CONTINUOUS INTERPOLATION SPACES WITH A FUNCTION PARAMETER : PROPERTIES AND APPLICATIONS

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Abstract. This paper investigates continuous interpolation spaces defined by a function parameter, a construct central to the theory of traces and the analysis of boundary value problems for partial differential equations. We explore their functorial interpretation and establish density results for specific cases. These spaces, characterized by asymptotic regularity properties, are instrumental in the analysis of operators in weighted functional spaces and in solving PDEs with precise boundary behavior. The paper also addresses limiting cases, $\theta = 0$ and $\theta = 1$, in the definitions of K_∞^θ and J_∞^θ. As illustrated, these techniques, when applied to traditional spaces, can give rise to new spaces.

1. Introduction

In the context of solving boundary value problems involving partial differential equations, the method of traces is frequently employed [9, 21, 28, 10]. This method is often applied to tackle problems with boundary value or non-homogeneous boundary conditions. Within this framework, trace theorems provide intriguing interpolation methods [14, 22, 25, 35].

Let us first recall the theory of real interpolation methods [1, 2, 15, 23, 36, 37] in order to introduce the continuous interpolation spaces. Let $\mathscr{N}$ denote the category of normed vector spaces, with morphisms being bounded linear operators. The theory of interpolation for normed spaces is summarized as follows (for further details, see [2, 18]): If $(A_0, \|\cdot\|_{A_0})$ and $(A_1, \|\cdot\|_{A_1})$ are normed topological vector spaces, A_0 and A_1 are compatible if they are subspaces of a Hausdorff topological vector space. In this case, $A_0 \cap A_1$ is a normed vector space for the norm

$$\|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

and $A_0 + A_1$ is a normed vector space for the norm

$$\|a\|_{A_0+A_1} := \inf_{a=a_0+a_1}\{\|a_0\|_{A_0}, \|a_1\|_{A_1}\}.$$

Let $\mathscr{C}$ be a sub-category of $\mathscr{N}$ and let $\mathscr{C}_c$ be a category of compatible couples $\mathbf{A} = (A_0, A_1)$, where both $A_0 \cap A_1$ and $A_0 + A_1$ are objects of $\mathscr{C}$. Morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathscr{C}_c$ are bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that both $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are morphisms in $\mathscr{C}$. The basic functors Σ and Δ from $\mathscr{C}_c$ to $\mathscr{C}$ are defined as follows: $\Sigma(T) = \Delta(T) = T$ and

$$\Delta(\mathbf{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\mathbf{A}) = A_0 + A_1.$$

2020 *Mathematics Subject Classification.* 46B70; 46E99.

Key words and phrases. Interpolation spaces; Boyd functions.

In the sequel, $\mathscr{C}$ represents any subcategory of $\mathscr{N}$ such that $\mathscr{C}$ is closed under the operations sum and intersection, and $\mathscr{C}_c$ denotes the category of all compatible couples $\mathbf{A}$ of spaces in $\mathscr{C}$. Given a couple $\mathbf{A} = (A_0, A_1)$ in $\mathscr{C}_c$, a space $A \in \mathscr{C}$ is an intermediate space between A_0 and A_1 (or with respect to $\mathbf{A}$) if

$$\Delta(\mathbf{A}) \hookrightarrow A \hookrightarrow \Sigma(\mathbf{A}).$$

Such a space A is called an interpolation space between A_0 and A_1 (or with respect to $\mathbf{A}$) if in addition, for every $T : \mathbf{A} \to \mathbf{A}$, it holds that $T : A \to A$. Now, if $\mathbf{B}$ is another couple in $\mathscr{C}_c$, two spaces A and B in $\mathscr{C}$ are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$ if they are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$, respectively, and if $T : \mathbf{A} \to \mathbf{B}$ implies $T : A \to B$. Two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$, respectively, are said to be exact of exponent $\theta \in (0,1)$ if for any $T : \mathbf{A} \to \mathbf{B}$,

$$\|T\|_{A,B} \leq \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}.$$

An interpolation functor (or interpolation method) on $\mathscr{C}$ is a functor $F : \mathscr{C}_c \to \mathscr{C}$ such that, for any two couples $\mathbf{A}$ and $\mathbf{B}$ in $\mathscr{C}_c$, $F(\mathbf{A})$ and $F(\mathbf{B})$ are interpolation spaces with respect to $\mathbf{A}$ and $\mathbf{B}$, and $F(T) = T$ for all $T : \mathbf{A} \to \mathbf{B}$. We say that F is exact of exponent θ if $F(\mathbf{A})$ and $F(\mathbf{B})$ are exact interpolation spaces of exponent θ.

