscillation frequency and Ω_{rot} is the stellar rotation rate. The behaviour of λ with respect to ν depends on both the spherical degree ℓ and the azimuthal order m . This is illustrated in Figure 5.21. As seen in Figure 5.21, retrograde modes in the corotating frame ($m > 0$) display a strong increase in λ with increasing spin parameter, while prograde modes in the corotating frame ($m < 0$) are much less affected and their λ values tend toward a low asymptote. Consequently, in fast-rotating stars, prograde

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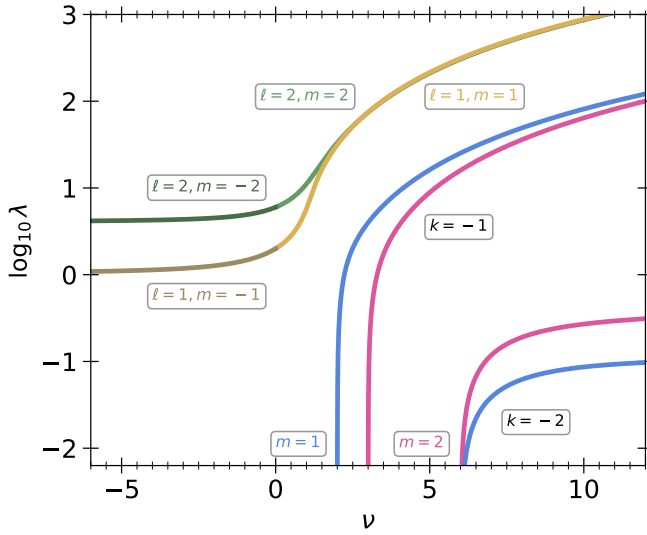


Figure 5.21: Evolution of λ as a function of ν for modes with different spherical degrees and azimuthal orders. Negative values of ν correspond to modes with $m < 0$. The red and blue curves show Rossby modes, while the green and yellow curves represent $\ell = 1, 2$ gravito-inertial modes.

modes are favoured observationally due to their higher visibility, while unstable retrograde modes are more difficult to observe. An interesting aspect about retrograde modes is that their inertial period (which corresponds to the frame of the observer), defined as,

$$P_{\text{in}} = \frac{P_{\text{co}}}{1 - m\nu/2}, \quad (5.163)$$

where P_{in} is the period in the inertial frame and P_{co} is the period in the corotating frame, can become negative at high spin parameters. In this case, modes are observed as prograde in the inertial frame, despite behaving as retrograde modes in the corotating frame. This is illustrated in Figure 5.22, which shows the inertial periods of retrograde $\ell = 1, m = 1$ modes in a $1.50 M_{\odot}$ γ -Doradus star rotating at $0.5\Omega_{\text{crit}}$. In this evolutionary sequence, unstable retrograde modes appear observationally as prograde, with periods in the classical γ -Doradus range. To further illustrate this, Figure 5.23 shows the period-spacing patterns for prograde and retrograde modes in fast-rotating γ -Doradus stars.

As seen in Figure 5.23, classical prograde modes occupy the low-period domain with rapidly decreasing period spacings, while retrograde modes exhibit increasing spacings at longer periods. This behaviour also applies to retrograde modes that appear as prograde (lower dark orange curve). The observed period spacings of many γ -Doradus stars (Li et al., 2020b) are well explained by the prograde modes, while retrograde modes are rarely detected. A systematic search for increasing-spacing patterns at longer periods could help identify retrograde modes and thus constrain the internal

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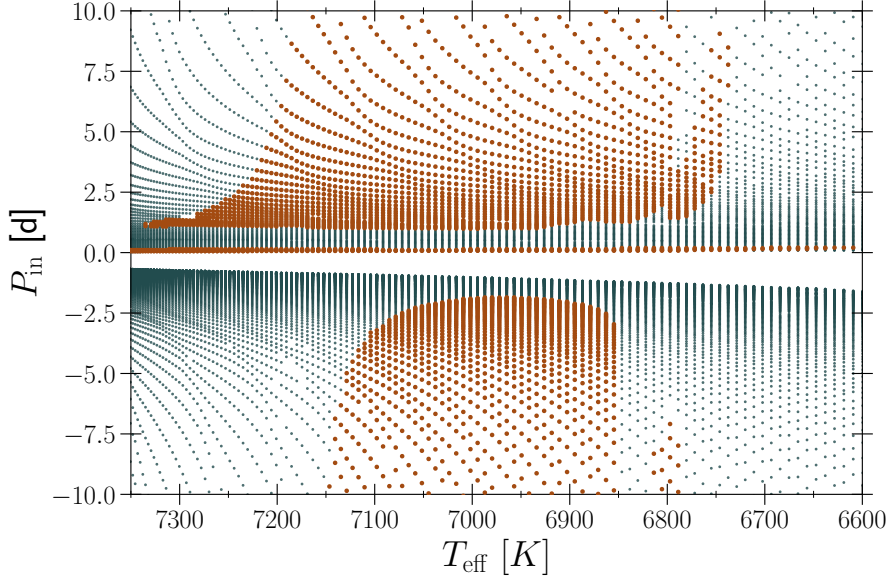


Figure 5.22: Inertial periods of retrograde modes ($\ell = 1, m = 1$) along the evolution of a $1.55 M_{\odot}$ γ -Doradus star with $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$. Grey dots correspond to stable modes, orange dots to unstable modes. Modes with negative inertial periods are retrograde modes appearing as prograde to the observer.

rotation of fast-rotating γ -Doradus stars. The $m = 0$ modes form decreasing spacing patterns, between the $m = -1$ and $m = +1$ modes, which is broadly consistent with the few observed cases (Li et al., 2020b).

To investigate the impact of rotation more generally, we extended the model grid presented in Subsection 5.5.2 to include rotation under the traditional approximation, assuming solid-body rotation. We considered initial rotation rates from $\Omega = 0$ to $0.5\Omega_{\text{crit}}$, in steps of $0.1\Omega_{\text{crit}}$. Here we focus on prograde modes, which dominate the observations of fast rotators. The full results, including all azimuthal orders, are made publicly available. In Figure 5.24 and Figure 5.25, we show the instability strips for prograde $\ell = 1$ and $\ell = 2$ modes respectively at $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$, for comparison with the reference grid (Figure 5.16). As seen in Figure 5.24, the overall position and width of the IS are not strongly affected by rotation, and the number of unstable modes is comparable to the non-rotating case. In contrast, the quadrupole ($\ell = 2, m = -2$) modes exhibit more excited modes, as expected from their higher mode density. An interesting impact of rotation is on the radial orders of the unstable modes. In Figure 5.26 and Figure 5.27, we illustrate the radial orders of excited modes as a function of effective temperature for $\ell = 1, m = -1$ and $\ell = 2, m = -2$ modes, respectively.

By comparing Figure 5.26 with the reference non-rotating case (Figure 5.10), we see that rotation shifts the excited modes to lower radial orders for $\ell = 1, m = -1$. For quadrupole prograde modes, the IS extends over $n \sim 17\text{--}84$, broader than for the

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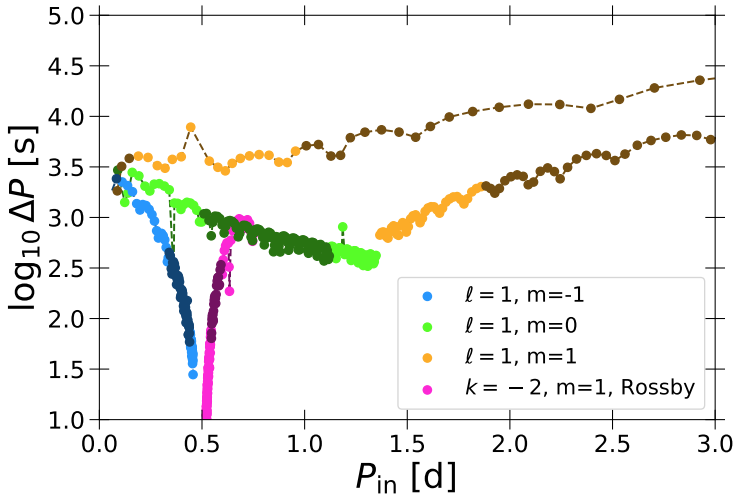


Figure 5.23: Inertial-frame period spacings as a function of period for a $1.55 M_{\odot}$ γ -Doradus star with $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$. Blue curves: prograde modes. Orange curve: retrograde modes. Green curve: $m = 0$ modes. Pink curve: $k = -2, m = 1$ Rossby modes. Darker colours correspond to unstable modes.

dipole modes. Observationally, this agrees with [Li et al. \(2020b\)](#), who found a broader radial-order distribution and higher median values for $\ell = 2$ modes compared to $\ell = 1$ modes. Overall, the blue edge of the IS tends to be characterised by the excitation of relatively low-order modes ($n \sim 15\text{--}40$), while the red edge favours higher-order modes ($n \sim 30\text{--}60$). Near the centre of the IS, modes with $n \sim 15\text{--}85$ are excited.

ROSSBY MODES

By including the effect of rotation on the oscillation modes, an additional class of gravito-inertial modes can appear in γ -Doradus stars: the Rossby modes. Rossby modes are low-frequency, large-scale gravito-inertial oscillations. Their theoretical instability strips were investigated for SPB stars by [Townsend \(2005\)](#), but have not previously been explored for γ -Doradus stars. Observationally, *Kepler* data revealed 83 γ -Doradus stars displaying Rossby modes ([Li et al., 2020b](#)).

The identification of Rossby modes requires two parameters, k and m . For details on k , see [Saio et al. \(2018\)](#). In [Figure 5.21](#), the λ values of Rossby modes are shown in blue and pink for the lowest k and m . Despite the higher spin parameters, the corresponding λ values are generally lower than those of retrograde gravito-inertial modes, and for $k = -2$ even lower than prograde modes. Since λ is a proxy for observability, $k = -2$ modes are more easily detected than $k = -1$. Similarly, $m = 1$ Rossby modes are favoured over higher- m values. Observations support this expectation, as nearly all Rossby modes detected by [Li et al. \(2020b\)](#) correspond to $k = -2, m = 1$. In [Figure 5.28](#) and [Figure 5.29](#), we illustrate the instability strips for the $k = -2, m = 1$ and

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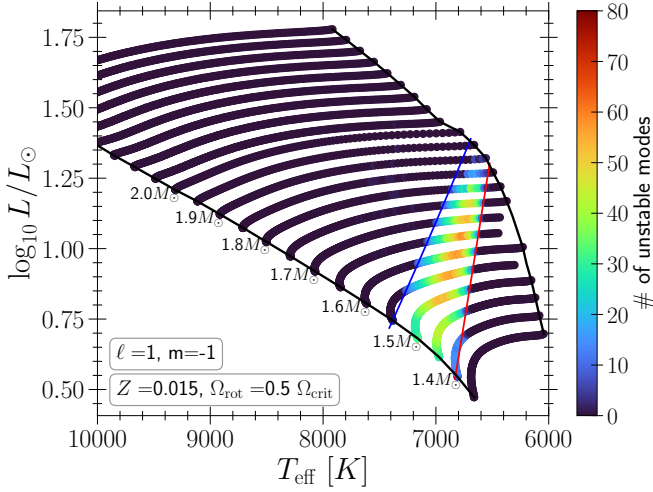


Figure 5.24: HR diagram of the γ -Doradus instability strip for prograde dipole modes ($\ell = 1, m = -1$) with $Z = 0.015$ and $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$. The colour code indicates the number of unstable modes. The blue and red edges are the same than in Figure 5.16.

$k = -2, m = 2$ Rossby modes, respectively. The theoretical IS for the $k = -2, m = 1$ Rossby modes is less extended in luminosity than that of the gravito-inertial modes, and the maximum number of unstable modes is shifted toward lower luminosities. This favours the detection of these modes in the lower-mass region of the γ -Doradus IS. By contrast, the $k = -2, m = 2$ Rossby modes occupy a similar HR-diagram region to the gravito-inertial modes. The main difference lies in the number of excited modes: a significantly larger number of $k = -2, m = 2$ modes are excited by the flux-blocking mechanism. However, only the $k = -2, m = 1$ modes have been detected in practice (Li et al., 2019). From these observations, Li et al. (2019) also constrained the radial-order range of the detected Rossby modes. In Figure 5.30, we show the theoretically predicted range for the $k = -2, m = 1$ modes. Near the blue edge of the IS, modes with radial orders $n \sim 13\text{--}28$ are excited, followed by a sharp transition to the main region where a flat distribution of modes with $n \sim 15\text{--}45$ is excited. Observations by Li et al. (2019) show a similar behaviour: a uniform distribution between $n = 10\text{--}15$ and $n = 45\text{--}50$, followed by a gradual decline at higher n . Overall the observed and predicted unstable range for Rossby modes seems consistent.

5.6.4 DISCUSSION AND CONCLUSION

In this article we have presented an updated view of the modelling of the γ -Doradus instability strip (IS), including the effects of metallicity, convective parameters, and rotation. We explored the γ -Doradus IS under variations of these parameters using a time-dependent convection (TDC) treatment of the convection–oscillation interaction (Dupret et al., 2005; Grigahcène et al., 2005). We found that these parameters have

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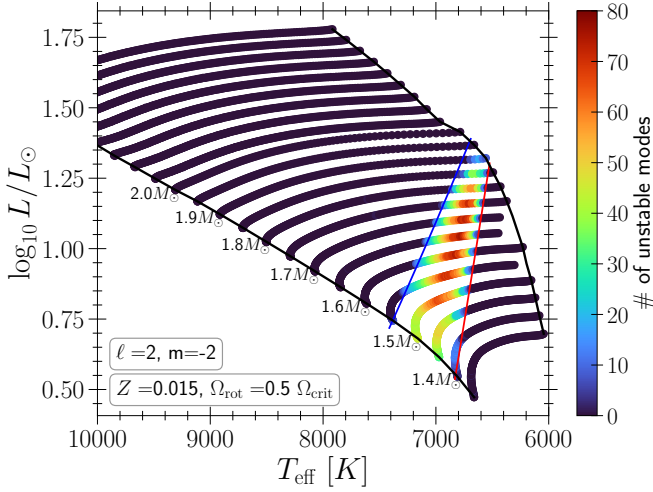


Figure 5.25: HR diagram of the γ -Doradus instability strip for prograde quadrupole modes ($\ell = 2, m = -2$) with $Z = 0.015$ and $\Omega_{\text{rot}} = 0.5 \Omega_{\text{crit}}$. The colour code indicates the number of unstable modes. The blue and red edges are the same than in Figure 5.16.

only a minor impact on the position of the IS in the HR diagram except for the MLT parameter. However, we showed that rotation affects the excitation of modes and the theoretical period ranges in which the different types of modes can be found. In particular, for fast rotators, retrograde $\ell = 1$ modes may appear as prograde in the observer’s frame, with periods close to 1.5 days and a distinctive ascending pattern in the period spacing, in line with previous theoretical and observational studies of gravito-inertial and Rossby modes in intermediate-mass stars (Bouabid et al., 2013; Van Reeth et al., 2015; Saio et al., 2018; Li et al., 2019). By including rotation we were also able to explore the Rossby-mode γ -Doradus IS, which is located in the same region of the HR diagram as the gravito-inertial γ -Doradus IS.

The theoretical exploration of the γ -Doradus IS also revealed interesting results concerning the radial orders driven by convective blocking when using the TDC treatment. When comparing the range of typically excited radial orders with the mode properties of the 611 clearly identified γ -Doradus stars observed by *Kepler* and analysed by Li et al. (2019), we found broad agreement between theory and observations. For prograde modes of both $\ell = 1$ and $\ell = 2$, our models reproduce the observed ranges of unstable modes, as well as global tendencies such as differences in the median radial orders depending on the mode type. For Rossby modes, where 83 pulsators have been observed, we also found good agreement between predicted and observed radial orders, consistent with previous theoretical expectations for r modes in upper main-sequence stars (Saio et al., 2018).

Concerning the position of the theoretical γ -Doradus IS compared to the observed

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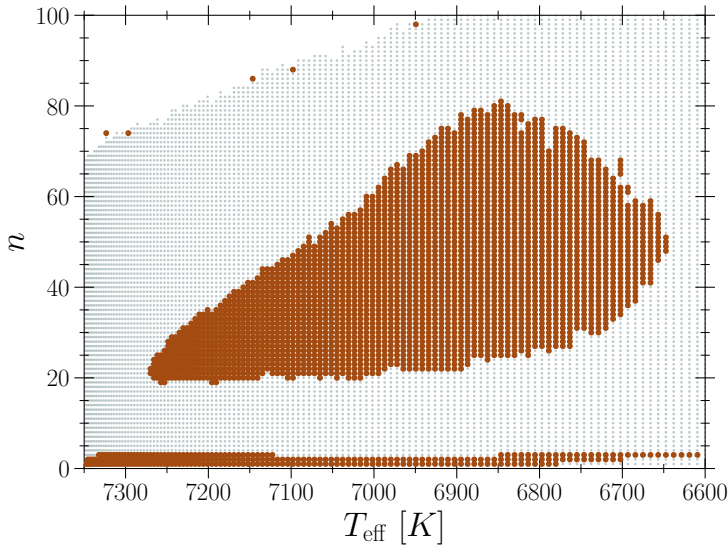


Figure 5.26: Radial orders of stable (black) and unstable (red) $\ell = 1, m = -1$ modes along the $1.50 M_{\odot}$ evolutionary track with $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$.

IS, we still find an extension towards the blue that is not predicted by our models, confirming earlier reports of hot or “outlier” γ -Doradus stars (Balona et al., 2016; Qian et al., 2019; Antoci et al., 2019; Li et al., 2019). Several phenomena could explain this discrepancy. The most straightforward is that the current TDC treatment is insufficient to accurately predict the blue edge of the γ -Doradus IS. More advanced descriptions of the convection–oscillation interaction may be required, potentially informed by new multi-dimensional convection formalisms and 3D simulations (Dupret et al., 2005; Grigahcène et al., 2005; Xiong et al., 2016; Lizin et al., 2024). However, developing such models is challenging, as it likely requires 3D convection modelling tailored to thin convective envelopes such as those of γ -Doradus stars. Other possible explanations for the extended blue edge include misplacement of stars in the HR diagram, for example due to unresolved binaries that make stars appear hotter or more luminous, and blending or contamination effects in space-based photometry, which can bias the inferred stellar parameters (Murphy et al., 2019).

By inspecting the γ -Doradus IS in more detail, we found that the blue edge predicted by the TDC treatment is characterised by the excitation of relatively low radial-order modes ($n \sim 15$ –40 for $\ell = 1$), while in the centre of the IS modes with $n \sim 20$ –80 are excited, and at the red edge only modes with $n \sim 30$ –60 are driven. An analysis of the radial orders of the excited modes of stars observed at the extreme blue edge of the γ -Doradus IS could be interesting. If such stars excite only low-order modes, they must indeed lie close to the theoretical blue edge, which models currently fail to reproduce. However, if they also excite high-order modes, then these stars could be misplaced in the HR diagram or the TDC treatment is insufficient to

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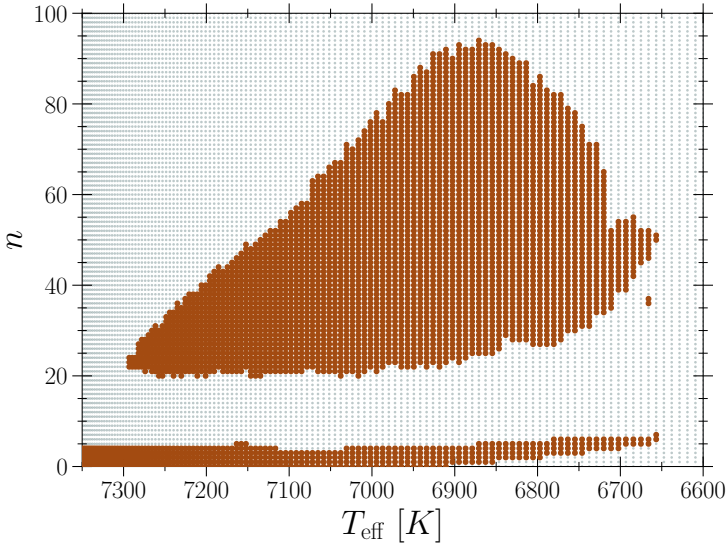


Figure 5.27: Radial orders of stable (black) and unstable (red) $\ell = 2, m = -2$ modes along the $1.50 M_{\odot}$ evolutionary track with $\Omega_{\text{rot}} = 0.5\Omega_{\text{crit}}$.

accurately predict the range of unstable modes in those regimes. All the computations made for this article and the full grids of γ -Doradus stars can be found at this adresse: <https://doi.org/10.58119/ULG/F80XLQ>.

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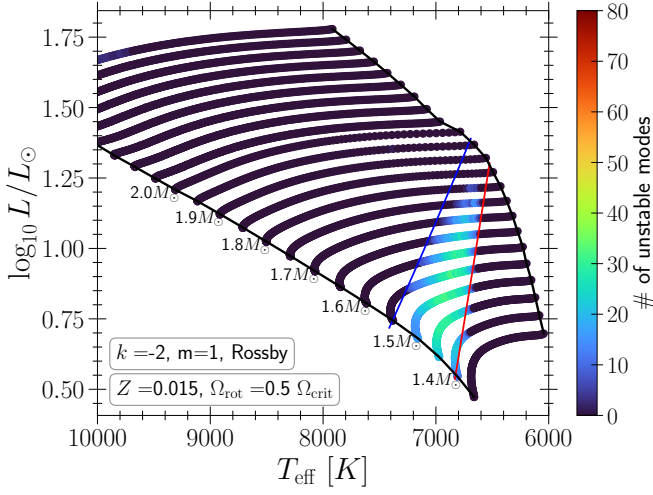


Figure 5.28: HR diagram of the γ -Doradus instability strip for Rossby modes with $k = -2, m = 1$ at $Z = 0.015$ and $\Omega_{\text{rot}} = 0.5 \Omega_{\text{crit}}$. The colour code indicates the number of unstable modes. The blue and red edges are the same than in Figure 5.16.

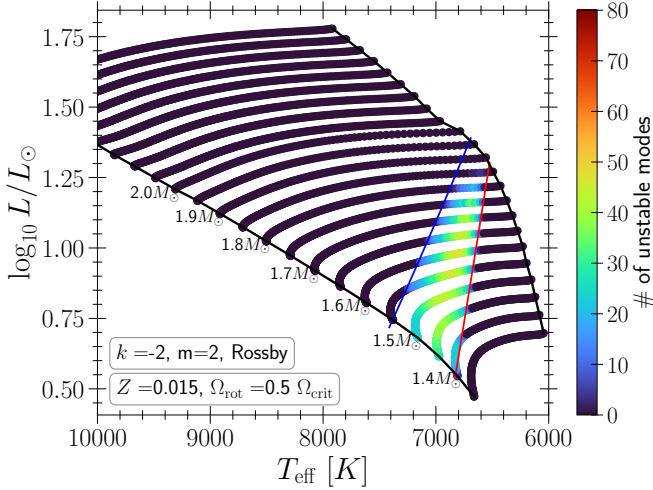


Figure 5.29: HR diagram of the γ -Doradus instability strip for Rossby modes with $k = -2, m = 2$ at $Z = 0.015$ and $\Omega_{\text{rot}} = 0.5 \Omega_{\text{crit}}$. The colour code indicates the number of unstable modes. The blue and red edges are the same than in Figure 5.16.

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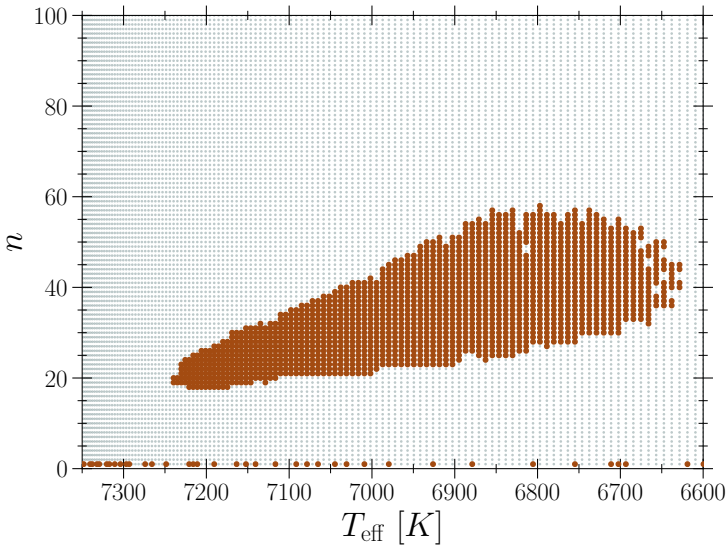


Figure 5.30: Radial orders of stable (black) and unstable (red) Rossby modes with $k = -2, m = 1$ along the $1.55 M_{\odot}$ reference track with $\Omega_{\text{rot}} = 0.5 \Omega_{\text{crit}}$.

TIDALLY EXCITED OSCILLATIONS IN FAST ROTATING STARS

this chapter is mostly extracted and an expansion from [Fellay & Dupret \(2025\)](#).

In recent years, detections of tidally excited stellar oscillations (TEOs) in multiple star systems have opened new opportunities to study stellar physics and understand stellar structural properties. TEOs have been observed across all types of classical pulsators, and particularly in low- and intermediate-mass binary stars located in the classical pulsator instability strips ([Li et al., 2024](#)).

From a physical perspective, stars can act as a forced oscillator. A given single star has resonances frequencies (which are the oscillation modes eigenfrequencies obtained with a stellar oscillation code) such as if a perturbation with frequency close to one of those resonance frequency is applied to a the star, an high amplitude mode is able to propagate into the star. In the case of TEOs, the perturbation is coming from the presence of the companion (and the tidal force) that will force the oscillation with forcing frequencies that are multiple of the orbital frequency of the system. Tidal forces can induce two primary effects: large-scale mass redistribution due to a global hydrostatic adjustment (i.e. equilibrium tides) and the excitation of internal oscillation waves (i.e. dynamical tides ([Zahn, 1970, 1975](#))). The periodic perturbation from a companion can excite a broad spectrum of low-frequency stellar oscillations, including gravity waves, inertial waves, and Rossby waves. However, for standing waves to be excited, the perturbation frequency must be in tune with a free oscillation frequency of the star, producing a resonance. Non-linear mode excitation, where a single mode excites multiple daughter modes, may also occur ([Weinberg et al., 2012](#); [Guo, 2020](#); [Guo et al., 2022](#)), as can the excitation of low-frequency travelling waves ([Burkart et al., 2012](#); [Bowman et al., 2019](#); [Ratnasingam et al., 2019](#)). Here, we focus on the case where the perturbation from the companion linearly excites a single standing wave.

Following the detection of TEOs, rapid progress was made in theoretical modelling methods to understand their behaviour. There have been two main approaches to mod-

6.1. MODELLING TIDALLY EXCITED OSCILLATIONS

elling TEOs: direct modelling (Smeyers et al., 1998; Willems, 2003; Burkart et al., 2012; Valsecchi et al., 2013; Sun et al., 2023) and modal decomposition (Zahn, 1970, 1975; Burkart et al., 2012; Fuller & Lai, 2012; Fuller, 2017). In the direct modelling approach, the properties of TEOs are obtained by modifying a stellar oscillation code to include the periodic perturbation from a companion. Two notable oscillation codes developed with this methodology are CAFein (Valsecchi et al., 2013) and gyre.tides (Sun et al., 2023), both of which account for non-adiabatic effects on the oscillation modes. In the modal decomposition approach first introduced by Zahn (1970, 1975), the perturbation from the companion is decomposed into a combination of free oscillation eigenfunctions with different weights. This methodology relies on the adiabatic assumption, and its validity becomes uncertain when considering non-adiabatic oscillation modes, leading to discrepancies in the upper layers of stars (see e.g. Sun et al. (2023)).

A key advantage of the modal decomposition method is the possibility to include the effects of rotation via the traditional approximation (Fuller, 2017). In contrast, incorporating rotational effects in direct modelling is less straightforward. Classical oscillation codes often rely on the traditional approximation coupled with the Cowling approximation (Cowling, 1941; Townsend, 2005), which neglects potential perturbation in the differential equation system and excludes the Poisson equation from the system to be solved. For dynamical tides, the potential perturbation is crucial, as the orbital migration due to the excitation of oscillation modes in the star is proportional to this perturbation (Smeyers et al., 1998; Willems, 2003; Valsecchi et al., 2013; Sun et al., 2023). When following the approach of Smeyers et al. (1998); Willems (2003); Valsecchi et al. (2013); Sun et al. (2023), the potential perturbations have to be treated consistently, necessitating the simultaneous solution of the Poisson equation and the system of differential equations. In Witte & Savonije (1999a,b, 2001); Savonije & Witte (2002); Savonije (2005), an equivalent formalism is presented that only requires the density perturbation to compute the dynamical tides, meaning that in this case the Cowling approximation can be used.

6.1 MODELLING TIDALLY EXCITED OSCILLATIONS

In the previous section, we described the methodology used in MAD to compute the free oscillations of single stars. In this section, we present the adaptations required in MAD to compute the oscillation modes following the direct modelling approach (without stellar rotation) excited by the periodic tidal perturbation from a nearby star. The modifications can be divided into two parts: the inclusion of the perturbation from the companion, presented in Subsection 6.1.1, and the modifications of the algorithm used to solve the system, presented in Subsection 6.1.3.

