An introduction to (alternate) numeration systems

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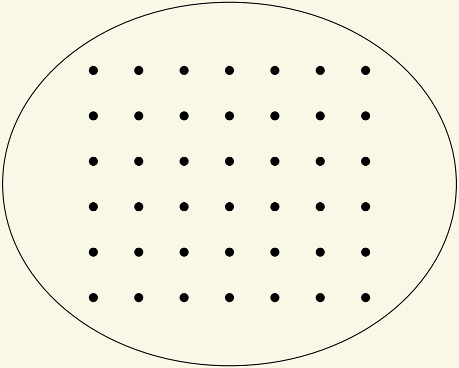
23 May 2026

1. Y
2. U
3. β
4. $\mathcal{B}$
5. $\exists$

Numbers and representations



Number of points above: 42, or XLII, or 00101010, or...

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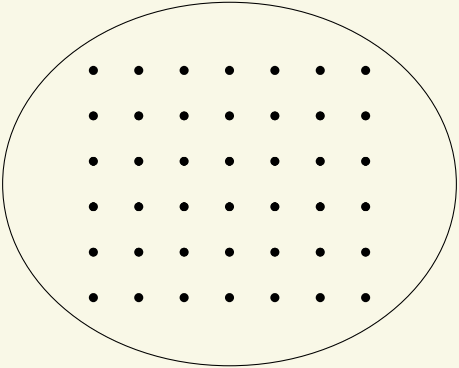

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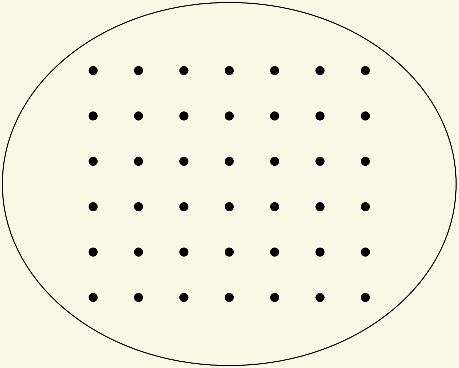

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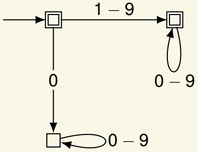
Distinction between *number* and *representation*.
Systems may have disadvantages: floating point binary cannot precisely represent $1/10$, binary chars can only represent numbers up to 255, roman numbers are a pain to multiply,...

Recognizing true representations

How do we know if a representation is correct? We can use *automata*.



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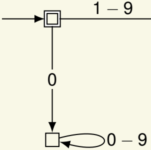

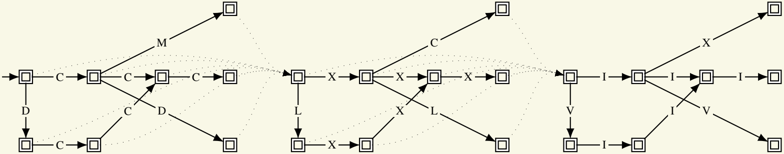

Wythoff's game: two piles of stones lay on a table. Two players alternatively remove either some stones from one pile, or the same number of stones from both piles. The one who cannot play loses.
Is the position $(42, 68)$ losing for the first player?

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Is the position $(42, 68)$ losing for the first player?
Understand this game with *Zeckendorf numeration*.

n	55	34	21	13	8	5	3	2	1
42	0	1	0	0	1	0	0	0	0
68	1	0	0	1	0	0	0	0	0

This position is losing (representations $(w0^{2k}, w0^{2k+1})$).

Numeration systems

Definition

A *numeration system* is a pair of maps between a set of numbers N (say, $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$) and a set of words W (say, A^* or $A^{\mathbb{N}}$), the *representation map*

$$\mathsf{rep}\colon N \to W : n \mapsto \mathsf{rep}(n)$$

and the *evaluation map*

$$\mathsf{val}\colon L \to N : w \mapsto \mathsf{val}(w),$$

where $\mathsf{val} \circ \mathsf{rep} = \mathrm{id}_N$ and $\mathsf{rep}(N) \subset L \subset W$.