For a couple $\mathbf{A}$ and $t > 0$, we define

$$K(t,a) = K(t,a;\mathbf{A}) := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),$$

for $a \in \Sigma(\mathbf{A})$. The function $t \mapsto K(t,a)$ is positive, increasing and concave. Moreover, we have $K(t,a) \leq \max\{1, t/s\} K(s,a)$. Now, for a couple $\mathbf{A}$ and $t > 0$, we set

$$J(t,a) = J(t,a;\mathbf{A}) := \max\{(\|a\|_{A_0}, t\|a\|_{A_1}),$$

for $a \in \Delta(\mathbf{A})$. It is easy to see that $K(t,a)$ and $J(t,a)$ are equivalent norms on $\Sigma(\mathbf{A})$ and $\Delta(\mathbf{A})$, respectively. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, let $K_q^\theta(\mathbf{A})$ denotes the space of $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^\theta} = (\int_0^\infty (\frac{K(t,a)}{t^\theta})^q \, \frac{dt}{t})^{1/q} < \infty,$$

with the usual modification if $q = \infty$. Now, let $J_q^\theta(\mathbf{A})$ be the space of elements $a \in \Sigma(\mathbf{A})$ which can be represented by

$$a = \int_0^\infty b(t) \, \frac{dt}{t},$$

with convergence in $\Sigma(\mathbf{A})$, where b is measurable with values in $\Delta(\mathbf{A})$ and such that

$$t \mapsto \frac{J(t,b(t))}{t^\theta} \in L_*^q,$$

equipped with the norm

$$\|a\|_{J_q^\theta} = \inf_b \| \frac{J(t,b(t))}{t^\theta} \|_{L_*^q},$$

where L_*^q denotes the Lebesgue space with measure dt/t. It can be shown that K_q^θ and J_q^θ are exact interpolation functors of exponent θ. Moreover, we have $J_q^\theta(\mathbf{A}) = K_q^\theta(\mathbf{A})$ with equivalence of the norms.

In many contexts, a definition of interpolation spaces which gives some intermediate spaces is very useful [25]. Let $K_\infty^{0,\theta}(\mathbf{A})$ denote the spaces of all $a \in K_\infty^\theta(\mathbf{A})$ such that

$$\lim_{t\to 0} \frac{K(t,a)}{t^\theta} = \lim_{t\to\infty} \frac{K(t,a)}{t^\theta} = 0.$$

The space is called a continuous interpolation space. These spaces are particularly valuable in contexts where asymptotic regularity is pivotal, such as the analysis of operators in weighted functional spaces, the study of solutions to partial differential equations with precise asymptotic behavior, or the theory of traces on manifolds.

In this paper, we delve into the characteristics of these continuous interpolation spaces defined by a function parameter. We demonstrate their functorial interpretation; in particular, we address the density issue for the case L^∞. Furthermore, we showcase their utility through several examples. We also consider the limiting case, where θ is equal to 0 or 1 in the generalized definitions of K_∞^θ and J_∞^θ.

2. Real interpolation

This section offers a concise overview of selected definitions and results from [18]. The theory of real interpolation methods has been generalized by incorporating a function parameter, as detailed in [5, 7, 11, 12, 26, 27, 31, 32, 33], among others. By function parameter, we refer to a Boyd function. A function $\phi : (0,\infty) \to (0,\infty)$ is termed a Boyd function if it satisfies the following conditions: it is continuous, $\phi(1) = 1$, and for all $t \in (0,\infty)$,

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty.$$

The set of Boyd functions is denoted by $\mathcal{B}$. In this context, Boyd indices have significant applications (see [3, 4, 13, 15, 27]). Specifically, the lower and upper Boyd indices of a Boyd function ϕ are defined as follows:

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\bar{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \bar{\phi}(t)}{\log t}.$$

The theory of interpolation can be generalized naturally. For $\phi \in \mathcal{B}$, we define the function $\phi_*(t) = t/\phi(t)$ for $t > 0$.

Definition 2.1. Let $\mathbf{A} = (A_0, A_1)$ and $\mathbf{B} = (B_0, B_1)$ be two couples in $\mathscr{C}_c$; two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$, respectively, are said to be of exponent $\phi \in \mathcal{B}$ if for any $T : \mathbf{A} \to \mathbf{B}$,

$$\|T\|_{A,B} \le C\bar{\phi}_*(\|T\|_{A_0,B_0})\bar{\phi}(\|T\|_{A_1,B_1}) \tag{1}$$

for some constant $C > 0$. If $C = 1$, we say that A and B are exact of exponent ϕ.

We will often assume that $\underline{b}(\phi) > 0$ and $\bar{b}(\phi) < 1$, which corresponds to the classical assumption $0 < \theta < 1$. The cases 0 and 1 are considered in Section 4.