6.1.1 INCLUSION OF THE PERTURBATION FROM THE COMPANION IN THE PERTURBATIVE FRAMEWORK

For non-rotating stars, the dynamical part of the system of equations to solve is given in Equation 5.142 and Equation 5.143. The impact of the companion appears by

6.1. MODELLING TIDALLY EXCITED OSCILLATIONS

adding the gradient of the potential generated by the companion in the motion equation, which gives:

$$\sigma^2 \xi_r = \frac{d(\Psi' + \Psi_2)}{dr} + \frac{d(g\xi_r)}{dr} + \frac{1}{\rho} \frac{d\delta P}{dr} + g \frac{\delta \rho}{\rho}, \quad (6.1)$$

and,

$$\frac{\delta \rho}{\rho} + \frac{1}{r^2} \frac{d}{dr}(r^2 \xi_r) - \frac{\ell(\ell+1)}{\sigma^2 r^2} \left(\Psi' + \Psi_2 + g\xi_r + \frac{\delta P}{\rho} \right) = 0, \quad (6.2)$$

where Ψ_2 is the potential from the secondary evaluated at each layer of the primary and projected onto a spherical harmonics basis. By adding the presence of the companion, the Poisson equation remains unmodified as Ψ_2 is a solution of the Laplace equation. The gravitational potential from the companion is classically obtained using the perturbative approach which assume that stellar deformations are small perturbation from the spherical symmetry compared to the radius of the star. In this work, as is classically done, the secondary is approximated by a point mass. Within this approximation and in the perturbative framework, the tidal perturbing potential from the companion can be written on the coordinate frame centred on the primary as:

$$\Psi_2(\mathbf{r}, t) = -\frac{GM_2}{r_{\text{sep}}} \sum_{\ell \geq 1} \left(\frac{r}{r_{\text{sep}}} \right)^\ell P_\ell(\lambda), \quad (6.3)$$

where r_{sep} is the instantaneous separation of the binary system with orbital eccentricity e , a semi-major axis a and a true anomaly ν at a given time, written as:

$$r_{\text{sep}} = \frac{a(1 - e^2)}{1 + e \cos(\nu)}. \quad (6.4)$$

$P_\ell(\lambda)$ are the Legendre polynomial with $\lambda = \cos \phi_0 \sin \theta$, θ begin the colatitude and ϕ_0 is the azimuthal angle of the element studied with respect to the line linking the two stars and the center of mass of the system. This angle can be expressed as a function of the true anomaly of the system ν and ϕ the azimuthal angle between the position of the element studied and the apsidal line:

$$\phi = \phi_0 + \nu. \quad (6.5)$$

In [Figure 6.1](#) we detail all those angle for a given binary system. The Legendre polynomials can be rewritten as a function of spherical harmonics with:

$$P_\ell(\lambda) = \sum_{-m}^m d_\ell^m Y_\ell^m(\theta, \phi_0), \quad (6.6)$$

where the d_ℓ^m are defined as:

$$d_\ell^m = \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(0), \quad (6.7)$$

where P_ℓ^m are the associated Legendre polynomials. The objective now is to decompose the perturbation from the companion with Fourier series as this perturbation is

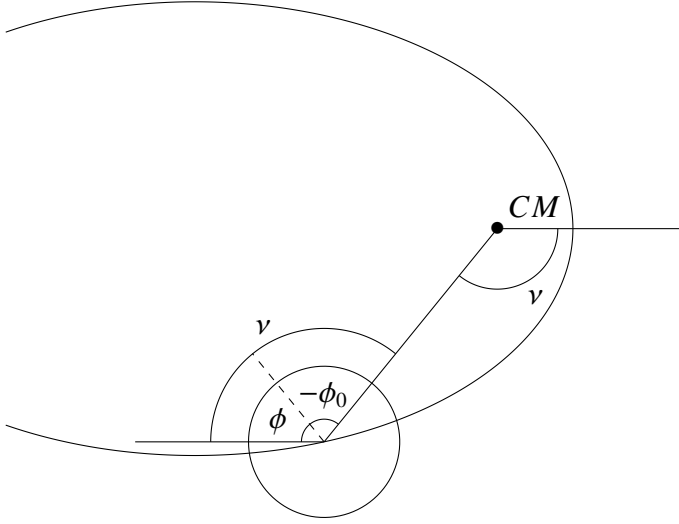


Figure 6.1: Illustration of the different angles used in the developments of the dynamical tides.

periodic over an orbit to obtain the effective forcing of the stars for each forcing frequency of the system. If a function $f(\mathcal{M})$ is 2π periodic and can be integrated then this function can be decomposed with Fourier series:

$$f(\mathcal{M}) = \sum_{k=-\infty}^{\infty} f_k e^{-ik\mathcal{M}}, \quad (6.8)$$

with,

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(r) e^{ik\mathcal{M}} d\mathcal{M}, \quad (6.9)$$

where the f_k uniformly converges toward $f(r)$. In the previous equations $\mathcal{M}$ is the mean anomaly ($\mathcal{M} = \Omega_{\text{orb}} t$, Ω_{orb} being the orbital angular frequency). Rewriting $P_\ell(\lambda)$ in [Equation 6.3](#), we obtain:

$$\Psi_2(\mathbf{r}, t) = -\frac{GM_2}{r} \sum_{\ell > 1} \sum_{m=-\ell}^{\ell} \left(\frac{r}{r_{\text{sep}}}\right)^{\ell+1} d_\ell^m P_\ell^m(\cos \theta) e^{im(\phi-\nu)}, \quad (6.10)$$

which can be rewritten by decomposing $\Psi_2(\mathbf{r}, t)$ in Fourier series:

$$\Psi_2(\mathbf{r}, t) = -\frac{GM_2}{r} \sum_{\ell > 1} \sum_{m=-\ell}^{\ell} \left(\frac{r}{a}\right)^{\ell+1} d_\ell^m P_\ell^m(\cos \theta) e^{im\phi} \sum_{k=-\infty}^{\infty} e^{-ik\mathcal{M}} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r_{\text{sep}}}\right)^{\ell+1} e^{i(k\mathcal{M}-m\nu)} d\mathcal{M}, \quad (6.11)$$

which can be rewritten as

$$\Psi_2(r, \mu, \phi, \mathcal{M}) = -\frac{GM_2}{r} \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{r}{a}\right)^{\ell+1} X_k^{-(\ell+1), -m} e^{i(k\mathcal{M}+m\phi)} d_\ell^m P_\ell^m(\cos \theta),$$

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by identifying the Hansen coefficients, defined as

$$X_k^{n,-m} = \frac{1}{P_{\text{orb}}} \int_0^{2\pi} \left(\frac{r_{\text{sep}}}{a} \right)^n \cos(mv + k\mathcal{M}) dt = \frac{(1 - e^2)^{n+3/2}}{2\pi} \int_0^{2\pi} \frac{\cos(mv + k\mathcal{M})}{(1 + e \cos v)^{n+2}} dv,$$

For a given oscillation mode, the radial component of the tidal perturbing potential is therefore given by:

$$\Psi_{2,\ell,k}^m(r) = -\frac{GM_2}{r} \left(\frac{r}{a} \right)^{\ell+1} X_k^{-(\ell+1),-m} d_\ell^m, \quad (6.12)$$

which is the expression of the potential to insert in Eqs. 6.1, and 6.2. Furthermore, for tidally excited oscillations, the time dependent part of the entire system of differential equations is decomposed into a Fourier series, as the perturbation from the companion is periodic and each response becomes periodic.

In the case of non-rotating stars, [Sun et al. \(2023\)](#) pointed out that all the modifications presented in this section can be achieved by only modifying the surface boundary condition of the Poisson equation and solving the other equations using $\Psi' + \Psi_2$ as the unknown related to the potential. Mathematically, the surface boundary condition of the Poisson equation has to be modified to

$$\frac{\partial(\Psi' + \Psi_2)}{\partial r} + \frac{\ell + 1}{r}(\Psi' + \Psi_2) = -4\pi G\rho\xi_r + \frac{\partial\Psi_2}{\partial r} + \frac{\ell + 1}{r}\Psi_2. \quad (6.13)$$

The previous equation was obtained by adding $\partial\Psi_2/\partial r + \Psi_2(\ell + 1)/r$ on each side of the Poisson equation surface boundary condition ([Equation 5.93](#)), for more details about this boundary condition see [Subsection 5.2.3](#). By modifying either the surface boundary condition of the Poisson equation or the entire system of differential equations, the system to solve becomes inhomogeneous. Consequently, the algorithm for solving the system of equations must be modified, as presented in the next section.

6.1.2 INCLUSION OF THE PERTURBATION FROM THE COMPANION IN THE NON-PERTURBATIVE FRAMEWORK

In the previous section, the tidal perturbation exerted by the companion was treated within a perturbative framework, in which the companion was approximated as a point mass. The second part of this thesis was devoted to the development of a non-perturbative method aimed at modelling the full gravitational and tidal potential generated by each distorted component of a binary system. One of the main results of this work was the identification of a dipolar tidal component in highly distorted configurations, together with significant modifications of the usual quadrupolar tidal potential.

Although the present formalism for tidally excited oscillations is not suited to systems that are so strongly distorted that the oscillation cavities themselves are significantly deformed, it remains applicable to wider binaries in which non-perturbative effects already modify the tidal forcing, or in which a strongly distorted secondary can still excite oscillation modes in the primary. In such cases, it is worth considering the tidal forcing produced by a companion described with the non-perturbative MoBiDICT

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approach. One practical advantage of this method is that it directly provides the perturbing potential of the companion projected onto modified spherical harmonics, here normalized differently from the standard convention. Within the MoBiDICT framework, the gravitational potential of the secondary can be written as

$$\Psi_2(\mathbf{r}, t) = \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \Psi_{2,\ell}^m(r, \nu) \tilde{Y}_{\ell}^m(\theta, \phi_0) = \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \Psi_{2,\ell}^m(r, \nu) \tilde{d}_{\ell}^m P_{\ell}^m(\cos \theta) \cos(m\phi_0), \quad (6.14)$$

where $\phi_0 = \phi - \nu$, and $\tilde{d}_{\ell}^m$ is a normalization coefficient that will be specified later. Using the complex exponential form of the cosine, this expression becomes

$$\Psi_2(\mathbf{r}, t) = \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \Psi_{2,\ell}^m(r, \nu) \tilde{d}_{\ell}^m P_{\ell}^m(\cos \theta) \left[e^{im(\phi-\nu)} + e^{-im(\phi-\nu)} \right]. \quad (6.15)$$

Following the same strategy as in the perturbative case, the tidal potential can then be expanded as a Fourier series in the mean anomaly $\mathcal{M}$:

$$\Psi_2(\mathbf{r}, t) = \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{d}_{\ell}^{|m|} P_{\ell}^{|m|}(\cos \theta) e^{im\phi} \sum_{k=-\infty}^{\infty} \Psi_{2,\ell,k}^m(r) e^{ik\mathcal{M}}, \quad (6.16)$$

where the radial Fourier coefficients are defined by

$$\Psi_{2,\ell,k}^m(r) = \frac{1}{2\pi} \tilde{d}_{\ell}^{|m|} \int_0^{2\pi} \Psi_{2,\ell}^{|m|}(r, \nu) e^{i(m\nu - k\mathcal{M})} d\mathcal{M}. \quad (6.17)$$

Because of the symmetry of the system, only the cosine part contributes in practice, so that one may equivalently write

$$\Psi_{2,\ell,k}^m(r) = \frac{1}{2\pi} \tilde{d}_{\ell}^{|m|} \int_0^{2\pi} \Psi_{2,\ell}^{|m|}(r, \nu) \cos(m\nu - k\mathcal{M}) d\mathcal{M}. \quad (6.18)$$

The case $m = 0$ must be treated separately to avoid double counting of the axisymmetric term. These coefficients can only be computed numerically, since they require the stellar deformation to be evaluated with MoBiDICT for a sequence of true anomalies, following a procedure similar to that used for the calculation of apsidal motion. The inclusion of MoBiDICT in the computation of dynamical tides will not be discussed further in this section, because the associated computational cost is high and large parameter studies remain difficult. Moreover, the aim of the present section is to introduce the basic principles of tidally excited oscillations. The impact of MoBiDICT on these oscillations is therefore left to the discussion in [Chapter 8](#); given the available time and the remaining physical developments required, this direction was not pursued further in the present work.

6.1.3 MODIFICATIONS OF THE ALGORITHM

In the case of free oscillations, the system of equations to be solved is presented in the previous section. In practice, we express these equations in a matrix representation. This representation allows us to write the system of equations as:

$$(\mathbf{A} - \sigma\mathbf{B})\mathbf{x} = 0, \quad (6.19)$$

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where $\mathbf{A}$ and $\mathbf{B}$ are complex square band matrices. The radial discretisation of the equations were made using staggered grids following the approach of [Castor \(1971\)](#), for the computations made in this article ten thousands points on each grids were used. In MAD, the eigenvectors $\mathbf{x}$ and eigenvalues σ are obtained using a generalised inverse iteration algorithm. More details about the algorithm employed to solve the system, the matrix representation of the problem, and its discretisation can be found in [Dupret \(2001\)](#).

In the case of dynamical tides, since the perturbation from the companion is periodic and the system is developed in a Fourier series, the forcing frequencies σ are determined by the orbital properties of the system and stellar rotation. Instead of solving the system of equations for a single frequency, a dependency on k appears, and the system must be solved independently for different values of k , both positive and negative. Mathematically, the forcing frequencies expressed in the corotating frame for a given orbital and stellar rotation rate, as:

$$\sigma_{m,k} = k\Omega_{\text{orb}} + m\Omega_{\text{rot}}. \quad (6.20)$$

It should be noted that in this article, we have adopted the sign convention for k of [Valsecchi et al. \(2013\)](#), whereas [Sun et al. \(2023\)](#) adopted the opposite sign convention. In this article, we also adopted the convention that $\sigma_{m,k}$ should remain positive. If a negative frequency is obtained, the same oscillation mode can be studied by considering an opposite of m and k . With our sign conventions, $k, m < 0$ corresponds to prograde waves and $k, m > 0$ to retrograde waves in the inertial frame. Prograde waves in the inertial frame have in general larger amplitudes because their Hansen coefficients are larger. Furthermore, $m > 0$ corresponds to retrograde waves and $m < 0$ to prograde waves in the corotating frame. In this paper, when not specified, "prograde/retrograde" means "prograde/retrograde in the corotating frame", what matters in term of mode physics.

The only unknown of the problem are the eigenfunctions of each oscillation mode. Moreover, as shown in [Subsection 6.1.1](#), the inclusion of the perturbation from the companion transforms the system to solve into an inhomogeneous set of differential equations. The new system to solve can be written as

$$(\mathbf{A} - \sigma_{m,k}\mathbf{B})\mathbf{x} = \mathbf{C}_k, \quad (6.21)$$

where $\mathbf{C}$ is a complex vector containing the inhomogeneous part of the system of differential equation. To solve [Equation 6.21](#) we used the LU factorisation already implemented in MAD but without the inverse iterative algorithm as the frequency is now an input parameter given by [Equation 6.20](#) and no longer the free oscillation frequency.

6.1.4 ITERATIVE SOLVING OF THE POISSON EQUATION: THE COUPLED SOLUTION

The novelty of our new methodology is that we treat the potential perturbation as an inhomogeneous term in [Equation 5.142](#) and [Equation 5.143](#), thereby separating the

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solving of the Poisson equation from that of the main system of differential equations. We then iteratively solve the system of differential equations for several Hough oscillation modes before combining them to solve the coupled Poisson equation and obtain the potential perturbations. As the coupling of the oscillations is arising from the non-separability of the Poisson equation, we call this mode coupling ‘gravitational coupling’.

In the case of the coupled oscillations, the Poisson equation, as written in [Equation 5.156](#) can be expanded using the projections presented in [Equation 5.6.2](#), and can be expressed as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Psi''}{dr} \right) - \frac{\hat{\ell}(\hat{\ell} + 1)}{r^2} \Psi'' = 4\pi G \hat{\rho}', \quad (6.22)$$

where we define

$$\hat{\rho}' \equiv \left(\delta\rho - \frac{d\rho}{dr} \xi_r \right) + \frac{1}{4\pi G r^2} \sum_{\ell'' \neq \ell} \sum_{\ell'} c_{\ell, \ell'} c_{\ell'', \ell'} \ell' (\ell' + 1) \Psi_{\ell''}^m. \quad (6.23)$$

With this formulation, solving the Poisson equation for a given oscillation mode requires the potential perturbation from modes with the same ℓ parity and same m . In theory, this equation should be solved simultaneously for those few oscillation modes. In our modelling method, we have chosen to iteratively solve the system of equations for each oscillation mode, coupling the oscillations through the Poisson equation at each iteration step. It is well known that the projections of the potential on spherical harmonics are solution of an integral equation. We note that a similar result is found for the projections on the Hough basis. The potential given by the following integral relation is the solution of the Poisson equation ([Equation 6.22](#)):

$$\Psi_{\ell}^m(r) = -\frac{4\pi G}{(2\hat{\ell} + 1)r^{\hat{\ell}+1}} \int_0^r \hat{\rho}_{\ell}^{m'}(r') r'^{\hat{\ell}+2} dr' - \frac{4\pi G r^{\hat{\ell}}}{2\hat{\ell} + 1} \int_r^{\infty} \hat{\rho}_{\ell}^{m'}(r') r'^{1-\hat{\ell}} dr'.$$

Our iterative process uses this method to solve the coupled Poisson equation. The iterative process is designed as follows:

1. We independently solve the system of differential equations for all the oscillation modes from $\ell' = m$ to $\ell' = L$ with the same parity as m using our uncoupled approximation. This provides us with an initial estimation of Ψ_{ℓ}^m , $\delta\rho$, and ξ_r .
2. For all the oscillation modes, we couple our oscillations by computing $\hat{\rho}'$ and independently solving the Poisson equation with [Equation 6.24](#) to obtain $\Psi_{\ell}^m(r)$.
3. We then solve the system of differential equations again, excluding the Poisson equation and treating the previously obtained potential perturbation as an independent term in the motion and continuity equations.
4. If the code has not converged, we go back to the second step with the newly obtained $\delta\rho$ and ξ_r .

6.1.5 NUMERICAL IMPLEMENTATION OF THE TIDALLY EXCITED OSCILLATIONS

As previously mentioned, including the effect of the rotation properly treating the Poisson equation is particularly tedious to include numerically due to the central discretisation of MAD which does not allow cases where $\hat{\ell} > \tilde{\ell}$. However, having an inhomogeneous term in the system of differential equation is a particularly useful to treat such cases as it allow for solving the Poisson equation outside of the main system of differential equations and introduce the perturbed potential in the inhomogeneous part of the matrix system. In the cases $\hat{\ell} > \tilde{\ell}$, this method is systematically used, we first solve the system of differential equation using $\hat{\ell} = \tilde{\ell}$, then solve the Poisson equation using the real value of $\hat{\ell}$, then introduce the obtained potential as an inhomogeneous term and iterate until convergence. In addition, obtaining the imaginary part of the potential perturbation at the stellar surface requires an iterative procedure, because this quantity is particularly sensitive to numerical imprecision. Similar numerical difficulties are also encountered in MAD, which is why the imaginary part of the eigenfrequency is often recomputed from the work integral rather than taken directly from the eigenvalue solution.

6.1.6 VERIFICATIONS OF THE METHODOLOGY

The numerical implementation of our method was made step by step to ensure the numerical stability and correctness of each portion of the code. Initially, we used a version of MAD that computed free gravito-inertial modes using the traditional approximation. We then implemented tidal forcing by modifying the surface boundary condition of the Poisson equation, as presented in [Subsection 6.1.1](#). We then implemented the tidal forcing through the modifications of the whole system of differential equations, as described in [Subsection 6.1.1](#), decoupled the Poisson equation and solved it independently. We verified that both version of the tidal forcing were giving the same results and eigenfunctions for a few oscillation modes for a given stellar model. We then proceeded to modify it to include the effect of the rotation through full iterative scheme described in [Subsection 6.1.4](#).

For the presentation of the results in this article, we define the following dimensionless variables:

$$\tilde{\Psi} = \frac{R}{GM}\Psi; \tilde{\xi}_r = \frac{\xi_r}{R}; \tilde{\xi}_h = \frac{\xi_h}{R}; x = \frac{r}{R}, \quad (6.24)$$

where M is the mass of the studied and R is its radius. To verify the validity of our new tidal oscillation method in a non-rotating case, we wanted to compare it with two existing tidal oscillation codes: CAFein ([Valsecchi et al., 2013](#)) and gyre tides ([Sun et al., 2023](#)). However, our version of MAD is currently optimised to work with our stellar evolution code, the Code Liegeois d'Evolution Stellaire (CLES, [Scuflaire et al., 2008b](#)), which guarantees high radial precision models to accurately compute stellar oscillations. Consequently, we cannot use models from MESA, as used in CAFein and gyre tides. This limitation means direct comparisons of oscillation results are not feasible due to differences in stellar structures. A comprehensive comparison of all three oscillation codes would require careful consideration and could be the focus of future research.

To avoid this problem, we decided to directly compare the published results, in particular all the results given in [Valsecchi et al. \(2013\)](#). We computed a stellar model equivalent to the model of the SPB star presented in [Valsecchi et al. \(2013\)](#) with CLES, ensuring it matched the characteristics of their twin binary system. Our stellar model has a mass of $5.0M_{\odot}$, an initial metallicity of 0.18, and we evolved our star until the central hydrogen reaches $X_C = 0.5$, we then verified with OSC that the free oscillation of the g-modes were close to ones of [Valsecchi et al. \(2013\)](#). For the stellar modelling we used the AGSS09 abundances ([Asplund et al., 2009](#)), the FreeEOS equation of state ([Irwin, 2012](#)), the OPAL opacities ([Iglesias & Rogers, 1996](#)), the $T(\tau)$ relation from Model-C of [Vernazza et al. \(1981\)](#) for the atmosphere, the mixing length theory of convection implemented as in [Cox & Giuli \(1968\)](#), and the nuclear reaction rates of [Adelberger et al. \(2011\)](#). We then used the same set-up for the binary as [Valsecchi et al. \(2013\)](#), namely we studied the quadrupolar tidally induced oscillation, with $m = 0$ and $k = 1$, for a twin binary system with an eccentricity of 0.4. We then reproduced different figures from [Valsecchi et al. \(2013\)](#), in particular we look at the surface radial displacement for our twin binary around two resonances (Fig.5 from [Valsecchi et al. \(2013\)](#)), [Figure 6.2](#) illustrates this displacement as a function of the dimensionless frequency σ_T ($\sigma_T = \sigma \sqrt{R^3/GM}$). For the same binary system, we also

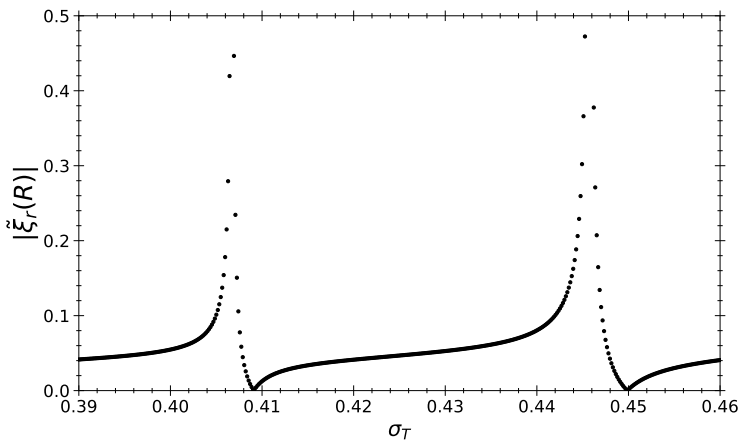


Figure 6.2: Surface radial displacement of the mode $\ell = 2, m = 0, k = 1$ for the twin binary system composed of $5.0M_{\odot}$ SPB stars with an eccentricity of 0.4. This figure is a reproduction of Fig. 5 of [Valsecchi et al. \(2013\)](#).

verified the eigenfunction of the tidally excited oscillations, illustrated in [Figure 6.3](#).

6.1.7 ON THE VALIDITY OF THE COWLING APPROXIMATION

In this section, we examine the validity of the Cowling approximation, which neglects the perturbation of the gravitational potential in the continuity, mass conservation, and radial motion equations. Theoretically, this approximation is expected to be valid in

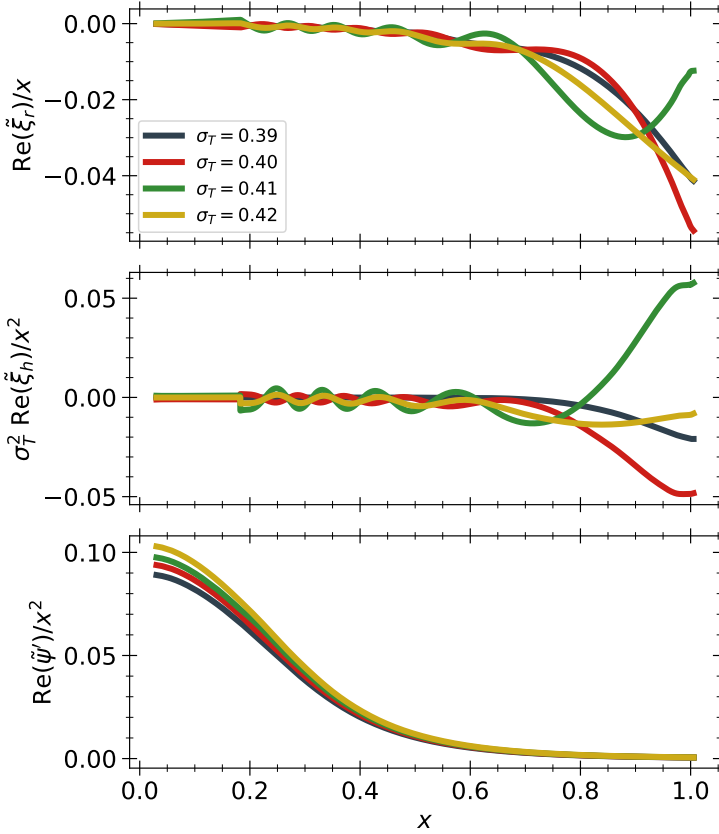


Figure 6.3: Eigenfunctions of the mode $\ell = 2, m = 0, k = 1$ for the twin binary system composed of $5.0M_{\odot}$ SPB stars with an eccentricity of 0.4. We chose eigenfunctions of oscillations modes out of resonance. The eigenfunction can directly be compared to Fig. 7 of [Valsecchi et al. \(2013\)](#)

the regime of high radial order and/or high spherical order oscillation modes ([Unno et al., 1989](#)). However, for low radial order and low spherical order oscillation modes, the validity of this approximation is not guaranteed. For dipolar modes, [Takata \(2005\)](#) already showed that this approximation is not always accurate.

The Cowling approximation, if not accurate, can induce both a shift in the frequencies of the oscillation modes and a difference in the amplitudes of the modes. Consequently, the validity of this approximation has to be verified with a global scan of the oscillation spectrum. While the formalism from ([Witte & Savonije, 1999a,b, 2001](#); [Savonije & Witte, 2002](#); [Savonije, 2005](#)) uses the integral of the imaginary part of the density perturbation over the volume of the star to study dynamical tides (which corresponds to the computation of the mechanical dissipation), our formalism uses the imaginary part of the gravitational perturbation, which can be obtained by integrating the imaginary part of the density perturbation over the radius of the star (see [Equa-](#)

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tion 6.24). Therefore, the verification of the validity of Cowling can be done with a global scan of the oscillation spectrum, looking at the imaginary part of the potential perturbation, consistently obtained with our methodology or obtained a posteriori, with the density perturbation in the formalism with the Cowling approximation.

The validity of this approximation can already be tested without including rotational effects. For the $3.5 M_{\odot}$ SPB star model presented in Section 6.3, in Figure 6.4, we compare the surface $\text{Im}(\Psi')$ obtained with and without the Cowling approximation around the $m = 0, n = 11$ and $n = 5$ oscillation modes. We find almost no shift

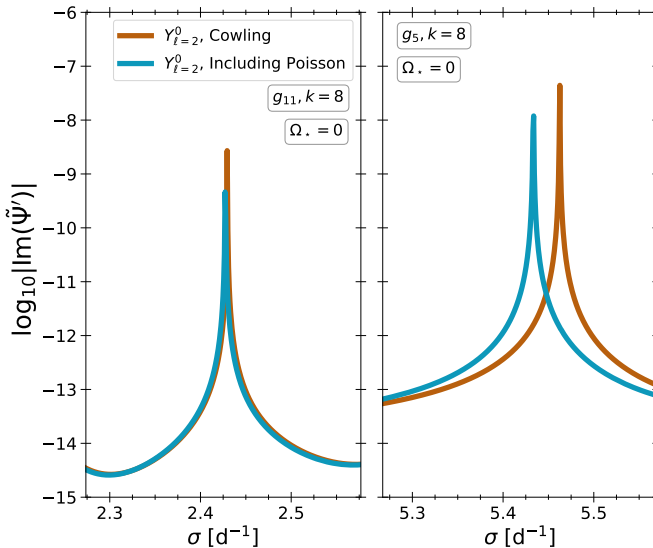


Figure 6.4: Imaginary part of Ψ' around the $m = 0, n = 11$ (on the left panel) and $n = 5$ (on the right panel) oscillation modes as a function of the forcing frequency. The orbital parameters of the system were we arbitrary chosen to be $k = 8$ and $e = 0.4$.

in frequency for the $n = 11$ oscillation mode due to the Cowling approximation. At the surface, we see an overestimation of approximately one order of magnitude in the imaginary part of the perturbed potential at the centre of the resonance line. Furthermore, we observe systematic difference in the amplitude of about one order of magnitude for low-radial oscillation modes, as seen for the $n = 5$ oscillation mode. In addition, for the $n = 11$ mode, we observe significant differences in the imaginary part of the potential perturbation, with an offset of the frequency of the maximum of about 0.030 d^{-1} . For the $m = 0$ modes, this frequency shift is more important for lower radial order modes. This highlights the limitations of the Cowling approximation for modes with relatively low radial order. A more detailed investigation could be the subject of a future work, would explore the impact of the gravitational coupling on the limitations of the Cowling approximation.

Given these limitations, there is a clear difference between dynamical tide treatments based on potential perturbations (as used in this work) and those based on density perturbations, such as in the formalism by (Witte & Savonije, 1999a,b, 2001; Savonije &

6.2. EXPLORATION OF GRAVITATIONAL COUPLING

Witte, 2002; Savonije, 2005). Although small frequency shifts and amplitude variations may have a minor impact on broad theoretical studies of the effect of dynamical tides on orbital evolution of binary system, they become important when attempting to model specific observed systems. This frequency shift can determine whether certain modes are theoretically resonating due to the companion forcing or not in tune and not observable.

Finally, while this test was done for $\ell = 2$ modes, stronger discrepancies can be expected for the $\ell = 1$. Asymptotic treatment which are widely used in classical asteroseismology strongly depend on this assumption, therefore a more global assessment of those model related uncertainties should be conducted. It should also be emphasized that the use of the traditional approximation of rotation together with the Cowling approximation will likewise introduce model-dependent systematic uncertainties, in particular for low-radial-order modes. For classical free oscillations, a more appropriate way to account for the Poisson equation would be to adopt the uncoupled approximation, while a fully coupled treatment would be considerably more difficult to implement.

6.2 EXPLORATION OF GRAVITATIONAL COUPLING

In this section, we explore the gravitational coupling of oscillation modes due to rotation. Since it is well known that $\ell = 2$ tides are dominating, we only consider modes with even values of ℓ .