Positional systems

A *positional system* is one where the evaluation map is a dot product with a given *weight vector* U:
$\mathrm{val}(w_{\ell-1} \cdots w_0) = \sum_{i=0}^{\ell-1} w_i U_i$.

For instance, base-10 and Zeckendorf use the sequence of powers of 10 and the Fibonacci sequence respectively.

Such a system is *greedy* if the representation map is done through a greedy algorithm.

Zeckendorf numeration

Here, the sequence $(U_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence, given by $U_n = U_{n-1} + U_{n-2}$ with the initial conditions $(1, 2)$.

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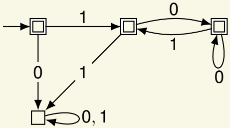

Squares

Here, the sequence $(U_n)_{n \in \mathbb{N}}$ is the sequence of squares, $U_n = (n+1)^2$.

n	81	64	49	36	25	16	9	4	1
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There is no automaton recognizing this language: seeing a 1 at position n forces us to see zeroes until some position $\sim \sqrt{2n}$, but automata are incapable of computing square roots.

Regularity

The *language* L_U of a numeration system U is the set $0^* \, \mathtt{rep}(\mathbb{N})$. It is *regular* if it is recognized by a finite automaton.

Question

Can we give necessary and sufficient conditions on U for L_U to be regular?

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Proposition

It is enough to look at the regularity of $\mathrm{Max}(L_U) = \{\operatorname{rep}(U_n - 1) : n \in \mathbb{N}\}$.

For instance, in the Zeckendorf numeration system, we have $\mathrm{Max}(L_U) = \{\varepsilon, 1, 10, 101, 1010, 10101, \dots\} = \{\mathrm{Pref}((10)^\omega)\}$.

β-numeration

The β-numeration system represents numbers in $[0, 1]$ by words in $A^{\mathbb{N}}$ and numbers in $[0, +\infty)$ by words in $A^{\mathbb{Z}}$. The evaluation map is given by

$$\mathrm{val}\colon a_{N-1} \cdots a_0 \bullet a_{-1} a_{-2} \cdots \mapsto \sum_{j=0}^{N-1} a_j \beta^j + \sum_{j=1}^{\infty} a_{-j} \beta^{-j}.$$

Example

Consider β equal to the golden ratio. Then, we have $\mathrm{val}(1 \bullet (010)^\omega) = 3/2$ as

$$1 + \frac{1}{\beta^2} + \frac{1}{\beta^5} + \frac{1}{\beta^8} + \cdots = 1 + \frac{1}{\beta^2} \frac{\beta^3}{\beta^3 - 1} = \frac{3}{2}.$$

The representation map is given by a greedy algorithm. For instance, with $x \in [0,1]$, we can start with $r_0 = x$ then set $a_i = \lfloor \beta r_{i+1} \rfloor$ and $r_i = \beta r_{i+1} - a_i$. We have $\mathrm{rep}_\beta(x) = a_{-1} a_{-2} \cdots$, which we call $d_\beta(x)$.

Example

Consider β equal to the golden ratio and let us represent $2/\beta^2$.

i	0	1	2	3	4
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Thus the representation of $2/\beta^2$ is $\bullet 10010^\omega$ (and not $\bullet 020^\omega$).

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βr_{-i}	$\frac{2}{\beta}$	$2 - \beta$	$\beta - 1$	1	0
a_{i-1}	1	0	0	1	0

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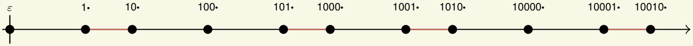
β-numeration

For numbers in $[1, +\infty)$, the representation is noted $\langle x \rangle$ and is obtained in much the same way, except that we start at a position i such that $\beta^{i-1} \leq x < \beta^i$ and we represent β^i.