It is important to note that A and B are of exponent ϕ if and only if they are of exponent $\check{\phi} := 1/\phi(1/\cdot)$. Similarly we say that a functor F is (exact) of exponent ϕ if $F(\mathbf{A})$ and $F(\mathbf{B})$ are (exact) of exponent ϕ. Clearly, Δ and Σ are exact interpolation functors on any sub-category of $\mathscr{N}$. Moreover, two interpolation spaces A and B with respect to $\mathbf{A}$ and $\mathbf{B}$, respectively, are of exponent $\phi \in \mathcal{B}$ if and only if

$$\|T\|_{A,B} \le C\bar{\phi}^*(\|T\|_{A_0,B_0})\bar{\phi}(\|T\|_{A_1,B_1})$$

for some constant $C > 0$, where ϕ^* is the Boyd function explicitly defined by $\phi^*(t) := t\,\phi(\frac{1}{t})$ for $t > 0$.

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $K_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ such that

$$\|a\|_{K_q^\phi} = \|a\|_{K_q^\phi(\mathbf{A})} := \Big(\int_0^\infty \big(\frac{1}{\phi(t)}\, K(t,a)\big)^q \,\frac{dt}{t}\Big)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Theorem 2.2. *For $\phi \in \mathcal{B}$, K_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$. Moreover, we have*

$$K(t,a) \le C\phi(t)\|a\|_{K_q^\phi}.$$

Proposition 2.3. *For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we have*

$$K_q^\phi(A_0, A_1) = K_q^{\phi^*}(A_1, A_0).$$

For $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, let $J_q^\phi(\mathbf{A})$ be the space of all $a \in \Sigma(\mathbf{A})$ that can be represented by $a = \int_0^\infty b(t)\,dt/t$, with convergence in $\Sigma(\mathbf{A})$, where b is a measurable function taking its values in $\Delta(\mathbf{A})$ for $t > 0$ and

$$t \mapsto \frac{J(t, b(t))}{\phi(t)} \in L^q_*. \tag{2}$$

This space is equipped with the norm

$$\|a\|_{J_q^\phi} = \|a\|_{J_q^\phi(\mathbf{A})} := \inf_b \|\frac{J(t,b(t))}{\phi(t)}\|_{L^q_*},$$

the infimum being taken on all measurable functions $b : (0, \infty) \to \Delta(\mathbf{A})$ such that $a = \int_0^\infty b(t)\,dt/t$.

Theorem 2.4. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$, and $q \in [1, \infty]$, we have*

$$J_q^\phi(\mathbf{A}) = K_q^\phi(\mathbf{A}).$$

Proposition 2.5. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$ and $q \in [1, \infty)$, $\Delta(\mathbf{A})$ is dense in $K_q^\phi(\mathbf{A})$.*

For $q = \infty$, $\Delta(\mathbf{A})$ is not dense in $K_\infty^\phi(\mathbf{A})$ (see Proposition 3.5).

Theorem 2.6. *Let $\mathbf{A} = (A_0, A_1)$ be a couple of Banach spaces such that $\Delta(\mathbf{A})$ is dense in both A_0 and A_1. For $1 \le q < \infty$, $0 < \underline{b}(\phi)$, and $\overline{b}(\phi) < 1$, we have*

$$K_q^\phi(\mathbf{A})' = K_{q'}^{\check{\phi}}(\mathbf{A}'),$$

where q' is the conjugate exponent to q.

For the case $q = \infty$, see Proposition 3.6.

3. Continuous interpolation spaces

For $\phi \in \mathcal{B}$, let $K_\infty^{0;\phi}(\mathbf{A})$ denote the space comprising all $a \in \Sigma(\mathbf{A})$ such that

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,a) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,a) = 0.$$

Since $t \mapsto K(t,a)$ is continuous on $(0,\infty)$, $K_\infty^{0;\phi}(\mathbf{A})$ is in $K_\infty^{\phi}(\mathbf{A})$. We endow $K_\infty^{0;\phi}(\mathbf{A})$ with the norm inherited from $K_\infty^{\phi}(\mathbf{A})$, and it is clear that $K_\infty^{0;\phi}(\mathbf{A})$ is a closed subspace of $K_\infty^{\phi}(\mathbf{A})$.

We now proceed with a few straightforward observations.