6.2.1 GENERALITIES ABOUT GRAVITATIONAL COUPLING DUE TO ROTATION

The gravitational coupling between oscillations modes is defined by the $c_{\ell',\ell,k}^m$ coefficients, which are highly dependent on the value of the spin parameter. According to Equation 6.22, the coupling between two oscillations modes is particularly dependent of the product $c_{\ell,\ell'}^m c_{\ell'',\ell'}^m$, and its value compared to $(c_{\ell,\ell}^m)^2$ (the coefficient linked to the uncoupled term). If $(c_{\ell,\ell}^m)^2 \gg c_{\ell,\ell'}^m c_{\ell'',\ell'}^m$, then the coupling is weak and the solution is dominated by the uncoupled term and the uncoupled approximation is valid. In opposition, if $(c_{\ell,\ell}^m)^2 \sim c_{\ell,\ell'}^m c_{\ell'',\ell'}^m$, then the coupling starts to play a role and impacts the oscillation modes. The $c_{\ell',\ell'}^m$ coefficients are model independent and only affected by the values of ℓ, ℓ', m , and the spin parameter ν . Therefore, model independent comparison of these coefficients as a function of the spin parameter is possible. In Figure 6.5 we show the evolution of the ratio $c_{\ell,\ell'}^m c_{\ell'',\ell'}^m / (c_{\ell,\ell}^m)^2$ in a case where we only considered the coupling between the $\ell = 2$ and $\ell = 4$ oscillation modes. This figure directly shows in which regime of spin parameter and for which oscillation modes the gravitational coupling has an effect. For all the oscillations modes, Figure 6.5 indicates that below a spin parameter of $\nu < 1.2$, the mode coupling do not impact the $\ell = 2$ oscillation modes. For the prograde modes ($m = -2$), even at high spin parameters, the coupling remains low and the uncoupled approximation is valid. This result was expected due to significant λ discrepancy between the $\ell = 2$ and $\ell = 4$ prograde modes (see Fig. 34.3 of Unno et al. (1989), $\lambda(\nu)$ tends towards a small value for $\ell = 2$ while it increases significantly with ν for $\ell = 4$). We still attempted to compute the coupling,

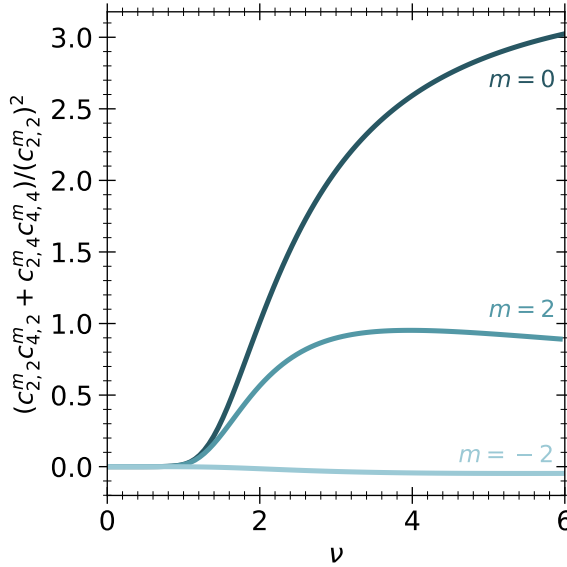


Figure 6.5: Ratio of the coupling coefficient for a coupling between the $\ell = 2$ and $\ell = 4$ oscillation modes as a function of the spin parameter and for different values of m .

but we found that not only is the coupling factor small, but the potential perturbation of the $\ell = 4$ prograde mode is several orders of magnitude smaller than the quadrupolar mode, making the coupling effectively null.

For the $m = 0$ modes, we saw that the coupling drastically increases after $\nu = 1.2$ and is the strongest overall. We explore the quantitative impact of this strong coupling on the oscillation modes in [Subsection 6.2.2](#). For the retrograde mode ($m = 2$), we saw that the coupling starts to become significant when $\nu > 1.2$, reaches a plateau around $\nu = 3$, and then slowly decreases. The quantitative effect of this coupling on the oscillation modes is explored in [Subsection 6.2.3](#).

Finally, by including the effect of the rotation through the traditional approximation, a fourth type of oscillation modes appear: the Rossby modes which are low frequency $\ell = 2, m = 2$ oscillation modes. The latitudinal index k of the $\ell = 2, m = 2$ Rossby modes according to the classification of [Lee & Saio \(1997\)](#) is $k = -2$ (not to be confused with the Fourier index k used in this paper). Following [Saio et al. \(2018\)](#), in order to propagate in radiative zones ($\lambda > 0$), Rossby modes must have spin parameters larger than $\nu > (m + |k|)(m + |k| - 1)/m$, which gives for the $\ell = 2, m = 2$ Rossby modes $\nu > 6$. Specifically, in Fig. 1 of [Saio et al. \(2018\)](#) we see that, for the $\ell = 2, m = 2$ Rossby modes, $\lambda > 0$ when $\nu > 6$, in their Fig. 7 we can see the geometry of those modes. Rossby modes have low λ , contrary to the classical retrograde gravito-inertial modes. Consequently, due to the significant difference in λ between the Rossby modes and the classical retrograde modes, the mode coupling is negligible for them. In this section, the coupling of oscillation modes were computed up to

6.2. EXPLORATION OF GRAVITATIONAL COUPLING

$\ell = 10$ for the retrograde and $m = 0$ oscillation modes, neglecting higher order modes and the coupling of the Rossby and prograde modes.

6.2.2 $\ell = 2, m = 0$ OSCILLATION MODES

In this section, we explore the impact of the gravitational coupling on the surface properties of the $\ell = 2, m = 0$ oscillations modes, focusing in particular on the surface imaginary part of the perturbed potential, as this quantity is directly impacting the evolution of the orbital parameters of the system (Smeyers et al., 1998; Willems, 2003; Valsecchi et al., 2013; Sun et al., 2023). From the previous section, we know that below a spin parameter of $\nu = 1.2$ the coupling is negligible. In this section, we focused on the cases where $\nu > 1.2$. We used the same SPB stellar model as in Subsection 6.1.6, and looked at the same scenario: a twin binary system composed of two of those SPB stars with an orbital eccentricity of 0.4. We first wanted to quantify the effect of mode coupling on unstable oscillation modes with different stellar rotation rates. Figure 6.6 illustrates the difference in the surface imaginary part of the perturbed potential between the uncoupled and coupled versions of our code for two values of the stellar rotation rate, and for the same gravito-inertial oscillation mode ($n = 11$).

From the left part of the figure, we observe that the coupling between oscillation modes, as expected, is low when $\nu < 1.2$. However, even in this case, we can see small peaks in the imaginary part of the perturbed potential, which are associated with the $\ell = 4, m = 0$ oscillation modes. These resonances impact the $\ell = 2$ modes even if the coupling is low. As the spin parameter increases, we observe an overall difference across the entire spectrum; in the case presented here, there is a factor 2 of difference between the solutions given by the uncoupled and coupled oscillation codes in the wings of the resonance. As expected, near the $\ell = 2$ resonance the two oscillation codes are equivalent since one mode dominates. We can also see that with the increase of the coupling, the effect of the $\ell = 4$ resonances on the $\ell = 2$ oscillations starts to become stronger. Finally, we can observe a change of sign on the imaginary part of the potential perturbation in the right panel of Figure 6.6. This change of sign corresponds to a transition from a damping toward an excitation of an oscillation mode.

We then wanted to verify the impact of the coupling on stable oscillation modes, as the coupled oscillation code shows the most differences when no single oscillation mode dominates and for modes with large radial orders where the spin parameter is high. For the same binary system, we chose the stable oscillation with $\ell = 2, m = 0, n = 25, k = 5$. Figure 6.7 illustrates the difference in the surface imaginary part of the potential perturbation around a stable oscillation mode for different values of the stellar rotation. For this mode, we already see a greater difference between the uncoupled and coupled oscillation codes. In this case, as the frequencies of the oscillation modes are smaller, the spin parameter is greater, making the coupling more significant for these oscillation modes. In the case of medium stellar rotation ($\Omega = 0.25 \Omega_{\text{crit}}$), the difference between the two codes is uniform, with about a factor of 2 difference. Here, we do not observe any impact from $\ell = 4$ resonances as in this regime of frequency the $\ell = 4, m = 0$ oscillation modes have too high radial orders. In the case of high rotation, the coupling of the oscillation uniformly increases the imaginary part of the perturbed potential by an order of magnitude. Importantly,

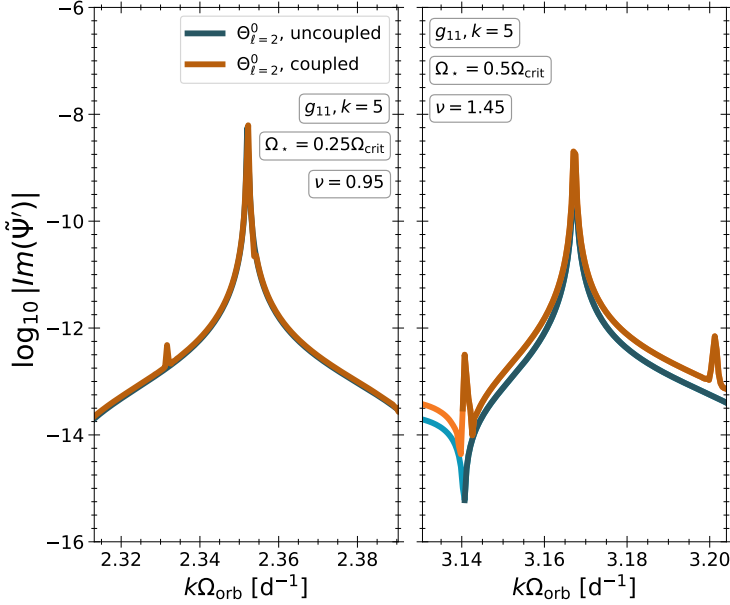


Figure 6.6: Surface imaginary part of the perturbed potential for the $\ell = 2, m = 0, n = 11, k = 5$ oscillation mode. The left panel correspond to a case where the stellar rotation frequency is $0.25 \Omega_{\text{crit}} = 1.15 \text{ d}^{-1}$, while in the right panel the stellar rotation frequency is $0.5 \Omega_{\text{crit}} = 2.29 \text{ d}^{-1}$. The orange curves correspond to the results of the coupled oscillation code while the blue curves correspond to the solution from the uncoupled oscillation code. The dark curves correspond to a negative value of the perturbed potential while the lighter curves corresponds to positive values.

at the resonance, the difference between the two oscillation codes also increases by an order of magnitude. This is explained by the retroactive effect of coupling when the spin parameter is significant. In more detail, the resonance of the $\ell = 2$ mode significantly impacts the $\ell = 4$ mode, that due to the high coupling, can retroactively impact the $\ell = 2$ modes.

6.2.3 RETROGRADE MODES ($\ell = 2, m = 2$)

In the case of the retrograde modes, from Figure 6.5, we expected a weaker gravitational coupling than for the $m = 0$ modes. To verify it, in Figure 6.8, we studied the model discrepancy as done in Subsection 6.2.2 for the $n = 11$ and $n = 25$ retrograde oscillation modes, when the stellar rotation frequency is $0.5 \Omega_{\text{crit}} = 2.29 \text{ d}^{-1}$. As expected, by comparing Figs. 6.6 (right panel), 6.7 and 6.8, we saw that for the $m = 2$ oscillation modes, the mode coupling plays a weaker role than for $m = 0$ oscillation modes. For stable modes at high spin parameters, we still see differences between the two modelling methods of about a factor of 5 (Figure 6.8, left panel). For the unstable modes (Figure 6.8, right panel), the discrepancies remain small.

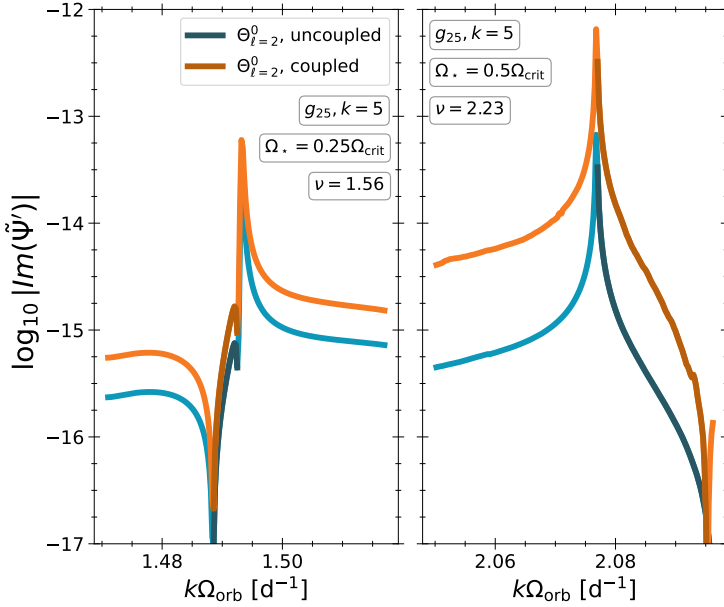


Figure 6.7: Surface imaginary part of the perturbed potential for the $\ell = 2, m = 0, n = 25, k = 5$ oscillation mode. The left panel correspond to a case where the stellar rotation frequency is $0.25 \Omega_{\text{crit}} = 1.15 \text{ d}^{-1}$, while in the right panel the stellar rotation frequency is $0.5 \Omega_{\text{crit}} = 2.29 \text{ d}^{-1}$. The orange curves correspond to the results of the coupled oscillation code while the blue curves correspond to the solution from the uncoupled oscillation code. The dark curves correspond to a negative value of the perturbed potential while the lighter curves corresponds to positive values.

6.3 RESONANCE SPECTRA OF FAST ROTATING STARS

In this section, we model and study the global resonance spectra of binary stars, including the effect of the rotation as described previously. In [Subsection 6.3.1](#), we present the global scan of the oscillation spectrum done with our code for a twin system composed of two fast rotating $3.5M_{\odot}$ SPB stars. Then, in [Subsection 6.3.2](#), we study the impact of the rotation on the spectra of this system. Finally, in [Subsection 6.3.3](#), we study the impact of different eccentricities on the spectra of the same binary system.

6.3.1 GLOBAL AND LOCAL OSCILLATION SPECTRA

In this section, we create and analyse the resonance spectra of a binary system. For this study, we computed stellar models of an SPB star with a mass of $3.5M_{\odot}$ with the same other ingredients than the stellar model described in [Subsection 6.1.6](#). One key application of including rotation effects into our oscillation code is the ability to model tidally excited Rossby oscillation modes. We chose to use a different stellar

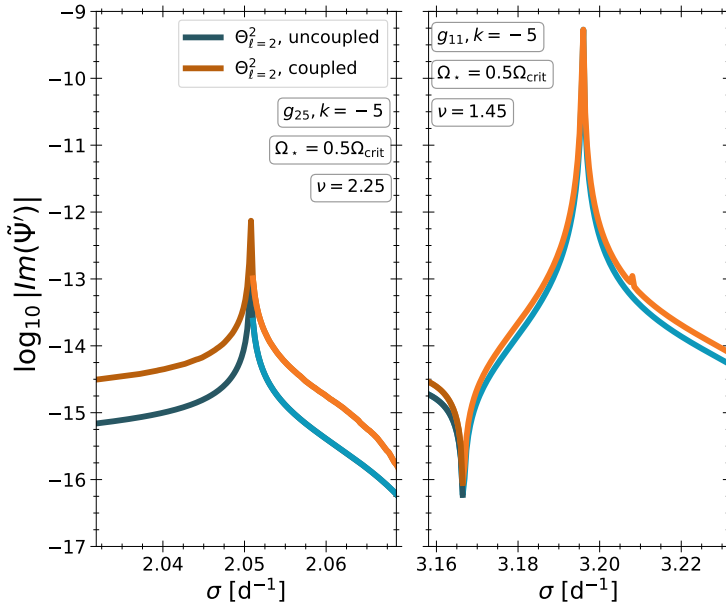


Figure 6.8: Surface imaginary part of the perturbed potential for the $\ell = 2, m = 2, n = 25, k = -5$ oscillation mode, on the left, and then $\ell = 2, m = 2, n = 11, k = -5$ oscillation mode, on the right. The orange curves correspond to the results of the coupled oscillation code while the blue curves correspond to the solution from the uncoupled oscillation code. The dark curves correspond to a negative value of the perturbed potential while the higher curves corresponds to positive values.

model for this study because the previous model lay outside the instability strip for Rossby modes (Townsend, 2005), whereas a $3.5M_{\odot}$ stars are inside this instability strip.

Efficiency is crucial for conducting a comprehensive scan of the oscillation spectrum, and include modes with different k and m values, as well as systems with varying orbital properties. For all results presented here, we used the uncoupled version of our stellar oscillation code due to its faster execution time. With this version, we implemented a scan strategy independent of the system's orbital parameters and k . The methodology capitalises on the linearity of the solutions with respect to Ψ_2 : a value of Ψ_2 proportional to x^{ℓ} is imposed during the oscillation code's execution, effectively ignoring the contribution from the Hansen coefficients, allowing for spectrum scanning relative to the corotation frequencies for each m value and a specified rotation frequency. Then the spectrum is computed for with each scan by scaling the solutions with the actual Ψ_2 value, as given in Equation 7.3, across all considered k values and desired orbital parameters. For quadrupolar modes, this approach requires four separate scans of the oscillation spectrum (one for each m , with the fourth for Rossby modes), with a subsequent reconstruction of all oscillations. This strategy significantly reduces the computation time by avoiding the need for hundreds of individual scans.

6.3. RESONANCE SPECTRA OF FAST ROTATING STARS

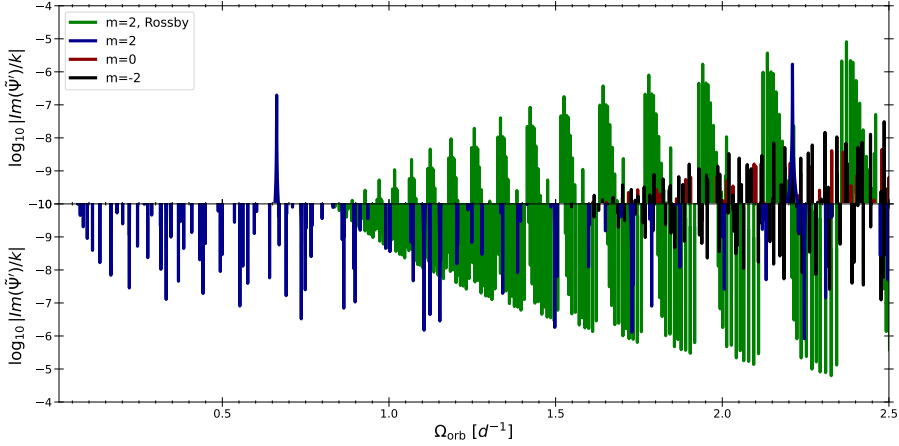


Figure 6.9: Scan of the surface imaginary part of the perturbed potential as a function of the orbital rotation rate. The system studied is a twin binary system composed of $3.5M_{\odot}$ stars with a stellar rotation frequency of $0.5\Omega_{\text{crit}} = 1.89\text{d}^{-1}$ and an orbital eccentricity of $e = 0.4$. The top panel corresponds to positive values of while the lower panel corresponds to negative values of the potential perturbation. The amplitude of all the oscillation modes have been divided by k to magnify the width of oscillation modes.

Moreover, this methodology works independently of the system's orbital parameters, allowing heavy computations to be executed autonomously. This methodology is particularly advantageous to include detailed dynamical tides into orbital evolution codes. To ensure coherence in the amplitude of the studied oscillation modes, we maintained a constant density of corotation period points in our scans, focusing only on regions near resonances identified with the free oscillations code.

Figure 6.9 illustrates the amplitude scan of various oscillation modes for a twin binary system composed of $3.5M_{\odot}$ stars, with a stellar rotation of $0.5\Omega_{\text{crit}}$ and an orbital eccentricity of $e = 0.4$. Given the density of the spectrum we chose to only illustrate the dominant modes with an amplitude higher than the threshold of 10^{-10} . In all such resonance spectra, we divided Ψ' by k . Indeed, one sees from Equation 6.20 that the widths of the resonance lines in terms of Ω_{orb} are proportional to k^{-1} . The impact of crossing a resonance on the orbital parameters of a system is proportional to the integral of the imaginary part of the potential taken over the width of the resonance. Therefore, by dividing Ψ' by k , we account for the effect of the widths of the resonance lines on the orbital evolution of the system.

This figure shows the dominance regime of each type of oscillation mode across different orbital separations. At lower orbital rotation rates ($\Omega_{\text{orb}} < 1\text{ d}^{-1}$), the spectrum is dominated by retrograde oscillation modes ($m = 2$). As the orbital rotation rate increases, Rossby modes, in green, begin to dominate the spectrum, although retrograde modes still exhibit high-amplitude resonances in this regime. Generally, prograde and

6.3. RESONANCE SPECTRA OF FAST ROTATING STARS

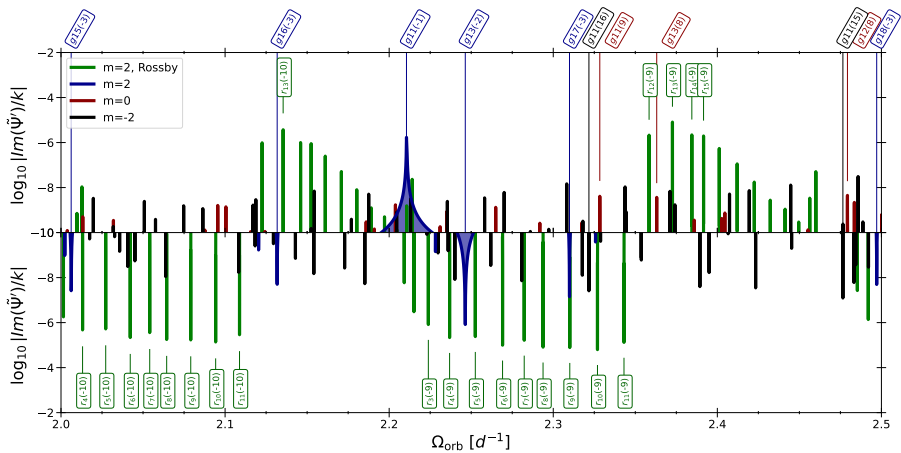


Figure 6.10: Scan of the surface imaginary part of the perturbed potential for a limited portion of the orbital rotation rate. The system studied is a twin binary system composed of $3.5M_{\odot}$ stars with a stellar rotation frequency of $0.5\Omega_{\text{crit}} = 1.89\text{d}^{-1}$ and an orbital eccentricity of $e = 0.4$. The top panel corresponds to positive values of while the lower panel corresponds to negative values of the potential perturbation. On the upper part of the figure the identification of the dominant oscillation can be found and is noted $gn(k)$. The amplitude of all the oscillation modes have been divided by k to magnify the width of oscillation modes.

6.3. RESONANCE SPECTRA OF FAST ROTATING STARS

$m = 0$ oscillation modes do not significantly influence the spectrum in this scan. As illustrated in Figure 6.9, the spectrum at high orbital rotation rate is very dense, in Figure 6.10 we decided to focus on the part at higher orbital rotation rate, to identify the dominant oscillation modes and study in more depth the oscillation spectrum.

The Rossby modes are $m = 2$ even modes with a dominating $\ell = 2$ component able to produce high amplitude resonances (see their geometry in bottom right panel of Fig. 2 in Saio et al. (2018)). They all have spin parameters larger than 6. For the stellar model and rotation considered, the Rossby modes of radial order 12 – 23 are unstable and they are stable out of this interval. In Figure 6.10, one can identify the unstable Rossby modes with positive $\text{Im}(\tilde{\Psi}')$ whereas stable modes have negative values. The spectrum reveals that Rossby modes form groups with varying radial orders but identical k . These separated groups come from the very low corotation frequencies of the Rossby modes. Imposing spin parameters larger than 6 and positive corotation frequencies gives:

$$2 \frac{\Omega_{\text{rot}}}{|k|} > \Omega_{\text{orb}} > \frac{5}{3} \frac{\Omega_{\text{rot}}}{|k|}, \quad (6.25)$$

where k has to be negative so that these modes are prograde in the inertial frame and retrograde in the corotation frame. In practice, Rossby modes of lower radial order are found at the lower frequency limit, while higher radial order modes accumulate closer to the higher frequency limit. It is noteworthy that the density of Rossby modes increases with $|k|$, this can be seen in Figure 6.9 for example.

The second major type of oscillation modes impacting the spectrum are the retrograde gravito-inertial oscillation modes ($m = 2$), in blue in Figure 6.10, with a few large resonances, particularly at low k values. In the case of prograde modes, their propagation is constrained solely by their requirement to be prograde modes in the inertial frame:

$$0 < \Omega_{\text{orb}} < 2 \frac{\Omega_{\text{rot}}}{|k|}, \quad (6.26)$$

where k negative for the same reason as the Rossby modes. These resonances dominate the low orbital frequency part of the spectrum, where they have similar radial orders and different k . Indeed, a null frequency in the observer frame corresponds to a given value of the dimensionless mode frequency in the corotating frame equal to 2 times the dimensionless rotation frequency. For this rotation rate, no unstable retrograde modes are found, the change in sign of the $g_{11}(-1)$ is due to the negative value of the Hansen coefficient when $k = -1$. As the modes are all stable, they all (except one) contribute in the same dissipative direction. In Figs. 6.9 and 6.10 we added the modes retrograde in both the inertial and corotation frame (with $k > 0$). However, those modes do not play a significant role and only a few of them can be found at high orbital frequency in Figure 6.9.

In the case of the prograde modes ($m = -2$), propagation is governed by the lower limit of the orbital frequency to ensure a positive corotation frequency:

$$\Omega_{\text{orb}} > 2 \frac{\Omega_{\text{rot}}}{|k|}, \quad (6.27)$$

6.3. RESONANCE SPECTRA OF FAST ROTATING STARS

where k has to be positive. In practice, high radial order prograde modes accumulate close to the lower limit of the orbital frequency previously given, while low radial order modes can be found at high orbital frequencies. In Figure 6.10, no low radial order low k prograde modes are found, as the required orbital frequency exceeds the orbital frequency leading to binary contact.

Finally, for the $m = 0$ oscillation modes, in the case of high rotation, low-order high orbital frequency oscillation modes require too high values of k to be visible. For this specific model at high rotation, we do not observe any unstable modes.

6.3.2 IMPACT OF THE STELLAR ROTATION

In the previous section, we studied the resonance spectrum for a fast-rotating star. In this section, we examine the impact of the stellar rotation rate on the oscillation spectrum of binary stars by decreasing the rotation. The results from this section can be anticipated using the excitation criteria for modes discussed earlier; reducing the rotation changes the distribution of oscillation modes on the diagram. We used the same model as in the previous section, reduced the rotation frequency to $0.25 \Omega_{\text{crit}} = 0.95 \text{d}^{-1}$, and performed the same global and local scans of the oscillation spectrum, shown in Figure 6.11 and Figure 6.12, respectively.

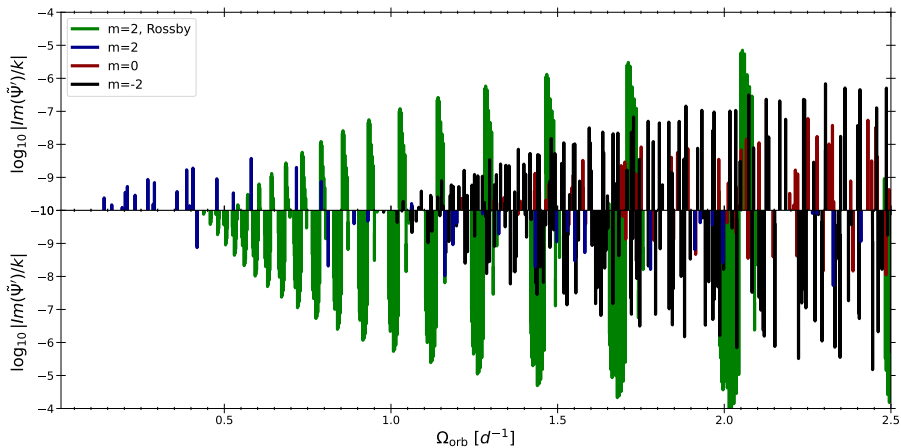


Figure 6.11: Scan of the surface imaginary part of the perturbed potential as a function of the orbital rotation rate. The system studied is a twin binary system composed of $3.5M_{\odot}$ stars with a stellar rotation frequency of $0.25\Omega_{\text{crit}} = 5.95 \text{d}^{-1}$ and an orbital eccentricity of $e = 0.4$. The top panel corresponds to positive values of while the lower panel corresponds to negative values of the potential perturbation. The amplitude of all the oscillation modes have been divided by k to magnify the width of oscillation modes.

By comparing the spectra from this section and Subsection 6.3.1, several effects of the rotation can be directly observed. First, by decreasing the rotation, the retrograde gravito-inertial are significantly depleted, even at low orbital frequencies. This depletion is a direct consequence of the more than 2 times larger radial orders of these

6.3. RESONANCE SPECTRA OF FAST ROTATING STARS

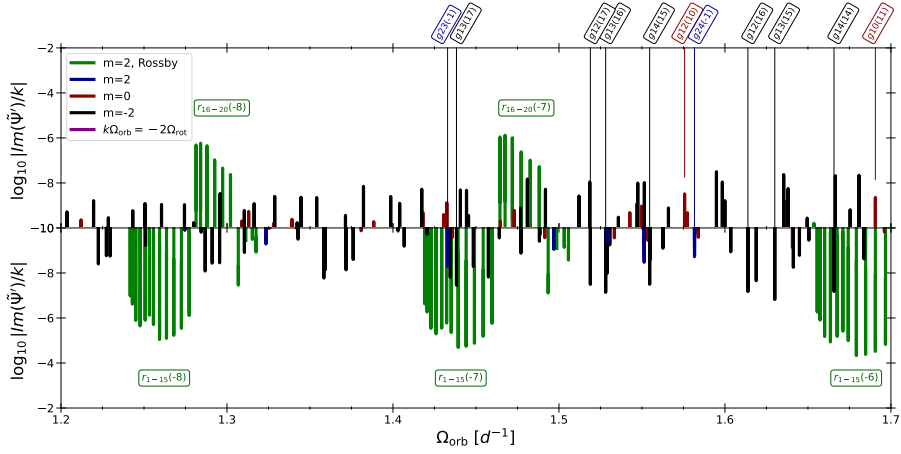


Figure 6.12: Scan of the surface imaginary part of the perturbed potential for a limited portion of the orbital rotation rate. The system studied is a twin binary system composed of $3.5M_{\odot}$ stars with a stellar rotation frequency of $0.25\Omega_{\text{crit}} = 5.95\text{d}^{-1}$ and an orbital eccentricity of $e = 0.4$. The top panel corresponds to positive values of while the lower panel corresponds to negative values of the potential perturbation. On the upper part of the figure the identification of the dominant oscillation can be found and is noted g radial order (k). In this figure, the amplitude of all the oscillation modes have been divided by k to magnify the width of oscillation modes.

modes at a given orbital frequency. This is evident from the identification of the retrograde modes in Figure 6.12; while in Subsection 6.3.1 the $n = 11$ could be found at high orbital frequency, here only the $n = 23 - 24$ are found in this regime.