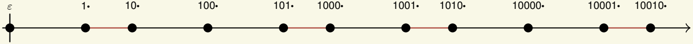
Example

Consider β equal to the golden ratio and let us represent 2. Since $\beta \leq 2 < \beta^2$ and since $d_\beta(2/\beta^2) = \bullet 10010^\omega$, we have $\langle 2 \rangle_\beta = 10 \bullet 010^\omega$.

The set of β-*integers* is the set of numbers whose representation has no fractional part.



When β is the golden ratio, this set is a Delone set that is aperiodic and has low complexity, The set of β-*integers* is the set of numbers whose representation has no fractional part.



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Quasicrystals

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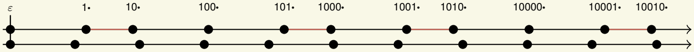

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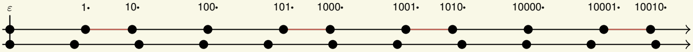

When β is the golden ratio, this set is a Delone set that is aperiodic and has low complexity, and it is also a Meyer set and is bounded displacement equivalent to a lattice, which connects it to the theory of quasicrystals.
We can notice a connection to the Zeckendorf numeration system.

n	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{rep}_U(n)$	ε	1	10	100	101	1000	1001	1010	10000	10001	10010

The Hollander connection

Another crucial connection: We have $\langle 1\rangle_\beta = 1 \bullet 0^\omega$ as we always have, but we find $d_\beta(1) = \bullet 110^\omega$. It follows that the words

$$\bullet 1011,\ \bullet 101011, \ldots, \bullet (10)^\omega$$

all have value 1. These are the *intermediate representations* of 1, ranging from the *greedy* to the *quasi-greedy* representations of 1.

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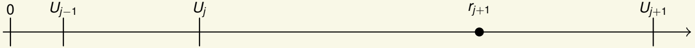
This is a strong instance of the following theorem.

Theorem (Hollander, 1998)

If the numeration system U is such that $\frac{U_{n+1}}{U_n} \to \beta > 1$ then, given ℓ, for n large enough $\mathrm{rep}_U(U_n - 1)$ shares a prefix of length at least ℓ with some intermediate representation of 1.

An intuition for Hollander's criterion

Let us sketch the algorithm that represents n in base U:



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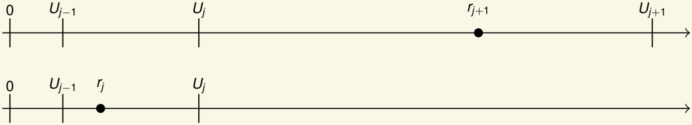

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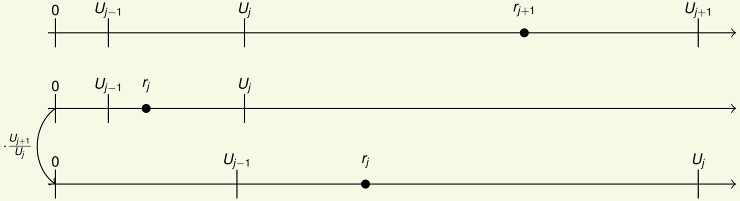

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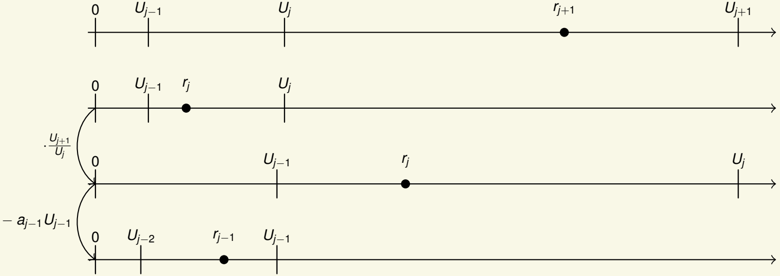
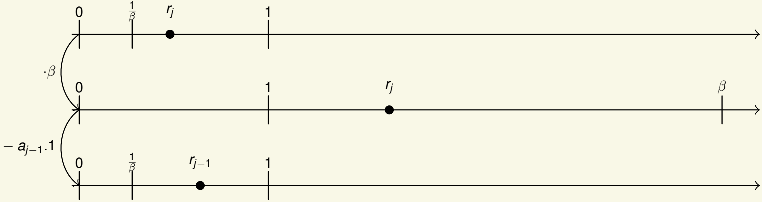
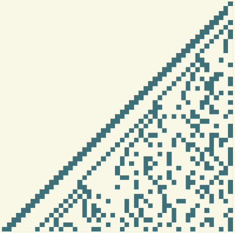