Remark 3.1. For reasons analogous to Proposition 2.3, we observe that

$$K_\infty^{0;\phi}(A_0,A_1) = K_\infty^{0;\phi^*}(A_1,A_0).$$

Remark 3.2. For $q<\infty$, $K_q^{\phi}(\mathbf{A})$ is continuously embedded in $K_\infty^{0;\phi}(\mathbf{A})$. Specifically, for $x \in K_q^{\phi}(\mathbf{A})$, we have

$$\begin{aligned}\|1/\bar{\phi}\|_{L_\star^q(1,\infty)} \frac{1}{\phi(t)} K(t,x) &\le C\Big(\int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s}\Big)^{1/q} K(t,x)\\ &\le C\Big(\int_t^\infty (\frac{1}{\phi(s)} K(s,x))^q \frac{ds}{s}\Big)^{1/q},\end{aligned}$$

which implies that $K(t,a)/\phi(t) \to 0$ as $t\to\infty$. Furthermore, using the fact that

$$K_q^{\phi}(A_0,A_1) = K_q^{\phi^*}(A_1,A_0),$$

we obtain

$$\begin{aligned}0 &= \lim_{t\to\infty} \frac{1}{\phi^*(t)} K(t,a;(A_1,A_0)) = \lim_{t\to\infty} \frac{1}{\phi(1/t)} K(1/t,a)\\ &= \lim_{t\to 0} \frac{1}{\phi(t)} K(t,a).\end{aligned}$$

Theorem 3.3. *For $\phi \in \mathcal{B}$, $K_\infty^{0;\phi}$ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.*

Furthermore, the inequality

$$K(t,a) \le C\phi(t)\|a\|_{K_\infty^{0;\phi}(\mathbf{A})}$$

holds.

Proof. We have the following embeddings:

$$\Delta(\mathbf{A}) \hookrightarrow K_1^{\phi}(\mathbf{A}) \hookrightarrow K_\infty^{0;\phi}(\mathbf{A}) \hookrightarrow K_\infty^{\phi}(\mathbf{A}) \hookrightarrow \Sigma(\mathbf{A}).$$

Let $\mathbf{A}$ and $\mathbf{B}$ be two couples in $\mathscr{N}_c$; for $T:\mathbf{A}\to\mathbf{B}$, it holds that

$$K(t,Ta;\mathbf{B}) \le \|T\|_{A_0,B_0} K\Big(t\frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}, a;\mathbf{A}\Big).$$

Thus, the conditions

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,a) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,a) = 0$$

imply

$$\lim_{t\to 0} \frac{1}{\phi(t)} K(t,Ta) = \lim_{t\to\infty} \frac{1}{\phi(t)} K(t,Ta) = 0.$$

Since K_∞^ϕ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$, the proof follows. ☐

As in the standard case, the following inclusion holds:

Proposition 3.4. *Given* $\phi_0, \phi_1 \in \mathcal{B}$ *such that* $\bar{b}(\phi_0) < \underline{b}(\phi_1)$, *if* $A_1 \hookrightarrow A_0$, *then* $K_\infty^{0,\phi_1}(\mathbf{A}) \hookrightarrow K_\infty^{0,\phi_0}(\mathbf{A})$.

The space $K_\infty^{0,\phi}(\mathbf{A})$ can be characterized as follows.

Proposition 3.5. *For* $\phi \in \mathcal{B}$ *such that* $0 < \underline{b}(\phi)$ *and* $\bar{b}(\phi) < 1$, *the closure of* $\Delta(\mathbf{A})$ *in* $K_\infty^\phi(\mathbf{A})$ *is* $K_\infty^{0,\phi}(\mathbf{A})$.

Proof. Let a be such that $K(t,a)/\phi(t)$ tends to 0 as t or $1/t$ tends to infinity. Without loss of generality, assume that $a = \sum_j b_j$ in $\Sigma(\mathbf{A})$, where $b_j \in \Delta(\mathbf{A})$ and $J(2^j, b_j) \leq CK(2^j, a)$ for all $j \in \mathbb{Z}$ (see the proof of Theorem 2.4). Consequently, we have

$$\|a - \sum_{|j|<j_0} b_j\|_{J_{\phi,\infty}(\mathbf{A})} \leq C \sup_{|j|\geq j_0} \frac{K(2^j, a)}{\phi(2^j)},$$

for $j_0 \in \mathbb{N}$, which implies that a is in the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$.

Now, if a is in the closure of $\Delta(\mathbf{A})$ in $K_\infty^\phi(\mathbf{A})$, for $\varepsilon > 0$, there exists $b \in \Delta(\mathbf{A})$ such that $\|a - b\|_{K_\infty^\phi(\mathbf{A})} < \varepsilon$. Thus, we obtain

$$K(t,a) \leq K(t, a-b) + K(t,b) \leq \phi(t)\|a-b\|_{K_\infty^\phi(\mathbf{A})} + \min\{1,t\} J(1,b).$$

This leads to

$$\frac{K(t,a)}{\phi(t)} \leq \varepsilon + \min\{1,t\}\frac{J(1,b)}{\phi(t)},$$

which suffices to conclude the proof. ☐

Let us also recall the duality theorem in this context, well-known and easily derived from Section 3.7 of [2].