On the other hand, the prograde modes have their lower orbital frequency limit shifted to lower frequencies, allowing modes with low radial orders and k to be found in the high orbital frequency regime. This results in an increase in the amplitude and importance of these modes at high orbital frequencies. At lower rotation frequencies, we also observe the appearance of unstable prograde modes, leading to both positive and negative $\text{Im}(\Psi')$, which could result in unpredictable effects from the dynamical tides. Decreasing the rotation frequency also impacts the Rossby modes. Fewer unstable Rossby modes are found in the spectrum, and there is a decrease in the spread of Rossby modes, creating small areas with a high density of Rossby modes. Additionally, the decrease in rotation reduces the values of $|k|$ at a given orbital frequency, which, with this eccentricity, increases the amplitude of the modes. From an orbital evolution standpoint, having well-localised areas with a high density of high-amplitude Rossby modes could significantly impact the system through dynamical tides.

Finally, for the $m = 0$ modes, lower rotation allows access to lower radial order modes with higher amplitudes (due to the decrease of λ) at high orbital frequencies. In this regime, $m = 0$ modes can become as significant as the prograde oscillation modes.

6.3.3 IMPACT OF ECCENTRICITY

The eccentricity of the binary system also impacts the scan of the oscillation spectrum by modifying the Hansen coefficients, thereby influencing the significance of the orbital harmonics k . In Figure 6.13, we illustrated the Hansen coefficient, as used in Equation 7.3, for the $\ell = 2, m = 2$ modes, for different values of eccentricity and as a function of k .

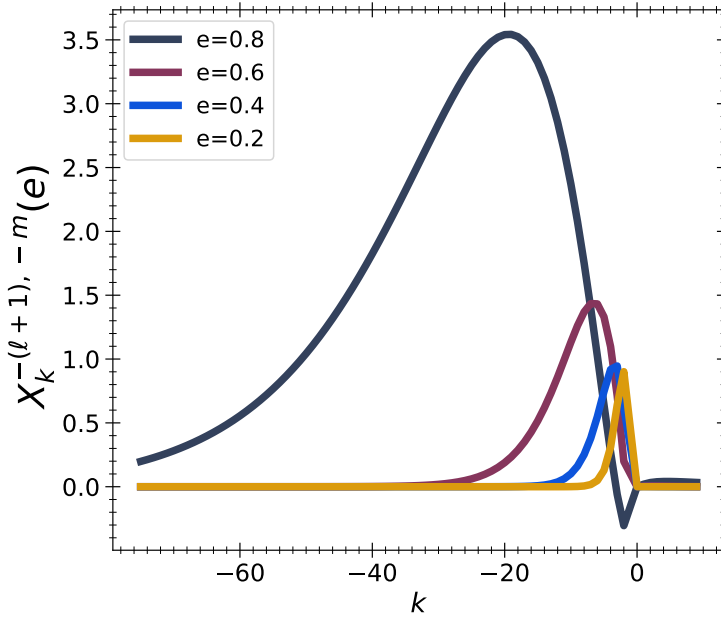


Figure 6.13: Evolution of the Hansen coefficient as function of k for the $\ell = 2, m = 2$ modes and different eccentricities.

Looking at the Hansen coefficients, we can see that higher eccentricity means higher values of k playing a key role. Consequently, in the oscillation spectrum of a binary system, higher eccentricity increases the amplitude of high k modes, potentially introducing new dominant oscillation modes and resulting in a denser spectrum. For the same binary system as in Subsection 6.3.1, Figure 6.14 illustrates the global oscillation spectrum when the eccentricity is 0.7. With increased eccentricity, oscillation modes with higher k values significantly impact the spectrum, making it denser. Specifically, Figure 6.14 shows that the Rossby modes become more dense. For the retrograde modes, having higher k available does not significantly change the oscillation spectrum; it only becomes denser at low frequencies. This is because the accumulation of retrograde modes occurs below the limit given in Equation 6.26, where higher k values correspond to lower orbital frequencies and thus higher orbital separations.

However, higher eccentricity significantly impacts the prograde and $m = 0$ modes. With higher k values, modes of low radial order can be reached at higher orbital frequencies. This effect is seen in Figure 6.14, where, at lower orbital eccentricity, the

6.4. DISCUSSION

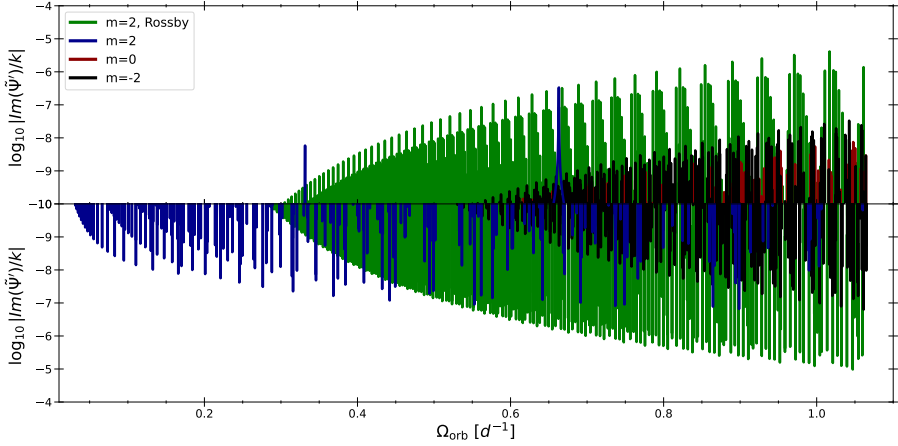


Figure 6.14: Scan of the surface imaginary part of the perturbed potential as a function of the orbital rotation rate. The system studied is a twin binary system composed of $3.5M_{\odot}$ stars with a stellar rotation frequency of $0.5 \Omega_{\text{crit}} = 1.89\text{d}^{-1}$ and an orbital eccentricity of $e = 0.7$. The top panel corresponds to positive values of while the lower panel corresponds to negative values of the potential perturbation. The amplitude of all the oscillation modes have been divided by k to magnify the width of oscillation modes.

prograde and $m = 0$ modes were almost non-existent, at higher eccentricities, these modes become significant at high orbital frequencies. However, their amplitudes remain significantly lower than those of Rossby modes.

6.4 DISCUSSION

In the previous sections, we introduced and tested a method to incorporate the effects of rotation into binary stellar oscillation models without relying on the Cowling approximation. Although developed for binary systems, this formulation can also be implemented in classical non-adiabatic oscillation codes. To account for rotation while solving the Poisson equation, we developed two distinct approaches: the first considers only the dominant term for each oscillation mode, while the second fully accounts for a new type of coupling that we called "gravitational coupling". We found that the gravitational coupling significantly affects the amplitude in the wings of resonances and for high spin parameters ($\nu > 1.5$). Accounting for the gravitational mode coupling is important for accurately modelling the amplitudes of specific oscillation modes and compare the models with observations.

By scanning the oscillation spectrum with the effects of rotation included, we identified patterns of mode accumulation and the dominant oscillation modes for various system configurations. Notably, Rossby modes emerged as dominant, occupying specific dense regions of the spectrum. From the perspective of orbital evolution, com-

6.4. DISCUSSION

puting these modes is critical, as these accumulation zones can significantly influence the orbital evolution of binary systems.

For prograde and $m = 0$ mode resonances, we expect their impact on orbital evolution to be weak due to their lower amplitudes and the opposing effects of stable and unstable modes. However, for Rossby modes, small regions in the orbital frequency domain could lead to rapid orbital evolution or even orbital trapping. Such trapping may also occur due to large, sparse resonances generated by retrograde oscillation modes. The rapid orbital evolution induced by Rossby modes could prevent the detection of binaries within these frequency domains.

By Newton's third law, the force exerted by the perturbed primary on the unperturbed secondary (proportional to the the perturbation of the potential generated by the primary) is equal and opposite to the force exerted by the unperturbed secondary on the perturbed primary (proportional to the perturbation of the density of the primary). Exploiting this property, [Witte & Savonije \(1999a,b, 2001\)](#); [Savonije & Witte \(2002\)](#); [Savonije \(2005\)](#) developed a method for the computation of the orbital evolution that only requires the density perturbation, allowing for the use of the Cowling approximation to study dynamical tides. This approach also consistently includes the effects of rotation without needing to solve the Poisson equation. In [Witte & Savonije \(1999a,b, 2001\)](#); [Savonije & Witte \(2002\)](#), a complete treatment of rotation with the Cowling approximation is adopted, while in [Savonije \(2005\)](#) the rotation is included with the traditional approximation still using the Cowling approximation.

A key difference between our approach and that of [Savonije \(2005\)](#) lies in the treatment of the Cowling approximation. In [Subsection 6.1.7](#), we assess the validity of the Cowling approximation and demonstrate that for low-radial order modes, it becomes problematic, with both a shift in frequencies of the maximums and a discrepancy of their amplitudes of about one order magnitude. In these cases, solving the Poisson equation is necessary to obtain accurate amplitudes and frequencies for low-radial order modes. In [Subsection 6.1.7](#), we only explored the validity of the Cowling approximation without including the effects of the rotation, we expect the gravitational coupling to also have an impact on the validity of the Cowling approximation, this could be the subject of another article.

In this work and in the literature ([Witte & Savonije, 1999a,b, 2001](#); [Savonije & Witte, 2002](#); [Ogilvie & Lin, 2004](#)) multiple types of mode coupling appear due to the rotation, based on different physical processes. A first type of linear coupling comes from the fact that the Coriolis and centrifugal forces make the problem of non-radial oscillations non-separable. Under the traditional approximation, the terms responsible for this non-separability are neglected and the problem can be separated in terms of the Hough functions. However, this approximation is not valid, for example, for p-modes and in convective zones. In this article we investigated a second type of coupling that we called gravitational coupling. This coupling arises from the fact that the Hough functions are not eigenfunctions of the Laplace operator in the Poisson equation. The gravitational coupling allows for interaction between modes, like the classical mode coupling. In particular, quadrupolar modes are directly influenced by the potential

6.5. CONCLUSION

perturbation of spherical modes of higher degrees, as shown in [Figure 6.6](#). When using the Cowling approximation, this gravitational coupling is neglected; however, we demonstrate that in the cases we studied, the Cowling approximation may not be fully valid. The validity of the Cowling approximation should also be impacted by the inclusion of the gravitational coupling.

Finally, there is another effect that should not be considered as a coupling per se. The tidal potential generated by the secondary is largely dominated by quadrupolar spherical harmonics ($\ell = 2$). However, for high values of the spin parameter, the Hough functions of higher degrees have a non-negligible quadrupolar component. The associated modes are therefore tidally excited by the quadrupolar component of the tidal potential, but they remain “uncoupled”. Under the uncoupled approximation, the gravitational coupling is entirely neglected, even though a single forcing term may excite multiple modes; separability is assumed. This approach is similar to that used in [Savonije \(2005\)](#) where the Cowling approximation was also employed. In contrast, our coupled formalism takes into account the non-separability of the modes.

The modelling method presented here is not limited to binary systems; it can also be applied to planetary systems. Our approach for scanning the oscillation spectrum is independent of the companion’s properties and orbital parameters, is well-suited for studying star-planet interactions. This method allows for precise calculations of planetary evolution, including the effects of dynamical tides on the system’s orbital parameters.

Recently [Papaloizou & Savonije \(2023, 2024\)](#), explored the excitation of Rossby modes by giant planets orbiting solar-type stars. Their formalism, closer to the one of [Ogilvie & Lin \(2004, 2007\)](#), includes a full treatment of rotation, a necessary feature when studying tidally excited gravito-inertial or inertial oscillations in such systems. The primary limitation of our model lies in the applicability of the traditional approximation. While we focused on SPB stars in this article, the traditional approximation also remains valid for γ Doradus stars and more massive stars. Although most planets are found around low-mass stars, several γ Doradus stars host planets, making them promising targets for applying our methodology. In addition, a significant portion of binary systems exhibiting tidally excited oscillations are located within the γ Doradus instability strip, further emphasising their relevance as targets for our model. However, our method is not suitable for low-mass stars with extensive convective envelopes, the type of stars studied in [Ogilvie & Lin \(2004, 2007\)](#); [Papaloizou & Savonije \(2023, 2024\)](#).

6.5 CONCLUSION

In this work, we developed a new theoretical modelling procedure to include the effect of rotation on tidally excited oscillation modes through the traditional approximation, without using the Cowling approximation. We first presented the basic formalism of tidally excited oscillations before introducing our new modelling procedure in [Section 6.1](#). Then, in [Section 6.2](#), we compared the two modelling methods we developed to identify the regime in which the uncoupled approximation is valid. In [Section 6.3](#),

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we examined the impact of including rotation on the oscillation spectrum of binary systems before discussing the implications of our results in [Section 6.4](#).

To include the effect of rotation on the oscillation modes, we developed two different methodologies. The first uses the uncoupled approximation, assuming no coupling between the oscillation modes in the Poisson equation. The second method iteratively solves the Poisson equation, including the effect of mode coupling without this assumption. We compared these two modelling methods for different types of oscillations. We determined that for prograde modes, mode coupling has no significant impact on the amplitude of the oscillation spectrum even at high spin parameters. The same conclusion applies to Rossby modes. For the retrograde ($l = 2, m = 2$) oscillation modes, gravitational coupling mode coupling starts to affect the amplitude when the spin parameter exceeds 1.2. In the case studied, involving an SPB star of $5.0M_{\odot}$, we observed a factor of 2 increase in amplitude due to mode coupling. For the $l = 2, m = 0$ oscillation modes, the impact of coupling is greater, with differences up to an order of magnitude in the amplitude of the oscillation waves within the same spin parameter regime as for the retrograde modes. Importantly, we found that the uncoupled version of the code is generally sufficient to model the amplitude of oscillations close to resonances, with most differences occurring out of resonance. The coupled version of the code is recommended for detailed modelling of the amplitudes of observed oscillation modes at high spin parameters, where accurately modelling the exact amplitude of the modes is essential.

Using the uncoupled version of our binary evolution code, we explored the total oscillation spectrum of a theoretical twin binary system composed of two $3.5M_{\odot}$ SPB stars. At high stellar rotation rates, we observed that tidally excited Rossby modes completely dominate the oscillation spectrum at high orbital frequencies, while retrograde modes dominate at lower orbital frequencies. Due to their high amplitudes, high density and predictable positions in the resonance spectrum (see [Equation 6.25](#)), they are expected to have a significant impact on the orbital evolution of binary systems. In contrast, at high rotation rates, the prograde and $m = 0$ modes play a less significant role, exhibiting lower resonance amplitudes.

Exploring the oscillation spectrum at lower stellar rotation rates revealed a contrasting scenario: prograde and $m = 0$ modes now play a key role with high-amplitude modes appearing at high orbital frequencies. Although Rossby modes still dominate overall, prograde and $m = 0$ modes exhibit comparable amplitudes, with less predictable positions. Decreasing the stellar rotation rate causes the retrograde modes to significantly reduce their amplitudes. This phenomenon can be attributed to increase of the modes radial order along with the decrease of the rotation for a given orbital frequency. Furthermore, we investigated the impact of the orbital eccentricity on the oscillation spectrum and observed that increasing the eccentricity leads to a denser resonance spectrum. Notably, prograde and $m = 0$ modes can become significant even at high stellar rotation rates.

Given the significant impact of rotation on the oscillation spectrum of binary systems, a more comprehensive study is warranted to explore how dynamical tides affect

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the orbital evolution of these systems. The densely populated orbital frequency regions by high-amplitude Rossby modes suggest the potential occurrence of resonance locking or rapid orbital evolutions at the accumulation points of these modes. Integrating our oscillation code with an orbital evolution code is essential and planned for future research, allowing a detailed investigation into how rotation influences the orbital dynamics of binary systems via dynamical tides.

CHAPTER 7

TREMOR (TIDAL AND ROTATIONAL EVOLUTION BY MODELLING OSCILLATIONS AND RESONANCES)

this chapter is mostly extracted and an expansion from [Fellay et al. \(2026\)](#).



Figure 7.1: Logo of TREMOR.

Unlike stars with convective envelopes, those with radiative envelopes dissipate most of their orbital energy through high-radial-order tidally excited oscillations. The excitation of oscillation modes by tides and their subsequent damping effects on the orbit is a process known as dynamical tides. In general, when the forcing frequency from the companion ($\sigma_{m,k}$) is small compared to the inverse of the stellar dynamical timescale ($1/\tau_{\text{dyn}}$), the tidally excited oscillations are damped in the near-surface layers by radiative damping ([Zahn, 1975, 1977](#)). This regime, for which the theory of

tidal dissipation formulated by Zahn (1975, 1977) is valid, predicts a dissipation of orbital energy into the star. The dissipation ultimately drives the system towards a minimum energy state, which corresponds to a circular orbit with the stellar rotation rate synchronised to the orbital motion. If the system has not yet reached a circularisation, a pseudo-synchronisation can occur where the rotation and orbital angular velocity at the periastron are near synchronisation (Hut, 1981). For higher forcing frequencies, tidally excited oscillations can resonate with lower-radial-order oscillation modes, induce non-linear mode excitations (Weinberg et al., 2012; Guo, 2020; Guo et al., 2022), or generate travelling waves (Burkart et al., 2012; Bowman et al., 2019; Ratnasingam et al., 2019). Each of these phenomenon results in a non-linear response of the companion to the forcing frequency (Zahn, 1977; Witte & Savonije, 1999a,b, 2001; Savonije & Witte, 2002; Savonije, 2005), making the classical theory of Zahn (1975, 1977) inapplicable. Furthermore, Zahn (1975, 1977) neglected the effects of rotation on the mode eigenfunctions, which may not be valid for rapidly rotating stars.

For high forcing frequencies relative to the inverse of the stellar dynamical timescale, Fuller (2021) points out that in stars with unstable modes, dynamical tides can extract energy and angular momentum from the star and transfer them to the orbit. This phenomenon, called ‘inverse tides’, allows for rapid variations in stellar rotation (as the stellar moment of inertia is significantly lower than that of the orbit), with a sign opposite to that predicted by Zahn (1975, 1977). Only stars within an instability strip can experience this process, as unstable modes must be excited by tides (Fuller, 2021). In particular, intermediate-mass stars (from $1.5 M_{\odot}$ to $8.0 M_{\odot}$ in this article) in δ -Scuti, γ -Doradus, and slowly pulsating B-type (SPB) stars, instability strips present an excellent opportunity to study inverse tides, given the extensive research on their structure, pulsations, and rotation (Guzik et al., 2000; Dupret et al., 2002, 2005; Townsend, 2005; Bouabid et al., 2013; Li et al., 2019, 2020a,b; Saio et al., 2021).

In the framework of inverse tides, the primary mechanism for long-term angular momentum exchange is resonance locking (Fuller, 2017, 2021). This mechanism occurs when a binary system maintains a state in which the variation of a forcing frequency matches the change in resonance frequency of an unstable mode due to stellar evolution. In Fuller (2021), resonance locking is assumed to take place; however, the effect of multiple simultaneous oscillation modes and the effects of rotation or evolution on mode properties are neglected as well as the modification of the forcing frequencies consequent to the impact of tides. These factors could significantly influence the feasibility of resonance locking, as additional oscillation modes might disrupt the locking process, and rotation could alter mode properties and resonance frequencies (Unno et al., 1989).

In general, tides remain a challenging aspect of binary evolution modelling. Observationally, constraints suggest that Zahn’s tidal theory agrees with the global trends of circularisation (Khaliullin & Khaliullina, 2010) and synchronisation (or pseudo-synchronisation) (Khaliullin & Khaliullina, 2007) in binary star populations, particularly among low-mass stars (Shporer et al., 2016; Lurie et al., 2017). For intermediate-mass stars, the limited number of studies suggests that the circularisation and synchronisation trends (or pseudo-synchronisation) observed in close-in binaries are mostly

consistent with Zahn’s theory (Khaliullin & Khaliullina, 2007, 2010; Zimmerman et al., 2017; Saio & Kurtz, 2022). In contrast, studies of open clusters often show significant discrepancies. For instance, reproducing the observed circularisation period in NGC 6819 with the standard prescription used in standard population synthesis codes such as BSE (Hurley et al., 2002) requires artificially increasing the convective damping efficiency by a factor of 100 (Meibom & Mathieu, 2005; Geller et al., 2013; Milliman et al., 2014). On the other hand, the inverse tides mechanism predicts the existence of two distinct sub-populations. The first consists of unsynchronised rapidly rotating stars (or a population of Be stars formed without prior mass transfer) exhibiting tidally forced oscillations even in close binaries, a prediction that seems inconsistent with observations, except for a few fast-rotating outliers, such as HD 9899216 (Thompson et al., 2012; Saio & Kurtz, 2022), Ψ Centauri, (Mantegazza et al., 2010), and KIC 4659476 (Shporer et al., 2016), for which synchronisation may not yet have occurred even with the formalism of Zahn (1975, 1977). These exceptions suggest that inverse tides may not play a dominant role, necessitating further investigation. However, a second population of slowly rotating stars is also predicted by Fuller (2021). This prediction appears to be more consistent with observations, as more stars with sub-synchronous rotation are detected (Zimmerman et al., 2017; Lampens et al., 2018; Guo et al., 2019; Li et al., 2020a,b), a feature that is not predicted in the framework of Zahn (1975, 1977) for close binary systems. In addition, Zanazzi & Wu (2021) points out that resonance locking may be a prominent mechanism occurring during the stellar pre-main sequence of solar-like and intermediate-mass stars to circularise the orbit. However, in this article we focus on resonance locking during the main sequence.

In this study, we conduct an in-depth analysis of resonance locking. We have developed a new computational tool, Tidal and Rotational Evolution by Modeling Oscillations and Resonances (TREMOR), capable of integrating the secular orbital evolution timescale due to dynamical tides. By computing tidally excited oscillations on the fly using our linear non-adiabatic oscillation code MAD (Dupret, 2001; Dupret et al., 2002; Bouabid et al., 2013; Fellay & Dupret, 2025), which includes the effects of rotation on mode eigenfunctions via the traditional approximation, we account for the contributions of all dynamically excited tides within the system. This approach allows us to accurately predict the feasibility of resonance locking and its long-term impact on the orbital and rotational evolution of binary systems.

In Section 7.1, we formulate the problem of dynamical tides using the formalism of Smeyers et al. (1998); Willems & Aerts (2002); Willems (2003); Willems et al. (2010); Valsecchi et al. (2013) and describe in detail the methodology implemented in TREMOR to solve these equations simultaneously for all system forcing frequencies. Section 7.2 presents the results from TREMOR’s full numerical integrations for a few scenarios where dynamical tides play a key role. Finally, we discuss our findings in and conclude in Section 7.5.

7.1 DYNAMICAL TIDES IN FAST-ROTATING BINARIES

7.1.1 FORMULATION OF THE PROBLEM

In this section, we present the mathematical formalism used to describe the forcing of oscillation modes in a star and its consequent impact on the rotational and orbital properties of a binary system following the approach of [Smeyers et al. \(1998\)](#); [Willems & Aerts \(2002\)](#); [Willems \(2003\)](#); [Willems et al. \(2010\)](#); [Valsecchi et al. \(2013\)](#). We adopted a direct modelling approach ([Savonije & Papaloizou, 1983, 1984](#); [Smeyers et al., 1998](#); [Willems, 2003](#); [Burkart et al., 2012](#); [Valsecchi et al., 2013](#); [Sun et al., 2023](#)) where a stellar oscillation code or a non-adiabatic oscillation code, **MAD** ([Dupret, 2001](#); [Dupret et al., 2002](#); [Bouabid et al., 2013](#); [Fellay & Dupret, 2025](#)) in our case, is modified to include the forcing from the companion (this forced non-adiabatic oscillation code is called **MAD_tides**). In **MAD_tides**, we included the time-dependent convection model of [Grigahcène et al. \(2005\)](#) for our non-adiabatic computations in γ -Doradus stars. In our modelling, we assumed a solid body rotation and included the effect of rotation in our oscillation code through the traditional approximation without employing the Cowling approximation. (For further details on our implementation of rotation, see [Fellay & Dupret \(2025\)](#).) In this work, we adopted the 'uncoupled approximation' formalism presented in [Fellay & Dupret \(2025\)](#). The amplitude of the tidally excited oscillations induced in a star by its companion is proportional to the tidal potential perturbation from the companion.

The gravitational potential from the companion is obtained classically using the perturbative approach, which assumes that stellar deformations are small perturbations from the spherical symmetry compared to the radius of the star. In this work, as is classically done, the secondary was approximated as a point mass. Within this approximation and in the perturbative framework, the tidal perturbing potential from the companion can be expressed as a Fourier series in a coordinate frame centred on the primary:

$$\Psi_2(r, \mu, \phi, \mathcal{M}) = -\frac{GM_2}{r} \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{r}{a}\right)^{\ell+1} X_k^{-(\ell+1), -m} e^{i(k\mathcal{M} + m\phi)} d_\ell^m P_\ell^m(\mu), \quad (7.1)$$

where $\mathcal{M}$ is the mean anomaly ($\mathcal{M} = \Omega_{\text{orb}} t$, with Ω_{orb} being the orbital angular frequency), G is the gravitational constant, a is the semi-major axis, M_2 is the mass of the companion, r is the radial coordinate of the primary star, μ is the cosinus of the co-latitude angle, ϕ is the azimuthal angle, ℓ and m are respectively the spherical degree and azimuthal order of the associated Legendre polynomials $P_\ell^m(\mu)$, k are the orbital harmonics, and d_ℓ^m is a normalisation coefficient defined as

$$d_\ell^m = \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(0). \quad (7.2)$$

The coefficients $X_{-k}^{-(\ell+1), -m}$ are the Hansen coefficients defined by

$$X_k^{n, -m} = \frac{(1 - e^2)^{n+3/2}}{2\pi} \int_0^{2\pi} \frac{\cos(mv + k\mathcal{M})}{(1 + e \cos v)^{n+2}} dv,$$

where v is the true anomaly and e is the orbital eccentricity. Within our formalism, a modified dimensionless radial potential perturbation from the companion, used as the inhomogeneous term in the forced oscillation problem (see [Savonije & Papaloizou \(1983, 1984\)](#); [Smeyers et al. \(1998\)](#); [Willems \(2003\)](#); [Burkart et al. \(2012\)](#); [Valsecchi et al. \(2013\)](#); [Sun et al. \(2023\)](#); [Fellay & Dupret \(2025\)](#)), is defined as

$$\Psi_{2,\ell,k}^m(r) = -\left(\frac{r}{R_1}\right)^\ell, \quad (7.3)$$

where R_1 is the radius of the primary star. Because this term is independent of the orbital parameters, the dimensionless measure of the star's response, $\text{Im}(F_{\ell,m,k})$, to the various forcing angular frequencies defined by [Smeyers et al. \(1998\)](#); [Willems \(2003\)](#); [Willems et al. \(2010\)](#); [Valsecchi et al. \(2013\)](#) is given by

$$\text{Im}(F_{\ell,m,k}) = -\frac{1}{2}\text{Im}(\Psi'_{\ell,m,k}(R_1)), \quad (7.4)$$

where $\Psi'_{\ell,m,k}(R_1)$ is the dimensionless Eulerian potential perturbation response evaluated at the stellar surface by the non-adiabatic oscillation code. All the physics of the oscillations, as well as their influence on the orbital evolution of the system, is contained in $\Psi'_{\ell,m,k}(R_1)$. The Fourier coefficients $c_{\ell,m,k}$ from [Smeyers et al. \(1998\)](#); [Willems \(2003\)](#); [Willems et al. \(2010\)](#); [Valsecchi et al. \(2013\)](#) are expressed as

$$c_{\ell,m,k}(e) = X_k^{-(\ell+1),-m} d_\ell^m \left(\frac{R_1}{a}\right)^{\ell-2}. \quad (7.5)$$

Using an inhomogeneous term that is independent of the orbital configuration is particularly advantageous, as it allows the stellar response to be computed separately from the system's orbital properties and later rescaled to match the specific parameters of the binary system. In the case of circular orbits, we verified our numerical computation of $c_{\ell,m,k}(e)$ with the values provided by [Willems et al. \(2010\)](#). The effect of the excitation of oscillation modes by the forcing of the companion on the orbital parameters of the system is then given by

$$\frac{da}{dt} = 4\Omega_{\text{orb}} \frac{M_2}{M_1} a \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{R_1}{a}\right)^{\ell+3} \text{Im}(F_{\ell,m,k}) G_{\ell,m,k}^{(2)}(e), \quad (7.6)$$

$$\frac{de}{dt} = 4\Omega_{\text{orb}} \frac{M_2}{M_1} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{R_1}{a}\right)^{\ell+3} \text{Im}(F_{\ell,m,k}) G_{\ell,m,k}^{(3)}(e), \quad (7.7)$$

$$\frac{1}{I} \frac{dL_{\text{rot}}}{dt} = \frac{d\Omega_{\text{rot}}}{dt} + \Omega_{\text{rot}} \frac{d \ln I}{dt} = 4 \frac{\Omega_{\text{orb}}}{I} \left(\frac{Ga}{M_1 + M_2}\right)^{1/2} M_2^2 \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{R_1}{a}\right)^{\ell+3} \text{Im}(F_{\ell,m,k}) G_{\ell,m,k}^{(4)}(e), \quad (7.8)$$

where M_1 is the mass of the primary, L_{rot} is the rotational angular momentum of the primary, I is its moment of inertia, and Ω_{rot} is its rotational angular frequency. The evolution of the stellar rotation rate can be decomposed into a tidal component $(d\Omega_{\text{rot}}/dt)_{\text{tides}}$ and an evolutionary component $(d\Omega_{\text{rot}}/dt)_{\text{evo}}$:

$$\frac{d\Omega_{\text{rot}}}{dt} = \left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{tides}} + \left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{evo}}, \quad (7.9)$$

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where the evolutionary component is given by the right-hand side of the second equality in Equation 7.10,

$$\left(\frac{d\Omega_{\text{rot}}}{dt} \right)_{\text{evo}} = -\frac{d \ln I_{\star}}{dt} \Omega_{\text{rot}}, \quad (7.10)$$

and the tidal component is given by the right-hand side of Equation 7.8. The coefficients $G_{\ell,m,k}^{(2)}(e)$, $G_{\ell,m,k}^{(3)}(e)$, and $G_{\ell,m,k}^{(4)}(e)$ are defined as

$$G_{\ell,m,k}^{(2)}(e) = \frac{2}{\pi(1-e^2)^{\ell+1}} c_{\ell,m,k}(e) P_{\ell}^{[m]}(0) \left[(\ell+1)e \int_0^{\pi} (1+e \cos v)^{\ell} \sin(mv+k\mathcal{M}) \sin v dv \right. \\ \left. - m \int_0^{\pi} (1+e \cos v)^{\ell+1} \cos(mv+k\mathcal{M}) dv \right],$$

$$G_{\ell,m,k}^{(3)}(e) = \frac{2}{\pi(1-e^2)^{\ell+1}} c_{\ell,m,k}(e) P_{\ell}^{[m]}(0) \left[(\ell+1)e \int_0^{\pi} (1+e \cos v)^{\ell} \sin(mv+k\mathcal{M}) \sin v dv \right. \\ \left. - m \int_0^{\pi} (1+e \cos v)^{\ell-1} \left[(1+e \cos v)^2 - (1-e^2) \right] \times \cos(mv+k\mathcal{M}) dv \right], \quad (7.11)$$

$$G_{\ell,m,k}^{(4)}(e) = \frac{e}{(1-e^2)^{1/2}} \left[G_{\ell,m,k}^{(3)} - \frac{1-e^2}{2e} G_{\ell,m,k}^{(2)} \right]. \quad (7.12)$$

Our computations of $G_{\ell,m,k}^{(2)}(e)$, $G_{\ell,m,k}^{(3)}(e)$, and $G_{\ell,m,k}^{(4)}(e)$ were verified with the values given in Willems et al. (2010) for circular orbits. Finally, for convenience, we introduced the characteristic timescales of orbital and rotational evolution, which were used to determine the time taken by tides to drive migration (τ_a), circularise the orbit (τ_e), modify the stellar rotation ($\tau_{\Omega_{\text{rot}}}$), and alter the angular momentum ($\tau_{L_{\text{rot}}}$):

$$\tau_a = \frac{a}{da/dt}; \quad \tau_e = \frac{e}{de/dt}; \quad \tau_{\Omega_{\text{rot}}} = \frac{\Omega_{\text{rot}}}{d\Omega_{\text{rot}}/dt}; \quad \tau_{L_{\text{rot}}} = \frac{L_{\text{rot}}}{dL_{\text{rot}}/dt}. \quad (7.13)$$

7.1.2 PRINCIPLE BEHIND RESONANCE LOCKING

One of the key phenomenon induced by dynamical tides that has to be reproduced by TREMOR is resonance locking. This mechanism has been introduced in several works (see e.g. Fuller (2017, 2021)). The primary contribution of this study is to provide a quantitative verification of the feasibility of resonance locking while accounting for the effects of rotation, the evolution of forcing frequencies, stellar evolution and interactions between different forced oscillation modes. Resonance locking occurs when the co-rotating-frame forcing frequency associated with a given (m, k) pair (see Equation 7.14) evolves at the same rate as the co-rotating frequency of the closest free oscillation mode with the radial order n . In such cases, the de-tuning, defined as the difference between the forcing and co-rotating frequency, remains nearly constant. More specifically, resonance locking occurs when the resonance exerts a repulsive effect that prevents the de-tuning from decreasing. In this regime, a smaller de-tuning leads to a stronger tidal response, which in turn pushes the forcing frequency towards a larger de-tuning. If the star's structural evolution tends to push the system away

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from resonance (e.g. through envelope expansion and associated spin-down), resonance locking can maintain the system in resonance. This mechanism can remain stable over stellar evolution timescales, as we later show, and dynamical tides provide a long-term source of angular momentum and energy exchange between the components of the system and their orbit. However, not all oscillation modes are capable of supporting resonance locking. In this section, we present the basic principles of resonance locking and identify the types of modes that can support it.