An intuition for Hollander's criterion

Let us sketch the algorithm that represents x in base β:



An example of a U where Hollander's criterion applies less strongly is the sequence U given by the recurrence relation $U_{n+4} = 2U_{n+3} + U_{n+2} - 2U_{n+1} - U_n$ and the initial conditions $(1, 2, 3, 6)$. This system is linked to the golden ratio.



This image represents the first 50 words in $\mathrm{Max}(L_U)$ (which have length 0 to 49). Each line represents a word, each teal square a digit 1 and each cream square a digit 0.

Back to regularity

Theorem (Hollander, 1998)

If the numeration system U is such that $\frac{U_{n+1}}{U_n} \to \beta > 1$ then, given ℓ, for n large enough $\mathtt{rep}_U(U_n - 1)$ shares a prefix of length at least ℓ with some intermediate representation of 1.

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YBC 7289

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Mixed bases

The numbers on YBC 7289 are written in a system that uses 60 as a primary base, but that can be understood as a *mixed* system where the bases 6 and 10 alternate.

Let $U = (1, 10, 60, 600, 3600, 36000, \dots)$ obtained by multiplying alternatively by 10 and 6. Then L_U is regular, but does not satisfy the condition $\frac{U_{n+1}}{U_n} \to \beta$ of Hollander's criterion.

Rather than being associated with a real base β, this system is associated with the *alternate base* $(6, 10)$.

Alternate bases

Recall that the evaluation in base β is given by

$$\mathrm{val}\colon a_{N-1}\cdots a_0 \bullet a_{-1}a_{-2}\cdots \mapsto \sum_{j=0}^{N-1} a_j\beta^j + \sum_{j=1}^{\infty} a_{-j}\beta^{-j}.$$

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Rather than a single base β, an alternate base is built from p base elements $(\beta_{p-1},\ldots,\beta_0)$. The evaluation map is given by

$$\mathrm{val}\colon a_{N-1}\ldots a_0 \bullet a_{-1}\cdots \mapsto \sum_{j=0}^{N-1} a_j\beta_{j-1}\beta_{j-2}\cdots\beta_0 + \sum_{j=1}^{\infty} \frac{a_{-j}}{\beta_{-1}\cdots\beta_{-j}},$$

where we have set $\beta_n = \beta_{n+p}$ making $(\beta_n)_{n\in\mathbb{Z}}$ periodic.

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where we have set $\beta_n=\beta_{n+p}$ making $(\beta_n)_{n\in\mathbb{Z}}$ periodic.

Example

Let $(\beta_1,\beta_0)=\left(\frac{1+\sqrt{13}}{2},\frac{5+\sqrt{13}}{6}\right)$. The word $10\bullet 101$ has value

$$\beta_0+\frac{1}{\beta_1}+\frac{1}{\beta_1\beta_0\beta_1}=2$$

Alternate bases

The representation is still computed with a greedy algorithm, except that we must alternate the various bases.

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Example

Let us compute $d_{\mathcal{B}}(1)$ where $\mathcal{B} = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$ as above.

i	0	-1	-2	-3
r_i	1			
$\beta_{i-1} r_i$				
a_{i-1}				

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a_{i-1}	2	0	1	0

Thus the representation of 1 in base $\mathcal{B}$ is $\bullet 201$. Note that its representation in base $\sigma(\mathcal{B})$ is $\bullet 11$.