Proposition 3.6. *Assuming* $\Delta(\mathbf{A})$ *is dense in* A_0 *and* A_1, *then*

$$K_\infty^{0,\phi}(\mathbf{A})' = K_1^{\check{\phi}}(\mathbf{A}').$$

It is straightforward to verify that for $q \in [1, \infty]$, the space $K_q^\phi(\mathbf{A})$ coincides with the space $X_q^\phi(\mathbf{A})$, consisting of all $a \in \Sigma(\mathbf{A})$ for which there exists a representation $a = a_0(t) + a_1(t)$ for almost all $t > 0$, with $t \mapsto 1/\phi(t)a_0(t) \in L_*^q(A_0)$ and $t \mapsto t/\phi(t)a_1(t) \in L_*^q(A_1)$, equipped with the norm

$$\|a\|_{X_q^\phi(\mathbf{A})} = \inf_{a=a_0(t)+a_1(t)} \|\frac{1}{\phi(t)}a_0(t)\|_{L_*^q(A_0)} + \|\frac{t}{\phi(t)}a_1(t)\|_{L_*^q(A_1)}.$$

Proposition 3.7. *For* $\phi \in \mathcal{B}$, $K_\infty^{0,\phi}(\mathbf{A})$ *coincide with the subspace* $X_\infty^{0,\phi}(\mathbf{A})$ *of* $X_\infty^\phi(\mathbf{A})$ *(equipped with the induced norm), consisting of those elements for which there exists a representation* $a = a_0(t) + a_1(t)$ *satisfying*

$$\lim_{t\to 0} \frac{1}{\phi(t)}\|a_0(t)\|_{A_0} = \lim_{t\to\infty} \frac{1}{\phi(t)}\|a_0(t)\|_{A_0} = 0$$

and

$$\lim_{t\to 0} \frac{t}{\phi(t)}\|a_1(t)\|_{A_1} = \lim_{t\to\infty} \frac{t}{\phi(t)}\|a_1(t)\|_{A_1} = 0.$$

Proof. If $a \in K_\infty^{0,\phi}(\mathbf{A})$, then for every $t > 0$, there exist $a_0(t) \in A_0$ and $a_1(t) \in A_1$ such that

$$\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \leq 2K(t,a),$$

which provides a valid representation for a.

Conversely, if $a \in X_\infty^{0,\phi}(\mathbf{A})$, using a valid representation, we have

$$K(t,a) \leq \|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1},$$

and the conclusion follows. $\square$

We now provide an equivalence theorem in this context. Let $J_\infty^{0,\phi}(\mathbf{A})$ denote the subspace of $J_\infty^{\phi}(\mathbf{A})$, equipped with induced norm, consisting of those elements q for which there exists a representation $a = \int_0^\infty b(t)\,dt/t$ satisfying

$$\lim_{t\to 0} \frac{1}{\phi(t)}\|b(t)\|_{A_0} = \lim_{t\to\infty} \frac{1}{\phi(t)}\|b(t)\|_{A_0} = 0$$

and

$$\lim_{t\to 0} \frac{t}{\phi(t)}\|b(t)\|_{A_1} = \lim_{t\to\infty} \frac{t}{\phi(t)}\|b(t)\|_{A_1} = 0.$$

Theorem 3.8. *For* $\phi \in \mathcal{B}$ *such that* $0 < \underline{b}(\phi)$ *and* $\overline{b}(\phi) < 1$, *we have*

$$J_\infty^{0,\phi}(\mathbf{A}) = K_\infty^{0,\phi}(\mathbf{A}).$$

Proof. By the preceding proposition, it suffices to show that $J_\infty^{0,\phi}(\mathbf{A}) = X_\infty^{0,\phi}(\mathbf{A})$. Suppose that $a = \int_0^\infty b(t)\,dt/t$ belongs to $J_\infty^{0,\phi}(\mathbf{A})$. Then, for $t > 0$, we set

$$a_0(t) = \int_0^1 b(t\tau)\,\frac{d\tau}{\tau} \quad \text{and} \quad a_1(t) = \int_1^\infty b(t\tau)\,\frac{d\tau}{\tau},$$

which provides a representation for a in $X_\infty^{\phi}(\mathbf{A})$. Moreover, since both

$$\int_0^1 \bar{\phi}(\tau)\,\frac{d\tau}{\tau} \quad \text{and} \quad \int_1^\infty \frac{\bar{\phi}(\tau)}{\tau}\,\frac{d\tau}{\tau}$$

are finite, the dominated convergence theorem implies that a belongs to $X_\infty^{0,\phi}(\mathbf{A})$.