For stars within the γ -Doradus or SPB instability strips, the co-rotation frequencies of gravito-inertial modes generally increase over time due to the rising Brunt–Väisälä frequency in the radiative zone near the contracting core. For a given orbital angular frequency and stellar rotation frequency, the forcing frequency is given by

$$\sigma_{m,k} = k\Omega_{\text{orb}} + m\Omega_{\text{rot}}. \quad (7.14)$$

where we adopt the sign convention for k from [Valsecchi et al. \(2013\)](#), which differs from that used in [Sun et al. \(2023\)](#). We ensured $\sigma_{m,k} > 0$ throughout. If a negative frequency is obtained, the same oscillation mode can be studied by changing the signs of both m and k . Under this convention, $km < 0$ corresponds to prograde waves in the inertial frame, and $km > 0$ corresponds to retrograde waves. Prograde inertial-frame waves tend to have higher amplitudes due to larger Hansen coefficients. Thus, in this work, we consider only prograde waves in the inertial frame. Additionally, in the co-rotating frame, $m > 0$ corresponds to retrograde waves, and $m < 0$ corresponds to prograde waves. Unless explicitly stated, references to ‘prograde’ or ‘retrograde’ denote the motion in the co-rotating frame, which is most relevant to the physics of modes.

For $\ell = 2$ prograde modes ($m = -2, k > 0$), the forcing frequency is always positive when $2\Omega_{\text{rot}} < \Omega_{\text{orb}}$. In this case, resonances with stable prograde modes mimic the behaviour of equilibrium tides, where angular momentum is transferred from the orbit to the star. This leads to spin-up of the star and orbital decay, driving the system towards synchronisation. On the contrary, resonances with unstable prograde modes result in angular momentum transfer from the star to the orbit, spinning down the star and increasing the semi-major axis.

During resonance locking, the forcing frequency $\sigma_{m,k}$ must evolve at the same rate as the co-rotating frequency σ of the closest free oscillation mode. Neglecting changes in Ω_{orb} (which is justified because the orbital moment of inertia is much greater than that of the star), the locking condition becomes

$$\frac{d\sigma}{dt} = m \left(\frac{d\Omega_{\text{rot}}}{dt} \right)_{\text{tides}} + m \left(\frac{d\Omega_{\text{rot}}}{dt} \right)_{\text{evo}}. \quad (7.15)$$

The locking condition can finally be rewritten to isolate the tidal contribution as

$$\left(\frac{d\Omega_{\text{rot}}}{dt} \right)_{\text{tides}} = \frac{1}{m} \frac{d\sigma}{dt} - \left(\frac{d\Omega_{\text{rot}}}{dt} \right)_{\text{evo}}. \quad (7.16)$$

In [Figure 7.2](#), we illustrate resonance locking for the prograde and retrograde modes in a twin $1.5 M_{\odot}$ γ -Doradus binary system at an age of 738 Myr (see [Subsection 7.2.1](#) for model details). The red curves show the right-hand side of [Equation 7.16](#), while the

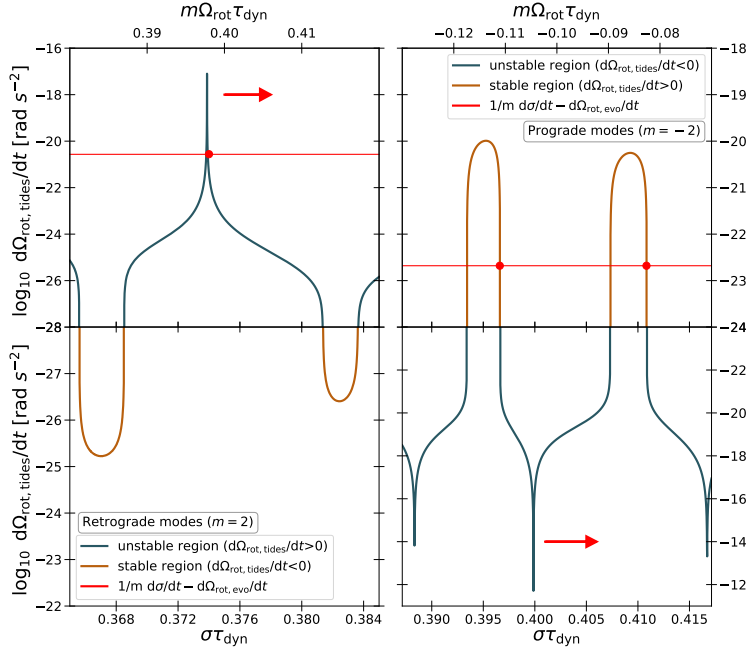


Figure 7.2: Comparison of $1/m d\sigma/dt - (d\Omega_{\text{rot}}/dt)_{\text{evo}}$ (red curve) with $(d\Omega_{\text{rot}}/dt)_{\text{tides}}$ for a retrograde mode ($\ell = 2, m = 2, k = -2, n = 30, \Omega_{\text{rot}} = 0.2 \Omega_{\text{crit}}$; left) and a prograde mode ($\ell = 2, m = -2, k = 2, n = 20, \Omega_{\text{rot}} = 0.05 \Omega_{\text{crit}}$; right). Forcing frequencies have been normalised by the dynamical timescale, τ_{dyn} . Blue segments represent unstable mode regions; orange curves represent stable regions. Arrows show the direction of mode evolution and spin trends. Red dots mark resonance locking points. Animated versions of this figure illustrating the various possible initial configurations and outcomes are provided in <https://dox.uliege.be/index.php/s/x1tGYkTIJHhb41J>.

tidal contribution $(d\Omega_{\text{rot}}/dt)_{\text{tides}}$ as obtained with `MAD.tides` and computed via Equation 7.8 is plotted for comparison. In practice, not all resonance lockings are stable. In the left panel of Figure 7.2, no stable locking is found to the left of the peak, even though the tidal amplitude reaches its maximum. In this region, if the system lies below the red threshold, the mode drifts away from the potential locking position. On the contrary, if the system lies above the red threshold, tides modify the forcing frequency more rapidly than the intrinsic mode evolution, and the system is driven towards the locking position on the right. An analogous reasoning applies to the right panel. For the case of retrograde modes, we provide in the supplementary material of this article, animations illustrating the locking process from different initial positions around the resonance. For retrograde modes, when $(d\Omega_{\text{rot}}/dt)_{\text{tides}} < 1/m d\sigma/dt - (d\Omega_{\text{rot}}/dt)_{\text{evo}}$, the motion of the modes resonances dominates the variation of the de-tuning until a locking is reached and the de-tuning remains constant. Resonance locking typically occurs in unstable regions near strong resonances. This leads to strong angular momentum deposition into the star and rotational spin-up. In contrast, prograde modes

undergo resonance locking in stable low-amplitude regions of the spectrum. However, such locking is only made possible by the presence of nearby unstable modes, which are essential for the locking process. Prograde modes are typically excited at high orbital frequencies, where the tidal amplitudes remain large, even far from resonance, and exceed the threshold required for resonance locking. Importantly, unstable prograde modes themselves cannot produce locking. Their influence is transient and instead helps drive the system into a stable locking configuration. While prograde locking also results in angular momentum deposition, the rate is typically two orders of magnitude lower than that of retrograde modes. As a result, stellar spin evolution in this regime is primarily governed by structural expansion. Nevertheless, depending on the rotation rate, age, and internal structure, locking can also occur in low-amplitude unstable prograde modes, leading instead to a slow angular momentum loss via tides.

7.1.3 NUMERICAL RESOLUTION OF THE PROBLEM

While the physical principle behind orbital evolution codes properly including the effect of dynamical tides has been known for a long time (see [Smeyers et al. \(1998\)](#); [Willems & Aerts \(2002\)](#); [Witte & Savonije \(1999a,b, 2001\)](#); [Savonije & Witte \(2002\)](#); [Willems & Aerts \(2002\)](#); [Willems \(2003\)](#); [Savonije \(2005\)](#); [Willems et al. \(2010\)](#)), the real challenge in developing such a tool is numerical, particularly in terms of computation time. Following the orbital evolution of binary systems induced by a single resonance or a few resonances was, for example, already accomplished in [Willems & Aerts \(2002\)](#); [Witte & Savonije \(1999a,b, 2001\)](#); [Savonije & Witte \(2002\)](#); [Savonije \(2005\)](#); [Fuller \(2021\)](#). However, tracking and including all the modes impacting the orbital evolution of the system at a given time is particularly challenging.

Four quantities are integrated with our method: a , e , and the angular momentum of the two stars. We thought that the best way to integrate these quantities was through what we refer to as the ‘scan method’. As introduced in [Subsection 7.1.1](#) and Eqs. [7.3](#) and [7.4](#), a normalised perturbation from the companion can be introduced to obtain the normalised tidal response ($\text{Im}(F_{\ell,m,k})$), independent of the orbital parameters of the system. Then, scans of the normalised tidal response can be performed and stored as a function of the forcing frequency. The properties of a mode at a given time and given forcing frequency are then obtained by interpolating between scans obtained at a given time step (dictated by the stellar model) and the following time step of the stellar model. The main advantage of this methodology is that all the heavy computations are performed outside the orbital evolution code, and once the scans are ready, the integration over the orbit can be quite fast.

We developed a version of the orbital evolution code exploiting this methodology; however, we faced various challenges that greatly limited its precision and computing time. The assumption that the problem is linear is not entirely justified when the rotation of the stars is included. It is well known that the effect of rotation on the frequencies of free oscillations is non-linear ([Unno et al., 1989](#)). Therefore, assuming a linear effect of rotation or totally neglecting it is not possible, except for slowly rotating stars. This led us to add an extra dimension in the scan of the oscillation spectrum. If we want to include the effect of rotation, we need to scan the oscilla-

tions both through the evolution of age and rotation. The main problem then became computation time. By adding a dimension to the scans, we drastically increased the computation time, as we needed to cover the entire parameter space in rotation at each time step of the model, effectively multiplying the scan time by the number of rotation steps chosen. Another concern was the interpolations themselves. While it was not a problem to develop interpolation methods that were reliable in the parameter space (age, rotation), many questions remained regarding the validity of interpolating oscillation modes, particularly in cases where modes or regions exhibit changes in behaviour (damping or driving) along with age or rotation.

To mitigate these drawbacks, we developed a method exploiting the advantages of the direct modelling method (Valsecchi et al., 2013; Sun et al., 2023; Fellay & Dupret, 2025), which we refer to as ‘on-the-fly computation’. The principle of this method is to compute the dynamical tides on the fly using our stellar oscillation code for a very limited number of modes. Instead of pre-computing a large number of scans for modes that are not necessarily used, we focused only on the dominant modes at a given time. The main challenge of this methodology is developing a highly efficient algorithm capable of detecting which oscillation modes to ignore and where the integration time step can be increased. We discuss the performance issues in more detail in Subsection 7.1.4. In this section, we focus on the global algorithm we used to integrate the tidal evolution equations on the fly.

Optimising the integration algorithm is central to reducing computation time. The primary objective of this optimisation is to control and monitor the integrator truncation errors while avoiding unnecessary oversampling of the dynamical tides. To achieve this, we used a fifth-order Runge-Kutta algorithm with step doubling coupled with a fourth-order Runge-Kutta algorithm to monitor truncation errors (Press et al., 1986). This setup enables an adaptive time step designed to keep truncation errors below a specified threshold by adjusting the integration step size accordingly. This algorithm is a standard approach for integrating the effects of tides on planetary or binary system orbital evolution (Jiang & Tremaine, 2010; Bolmont et al., 2015; Luna et al., 2020). It ensures efficient integration with minimal sampling of the forced oscillations while maintaining high precision, even near resonances. Additionally, we imposed an artificial upper limit on the time step to ensure that no sharp resonance contributions were missed. We also explored other integration methods, such as the Bulirsch-Stoer integration method (Press et al., 1986), that are standard integrators to compute the evolution of planetary systems (Chambers, 1999; Mardling & Lin, 2002; Fabrycky et al., 2014). However, these methods are rarely well-suited for functions with sharp transitions, such as tidally excited oscillations (Press et al., 1986).

As previously mentioned, the on-the-fly method requires drastically reducing the number of oscillation modes tracked. To ensure that the minimum number of modes are integrated while maintaining high precision, we developed a multi-layer filter, the principle of which is described in Figure 7.3. This multi-layer filter contains three filters, each called at different frequencies. The high tolerance filters, which are computationally expensive, are called less often than the low tolerance filters, which are less computationally demanding. When integrating the orbital evolution of the sys-

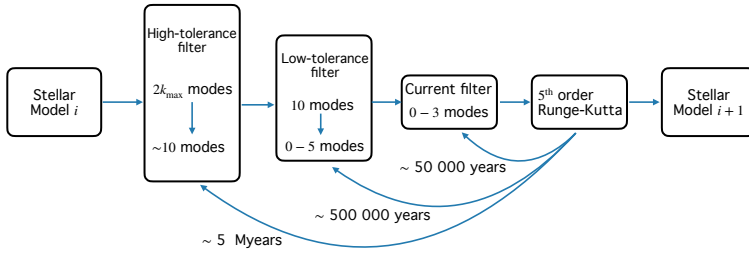


Figure 7.3: Illustration of the principle behind the multi-layer filter system used to compute the dynamical tides with the on-the-fly solving method for a γ -Doradus star with a mass of $1.5M_{\odot}$.

tem over the time span between two successive stellar models, the first filter applied to decide which modes are worth tracking is the high tolerance filter. In this step, the impact of dynamical tides at a given time is computed using our stellar oscillation code for each mode with a sufficiently low radial order (for all the computation in this article, we considered modes with a radial order up to $n = 200$). The exact upper limit in terms of co-rotating frequency depends on the mode type. In general, around $2k_{\max}$ (where $k_{\max}$ is the maximum number of orbital harmonics to consider, which varies as a function of the eccentricity due the shift of the peak in amplitude of the Hansen coefficients depending on k (Kołaczek-Szymański & Różański, 2023)) modes were computed at this step. We then selected only the modes for which the orbital and rotational characteristic timescales (as given in Equation 7.13) are below a given threshold, which is quite high (hence the high tolerance), and arbitrarily taken as a thousand times the nuclear timescale of the system. Only the few selected modes are passed to the low-tolerance filter, where we computed, in addition to the characteristic timescales, their derivatives with respect to time to detect which oscillation modes are close to resonance (which can drastically increase their effect on the orbital and rotational parameters). Only modes with sufficiently low characteristic timescales (which were arbitrarily chosen to be a hundred times the stellar nuclear timescale) or high characteristic timescale derivatives were selected and saved. For the few modes selected through this process, a final, frequently updated filter was applied based only on the current characteristic timescales (arbitrarily chosen to be ten times the stellar nuclear timescale), to ensure that only the dominant modes are integrated at a given time step. This multi-layer system is directly embedded in the Runge-Kutta algorithm to avoid wasting computation time. The refresh frequencies of all filters were parameterised based on the nuclear timescale.

7.1.4 PERFORMANCES AND PRECISION

As discussed previously, the principal difficulty in fully including dynamical tides into orbital evolution codes is the computational time. In this section, we examine the performance and precision of the two solving methods presented in Subsection 7.1.3. For both methodologies, the execution time is mainly dictated by the number of calls to `MAD_tides`, which represents by far the most computationally expensive area of

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the code. In the scanning method, the number of calls to the oscillation code cannot be significantly reduced without decreasing precision. The main drawback of this methodology is that during the scans, a large portion of the computations remains unused, but this cannot be avoided.

For the scans, we determined that around 200 sampling points (and uses of the oscillation code) per resonance are sufficient to guarantee an acceptable resolution of the function to allow for precise interpolations. When accounting for rotational effects, we needed to sample four types of modes (one for each m , plus Rossby modes). If we focus solely on low-radial-order modes (expected to have the highest amplitudes and to be unstable), which we assumed to be those with radial orders up to $n = 100$, we end up with 200 modes to sample at each stellar age and rotation rate. Over the course of an entire stellar evolution sequence, requiring 300 stellar models (for a γ -Doradus or SPB star), we must call our stellar oscillation code approximately 24 million times (solving 24 million instances of the forced non-adiabatic oscillation differential equation system). After optimising and parallelising `MAD.tides` (using a processor with 24 cores (eight performance cores plus 16 efficiency cores) and 32 threads), we achieved a computing speed of approximately 12000 forced oscillations per minute. This means that scanning the oscillations of an evolutionary sequence takes about 34 hours, plus an additional five hours are required to identify the free resonance frequencies. Ultimately, each scan for a given rotation requires approximately an equivalent of 27.5 million calls to the forced stellar oscillation code. To encompass the possible initial stellar rotation rates, at least two such scans with different rotation rates are necessary, bringing the total to around 55 million calls to our stellar oscillation code. Additionally, integrating the system's evolution requires several hours due to the extensive number of interpolations, although we note that this part of the code has not been fully optimised. The objective of the on-the-fly method is to drastically reduce the computational cost compared to the scan method or to increase the precision for the same cost.

With on-the-fly computation, many quantities can be precomputed and stored, leaving only the problem of solving the forced oscillation differential equation system. The primary advantage of this approach is that it significantly reduces the number of required forced oscillation computations. This efficiency is achieved by the algorithm itself and, in particular, the various filters presented in [Subsection 7.1.3](#) and their combinations. Since the number of modes included in the computation is determined dynamically during integration rather than being fixed, an exact estimation of computational time is not feasible and depends on the number of modes being tracked. However, we estimate that the computation time is roughly eight hours per billion years if only a few modes require integration.

For γ -Doradus stars, which have a typical nuclear timescale of approximately three billion years, the on-the-fly method costs already less than a single scan from scanning method per integration and achieves a higher overall precision. For SPB stars, which have nuclear timescales on the order of a few hundred million years, multiple integrations using the on-the-fly method cost less than a single scan. It should be noted that the scan method can operate at a lower overall precision than the on-the-fly

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method. The advantage of the scan method is that once the scan are done, multiple low cost integrations can be performed with various initial orbital parameters. However, most errors and imprecisions arise from interpolation and scan resolution (rather than the integrator itself), making them difficult to quantify. It is also worth noting that for rapid rotators, a large number of scans may be required to precisely interpolate the effects of rotation.

7.2 FULL INTEGRATION OF THE EFFECT OF DYNAMICAL TIDES

Before presenting our results, it is important to highlight that the theory of the dynamical tides developed by Zahn (1975, 1977) is not expected to remain valid in the forcing frequency or stellar rotational regimes considered in this article. In particular, in the regime of low-radial-order modes the condition $\sigma\tau_{\text{dyn}} \ll 1$ is not fulfilled and the proportionality of the tidal dissipation with respect to the forcing frequency studied by Zahn (1975, 1977) does not apply. Moreover, in fast rotating stars, the rotation has a strong effect on the oscillations eigenfunctions, which was not considered by Zahn (1975, 1977). In the limit of low forcing frequencies ($\sigma\tau_{\text{dyn}} < 0.1$), we verified that TREMOR's results are consistent with the prescriptions of Zahn (1975, 1977). We found that TREMOR and Zahn's prescription yields the same order of magnitude for dynamical tidal effects in this low forcing frequencies regime. However, a more detailed study is needed to fully explore the synergies and limitations of our approach compared to Zahn's prescription. In this article, we focus exclusively on the regime where Zahn (1975, 1977) is not expected to be valid or dominate, specifically, cases with high forcing frequencies ($\sigma\tau_{\text{dyn}} > 0.1$), where low-radial-order oscillation modes are excited by tidal interactions.

7.2.1 REFERENCE MODEL FOR THIS STUDY

Our goal is to investigate the impact of both stable and unstable oscillation modes on the orbital and rotational evolution of binary systems. For this reason, we focus on stars within the δ -Scuti, γ -Doradus, and SPB instability strips, as their evolution, modelling, rotation, and pulsation properties are well studied (Townsend, 2005; Bouabid et al., 2013; Li et al., 2019, 2020b; Saio et al., 2021). Due to the assumptions made in our stellar oscillation code, particularly the use of the traditional approximation, we were unable to reliably compute oscillation properties for stars with deep convective envelopes (Ogilvie & Lin, 2004, 2007) or for pressure modes. As a result, our study is limited to stars within the γ -Doradus and SPB instability strips. For this study, we chose one reference sequence of evolution of γ -Doradus stars. This sequence of evolution was computed with the Code Liégeois d'Évolution Stellaire (CLÉS; Scuflaire et al., 2008b). We chose a sequence in the lower end of the mass regime, and we chose a star with a mass of $1.5 M_{\odot}$. For this model we used an initial hydrogen mass fraction of $X = 0.72$, a solar metallicity $Z = 0.015$, a mixing length convective parameter of $\alpha = 2.0$, and a step-like overshooting of $\alpha_{\text{over}} = 0.2$. The model was stopped when the central hydrogen mass fraction reached $X_C = 0.05$, and we reached a final age of 2.7 Gyr for this evolutionary sequence. The parameters of the reference model were chosen to have unstable modes at the zero age main sequence and during a significant

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part of the main sequence. For all the stellar modelling, we used the AGSS09 abundances (Asplund et al., 2009), the FreeEOS equation of state (Irwin, 2012), the OPAL opacities (Iglesias & Rogers, 1996), the $T(\tau)$ relation from Model-C of Vernazza et al. (1981) for the atmosphere, the mixing length theory of convection implemented as in Cox & Giuli (1968), and the nuclear reaction rates of Adelberger et al. (2011).

To highlight the impact of resonance locking (or tidal forces in general) on the evolution of stellar rotational properties in our models, we compared them to the evolutionary tracks of isolated stars (i.e. assuming no tidal interactions at all) with the same initial parameters. Since CLÉS does not include a treatment of angular momentum transport, we computed reference models with Modules for Experiments in Stellar Astrophysics (MESA, version 24.08.1 Paxton et al., 2011, 2013, 2015, 2018, 2019; Jermyn et al., 2023), compiled using the MESA Software Development Kit for Linux (version 24.7.1 Townsend, 2024), which accounts for those processes. We then compared the surface velocities predicted by MESA models (considering only single stars) with those obtained from our CLÉS models including the effects of dynamical tides. This comparison enabled us to evaluate whether tidal effects can produce observationally significant differences. To ensure maximum consistency with the simulations run using TREMOR, we adopted the same initial stellar mass, chemical composition, opacity tables, mixing length and overshooting prescriptions, and termination condition. After evolving a non-rotating pre-main sequence model up to just before the zero age main sequence, we relaxed the model into solid-body rotation with a rate matching that in the TREMOR simulations. We then allowed differential rotation to develop between stellar layers, assuming shellular approximation. All five rotational mixing processes implemented in MESA were included, along with the associated angular momentum diffusion in the radial direction. These rotationally induced mixing mechanisms include dynamical shear instability, Solberg-Høiland instability, secular shear instability, Eddington-Sweet circulation, and the Goldreich-Schubert-Fricke instability. A detailed description of how rotation and associated mixing are treated in MESA can be found in Paxton et al. (2013) and Heger et al. (2000). In our MESA models, we neglected stellar winds, which are negligible for the stellar mass regime of interest here. Atmospheric boundary conditions were taken from Castelli & Kurucz (2003) model atmospheres, and convective boundaries were determined using the Ledoux criterion with the so-called convective premixing algorithm (Paxton et al., 2019, their Sect. 5.2). The micro- and macrophysics data used by the MESA modules are briefly summarised in the following section (Sect. 7.2.1).

MICRO- AND MACROPHYSICS DATA IN MESA

The equation of state utilised by MESA is a blend of equations of state from the following projects, OPAL (Rogers & Nayfonov, 2002), SCVH (Saumon et al., 1995), FreeEOS (Irwin, 2004), HELM (Timmer & Swesty, 2000), PC (Potekhin & Chabrier, 2010), and Skye (Jermyn et al., 2021). Radiative opacities are taken primarily from OPAL (Iglesias & Rogers, 1993, 1996), with low-temperature data from Ferguson et al. (2005) and the high-temperature, Compton-scattering dominated regime by Poutanen (2017). Electron conduction opacities are taken from Cassisi et al. (2007) and Blouin et al. (2020). Nuclear reaction rates are from JINA REACLIB (Cyburt et al.,

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2010), NACRE (Angulo et al., 1999) and additional tabulated weak reaction rates Fuller et al. (1985); Oda et al. (1994); Langanke & Martínez-Pinedo (2000). Screening is included via the prescription of Chugunov et al. (2007). Thermal neutrino loss rates are from Itoh et al. (1996).

7.2.2 FAST-ROTATING STARS (RETROGRADE MODES)

In γ -Doradus stars, a large instability strip exists due to the convective blocking, where the flux is periodically blocked at the base of the near surface convective zone (Guzik et al., 2000; Dupret et al., 2005). This extended instability strip results in a significant fraction of overstable oscillation modes, which can create resonance locking. We divided this study of γ -Doradus stars into two parts. First the cases of fast-rotating stars dominated by the effect of retrograde modes is described in this section, and then the case of slow-rotating stars in tight orbits, where the tidal interactions are dominated by prograde modes, is described in Subsection 7.2.3.