Applications to regularity

Proposition (Charlier, Cisternino, c.2021)

If L_U is regular, U is associated to an alternate base: there p and $\beta_{p-1}, \dots, \beta_0$ such that $\frac{U_{np+i+1}}{U_{np+i}} \to \beta_i$ for $i \in \{0, \dots, p-1\}$.

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If L_U is regular, U is associated to an alternate base: there p and $\beta_{p-1}, \ldots, \beta_0$ such that $\frac{U_{np+i+1}}{U_{np+i}} \to \beta_i$ for $i \in \{0, \ldots, p-1\}$.

Hollander's criterion generalizes to alternate bases.

Theorem (Charlier, K. 2025)

If the numeration system U is associated with the alternate base $(\beta_{p-1}, \ldots, \beta_0)$ then, given ℓ, for n large enough $\mathrm{rep}_U(U_{np+i} - 1)$ shares a prefix of length at least ℓ with some intermediate representation of 1 in the base $(\beta_{i-1}, \beta_{i-2}, \ldots, \beta_{i-p})$.

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We can use this theorem to study the regularity of L_U in general (way too technical to present here).

Existence of alternate bases with given representations of 1

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$$\psi \colon \mathbb{R}^n \to \mathbb{R}^n : (\beta_{p-1}, \ldots, \beta_0) \mapsto \left(\sum_{j=1}^{\infty} d_{i,j} \prod_{k=1}^{j} \beta_{i-k}^{-1} \right)_{i=0,\ldots,p-1}$$

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Theorem (K., Masáková, Pelantová '25 ; Šťovíček, Pelantová '26)

Given p words $\mathbf{d}_{p-1}, \ldots, \mathbf{d}_0$ satisfying "reasonable lexicographic conditions", there exists a unique alternate base of length p that has these words as representations of 1.

Sketch of proof (existence)

Consider a sequence of nonnegative matrices $(A_n)_{n\in\mathbb{Z}}$. If this sequence is "positive enough", there exist unique normalized vectors f_j such that

$$\bigcap_{n\geq 1} \mathbb{R}^n_{\geq 0} A_{j+n}\cdots A_{j+1} = \mathbb{R}^n_{\geq 0} f_j.$$

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Then there exists a unique sequence (γ_i) of numbers such that $\gamma_j f_{j-1} = f_j A_j$ for all j.

Sketch of proof (existence)

$$f_{j,1} = 1 \text{ and } \gamma_j f_{j-1} = f_j A_j \text{ for all } j.$$

Assume every matrix is of the form

$$A_n = \begin{pmatrix} a_{n,1} & \cdots & a_{n,k-1} & a_{n,k} \\ 1 & & & \vdots \\ & \ddots & & 1_{h+1} \\ & & 1 & \vdots \end{pmatrix}.$$

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$$\begin{aligned} 1 = f_{n-1,1} &= \frac{1}{\gamma_n}\left(a_{n,1} + \frac{1}{\gamma_{n+1}}(a_{n+1,2} + \ldots)\right) \\ &= \mathrm{val}_{(\frac{1}{\gamma_n}, \frac{1}{\gamma_{n+1}}, \ldots)}(a_{n,1} a_{n+1,2} a_{n+2,3} \cdots) \end{aligned}$$

Sketch of proof (existence)

This means that if the target words $\mathbf{d}_i$ are all ultimately periodic with the same preperiod and period, we can construct matrices and obtain base elements from the values of their generalized fixed points.
The general case for existence is deduced from this one by a continuity argument.

Other aspects of my research

We can also study alternate base systems for themselves, investigating:

- The structure of the $\mathcal{B}$-integers in the alternate base $\mathcal{B}$.
- What numbers have periodic expansions in these systems (following a paper of Schmidt in 1980).
- What Bertrand numeration systems transform into in this new setting.
- Equivalents of the notion of confluent Pisot number, a family of particularly well-behaved β to which the golden ratio belongs.
- ...

Further reading

If you want to find out more, check out this new pdf:

Université de Liège
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Linear numeration systems without a dominant root,
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Otherwise, **thank you for your attention !**