For $a \in X_\infty^{0,\phi}(\mathbf{A})$, suppose that a_0 and a_1 form a valid representation of a, and let φ be a test function such that $\int \varphi(\tau)d\tau/\tau = 1$. Define

$$\tilde{a}_0(t) = \int_0^\infty \varphi(\frac{t}{\tau})a_0(\tau)\,\frac{d\tau}{\tau} \quad \text{and} \quad \tilde{a}_1(t) = \int_0^\infty \varphi(\frac{t}{\tau})a_1(\tau)\,\frac{d\tau}{\tau},$$

so that $a = \tilde{a}_0(t) + \tilde{a}_1(t)$ for all $t > 0$. By defining $b(t) = tD_t\tilde{a}_0(t) = -tD_t\tilde{a}_1(t)$, we obtain

$$\int_0^\infty b(t)\frac{dt}{t} = \int_0^1 D_t\tilde{a}_0(t)dt - \int_1^\infty D_t\tilde{a}_1(t)dt = \tilde{a}_0(1) + \tilde{a}_1(1) = a.$$

Using the fact that the support of $D\varphi$ is contained in a compact set, the dominated convergence theorem allows us to conclude. $\square$

4. Limiting cases

One might question how the Equivalence Theorems 2.4 and 3.8 behave when $0 = \underline{b}(\phi)$ or $\overline{b}(\phi) = 1$. In [29], they have conditions on weights defined by

$$t \in (0,\infty) \mapsto t^{-\theta}\phi(t),$$

where ϕ is a slowly varying function on $(0,\infty)$, meaning that for each $\varepsilon > 0$, the function $t \mapsto t^{\varepsilon}\phi(t)$ is equivalent to a non-decreasing function, and $t \mapsto t^{-\varepsilon}\phi(t)$ is equivalent to a non increasing function. In our context, this corresponds to Boyd functions with Boyd indices equal to 0. Following the approach in Example 4.3 of [17], we obtain more general weights and thus can analyze how the conditions adapt.

Proposition 4.1. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$ or $\overline{b}(\phi) \geq 1$, and $q \in [1,\infty]$.*

1. *If*
$$\int_0^1 (\frac{t}{\phi(t)})^q \frac{dt}{t} = \infty \quad \text{or} \quad \int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t} = \infty \tag{3}$$
with the usual modification if $q = \infty$, then $K_q^\phi(\mathbf{A}) = \{0\}$.
2. *If*
$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1,t\})^q \frac{dt}{t} < \infty, \tag{4}$$
with the usual modification if $q = \infty$, then K_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Proof. For the first point, let $a \in \Sigma(\mathbf{A})\backslash\{0\}$. Since $K(\cdot,a)$ is non-decreasing, we have

$$\|a\|_{K_q^\phi(\mathbf{A})} \geq K(a,1)(\int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t})^{1/q}$$

and since $K(\cdot,a)/\cdot$ is non-increasing,

$$\|a\|_{K_q^\phi(\mathbf{A})} \geq K(a,1)(\int_0^1 (\frac{t}{\phi(t)})^q \frac{dt}{t})^{1/q},$$

Thus, the conclusion follows.

Now, for $a \in \Delta(\mathbf{A})$, we have

$$K(t,a) \leq \min\{1,t\}\|a\|_{\Delta(\mathbf{A})},$$

and for $a \in \Sigma(\mathbf{A})$,

$$K(t,a) \geq \min\{1,t\}\|a\|_{\Sigma(\mathbf{A})}.$$

Since $K_q^\phi(\mathbf{A})$ is an intermediate space between A_0 and A_1, we can conclude as in the usual case. □

Remark 4.2. Condition (3) is satisfied if and only if condition (4) is not satisfied.

Remark 4.3. In the first case, if $\Delta(\mathbf{A}) \neq \{0\}$, $K_q^\phi(\mathbf{A})$ is not an intermediate space between A_0 and A_1, and K_q^ϕ is not an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Remark 4.4. If $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, then

$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1,t\})^q \frac{dt}{t} < \infty$$

and we recover the usual case. If $q = \infty$ and $\phi = 1$ or $\phi = \mathrm{id}$, then (3) is also verified, since in this case

$$\sup_{t\in(0,\infty)} \frac{\min\{1,t\}}{\phi(t)} = 1.$$

Example 4.5. As noted in [8], if

$$\phi(t) = \begin{cases} t^\theta(1+|\log t|)^{\alpha_0} & \text{if } t \in (0,1], \\ t^\theta(1+|\log t|)^{\alpha_\infty} & \text{if } t \in (1,\infty), \end{cases},$$

then condition (4) is equivalent to the validity of one of the following conditions:

- $0 < \theta < 1$,
- $\theta = 0$ and $\alpha_\infty + \frac{1}{q} < 0$,
- $\theta = 0$, $q = \infty$ and $\alpha_\infty = 0$,
- $\theta = 1$ and $\alpha_0 + \frac{1}{q} < 0$,
- $\theta = 1$, $q = \infty$ and $\alpha_0 = 0$;

We now establish the adapted conditions for the continuous interpolation spaces.