LOW-ECCENTRICITY ORBITAL EVOLUTION

For low-eccentricity systems ($e < 0.3$), the problem of dynamical tides is significantly simplified. In such cases, only the first few orbital harmonics (k) can effectively excite oscillation modes with an amplitude sufficient to impact the orbital and rotational evolution of the star. This reduces the number of modes that must be tracked and minimises mode-mode interactions. We explored the effect of dynamical tides on the orbital and rotational evolution of the system by considering a twin system composed of two $1.5M_{\odot}$ γ -Doradus stars with an initial orbital angular frequency of $\Omega_{\text{orb}} = 2 \text{ rad d}^{-1}$, initial stellar rotation rates of $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2, 0.3$, and an eccentricity of 0.1. In Figure 7.4, we illustrate the resulting evolution of the stellar angular momentum, semi-major axis, and eccentricity. Additionally, Figure 7.5 presents the evolution of the rotation rate, rotation to critical rotation ratio, and surface rotation velocity for the same systems. Figure 7.4 shows that throughout most of the evolution, angular momentum is deposited into the stars, as indicated by $L_{\text{rot}}/L_{\text{rot,ini}} > 1$ in panel 3 of Figure 7.4. This deposition occurs due to the inverse tides. For the system with the fastest initial rotation, the deposition of angular momentum reaches a maximum increase of 35% relative to its initial spin angular momentum at 2 Gyr, before undergoing a rapid decline due to the influence of stable oscillation modes, which have high amplitudes given the low orbital period. This sudden drop is an evolutionary effect, as most of the unstable modes present in the early main sequence transition into stable modes in later stages. For an initial rotation of $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$ the increase of angular momentum reaches 70% of the initial angular momentum at the end of the main sequence. As expected, the effect of this angular momentum exchange on the orbital parameters is minimal since the orbital moment of inertia is significantly larger than the stellar moment of inertia, except near the end of the main sequence, where stable modes efficiently damp the eccentricity. To further investigate the rapid angular momentum deposition, Figure 7.6 illustrates the evolutionary timescales associated with angular momentum exchange, broken down by individual oscillation modes. The figure shows that for most of the system's evolution, a single dominant retrograde mode ($\ell = 2, m = 2, k = -2, n = 27$ for $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$ and $\ell = 2, m = 2, k = -2, n = 17$

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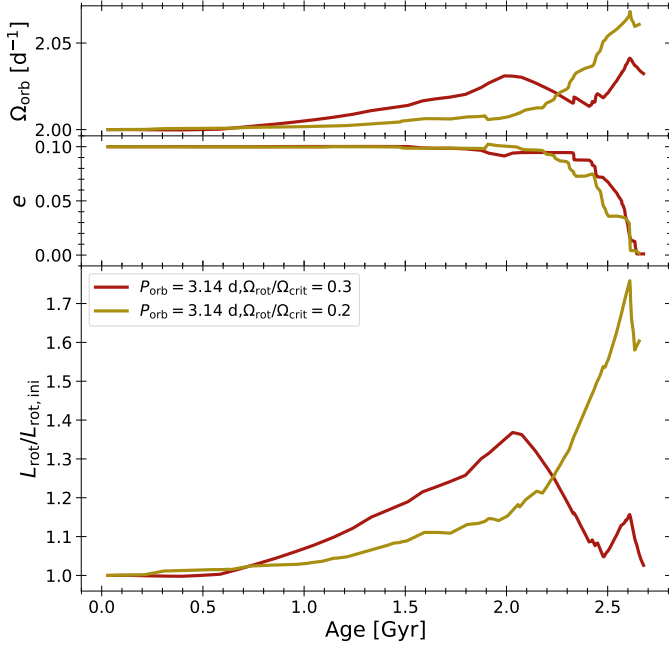


Figure 7.4: Evolution of the orbital frequency (top panel), eccentricity (middle panel), and stellar angular momentum (bottom panel) as a function of system age obtained with TREMOR. Two initial stellar rotation rates were considered: $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$ and 0.3 . The initial eccentricity was set to 0.1 , and the initial orbital period of the system is 3.14 days.

for $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.3$) is responsible for long-term resonance locking and sustained angular momentum deposition. Due to the stability of resonance locking and the small number of high-amplitude stable oscillation modes (resulting from the system’s low eccentricity), mode crossings are rare throughout the evolution.

In low-eccentricity systems, only a few non-negligible forcing frequencies occur, typically associated with low values of k . For rapidly rotating stars in wide systems, [Figure 7.6](#) illustrates the detailed response of the star to the different forcing frequencies of the system. This figure shows that for most of the main sequence, a resonance locking occurs with the $k = -2$ mode, as evidenced by the plateau in the stellar response. Outside of this locking, numerous resonance crossings are observed, corresponding to the peaks in the response. Most of these crossings occur towards the end of the main sequence, when changes in the stellar structure accelerate, leading to a faster evolution of the oscillation spectrum.

HIGH-ECCENTRICITY ORBITAL EVOLUTION

In the case of high-eccentricity binaries, more resonances with oscillation modes are expected, and resonance locking is likely to be shorter-lived. We explored the effect

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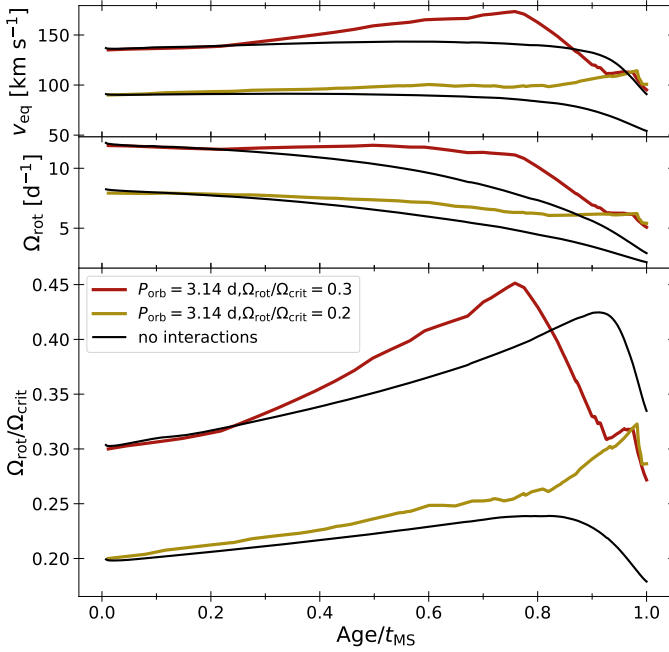


Figure 7.5: Evolution of the surface rotation velocity (top panel), the stellar rotation rate normalised by the critical rotation rate (middle panel), and the stellar angular momentum (bottom panel) as a function of system age, as computed with TREMOR. Two initial stellar rotation rates are considered: $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$ and 0.3 . The initial eccentricity was set to 0.1 , and the initial orbital period is 3.14 days. The black track corresponds to the evolution predicted by single-star models.

of dynamical tides on the orbital and rotational evolution of the system by considering a twin system composed of two $1.5M_{\odot}$ γ -Doradus stars with an initial orbital angular frequency of $\Omega_{\text{orb}} = 0.5 \text{ d}^{-1}$, initial stellar rotation rates of $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2, 0.3$, and an eccentricity of 0.55 . In Figure 7.7, we illustrate the evolution of angular momentum exchange induced by dynamical tides and its impact on stellar rotation. The same behaviour as in the low-eccentricity case is exhibited in Figure 7.7 but amplified. In particular, for the case with an initial rotation of $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$, the stellar angular momentum increases by 50% . In Figure 7.8, we further illustrate the timescales of angular momentum exchange for each tracked oscillation mode. When comparing Figs. 7.6 and 7.8, we observed that while for low-eccentricity cases, a single resonance locking is responsible for all the deposition of angular momentum, for higher-eccentricity cases a succession of short-lived locking occurs. Despite their short-term nature, these lockings effectively sustain resonance locking throughout most of the main sequence. The primary difference in the efficiency of angular momentum deposition between low- and high-eccentricity systems is determined by the probability of achieving a resonance lock. In high-eccentricity systems, the likelihood of encountering a resonance capable of locking the system is significantly higher. This is because unstable oscillation modes can be excited by a greater number of orbital harmonics,

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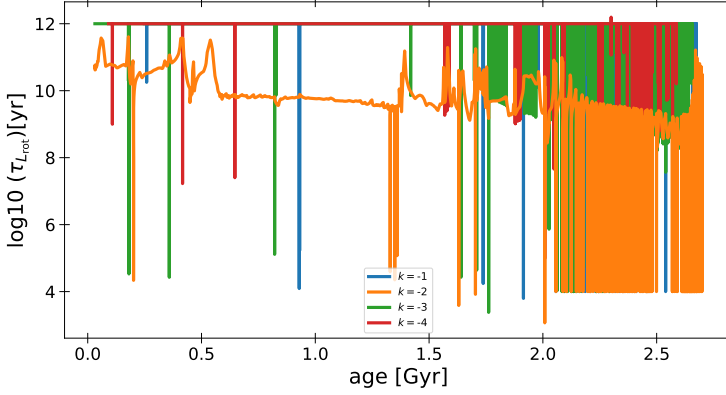


Figure 7.6: Timescale associated with the stellar rotational angular momentum as a function of age for a twin system with low eccentricity ($e = 0.1$) and high initial rotation rate ($\Omega_{\text{rot}} = 0.3\Omega_{\text{crit}}$), corresponding to the case shown in Figure 7.4. The plotted modes are retrograde, with orbital harmonics k distinguished by colour. A long-term resonance locking with the $k = -2$ mode is visible as a plateau, while the peaks correspond to resonance crossings.

allowing the system to rapidly re-lock after unlocking.

In the case of high-eccentricity, wide systems, many non-negligible forcing frequencies arise, which can resonate with free oscillation modes. In Figure 7.7, we show the detailed response of the star to a subset of these forcing frequencies. In this case, each forcing frequency is able to produce a short-lived resonance locking. The system remains locked until another, faster-moving resonance takes over, leading to a succession of brief resonance episodes. The immediate consequence is that a significant amount of angular momentum can be transferred over the course of the system's evolution, comparable to the effect of a single long-lived locking episode.

7.2.3 CLOSE-IN SLOW-ROTATING SYSTEMS (PROGRADE MODES RESONANCES)

For prograde modes, we required that positive forcing frequency for all $k > 0$ ($2\Omega_{\text{rot}} < \Omega_{\text{orb}}$) to ensure a positive forcing frequency. Furthermore, to study unstable prograde modes, we needed to reach high forcing frequencies (i.e. $\sigma\tau_{\text{dyn}} > 0.1$). Consequently, we focused on close-in systems with slowly rotating components. For the same twin system composed of two $1.5M_{\odot}$ γ -Doradus stars, we considered the case where the initial orbital angular frequency is $\Omega_{\text{orb}} = 5 \text{ d}^{-1}$, the initial stellar rotation rate is $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.05$, and the eccentricity is 0.1. Figure 7.9 illustrates the evolution of angular momentum exchange induced by dynamical tides and its impact on stellar rotation for this system. Figure 7.9 also shows that even at low orbital frequency, the stellar rotation is not significantly affected by dynamical tides during the first ~ 1.5 Gyr of the evolution. Subsequently, the dynamical tides have their classical dissipative effect, and the system is driven towards synchronisation. In the same figure, we

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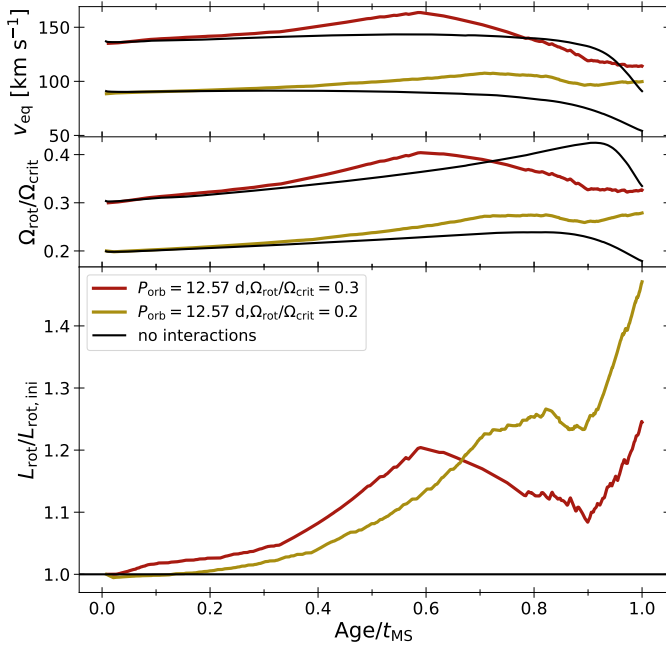


Figure 7.7: Evolution of the surface rotation velocity (top panel), the stellar rotation rate over the critical rotation rate (middle panel), and stellar angular momentum (bottom panel) as a function of system age obtained with TREMOR. Two initial stellar rotation rates were considered: $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.2$ and 0.3 . The initial eccentricity was set to 0.55 , and the initial orbital period of the system is 12.57 days. The black track corresponds to the evolution predicted by single stars models.

include a track based on the prescription from Zahn (1975, 1977), which suggests that given the separation, the system should synchronise within approximately 500 Myr. Examining the amplitude of each oscillation mode forced by tides (illustrated in Figure 7.10), we observed that multiple modes produce simultaneous short-timescale resonances. However, their net effect on rotation remains negligible. In this particular case, we identified an instance of four modes ($\ell = 2, m = -2, k = 1, 2, 3, 4$ with various radial orders changing through time) resonance locking, which decreases the angular momentum of the star. For this locking to occur, we required the presence of multiple modes with opposite signs that become locked together for extended periods. Across our simulations, we consistently observed that resonance locking is particularly stable. When a resonance locking event occurs with a mode, the relative motion of oscillation mode frequencies slows down significantly, or in some cases, their relative motion becomes fixed because the rotation frequency is nearly frozen. This form of resonance locking exploits such a behaviour to maintain stability over long timescales. Given our results, a mathematical explanation of this phenomenon is necessary to determine why certain resonances are particularly stable and up to which orbital frequency this phenomenon can occur.

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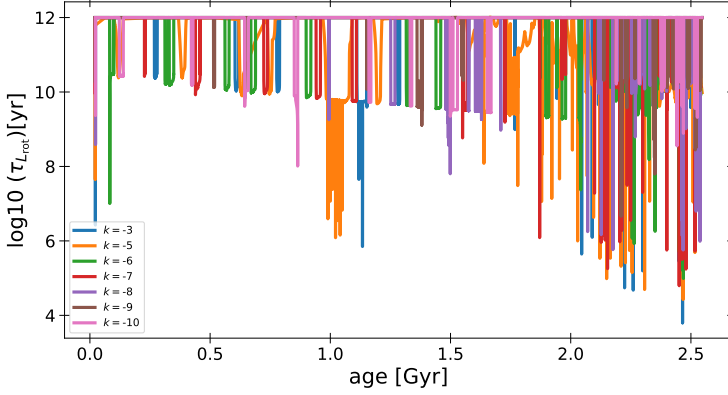


Figure 7.8: Timescale associated with the stellar rotational angular momentum as a function of system age for a twin system with high eccentricity ($e = 0.55$) and rapid initial rotation ($\Omega_{\text{rot}} = 0.2\Omega_{\text{crit}}$), as presented in Figure 7.7. The displayed modes are retrograde, with orbital harmonics k indicated by colour. Resonance locking occurs successively with modes of different orbital harmonics.

In Figure 7.10, we illustrate the four-mode locking occurring in a twin system with high orbital frequency and low rotation frequency, as discussed in Figure 7.9. Although the amplitudes of the four prograde modes remain relatively high throughout the evolution, the total angular momentum exchange remains low. Notably, between 0.4 and 0.5 Gyr, two modes with $k = 1$ and $k = 2$ become locked simultaneously, and their contributions to the stellar angular momentum timescale nearly cancel each other out, resulting in minimal net angular momentum transfer. At other stages, all four modes participate in resonance locking, compensating each other's effects over extended timescales.

7.2.4 CONSERVATION OF THE ANGULAR MOMENTUM IN TREMOR

In the case where only tidal effects are considered, the total angular momentum of the binary system must be conserved throughout its evolution. Ensuring angular momentum conservation is therefore a crucial test when numerically integrating the effects of tides.

With the Runge-Kutta integrator used in TREMOR, a gradual accumulation of numerical errors leading to slight angular momentum non-conservation is expected due to truncation errors. However, we control this accumulation through our adaptive time step, which ensures that the truncation error remains below a predefined threshold. For all computations presented in this article, we set this threshold to 10^{-10} for each integrated quantity, namely, the semi-major axis, the eccentricity, and the stellar rotational angular momentum of each star in the system.

For the evolution of the low-eccentricity fast-rotating γ -Doradus twin system, whose

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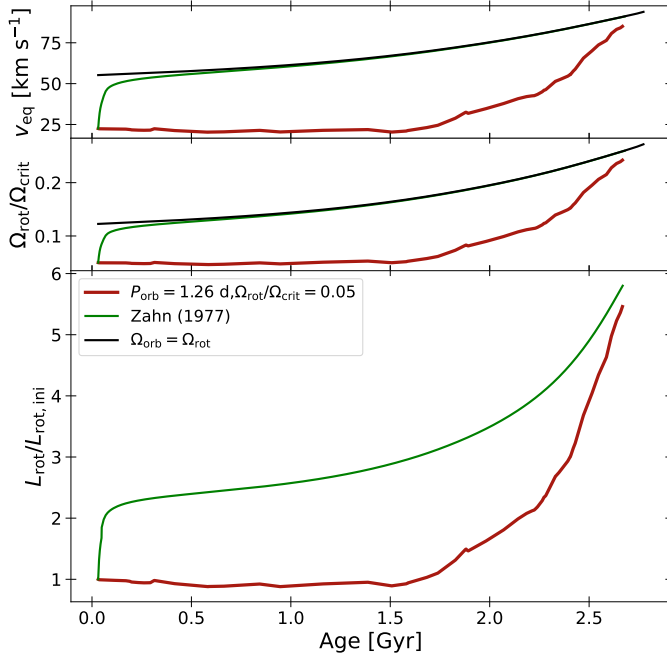


Figure 7.9: Evolution of the surface rotation velocity (top panel), the stellar rotation rate over the critical rotation rate (middle panel), and stellar angular momentum (bottom panel) as a function of system age obtained with TREMOR. The initial stellar rotation rate is $\Omega_{\text{rot}}/\Omega_{\text{crit}} = 0.05$, the initial eccentricity was set to 0.1, and the initial orbital period of the system is 1.26 days. The green track corresponds to the evolution predicted with the formalism from Zahn (1975, 1977). The black line corresponds to the rotation frequency for which the system is pseudosynchronised.

evolution is illustrated in Figure 7.4, we analyse the conservation of angular momentum in Figure 7.11.

As shown in these figures, angular momentum conservation is maintained with a precision of $\sim 10^{-10}$. In our case, no cumulative discrepancy arises over time, as the dynamical effects are non-monotonous.

7.2.5 STABILITY OF RESONANCE LOCKING

We initially assumed that different oscillation modes would evolve at different rates, making resonance locking stable only over short timescales, limited by the time it takes for a stronger or faster-moving resonance to either unlock or capture the system. However, if we express the condition for resonance locking as

$$\frac{d\sigma}{dt} = k \frac{d\Omega_{\text{orb}}}{dt} + m \frac{d\Omega_{\text{rot}}}{dt}, \quad (7.17)$$

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

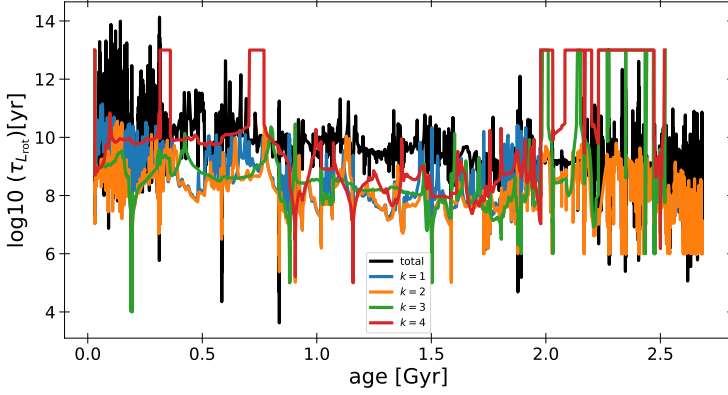


Figure 7.10: Timescale associated with the stellar rotational angular momentum as a function of system age for a twin system with high orbital frequency and low rotation frequency, as shown in Figure 7.9. The plotted modes are prograde, with individual orbital harmonics k shown in different colors. The black line represents the total contribution from all modes.

it can be observed that if $\frac{d\sigma}{dt}$ does not vary significantly for modes with different radial orders and given that $k \frac{d\Omega_{\text{orb}}}{dt} \ll m \frac{d\Omega_{\text{rot}}}{dt}$, then when this condition is satisfied for one mode, it is also satisfied for modes with nearby radial orders. In such a scenario, the relative positions of resonances become effectively fixed over time or evolve much more slowly than under normal conditions. For ‘high’ radial order modes ($n > 20$), which is the typical range where unstable modes are found in γ -Doradus stars (Bouabid et al., 2013), this condition is met. As a result, resonance locking can be sustained over timescales comparable to the stellar main sequence since resonance crossings (which could disrupt the locking) become much less frequent. This behaviour was first identified in our numerical simulations using TREMOR (see Section 7.2), confirming that it is a robust phenomenon.

7.3 PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

In this section, we aim at computing the impact of tides on the orbital and orbital parameters of binary systems using the asymptotic approximation (Tassoul, 1980). Including the effect of rotation through the traditional approximation, the asymptotic expression for the corotating frequency of an oscillation mode (Bouabid et al., 2013) is given by:

$$\sigma = \frac{\sqrt{\lambda} \int N/r \, dr}{\pi(n + 1/2)} = \frac{\sqrt{\lambda} N}{\pi(n + 1/2)}, \quad (7.18)$$

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

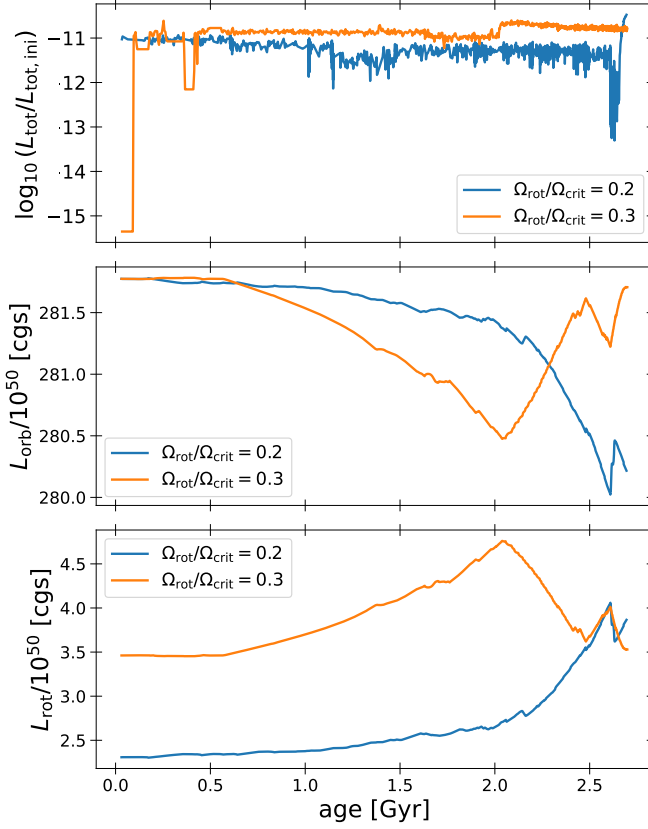


Figure 7.11: The top panel: conservation of angular momentum for the low-eccentricity fast-rotating γ -Doradus twin system, each line corresponds to a different initial rotation. The middle panel shows the evolution of the orbital angular momentum in cgs as a function of the time and the lower panel shows the evolution of the evolution of the total rotational angular momentum.

where λ is the eigenvalue associated with the Hough function of degree ℓ, m for a given spin parameter ν_H ($\nu_H = 2\Omega_{\text{rot}}/\sigma$), N is the Brunt-Väisälä frequency, r is the stellar radial coordinate, and n is the radial order of the mode. We define $\mathcal{N}$ as

$$\mathcal{N} = \int \frac{N}{r} dr. \quad (7.19)$$

During resonance locking, the rate of change of the forcing frequency (given by taking the time derivative of Equation 7.14) is the same as the one of the corotating oscillation frequency. The time derivative of the forcing frequency of a given mode is thus

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + \frac{\partial\sigma}{\partial\Omega_{\text{rot}}} \frac{d\Omega_{\text{rot}}}{dt} = k \frac{d\Omega_{\text{orb}}}{dt} + m \frac{d\Omega_{\text{rot}}}{dt}. \quad (7.20)$$

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

Then, we decompose the time derivative of the stellar rotation rate into a tidal component $(d\Omega_{\text{rot}}/dt)_{\text{tides}}$ and an evolutionary component $(d\Omega_{\text{rot}}/dt)_{\text{evo}}$ as done in Equation 7.9. Using Kepler third law we rewrite da/dt given in Equation 7.6 in $d\Omega_{\text{orb}}/dt$. Taking the ratio of Eqs. 7.6 and 7.8 $(d\Omega_{\text{orb}}/d\Omega_{\text{rot,tides}})$ we can remove their dependency to $\text{Im}(F_{\ell,m,k})$ (this removing is very useful because $\text{Im}(F_{\ell,m,k})$ is extremely sensitive to the detuning) to substitute $d\Omega_{\text{orb}}/dt$ in Equation 7.20 with $(d\Omega_{\text{orb}}/d\Omega_{\text{rot}})(d\Omega_{\text{rot}}/dt)$ and isolate the rate of change of stellar rotation due to dynamical tides, which is given by

$$\left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{tides}} = \frac{-(m - \partial\sigma/\partial\Omega_{\text{rot}})\left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{evo}} + \partial\sigma/\partial t}{m + k d\Omega_{\text{orb}}/d\Omega_{\text{rot}} - \partial\sigma/\partial\Omega_{\text{rot}}}. \quad (7.21)$$

Similarly, the semi-major axis and eccentricity evolution during a resonance locking with a single oscillation mode can be obtain with

$$\frac{da}{dt} = \frac{2}{3} \left(\frac{a^5}{G(M_1 + M_2)} \right)^{1/2} \frac{(m - \partial\sigma/\partial\Omega_{\text{rot}})\left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{evo}} + \partial\sigma/\partial t}{k + (m - \partial\sigma/\partial\Omega_{\text{rot}})(d\Omega_{\text{orb}}/d\Omega_{\text{rot}})^{-1}}. \quad (7.22)$$

$$\frac{de}{dt} = \frac{(m - \partial\sigma/\partial\Omega_{\text{rot}})\left(\frac{d\Omega_{\text{rot}}}{dt}\right)_{\text{evo}} + \partial\sigma/\partial t}{k(d\Omega_{\text{orb}}/de) + (m - \partial\sigma/\partial\Omega_{\text{rot}})(d\Omega_{\text{rot}}/de)}. \quad (7.23)$$

The expressions of $\partial\sigma/\partial t$ and $\partial\sigma/\partial\Omega_{\text{rot}}$ are obtained by taking the time and rotation frequency derivatives of Equation 7.18 and are given by

$$\frac{\partial\sigma}{\partial t} = \frac{\partial \ln N / \partial t}{1 + x/2} \sigma, \quad (7.24)$$

$$\frac{\partial\sigma}{\partial\Omega_{\text{rot}}} = \frac{1}{\nu_H} \frac{x}{1 + x/2}, \quad (7.25)$$

with,

$$x = \frac{d \ln \lambda}{d \ln \nu_H}. \quad (7.26)$$

Finally, $(d\Omega_{\text{orb}}/d\Omega_{\text{rot}})$ is obtained by taking the ratio of Eqs. 7.6 and 7.8:

$$\frac{d\Omega_{\text{orb}}}{d\Omega_{\text{rot,tides}}} = \frac{G_{\ell,m,k}^{(2)}(e)}{G_{\ell,m,k}^{(4)}(e)} \frac{M_1 + M_2}{M_1 M_2} I a^{-2}, \quad (7.27)$$

and similarly ,

$$\frac{d\Omega_{\text{orb}}}{de} = \frac{3}{2} \frac{G_{\ell,m,k}^{(2)}(e)}{G_{\ell,m,k}^{(3)}(e)} \left(\frac{G(M_1 + M_2)}{a^3} \right)^{1/2}, \quad (7.28)$$

$$\frac{d\Omega_{\text{rot}}}{de} = \frac{G_{\ell,m,k}^{(4)}(e)}{G_{\ell,m,k}^{(3)}(e)} \left(\frac{Ga}{M_1 + M_2} \right)^{1/2} \frac{M_1 M_2}{I} \quad (7.29)$$

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

Equation 7.21 provides a lower limit on the amplitude of dynamical tides required at a given stage of stellar evolution to establish a stable resonance locking. If the peak amplitude of an unstable resonance mode exceeds this threshold, determined by the radial order of the mode and k , then resonance locking will occur at the corresponding orbital parameters. It should be noted that in Equation 7.21 $m - \partial\sigma/\partial\Omega_{\text{rot}} \gg k\partial\Omega_{\text{orb}}/\partial\Omega_{\text{rot}}$ due to the discrepancy between stellar rotational and orbital moments of inertia. This allows us to neglect the dependency on k . Consequently, we find that the fundamental properties of resonance locking do not significantly depend on the orbital parameters of the system (provided that the amplitude of dynamical tides is sufficiently large), but rather on the oscillation and rotational characteristics of the star. Beyond that point, a distinction has to be made between prograde and retrograde modes.

7.3.1 LIMIT OF VALIDITY OF THE PRESCRIPTION IN γ -DORADUS STARS

The prescription developed in the previous section is valid as long as resonance locking can occur. In Fellay et al. (2026), we identified two distinct regimes in which resonance locking operates in γ -Doradus binaries, involving two different families of modes: prograde modes and retrograde modes.