Proposition 4.6. *Let $\phi \in \mathcal{B}$ such that $0 \le \underline{b}(\phi)$ or $\overline{b}(\phi) \ge 1$ and $q \in [1,\infty]$.*

(1) *If*

$$\lim_{t\to 0^+} \frac{t}{\phi(t)} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{1}{\phi(t)} > 0, \tag{5}$$

then $K_\infty^{0,\phi}(\mathbf{A}) = \{0\}$.

(2) *If*

$$\lim_{t\to 0^+} \frac{t}{\phi(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{\phi(t)} = 0, \tag{6}$$

then $K_\infty^{0,\phi}$ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Proof. For the first point, let $a \in \Sigma(\mathbf{A})\backslash\{0\}$. Since $K(\cdot, a)$ is non-decreasing, we get

$$\frac{K(t,a)}{\phi(t)} \ge K(a,1)\frac{1}{\phi(t)} \quad \text{for all } t \ge 1$$

and since $K(\cdot,a)/\cdot$ is non-increasing,

$$\frac{K(t,a)}{\phi(t)} \ge K(a,1)\frac{t}{\phi(t)} \quad \text{for all } t \le 1.$$

Hence, the conclusion follows.

Let us consider the second point. For $a \in \Delta(\mathbf{A})$, we have

$$K(t,a) \le \min\{1,t\}\|a\|_{\Delta(\mathbf{A})},$$

and for $a \in \Sigma(\mathbf{A})$,

$$K(t,a) \ge \min\{1,t\}\|a\|_{\Sigma(\mathbf{A})}.$$

Thus, the proof follows □

Remark 4.7. If $\phi = 1$ or $\phi = \mathrm{id}$, then (5) is satisfied and $K_\infty^{0,\phi}(\mathbf{A}) = \{0\}$.

We now present similar conditions for the J_q^ϕ method.

Proposition 4.8. *Let $\phi \in \mathcal{B}$ such that $0 \le \underline{b}(\phi)$ or $\overline{b}(\phi) \ge 1$, $q \in [1, \infty]$ and q' its conjugate exponent.*

1. *If*

$$\int_0^1 (\frac{1}{\phi(t)})^{q'} \frac{dt}{t} = \infty \quad or \quad \int_1^\infty (\frac{1}{t\phi(t)})^{q'} \frac{dt}{t} = \infty \tag{7}$$

with the usual modification if $q' = \infty$, then $\| \cdot \|_{J_q^\phi(\mathbf{A})}$ equals zero on $\Delta(\mathbf{A})$.

2. *If*

$$\int_0^\infty (\frac{1}{\phi(t)} \min\{1, \frac{1}{t}\})^{q'} \frac{dt}{t} < \infty, \tag{8}$$

with the usual modification if $q' = \infty$, then J_q^ϕ is an exact interpolation functor of exponent ϕ on $\mathscr{N}$.

Proof. One can easily adapt arguments of the appendix of [29]. □

Remark 4.9. Condition (7) is satisfied if and only if condition (8) is not satisfied.

The following lemma, known as the Fundamental Lemma, is established in [2], for instance.

Lemma 4.10. *Let $a \in \Sigma(\mathbf{A})$ and assume that the condition*

$$\lim_{t \to 0^+} \min\{1, \frac{1}{t}\} K(t, a) = \lim_{t \to \infty} \min\{1, \frac{1}{t}\} K(t, a) = 0. \tag{9}$$

holds. Then, for any $\varepsilon > 0$, there exists a representation $a = \int b(t) dt$ with convergence in $\Sigma(\mathbf{A})$ such that

$$J(t, b(t)) \le (C + \varepsilon) K(t, a)$$

for all $t > 0$.

Remark 4.11. If $\phi \in \mathcal{B}$ is such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) > 1$, then condition (9) is satisfied for $a \in K_q^\phi(\mathbf{A})$, since for these a and for all $t > 0$,

$$K(t, a) \le C \phi(t) \|a\|_{K_q^\phi(\mathbf{A})}.$$

This is the foundation for the equivalence theorem.

For the limiting cases, it is not always possible to verify condition (9), and the equivalence theorem cannot be proved in this way. We must adapt the Fundamental Lemma.

Lemma 4.12 (Modified Fundamental Lemma)**.** *Let $a \in \Sigma(\mathbf{A})$ and $q \in [1, \infty)$. Assume that condition (9) holds. Let $\phi \in \mathcal{B}$ such that*

$$\int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t} < \infty$$

and

$$\int_0^1 (\frac{1}{\phi(t)})^q \frac{dt}{t} = \infty.$$

Then, the function ψ, defined for all $s > 0$ by

$$\psi(s) = \phi(s)^{q/q'} \int_s^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t}$$

is a Boyd function, and there is a representation $a = \int_0^\infty b(s)\frac{ds}{s}$ *with convergence in* $\Sigma(\mathbf{A})$*, such that*

$$\psi(t)J(t, b(t)) \le C\frac{1}{\phi(t)}K(\xi(\frac{\eta(t)}{4}), a)$$

for all $t > 0$*, where* $\eta(t) = \int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s}$ *and* $\xi = \eta^{-1}$*.*

Proof. It is clear that $\eta \in \mathcal{B}$, since $\overline{\eta} \le \bar{\phi}$, and therefore $\psi \in \mathcal{B}$. The rest of the proof follows from the calculations in Lemma 4.3 of [29]. $\square$

Remark 4.13. If $\underline{b}(\phi) > 0$, then $1/\phi \in L_*^q(1, \infty)$, since $\overline{b}(1/\phi) = -\underline{b}(\phi) < 0$.