RETROGRADE MODES

In Section 7.3, we established a criterion based on $(d\Omega_{\text{rot}}/dt)_{\text{tides}}$ to determine the amplitude of dynamical tides required for resonance locking to occur. In this section, we compare this criterion with the amplitudes of retrograde dynamical tides computed for different orbital and stellar parameters in order to determine the limiting orbital period up to which resonance locking is possible. For our $1.5 M_{\odot}$ γ -Doradus model, applying this criterion is relatively straightforward, since a large fraction of the low-radial-order modes are unstable and are therefore expected to favour resonance locking. In Figure 7.12, we compare the amplitude required by the theoretical locking criterion with the amplitudes of the tidally excited oscillations obtained from scans of the oscillation spectrum. For this comparison, we consider a system with an orbital eccentricity of 0.4, a stellar rotation rate of $0.3 \Omega_{\text{crit}}$, and an age of 738 Myr. The results shown in Figure 7.12 include the contribution of $k \partial\Omega_{\text{orb}}/\partial\Omega_{\text{rot}}$ to Equation 7.21. As expected, however, the theoretical timescale criterion itself does not depend on k or on the orbital period. By comparing the locking threshold with the actual amplitudes of the dynamical tides, we find that, for the particular system shown in Figure 7.12, resonance locking remains possible up to an orbital period of about 70 days. Using the same scan-based method, we repeated this comparison for a range of stellar rotation rates and orbital eccentricities in order to construct a map of the forced-oscillation spectrum. For the $1.5 M_{\odot}$ γ -Doradus model at an age of 738 Myr, Figure 7.13 shows the maximum periastron separation at which resonance locking can occur as a function of eccentricity and stellar rotation rate. For the $1.5 M_{\odot}$ γ -Doradus model, resonance locking is not expected for rotation rates below about 15% of the critical rotation, because unstable modes that are retrograde in the corotating frame and prograde in the inertial frame become too scarce. At higher rotation rates, resonance locking remains possible, but its domain of occurrence changes. For rotation rates above about

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

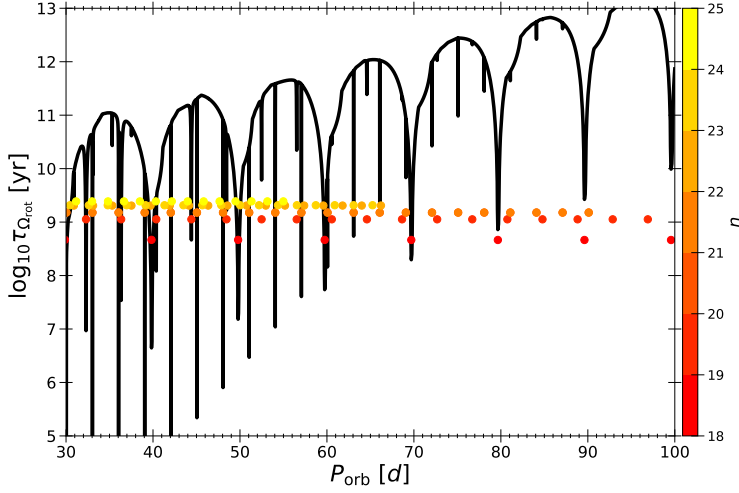


Figure 7.12: Comparison between the theoretical threshold for resonance locking and the amplitudes of tidally excited oscillations for a twin system composed of two $1.5 M_{\odot}$ γ -Doradus stars. The system has an orbital eccentricity of 0.4, a stellar rotation rate of $0.3 \Omega_{\text{crit}}$, and an age of 738 Myr. The colour scale indicates the radial orders of the oscillation modes.

40% of the critical rotation, locking is expected only in systems with close periastron passages, approximately

$$a(1 - e)/R_1 \lesssim 13,$$

because the forcing then resonates mainly with low-radial-order stable modes when $\Omega_{\star}/\Omega_{\text{crit}} \gtrsim 0.4$. This behaviour explains the unlocking observed in the binary system shown in Figure 7.7 when the rotation rate reaches $\Omega_{\star}/\Omega_{\text{crit}} \simeq 0.4$. By contrast, in the intermediate regime

$$0.15 \lesssim \Omega_{\star}/\Omega_{\text{crit}} \lesssim 0.40,$$

resonance locking can occur at much larger periastron separations, up to several tens of R_1 . Although locking is therefore possible even in relatively wide systems, smaller separations are more favourable because they allow access to a larger number of high-amplitude modes capable of maintaining the lock. In Figure 7.13, the maximum separation at which resonance locking occurs is clearly non-monotonic and displays a stripe-like pattern. This behaviour arises because high-amplitude modes become accessible at low orbital frequencies when the stellar rotation frequency lies close to a free oscillation eigenfrequency. To illustrate this behaviour, Figure 7.14 shows the range of modes that can be reached for a rotation rate of $0.40 \Omega_{\text{crit}}$, while Figure 7.15 shows the corresponding range for a rotation rate of $0.25 \Omega_{\text{crit}}$. The strip-like pattern seen in Figure 7.13 is directly related to the accessibility of these modes. Each time an unstable mode becomes reachable at low orbital frequency, resonance locking can in principle occur in that low-frequency regime. As the rotation rate increases, the accessible modes become more sparse. Since high-order g-modes are approximately equally spaced in period, new unstable modes are encountered less frequently

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

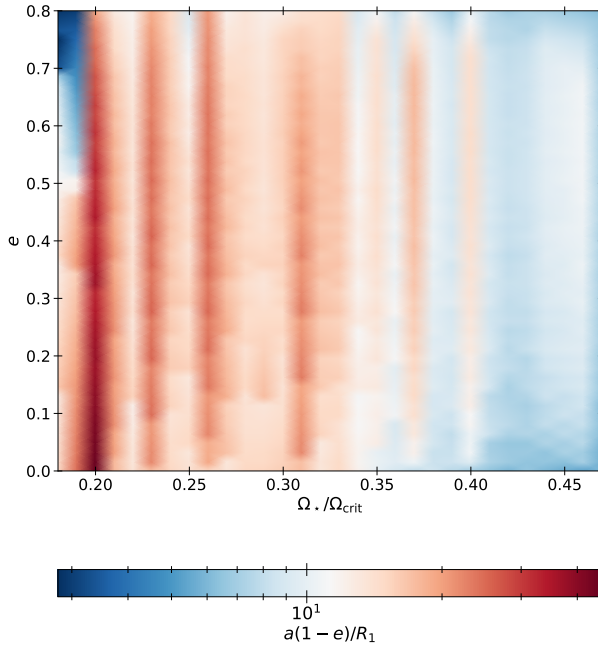


Figure 7.13: Map of the conditions under which resonance locking can occur. The colour scale indicates the maximum periastron separation at which locking is possible, as a function of stellar rotation rate (x-axis) and eccentricity (y-axis), for a $1.5 M_{\odot}$ γ -Doradus star. For rotation rates below 15% of the critical rotation, resonance locking with retrograde modes is not expected.

as the rotation increases, which explains the strip-like pattern seen in the high-rotation regime of Figure 7.13. In addition, whereas at low rotation rates the accessible modes are mostly unstable, at higher rotation rates the accessible modes increasingly become stable. This explains why resonance locking can no longer be found at low orbital frequencies once the rotation rate approaches roughly $0.40 \Omega_{\text{crit}}$. Overall, these results suggest that resonance locking, and more generally inverse-tide effects involving unstable modes, may influence the evolution of γ -Doradus binaries over a broad range of orbital separations, including systems that are not extremely compact.

PROGRADE MODES

The key difference with respect to retrograde modes is that, in this case, resonance locking occurs near amplitude dips in the dynamical-tide spectrum. The main effect of such locking is to impose a slow secular evolution of the stellar rotation, thereby preventing the rapid synchronisation that would otherwise be expected in close-in systems (Fellay et al., 2026). Providing a precise quantitative criterion for resonance locking with prograde modes is difficult, because several modes can contribute simultaneously to the locked state. This is illustrated in Figure 7.10, where the system undergoes a four-mode locking. Nevertheless, this type of locking is expected to oc-

7.3. PRESCRIPTION FOR THE ROTATIONAL AND ORBITAL EVOLUTION DURING RESONANCE LOCKING

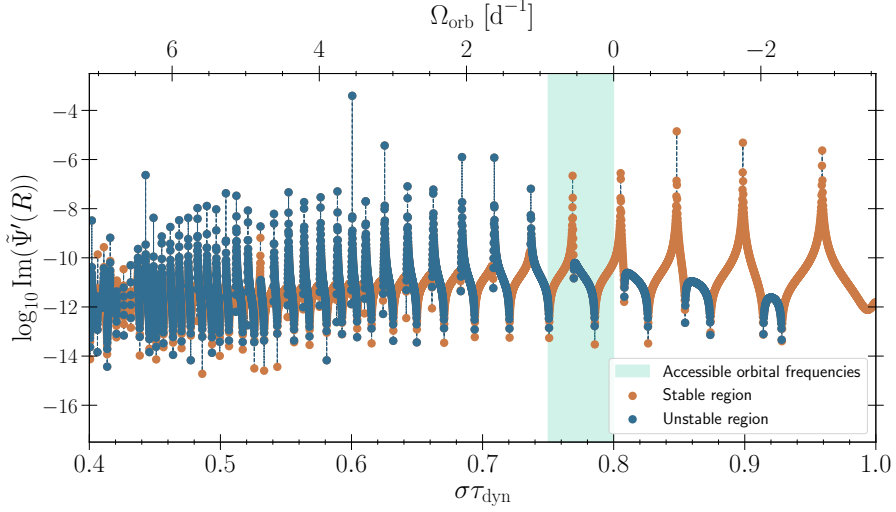


Figure 7.14: Range of retrograde modes accessible at a rotation rate of $0.40 \Omega_{\text{crit}}$. The accessible range is defined by requiring orbital frequencies low enough to be reached without contact or excessive tidal deformation. Here, we adopt an arbitrary upper limit of 1 d^{-1} for the orbital frequency, although this limit is probably already somewhat overestimated. The system considered is a twin $1.5 M_{\odot}$ γ -Doradus binary at an age of 738 Myr. To simplify the discussion, we assume a circular orbit and take $k = -m = -2$. The y-axis shows the stellar response, quantified by the imaginary part of the dimensionless potential perturbation, as a function of the dimensionless forcing frequency on the lower x-axis and the orbital frequency on the upper x-axis. Blue dots correspond to unstable responses, which are able to induce resonance locking, whereas orange dots correspond to stable responses, which are unable to do so.

cur preferentially in close-in systems, for which the orbital frequency is no longer negligible compared with the inverse of the stellar dynamical timescale. In practice, this corresponds approximately to $\Omega_{\text{orb}} \tau_{\text{dyn}} \gtrsim 0.1$. Such systems can have circular orbits, but they must also be slowly rotating, typically with $\Omega_{\text{rot}} \tau_{\text{dyn}} \lesssim 0.05$, so that the tidal forcing frequency, defined in Equation 7.14, remains sufficiently high to reach the frequency domain where unstable prograde modes can be excited. Because the tidal forcing is strong in these close-in systems, the absolute amplitudes of the tidally excited oscillations are not expected to be the limiting factor for the occurrence of locking, even when the lock occurs near local minima of the response spectrum. The relevant condition is instead whether the forcing frequency can access a region of the mode spectrum containing unstable prograde modes. This is therefore the criterion adopted here to determine whether resonance locking with prograde modes can occur.

7.4. COMPARISON WITH THE FULL IMPACT OF RESONANCE LOCKING

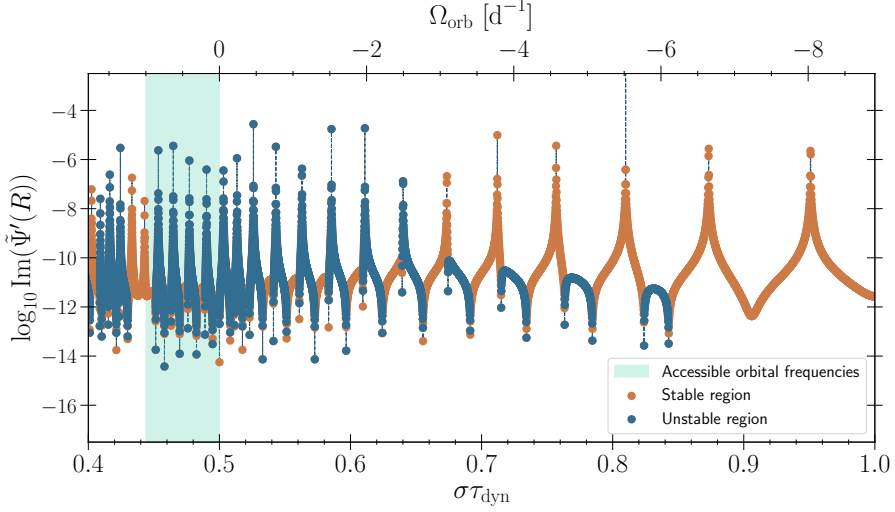


Figure 7.15: Range of retrograde modes accessible at a rotation rate of $0.25 \Omega_{\text{crit}}$. The accessible range is defined by requiring orbital frequencies low enough to be reached without contact or excessive tidal deformation. Here, we adopt an arbitrary upper limit of 1 d^{-1} for the orbital frequency, although this limit is probably already somewhat overestimated. The system considered is a twin $1.5 M_{\odot}$ γ -Doradus binary at an age of 738 Myr. To simplify the discussion, we assume a circular orbit and take $k = -m = -2$. The y-axis shows the stellar response, quantified by the imaginary part of the dimensionless potential perturbation, as a function of the dimensionless forcing frequency on the lower x-axis and the orbital frequency on the upper x-axis. Blue dots correspond to unstable responses, which are able to induce resonance locking, whereas orange dots correspond to stable responses, which are unable to do so.

7.4 COMPARISON WITH THE FULL IMPACT OF RESONANCE LOCKING

In [Section 7.3](#), we introduced a simple prescription for the evolution of stellar rotation driven by dynamical tides during resonance locking. In this section, we use this prescription to integrate the tidal evolution of binary systems and compare the results with the full integrations performed with TREMOR. This comparison allows us to assess whether the simplified prescription can reproduce the main secular effects of resonance locking without computing the full non-adiabatic tidal response at each evolutionary step. We consider both regimes identified in [Fellay et al. \(2026\)](#): rapidly rotating systems, in which resonance locking occurs mainly with retrograde modes, and close-in slowly rotating systems, in which locking with prograde modes can take place. In that work, we showed from full TREMOR integrations that resonance locking can strongly modify the rotational and orbital evolution of γ -Doradus binaries. Here, we compare those full integrations with the prescription derived in the previous section in order to determine which aspects of the evolution can be captured by the simplified

7.4. COMPARISON WITH THE FULL IMPACT OF RESONANCE LOCKING

model and where a complete integration of the dynamical tide remains necessary.

7.4.1 LOCKING WITH RETROGRADE MODES

For the high-eccentricity systems presented in Figure 7.2.2, we integrated the prescription for the evolution of the stellar rotation, semi-major axis, and eccentricity. In the regime where resonance locking does not occur, we assumed that the tidal dissipation is dominated by non-resonant dynamical tides described by the formalism of Zahn (1975, 1977). We considered a twin binary system composed of two $1.50 M_{\odot}$ γ -Doradus stars, with an initial orbital period of 12.28 days, an initial eccentricity of 0.55, and an initial rotation rate for each star equal to 30% of the critical rotation as in Figure 7.7. The comparison between the integration of the prescription and the results obtained with TREMOR is shown in Figure 7.16. By comparing the prescrip-

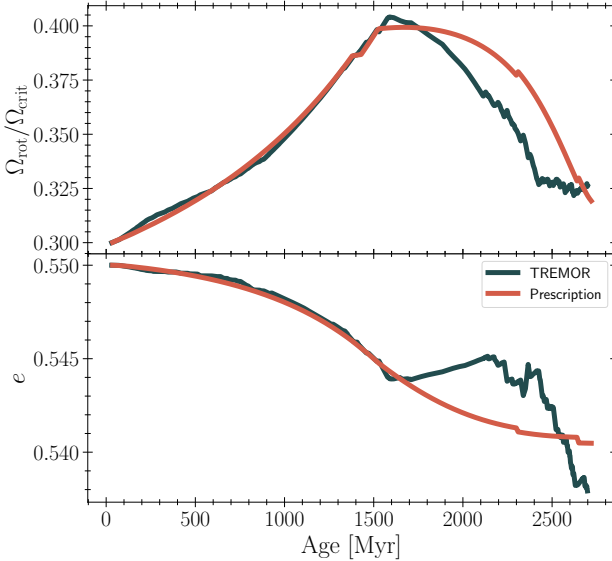


Figure 7.16: Comparison between the stellar-rotation and eccentricity evolution obtained from the full TREMOR integration (blue curve) and the resonance-locking prescription (orange curve) for a twin binary composed of two $1.5 M_{\odot}$ γ -Doradus stars. The system has the same orbital properties as in Figure 7.7, but with an initial stellar rotation rate of $0.3 \Omega_{\text{crit}}$.

tion with the results of the full TREMOR integration, we find that the two approaches agree reasonably well during the first part of the evolution, which corresponds to the resonance-locking phase. During this stage, the prescription is able to reproduce both the rotational and eccentricity evolution of the system. We also note that, by combining the results of the non-adiabatic computations with the predicted range of unstable

7.4. COMPARISON WITH THE FULL IMPACT OF RESONANCE LOCKING

modes, we are able to identify the part of the evolution during which resonance locking is expected to occur. When the locking phase ends, at around 1.5 Gyr, differences begin to appear between the full integration and the prescription. In this later stage of the evolution, we adopt the prescriptions of Zahn (1975, 1977) to describe the tidal evolution. However, the full integration of tidally excited oscillations predicts a significantly stronger tidal interaction than the one obtained from the Zahn (1975, 1977) formalism. This discrepancy is likely related to the regime of validity of these prescriptions, which were derived for low forcing frequencies. In the systems considered here, the higher forcing frequencies produce a tidal response that lies outside this regime and is therefore more accurately captured by TREMOR. As a consequence, at the end of the evolution, the prescription underestimates the changes in both eccentricity and stellar rotation. These corrections can only be recovered through a full integration of the effect of tidally excited oscillations.

In the case of more slowly rotating stars, we also explored the differences between the full integration and the prescription. We considered a twin binary system composed of two $1.50 M_{\odot}$ γ -Doradus stars with the same initial orbital properties as in the previous case, but with an initial rotation rate of 20% of the critical rotation. The comparison between the full integration and the prescription is shown in Figure 7.17. As illustrated in Figure 7.17, the differences between the prescription and the full integration are expected to be larger over the whole evolution. In the case of slower rotation, resonance locking appears to be less efficient, or at least less frequent. This can be understood from the fact that such a rotation rate only allows the system to reach unstable modes with very high radial orders. As a result, a large fraction of the theoretical instability strip becomes inaccessible, making locking less likely. Consequently, when the system becomes unlocked, it typically takes longer to re-enter a locking phase, and dissipative tides are able to act during these intervals. As in the faster-rotation case, we also observe strong discrepancies at the end of the evolution, owing to the growing influence of dissipative tides and to the limitations of the prescription of Zahn (1975, 1977) in the high-forcing-frequency regime. This behaviour is expected, since these prescriptions were not derived for such conditions. Overall, the comparison confirms that, for slowly rotating systems, the simplified prescription is less able to reproduce the detailed secular evolution obtained from the full TREMOR integration.

7.4.2 LOCKING WITH PROGRADE MODES

In the case of prograde modes, resonance locking is expected to occur in close binary systems with initially slow stellar rotation. When such locking takes place, the rotational evolution of the system is expected to be significantly slowed down with respect to that predicted by classical tidal prescriptions (Zahn, 1975, 1977), as discussed in the previous section. Using the prescription, we therefore attempted to reproduce the results of the full time integration. The comparison between the full TREMOR calculations and the prescription is shown in Figure 7.18. As shown in Figure 7.18, the prescription reproduces the rotational evolution reasonably well during the locking phase, which extends over approximately the first 1.5 Gyr. In this phase, both the prescription and the full TREMOR integration predict a slow secular evolution of the

7.4. COMPARISON WITH THE FULL IMPACT OF RESONANCE LOCKING

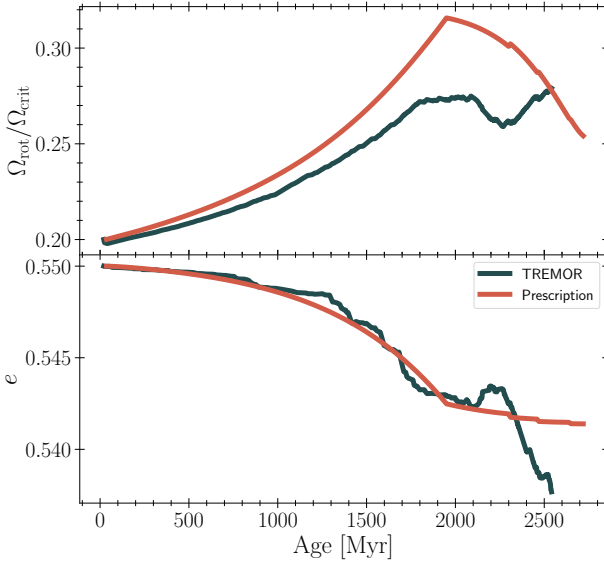


Figure 7.17: Comparison between the stellar-rotation and eccentricity evolution obtained from the full TREMOR integration (blue curve) and the resonance-locking prescription (orange curve) for a twin binary composed of two $1.5 M_{\odot}$ γ -Doradus stars. The system has the same orbital properties as in Figure 7.7, but with an initial stellar rotation rate of $0.2 \Omega_{\text{crit}}$.

stellar rotation, consistent with the idea that resonance locking can strongly reduce the efficiency of tidal spin evolution (Fuller, 2017). At later times, however, strong discrepancies appear between the TREMOR results and Zahn’s prescription. This is not unexpected, since classical prescriptions are not designed to account for the detailed behaviour of resonance locking, and the strong tidal interactions in this regime amplify the differences between the two approaches. For the eccentricity evolution, the agreement is poorer over the whole integration. In that case, discrepancies build up progressively because of the repeated unlocking episodes experienced by the system. Each of these events has a significant impact on the eccentricity, owing to the strength of the tidal interaction in this regime. We therefore expect the prescription to perform less well when applied to the orbital evolution, especially in close systems where the tides are strong and where each unlocking episode can produce a non-negligible change in the orbital parameters. By contrast, for the stellar rotation, the prescription still captures the overall secular behaviour and reproduces the inhibition of the strong tidal spin evolution that would otherwise be expected in these systems.

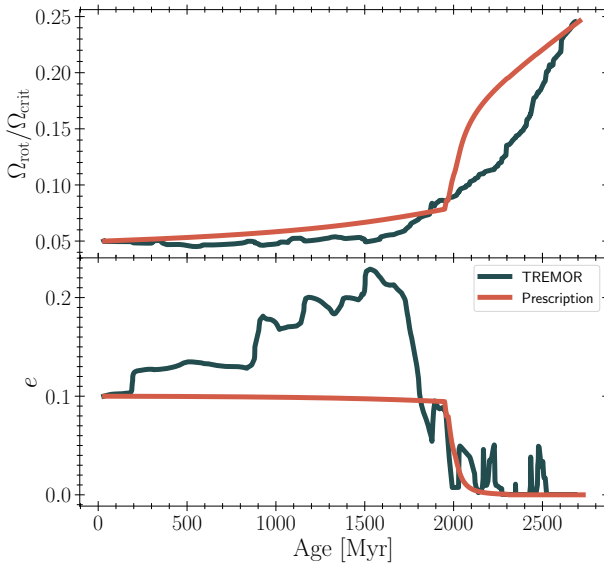


Figure 7.18: Comparison between the stellar-rotation and eccentricity evolution obtained from the full TREMOR integration (blue curve) and the resonance-locking prescription (orange curve) for a twin binary composed of two $1.5 M_{\odot}$ γ -Doradus stars. The system has initial orbital eccentricity of 0.1, a stellar rotation rate of $0.05 \Omega_{\text{crit}}$.

7.5 DISCUSSION AND CONCLUSION

In this study, we have developed, for the first time, a numerical method called TREMOR (introduced in [Section 7.1](#)), capable of integrating the simultaneous impact of all stellar oscillation modes excited by the presence of a companion on the orbital and rotational evolution of a system. Using TREMOR, we performed numerical integrations for twin binary systems with various initial parameters. Our results show that if the rotation is above approximately 15% of the critical rotation, resonances with unstable retrograde modes in the co-rotating frame and prograde in the observer frame lead to to an increase in stellar rotation and the deposition of angular momentum into the star. We find that resonance locking is a particularly stable state that can persist over long timescales. Throughout the main sequence, resonance locking with inverse tides can increase stellar angular momentum by up to 70% in both low- and high-eccentricity systems.

In close-in systems ($\Omega_{\text{orb}} \tau_{\text{dyn}} \gtrsim 0.1$) with slowly rotating components ($\Omega_{\text{rot}} \tau_{\text{dyn}} \lesssim 0.05$), prograde modes can also induce resonance locking. The effect of this locking is to enforce a slow evolution of stellar rotation, preventing the rapid synchronisation expected in close-in systems. In regimes where the forcing frequencies are small compared to the inverse of the dynamical timescale, we expect tidal dissipation mech-

7.5. DISCUSSION AND CONCLUSION

anisms described by Zahn (1975, 1977) to dominate.

A key distinction between our approach and that of Fuller (2021) is that we account for the evolution of the forcing frequency due to stellar rotation changes, include the effects of rotation on oscillation modes, and incorporate stellar evolution. Additionally, we do not impose a resonance locking. As a result, our findings differ significantly from those of Fuller (2021). Namely, our results suggest that resonance locking is not a violent process capable of forming Be stars. Instead, it is a low-acting, long-lasting mechanism that can modify the stellar rotation over most of the main sequence. This mechanism allows stars to retain a trace of their initial rotation, even in close-in systems, where strong tidal interactions would otherwise be expected to erase it.

One of the primary limitations of our model is the assumption of solid-body rotation. Under this assumption, we demonstrated that resonance locking provides an efficient angular momentum exchange mechanism capable of modifying global rotation. In reality, for high-radial-order oscillation modes (such as those considered in Zahn (1975, 1977)), most of the tidal dissipation likely occurs in the near-surface layers (Goldreich & Nicholson, 1989). This leads to surface synchronisation before angular momentum can be deposited into deeper layers. On the other hand, a few observations (Kallinger et al., 2017; Sowicka et al., 2017) suggest that large differential rotation can exist in stars that appear synchronised at the surface. This raises the possibility that inverse tides may induce asynchronisation in the deeper stellar interior, while the outer layers remain synchronised due to equilibrium tides or high-radial-order forced oscillation modes. The exact processes governing inverse tidal dissipation remain uncertain. However, in γ -Doradus stars, the adiabatic approximation breaks down at deeper layers for low-radial-order modes (Dupret et al., 2005), such as our overstable modes. This suggests that dissipation may occur deeper in the star. Other non-linear mechanisms may also contribute to deep angular momentum deposition via inverse tides. Additionally, angular momentum transport processes, such as magnetic torques, are expected to enforce near solid-body rotation (Spruit, 2002; Fuller et al., 2019), except in the near-surface layers, where a shear can develop. Including differential rotation and its evolution in our models would be a significant improvement, though the physical mechanisms underlying such an implementation remain uncertain, particularly for tidal dissipation. The major results of our study on the evolution of the global rotational angular momentum should not be fundamentally modified by taking into account differential rotation. What remains uncertain is the evolution of the surface rotation.

From an observational perspective, Zahn's tidal theory generally agrees with observed surface synchronisation (or pseudo-synchronisation) trends in close-in intermediate-mass star populations (Khaliullin & Khaliullina, 2007). However, a few observational studies (Zimmerman et al., 2017; Lampens et al., 2018; Guo et al., 2019; Li et al., 2020a,b) have reported the existence of slowly rotating close-in binaries. For instance, Zimmerman et al. (2017) report that approximately 20% of their sample consists of strongly sub-synchronous binaries, while the majority are only mildly sub-synchronous. This distribution appears to be inconsistent with the expected tendency towards pseudosynchronous rotation rates predicted by theoretical modelling

(Townsend & Sun, 2023). However, we did not observe a population of fast-rotating close-in binaries, except for a few outliers or systems where synchronisation may not have been expected to occur yet (Thompson et al., 2012; Mantegazza et al., 2010; Shporer et al., 2016; Li et al., 2020a; Saio & Kurtz, 2022). For example, Li et al. (2020a) reported six super-synchronous systems located near the transition between synchronised and unsynchronised binaries. However, it remains unclear whether these systems were born as fast rotators and are currently synchronising or whether inverse tides are at play. It should be noted that for fast-rotating stars exhibiting tidally forced oscillations, identifying the oscillation modes and deducing the near-core rotation rate may be challenging. It is also worth noting that non-linear effects, as investigated by Fuller (2021), may inhibit or make unlikely resonance locking with retrograde modes since such lockings occur close to resonance, where non-linear effects are expected to be significant. In contrast, lockings involving prograde modes occur away from resonance, where non-linear effects are expected to be weaker, making them more likely to operate. The study of these non-linear effects, however, remains a significant challenge, and current approaches rely on simplified adiabatic modelling. Further theoretical developments or direct non-adiabatic simulations would be highly valuable for improving our understanding of these phenomena, but this issue is beyond the scope of the present work.

In Zahn’s tidal theory, angular momentum deposition is expected to occur primarily near the surface (Goldreich & Nicholson, 1989). Thus, classical observation, limited to the outer stellar layers, should be consistent with Zahn’s model provided that low-frequency forcing is present in the system. We also do not exclude other processes, such as equilibrium tides with viscous dissipation (Darwin, 1880; Zahn, 1977; Goldreich & Nicholson, 1977; Zahn, 1989; Duguid et al., 2020; Vidal & Barker, 2020; Barker & Astoul, 2021) to still be able to synchronise the surface layers, as γ -Doradus stars possess an outer thin convective layer. If inverse tidal angular momentum deposition occurs deeper within the stellar interior, conventional observational methods (excluding asteroseismology or apsidal motion studies) may fail to detect the rapid rotation of deep radiative zones. Another possible explanation for the lack of observed fast-rotating close-in binaries is that such systems may not form frequently. While fast-rotating single stars are common (Royer et al., 2007), studies such as Khaliullin & Khaliullina (2007) and Li et al. (2020a) show that in binary systems, the average stellar rotation rate is significantly lower than that of single stars.

Future investigations should focus on detailed asteroseismic studies of pseudo-synchronised binary stars, particularly those with well-characterised surface rotation rates and oscillation frequencies. The key objective of such studies will be to determine whether the near-core rotation is consistent with the observed surface rotation rate. These observations, combined with further theoretical developments, will help refine our understanding of inverse tides, resonance locking, and angular momentum transport in binary stars.

CONCLUSION, DISCUSSION AND PERSPECTIVE

8.1 DEFORMATIONS IN CLOSE BINARY SYSTEMS

The first part of this thesis was dedicated to the modelling of stellar deformations in close binary systems. We developed a new non-perturbative modelling technique able to compute the structure of highly distorted stars while consistently accounting for the internal redistribution of mass. Such a methodology is required to determine the domain of validity of classical tidal theories, which generally rely on perturbative assumptions, and to quantify the corrections expected when these assumptions break down (Sterne, 1939; Eggleton, 1983; Sirotkin & Kim, 2009; Fellay et al., 2024). Although perturbative models are widely used in binary-star studies, a systematic assessment of their limitations in strongly deformed systems had not previously been carried out. We found that non-perturbative effects generally become significant when the periastron separation satisfies $\frac{r_{\text{peri}}}{R} = \frac{a(1-e)}{R} \lesssim 5$, where a is the semi-major axis, e the eccentricity, and R the stellar radius. In this regime, non-perturbative models predict stronger deformations and measurable differences in the tidal interaction compared with perturbative calculations. These differences directly affect observable quantities, in particular apsidal motion, which is a sensitive probe of the internal mass distribution and tidal distortion of the stellar components (Giménez & García-Pelayo, 1983; Claret & Giménez, 2010; Claret, 2019). For observed systems with measurable apsidal motion, we showed that the non-perturbative correction can reach an amplitude comparable to the relativistic contribution. This demonstrates that non-perturbative effects must be taken into account when using apsidal motion to test stellar structure and tidal interactions in close eccentric binaries (Claret & Giménez, 2010; Fellay et al., 2024).

For pure deformations, this thesis opens several promising directions that could not be explored within the scope of the present work, either because of time constraints or because priority was given to the developments that were the most coherent and relevant for the overall structure of the manuscript. In particular, the first part of this

thesis naturally suggests several extensions. The most immediate and arguably the most important one is the inclusion of orbital inclination, or equivalently spin–orbit misalignment, in the non-perturbative structural modelling. Throughout this thesis, the rotation axis of the system was assumed to be normal to the orbital plane. This hypothesis was initially adopted in order to simplify the problem, and it is physically motivated by the fact that tidal interactions are generally expected to drive binary systems toward an aligned configuration. Although this assumption remains justified in many cases, non-aligned systems represent realistic, non-idealized configurations that must ultimately be considered if one aims to reproduce observed binary systems more faithfully. From a formal point of view, introducing inclination is relatively straightforward. In many equations, the main modifications would simply take the form of additional geometric multiplicative factors, such as $\sin i$ or $\cos i$. Its numerical consequences, however, are much more significant. Indeed, the presence of inclination breaks the two symmetry planes currently exploited by MoBiDICT. As a result, the number of spherical harmonics that must be considered increases drastically, since one can no longer reduce the expansion by symmetry arguments and must instead consider the full spectral decomposition. Even though the spectral decomposition is not the most computationally expensive part of the method, this extension would still substantially increase the number of quantities that must be stored and manipulated, while also reducing the numerical precision that can be achieved at fixed computational cost. Consequently, including inclination would likely require adaptations to the numerical architecture of MoBiDICT. Such developments would provide a natural foundation for a more comprehensive and realistic study of binary systems.

A second direct application of the first part of this thesis concerns light-curve modelling, which was in fact part of the initial motivation of the project. Building on the classical light-curve synthesis framework introduced by [Wilson & Devinney \(1971\)](#) and later extended to more general binary configurations by [Wilson \(1979\)](#), and on modern implementations such as PHOEBE ([Prša & Zwitter, 2005](#); [Prša et al., 2016](#); [Conroy et al., 2020](#)), the modelling procedure developed here is expected to produce significant corrections to the stellar shape and surface properties in the regime of strong deformations. These structural changes should leave an observable imprint on the light curves of eclipsing binaries, especially in systems close to contact. A particularly promising extension would be to couple our structural deformation models to stellar-atmosphere calculations, for instance with CMFGEN ([Hillier & Miller, 1998](#)). In practice, this would consist of attaching the atmosphere locally to the deformed stellar surface by enforcing continuity of the pressure at the interface. Such a strategy would provide a more realistic atmospheric structure and would therefore make it possible to predict the local temperature consistently, including the effects of deformation. In this context, the treatment of gravity darkening is especially important, from the classical formulation of [von Zeipel \(1924\)](#) to the more recent prescriptions developed for rotating and binary stars by [Espinosa Lara & Rieutord \(2011\)](#) and [Espinosa Lara & Rieutord \(2012\)](#). Similar couplings between surface deformation and atmosphere or spectral modelling have already proved useful in the study of distorted massive binaries ([Palate & Rauw, 2012](#)). With such a modelling strategy, it should become possible to predict the light curves of near-contact eclipsing binaries while fully retaining the non-perturbative effects identified in this thesis. This would in turn provide a new way

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to constrain the observed properties of close binary systems. Although this work has not yet been carried out, it represents a natural continuation of the present study, even if, personally, I would prefer to focus first on other directions that I discuss later in this chapter.

Another direct outcome of the first part of this thesis concerns the last major numerical aspect implemented in MoBiDICT, namely the treatment of the critical direction on which this family of modelling methods relies. Although MoBiDICT was designed for binary systems, it builds on an approach that was initially proposed by [Roxburgh \(2004\)](#). The method developed here to handle this issue appears to be new and has not yet been published. As such, it could by itself constitute the basis of a separate article. This development could also have applications beyond the specific framework of MoBiDICT. In particular, it may be possible to implement a similar strategy in existing two-dimensional stellar structure codes in order to improve their numerical robustness and stability. Such an extension would nevertheless be more challenging, since the problem is more intricate in genuinely two-dimensional configurations, especially when the stellar structure is no longer barotropic ([Roxburgh, 2004](#); [Manchon, 2021](#); [Houdayer & Reese, 2023](#)). Finally, this method could serve as a starting point for a future extension of MoBiDICT to differential rotation. This would be especially valuable for studying angular-momentum transport and the large-scale flows that may be induced by tides and strong deformations in close binary systems ([Maeder & Meynet, 2012](#); [Manchon, 2021](#)). Altogether, these projects would provide a solid basis for investigating the internal structure of strongly deformed binaries, predicting their observational signatures, and understanding how close companionship modifies the structural and rotational evolution of stars.

Finally, one of the long-term projects made possible by this thesis concerns the modelling of contact binaries. Although the deformed structures developed here are formally valid only for detached binary systems, we quickly found that a limited number of additional assumptions and adaptations could provide the basis for a non-perturbative method able to derive self-consistent structures for contact configurations. Given the level of structural readjustment already obtained for very close detached systems, and given the importance of tidal deformation in overcontact evolution, a non-perturbative treatment appears necessary if one aims to model contact binaries realistically. Such a development would represent a major step forward in the theoretical understanding of binary evolution during these phases. Before reaching that stage, another major application of the work conducted in this thesis is the modelling of binary evolution while explicitly accounting for deformation effects. In the one-dimensional framework, I have presented a methodology that includes the impact of deformation directly into the stellar structure equations through correction factors such as f_T and f_P . As a first approximation, implementing these effects in an evolution code is therefore relatively straightforward ([Siess et al., 2013](#); [Fabry et al., 2022](#)). In practice, I have already coupled MoBiDICT to the binary stellar evolution code BINSTAR ([Siess et al., 2013](#)). Our first study, restricted to massive stars, suggests that the resulting modifications of the coupling coefficients are not sufficient to drastically alter the evolution, mainly because the evolutionary timescales of these systems are too short for the effects to accumulate strongly. By contrast, lower-mass systems, which remain

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on the main sequence for much longer, are expected to be more sensitive to these deformation effects. Investigating this regime will therefore be an important step toward a more complete and consistent description of binary star evolution.

8.2 INSTABILITY STRIPS OF INTERMEDIATE-MASS STARS.

In my work on the γ -Doradus instability strip, I computed and published modern grids of non-adiabatic stellar oscillation models, with particular emphasis on the effects of rotation and metallicity. This study provided a detailed framework for understanding the physical mechanisms that drive and damp modes in intermediate-mass stars, and it opens the way to several direct applications. One of the most natural extensions is the revision of theoretical instability strips. I have already started such a study for γ -Doradus stars, and a similar approach could be applied to slowly pulsating B stars, δ -Scuti stars, and more peculiar classes of pulsators. In this context, the classical instability domains established for upper main-sequence pulsators (Pamyatnykh, 1999) and the later developments based on non-adiabatic computations with time-dependent convection (Dupret et al., 2004; Grigahcène et al., 2005; Dupret et al., 2005) provide a strong theoretical foundation. A key technical development that I plan to pursue is the modification of MAD, and in particular of the `intrnad` program, so that non-adiabatic corrections can be computed from equilibrium models produced by stellar-evolution codes other than CLES. This extension is important because modern stellar-evolution codes now allow a much broader range of internal structures and evolutionary histories to be explored. For instance, MESA can model stars affected by mass loss, accretion, binary interaction, and non-standard evolutionary pathways (Paxton et al., 2011, 2015), while accretion-induced mixing and transport processes can strongly modify the internal chemical stratification of stars (Deal et al., 2015, 2016). Other structural codes, such as `Cesam2k20` (Morel, 1997; Morel & Lebreton, 2008; Marques et al., 2013), include detailed treatments of rotationally induced transport, atomic diffusion, and radiative accelerations, all of which can alter the distribution of heavy elements and therefore the pulsational properties of stars (Marques et al., 2013; Deal et al., 2018). In all these cases, access to a non-adiabatic oscillation code including time-dependent convection is essential for studying how structural peculiarities affect the instability strips of intermediate-mass stars. Such a tool would make it possible to test the seismic consequences of a wide variety of evolutionary scenarios, including chemically peculiar stars, accretion-modified stars, stripped stars, and products of binary interaction. In turn, these calculations would provide stronger constraints on the physical properties, internal transport processes, and formation channels of these stars.

8.3 TIDALLY EXCITED OSCILLATIONS AND FUTURE STELLAR OSCILLATIONS CODES.

For the work devoted to the asteroseismology of tidally excited oscillations, we first focused on the treatment of the Poisson equation when the Coriolis force is included. In particular, we investigated the gravitational coupling that arises when the perturba-

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tion of the gravitational potential is treated consistently rather than neglected through the Cowling approximation (Cowling, 1941; Townsend, 2003, 2005; Fellay & Dupret, 2025). We presented a method to include this coupling in the modelling of tidally excited oscillations and showed that it can have a significant impact on the perturbation of the gravitational potential, especially in regimes where the Cowling approximation is not necessarily valid (Townsend & Teitler, 2013; Sun et al., 2023; Fellay & Dupret, 2025). This coupling affects different families of modes in different ways. Modes with $m = 0$ are the most strongly affected, whereas prograde modes are expected to be only weakly modified. For retrograde, we showed that gravitational coupling can increase the predicted amplitudes of tidally excited oscillations by up to several orders of magnitude. This amplification originates from two related effects. First, the internal gravitational-potential perturbation is coupled when the spherical-harmonic expansion is projected onto the Hough-function basis used to describe the Coriolis-modified angular structure of the modes (Townsend, 2003; Bouabid et al., 2013; Fellay & Dupret, 2025). Second, an additional coupling appears when the tidal forcing itself is projected from the spherical-harmonic basis onto the Hough functions in order to obtain the appropriate forcing term. These two mechanisms explain both the local differences in the perturbed gravitational potential and the global discrepancies found in the predicted amplitudes of tidally excited oscillations.

In the context of the forced stellar oscillations, this work led me to develop a new type of one-dimensional oscillation code, based on an iterative treatment of Poisson's equation outside the main system of differential equations, while the correction associated with the perturbed gravitational potential is reintroduced as an inhomogeneous term in the matrix problem. More generally, this development highlighted an important methodological point: from the point of view of code design, handling specific equations through separate solution procedures is particularly convenient. It makes the implementation more modular, facilitates the introduction of modifications or numerical tests, and more generally improves the flexibility of the whole framework whenever additional physics has to be included. I should also stress that, in practice, the iterative treatment of Poisson's equation proved to be more accurate when non-adiabatic effects are included. In a fully direct solution of the differential system, a numerical precision floor is reached rather quickly when the contrast between the real and imaginary parts of the eigenfunctions or eigenfrequencies becomes too large, typically of the order of five to eight orders of magnitude. This limitation is already well known in MAD; for example, damping or growth rates are re-evaluated through the work integral in order to bypass the loss of precision affecting the direct determination of the imaginary part. In the case of Poisson's equation, solving it independently made it possible to reach a much more reliable precision for the perturbed potential near the stellar surface, which in turn strengthens confidence in the numerical robustness of the method. For this reason, if I had to develop a new oscillation code today, I would probably not choose a fully direct solution of the complete differential system, as is commonly done in many existing oscillation codes. Instead, I would favour a hybrid strategy: a direct solution for the core oscillation equations, such as the momentum and continuity equations, combined with iterative treatments for additional or non-standard physics, such as self-gravity or non-adiabatic corrections. Although the work presented in this thesis was developed in a one-dimensional oscillation framework, it

is also worth noting that the structures obtained with MoBiDICT could in principle be used as a basis for studying oscillations in deformed stars with strongly distorted propagation cavities. Non-perturbative calculations in rapidly rotating and distorted stars have already shown that centrifugal deformation and cavity geometry can profoundly modify the mode spectrum and lead to new mode families and propagation regimes (Reese et al., 2009; Ouazzani et al., 2012, 2015; Prat et al., 2016, 2018). Extending such approaches to the three-dimensional deformed structures considered here would be an extremely challenging long-term project, but it is also one of the clearest direct applications of the broader methodological developments achieved during this thesis. In that perspective, I would clearly recommend relying on an iterative architecture of the type developed here.

8.4 DYNAMICAL TIDES

Once a version of `MAD_tides` was available to compute the stellar response to an imposed tidal forcing, the natural next step was to integrate the cumulative effect of dynamical tides over evolutionary timescales. I therefore began the development of `TREMOR`, with the goal of treating the tens to hundreds of simultaneous forcing frequencies that arise in eccentric intermediate-mass binaries. The purpose of this framework is to identify which forcing components produce a non-negligible stellar response and to integrate their secular impact on the rotational and orbital evolution of the system. This objective is motivated by previous studies showing that resonant dynamical tides and resonance locking can substantially modify tidal evolution, but that most applications have either focused on individual resonances or relied on simplified prescriptions (Witte & Savonije, 2001; Fuller & Lai, 2012; Burkart et al., 2014; Fuller, 2017; Sun et al., 2023). To achieve this, we developed a new numerical framework based on a multi-layer filtering procedure coupled to an adaptive-timestep orbital integrator. This setup makes it possible to follow the combined effect of many dynamical-tide components and to determine which resonances dominate the exchange of energy and angular momentum between the stellar rotation and the orbit. The integrations performed with `TREMOR` suggest that resonance locking plays a central role in the tidal evolution of intermediate-mass binaries. While resonance locking had previously been proposed as a mechanism capable of accelerating tidal evolution and sustaining strong tidal torques (Burkart et al., 2014; Fuller, 2017, 2021), our calculations indicate a more nuanced behaviour in the systems explored here. In particular, we find that resonance locking does not necessarily act as a rapid mechanism for depositing large amounts of angular momentum into the star. Instead, it often appears as a slow and long-lived state that regulates the tidal interaction and helps maintain the stellar rotation close to its initial value throughout much of the evolution. This is especially clear for locking with prograde modes, which are expected to occur preferentially in short-period systems with slowly rotating stars. In these cases, resonance locking can delay the onset of stronger tidal interactions by keeping the system in a quasi-stable configuration where the stellar response is large but the secular evolution remains slow. In this sense, resonance locking acts less as a rapid spin-up mechanism and more as a stabilising channel for the rotational evolution. We also tested several numerical strategies designed to avoid or disrupt locking. However, the locking behaviour appears to be

robust: in many cases, the system eventually enters a locked state, even when this requires a multi-mode resonance locking. For retrograde modes, resonance locking can lead to a slow deposition of angular momentum into the star, gradually increasing the rotation rate over the main-sequence evolution. However, even then, the spin-up remains low in the models considered here, and the effect is again closer to the maintenance of the stellar rotation than to a rapid acceleration toward critical rotation. These results therefore suggest that, in intermediate-mass binaries, resonance locking may be a common and stable outcome of dynamical tides, but its evolutionary role is more subtle than a simple enhancement of tidal dissipation or angular-momentum transfer (Li et al., 2020b; Fuller, 2021; Kołaczek-Szymański & Różański, 2023; Townsend & Sun, 2023; Dewberry & Wu, 2024).

It is important, however, to state clearly the limitations of the formalism used in this work, since the long-term modelling of dynamical tides remains a difficult problem. First, near exact resonance, non-linear effects are expected to become important and may substantially modify the tidal behaviour predicted by linear calculations (Weinberg et al., 2012; Burkart et al., 2014; Barker et al., 2016; Fuller, 2017; Dewberry & Wu, 2024). For locking with prograde modes, this limitation is probably less severe in the regimes explored here, because the locking can involve several modes and may occur even when the system is not extremely close to a single resonance. In contrast, the case of retrograde modes is less straightforward. Non-linear saturation may occur before the mode amplitude required for locking is reached, in which case the prescriptions used here would no longer be valid. In that regime, retrograde-mode locking might be restricted to the closest systems, where the tidal response is strong enough before reaching the non-linear saturation effects. A second important uncertainty concerns the physics of tidal dissipation and its impact on the internal rotation profile of the star. In this work, we assume solid-body rotation. In reality, however, dynamical tides deposit energy and angular momentum in specific regions of the stellar interior, depending on the excited modes, their damping, and the structure of the propagation cavity. This local deposition can generate differential rotation, internal flows, shear, and subsequent angular-momentum redistribution (Rogers et al., 2013; Rogers, 2015; Aerts et al., 2019). For prograde-mode locking, the angular-momentum exchange found in our calculations is very slow and mainly acts by inhibiting the expected secular evolution of the stellar spin. We therefore do not expect the neglect of differential rotation to be the dominant source of uncertainty in that case. For retrograde-mode locking, the situation is more delicate: angular momentum can be deposited more efficiently in the envelope, potentially driving internal flows even if the net effect is primarily to maintain the initial rotation rate rather than to produce strong spin-up. These limitations indicate that the results presented here should be interpreted as a first step toward a more complete description of resonant dynamical tides in intermediate-mass binaries. Future work should investigate non-linear mode saturation, wave breaking, the spatial deposition of angular momentum, and the feedback between tidal torques and differential rotation. Such developments will be necessary to determine whether resonance locking remains as robust when the assumption of solid-body rotation is relaxed and when the internal transport of angular momentum is treated self-consistently.

8.5. PERSPECTIVES FOR THE DYNAMICAL-TIDES WORK

In this context, observations may be key to constraining the physical processes at work in close binary stars. This is especially true with the advent of space photometry, which has begun to provide constraints on the population of intermediate-mass stars in close binary systems (Li et al., 2020a). Motivated by this, we submitted a PLATO Guest Observer proposal, led by Gang Li as PI, to obtain observations of a selected sample of γ -Doradus stars in close binary systems showing strong asynchronism. For such stars, comparing the surface rotation rate with the near-core rotation rate inferred from asteroseismology could help constrain tidal dissipation processes in close binaries. It could also help disentangle the different mechanisms that may operate in these systems.

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The oscillation models developed in this thesis can be applied in more specific regimes, namely in systems where the deformation of the oscillating star remains moderate. Another particularly interesting configuration is one in which the primary remains only weakly deformed, while the secondary becomes strongly distorted as it evolves off the main sequence. In such a case, the secondary should be described with the non-perturbative approach implemented in MoBiDICT, while the primary could still be treated with TREMOR. Such configurations are especially relevant because post-main-sequence expansion naturally brings the secondary closer to its Roche lobe. Such systems provide a particularly appealing framework in which to study the competition between different channels of tidal dissipation. In the evolved and strongly distorted star, dissipation is expected to be dominated by the equilibrium tide acting through convective friction, as described by Zahn (1977). By contrast, in the main-sequence intermediate-mass companion, the dynamical tide in radiative regions may remain important. For these reasons, this class of systems appears to be a very promising application of the tools developed in this thesis.

Although all the formalism presented in the second part of this thesis was developed in the context of binary interactions, the same framework can in principle be applied to star–planet systems when considering dynamical tides. The underlying formalism remains essentially unchanged, the main difference being that the tidal perturbation is generated by a planetary companion rather than by a stellar one. This makes star–planet systems around intermediate-mass pulsators a particularly interesting extension of the present work. From an observational point of view, such an application is well motivated. In the case of the δ -Scuti host WASP-33, suggestive evidence for a coupling between the stellar pulsations and the orbital motion of its close-in giant planet has already been reported (Herrero et al., 2011). In addition, the directly imaged exoplanet host 51 Eridani has recently been identified as a γ -Doradus pulsator (Sepulveda et al., 2022). Studying the orbital evolution of such systems would therefore be very valuable, since significant tidal migration may occur and could eventually become observable. If so, these systems would offer a way to constrain the efficiency of dynamical tides and to calibrate the models developed here. More generally, this line of work should also be extended to other classes of pulsators. In the oscillation part of this thesis, I only considered binaries composed of γ -Doradus stars, but the role of

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resonance locking should also be explored for SPB stars. This extension is natural because γ -Doradus and SPB stars are both gravity-mode pulsators and can be analysed within closely related seismic frameworks. Moreover, resonance locking in massive main-sequence binaries has already been shown to be a potentially important mechanism for accelerating tidal evolution. Even if the prescriptions derived in the previous section remain formally valid, their quantitative impact in the SPB regime still needs to be investigated in detail.

In this work, we focused on intermediate-mass stars for several reasons. First, most of our modelling tools were already adapted to the internal structure of these stars. Second, rotation can be treated within the traditional approximation of rotation, and our non-adiabatic oscillation code includes time-dependent convection. Finally, recent observational surveys have made it possible to detect and characterize the asteroseismic properties of γ -Doradus stars in binary systems, including constraints on their rotational properties. Dynamical tidal interactions in cooler stars with deeper convective envelopes also represent an important topic, especially because planetary orbital evolution may depend strongly on tidal dissipation in such stars. However, this regime lies outside the scope of the present work. In cool stars, the traditional approximation used here is no longer appropriate for the dominant tidal response in convective regions, where inertial waves and modes are expected to play a central role. Moreover, the modelling of convection, and in particular of time-dependent convection, remains subject to significant uncertainties. Although extensive work has already been devoted to this regime (Ogilvie & Lin, 2007; Barker & Ogilvie, 2009, 2010; Ogilvie & Lesur, 2012; Ogilvie, 2014; Barker, 2020; Barker & Astoul, 2021; Barker, 2022), developing a non-adiabatic oscillation code capable of treating inertial modes beyond the traditional approximation, possibly in a non-perturbative framework, would constitute a substantial long-term project. For this reason, we do not prioritize this work.

Finally, in eccentric binaries, tides can affect the apsidal-motion rate not only through the classical static tidal and rotational distortions, but also through the oscillatory response of the stars to the time-dependent tidal forcing. The classical framework established by Cowling (1938) and Sterne (1939) treats the tidal deformation as quasi-static and relates the secular apsidal advance to the internal structure constants of the stellar components. When the forcing excites stellar oscillation modes, however, the induced dynamical distortions modify the effective quadrupole moment in a frequency-dependent way, so that the apsidal motion receives additional contributions beyond those predicted by equilibrium-tide theory alone (Quataert & Kumar, 1996; Smeyers & Willems, 2001). In particular, resonant dynamic tides can generate apsidal-motion terms that become significantly larger than the static contribution and may even change sign depending on the mode involved and on the forcing frequency (Smeyers et al., 1998). In this context, the formalism developed by Smeyers & Willems (2001) and extended by Willems & Aerts (2002) provides a direct way to connect the delayed stellar response to the resulting apsidal-motion rate. This is precisely the same theoretical framework that was used in the dynamical-tide part of this thesis. Being able to model the apsidal motion coherently while including these effects would therefore represent the final missing element of the broader apsidal-motion modelling undertaken in this thesis. This extension is particularly important because the corrections

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induced by dynamic tides are not negligible: for sufficiently short orbital periods and near resonances, they can reach several tens of percent relative to the classical prediction ([Smeyers & Willems, 2001](#); [Willems & Aerts, 2002](#)).

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APPENDIX A

PUBLICATIONS AND PRESENTATION DONE

In this section, I list the publications and presentations completed during my PhD. For the publications, I excluded non-written conference proceedings, such as contributions published only as presentation slides.

A.1 PUBLICATIONS

A.1.1 1st & 2nd AUTHOR

1. **Fellay, L.**; Buldgen, G.; Eggenberger, P.; Khan, S.; Salmon, S. J. A. J.; Miglio, A.; Montalbán, J., (2021), Asteroseismology of evolved stars to constrain internal angular momentum transport. IV. Internal rotation of Kepler-56 from an MCMC analysis of rotational splittings, *Astronomy & Astrophysics*, 654, A133.
2. **Fellay, L.**; Dupret, M.-A., (2023), MoBiDICT: New 3D static models of close synchronized binaries in hydrostatic equilibrium, *Astronomy & Astrophysics*, 676, A22.
3. **Fellay, L.**; Pezzotti, C.; Buldgen, G.; Eggenberger, P.; Bolmont, E., (2023), Constraints on planetary tidal dissipation from a detailed study of Kepler-91b, *Astronomy & Astrophysics*, 669, A2.
4. **Fellay, L.**; Dupret, M.-A.; Rosu, S., (2024), Underestimation of the tidal force and apsidal motion in close binary systems by the perturbative approach: Comparisons with non-perturbative models, *Astronomy & Astrophysics*, 683, A210.
5. Buldgen, G. ; **Fellay, L.** ; Betrisey, J. ; Deheuvels, S. ; Farnir, M. ; Farrell, E., (2024), MCMC inversions of the internal rotation of Kepler subgiants; *Astronomy & Astrophysics*.

A.2. PRESENTATIONS AT INTERNATIONAL OR NATIONAL CONFERENCE

6. **Fellay, L.**; Dupret, M.-A., (2025), Direct non-adiabatic modelling of gravito-inertial and Rossby tidally excited oscillations with the traditional approximation, *Astronomy & Astrophysics*, 694, A51.
7. **Fellay, L.**; Dupret, M.-A.; Kołaczek-Szymański, P. A., (2026), Resonance locking and tidal evolution in rotating γ -Doradus binaries, *Astronomy & Astrophysics*.
8. **Fellay, L.**; Dupret, M.-A., (submitted), On the theoretical instability strips of γ -Doradus stars including the effect of the metallicity and rotation, *Astronomy & Astrophysics*.
9. **Fellay, L.**; Dupret, M.-A., (in prep), Prescription for the rotational and orbital evolution of binaries undergoing resonance locking, *Astronomy & Astrophysics*.

A.1.2 OTHER PUBLICATIONS

1. V. Antoci, J. Labadie-Bartz , M. Swiech, O. Dürfeldt-Pedros, S.J. Murphy, D.W. Kurtz, T. Bedding, G. Handler, J. Fuller, R. Ouazzani , H. Kjeldsen, **L. Fellay**, M. Gade Pedersen, E. Niemczura, M. Deal, and P. Mani,; (submitted), Wildly Oscillating stars Unexplained dense ridge-like frequency agglomerations in A and F type pulsators, *Astronomy & Astrophysics*.

A.2 PRESENTATIONS AT INTERNATIONAL OR NATIONAL CONFERENCE

A.2.1 INVITED TALKS AT INTERNATIONAL CONFERENCES

1. 06/11/2025: Theoretical instability strips of δ -Scuti, γ -Doradus and SPB stars, including the effect of the rotation and metallicity, BINA conference on the collaboration between India and Belgium, Kalicut, India
2. 16/04/2026: Non-adiabatic study of γ -Doradus stars, application to binary interactions, ISSI-2026 Beijing conference, Beijing, China.

A.2.2 CONTRIBUTED TALKS IN INTERNATIONAL CONFERENCES

1. 28/03/2022: Asteroseismology of evolved stars to constrain the internal transport of angular momentum IV. Internal rotation of Kepler 56 from an MCMC analysis of rotational splittings, Workshop on the transport of angular momentum in stars, Toulouse, France.
2. 29/06/2022: 3D static models of close, synchronized binaries in hydrostatic equilibrium with MoBiDICT, AIPS Workshop, Paris, France.
3. 29/06/2023: Non-perturbative static 3D modelling of close binaries and its impact on the observations of low-mass and RGB binaries, Plato Science conference, Milazzo, Italia.

A.2. PRESENTATIONS AT INTERNATIONAL OR NATIONAL CONFERENCE

4. 18/07/2024: First direct modelling of gravito-inertial tidally excited non-adiabatic oscillations with the traditional approximation, TASC/KASC conference, Porto, Portugal
5. 21/08/2024: Non-adiabatic modelling of gravito-inertial and Rossby tidally excited oscillations, BRITE Side of Stars, Vienna, Austria
6. 23/09/2024: First direct non-adiabatic modelling of gravito- inertial and Rossby tidally excited oscillations, Cefalu, Italia.
7. 09/07/2025: Resonance locking and full integration of the impact of forced oscillations on the rotational and orbital evolution of rotating γ -Doradus binaries, Vienna, Austria.

A.2.3 TALKS IN NATIONAL CONFERENCES/MEETINGS

1. 09/05/2022: 3D modelling of the structure and oscillations of close binaries, PhD Day Uliege 2022.
2. 02/02/2023: Modélisation 3D statique de binaires synchronisées en équilibre hydrostatique avec MoBiDICT, annual presentation to the thesis comity.
3. 07/05/2024: 3D modelling of the structure and oscillations of close binaries, annual presentation to the thesis comity.
4. 27/05/2024: 3D modelling of the structure and oscillations of close binaries, PhD Day Uliege 2024.
5. 22/04/2025: Resonance locking and full integration of the impact of forced oscillations on the rotational and orbital evolution of rotating γ -Doradus binaries, annual presentation to the thesis comity.
6. 14/10/2025: Resonance locking and tidal evolution in rotating γ -Doradus binaries, FNRS contact group, Brussels.
7. 11/03/2026: Non-adiabatic study of γ -Doradus stars, application to low metallicity stars and binary interactions, annual presentation to the thesis comity.

A.2.4 SEMINARS

1. 03/02/2026: Theoretical instability strips of δ -Scuti, γ -Doradus and SPB stars, including the effect of the rotation and metallicity, seminar to the local stellar group, DTU Copenhagen, Denmark.
2. 12/02/2026: Theoretical instability strips of δ -Scuti, γ -Doradus and SPB stars, including the effect of the rotation and metallicity, Good Vibrations Seminars, online.

A.2. PRESENTATIONS AT INTERNATIONAL OR NATIONAL CONFERENCE

A.2.5 POSTERS

1. 06/2022: 3D static models of close, synchronized binaries in hydrostatic equilibrium with MoBiDICT, sdOB10, Liège, Belgium.
2. 07/2022: Internal rotation of Kepler-56 from an MCMC analysis of the rotational splittings, KASC/TASC 2022, Leuven, Belgium.
3. 09/2022: 3D static models of close, synchronized binaries in hydrostatic equilibrium with MoBiDICT, Gaia science conference, Naples, Italy.
4. 07/2023: MoBiDICT: first 3D non-perturbative static models of close binaries, 321 multiple systems, Leuven, Belgium.
5. 07/2023: Impact of non-perturbative modelling on the apsidal motion of massive binaries, 321 multiple systems, Leuven, Belgium.
6. 07/2025: Theoretical instability strips of δ -Scuti, γ -Doradus and SPB stars including the effect of the metallicity and rotation., KASC/TASC, Vienna, Austria.