Proposition 4.14. *Let* $\phi \in \mathcal{B}$ *such that* $0 \le \underline{b}(\phi)$ *and* $\overline{b}(\phi) < 1$*, and let* $q \in [1, \infty)$ *with its conjugate exponent* q'*. Suppose that*

$$\int_1^\infty (\frac{1}{\phi(t)})^q \frac{dt}{t} < \infty$$

and

$$\int_0^1 (\frac{1}{\phi(t)})^q \frac{dt}{t} = \infty.$$

Then, ψ *defined for all* $s > 0$ *by*

$$\psi(s) = \frac{1}{\phi(t)^{q/q'}} \int_s^\infty \phi(t)^q \frac{dt}{t}$$

is a Boyd function, and

$$K_q^\phi(\mathbf{A}) = J_q^{1/\psi}(\mathbf{A}).$$

Proof. Let $a = \int_0^\infty b(s)\frac{ds}{s} \in J_q^{1/\psi}(\mathbf{A})$. Then,

$$K(t, a) \le \int_0^\infty \min\{1, \frac{t}{s}\}J(s, b(s))\frac{ds}{s}.$$

We obtain

$$\|a\|_{K_q^\phi(\mathbf{A})} \le I_1 + I_2,$$

where

$$I_1 = (\int_0^\infty (\int_0^t J(s, b(s))\frac{ds}{s}\frac{1}{\phi(t)})^q \frac{dt}{t})^{1/q}$$

and

$$I_2 = (\int_0^\infty (\int_t^\infty \frac{J(s, b(s))}{s}\frac{ds}{s}\frac{t}{\phi(t)})^q \frac{dt}{t})^{1/q}.$$

By using Hardy-type inequalities (see [30]), we find

$$I_1 \le (\int_0^\infty (\frac{t^{1/q'}J(t, b(t))\psi(t)}{t})^q dt)^{1/q} = \|a\|_{J_q^{1/\psi}(\mathbf{A})}$$

and

$$I_2 \le (\int_0^\infty (\frac{t^{1+1/q'}J(t, b(t))\psi(t)}{t^2})^q dt)^{1/q} = \|a\|_{J_q^{1/\psi}(\mathbf{A})}.$$

Thus, we have $J_q^{1/\psi}(\mathbf{A}) \hookrightarrow K_q^\phi(\mathbf{A})$. Now, if $a \in K_q^\phi(\mathbf{A})$, for all $t > 0$,

$$K(t, a)(\int_t^\infty (\frac{1}{\phi(s)})^q \frac{ds}{s})^{1/q} \le \|a\|_{K_q^\phi(\mathbf{A})} < \infty,$$

so that, using our hypothesis, we obtain

$$\lim_{t\to 0^+} K(t,a) = 0.$$

Moreover, for all $t > 0$, we also have

$$\frac{K(t,a)}{t}\frac{t}{\phi(t)} \leq C\,\frac{K(t,a)}{t}\Big(\int_0^t \Big(\frac{s}{\phi(s)}\Big)^q \frac{ds}{s}\Big)^{1/q} \leq \|a\|_{K_q^{\phi}(\mathbf{A})} < \infty,$$

and thus

$$\lim_{t\to\infty} \frac{K(t,a)}{t} = 0,$$

since $\lim_{t\to\infty} \frac{t}{\phi(t)} = \infty$. Therefore, condition (9) is satisfied, and applying the modified Fundamental Lemma, we conclude that $K_q^{\phi}(\mathbf{A}) \hookrightarrow J_q^{1/\psi}(\mathbf{A})$, using $s = \eta(t)/4$ and then $u = \eta(s)$. ☐

For the case $q = \infty$, we also need a modified Fundamental Lemma.

Lemma 4.15 (Second Modified Fundamental Lemma)**.** *Let $a \in \Sigma(\mathbf{A})$. Assume that condition (9) is satisfied. Let $\phi \in \mathcal{B}$ be a non-decreasing bijection and define $\psi(t)$ for all $t > 0$ by*

$$\psi(t) = -\frac{\eta^2(t)}{t\eta'(t)},$$

where $\eta(t) = \frac{1}{t}\int_0^t \frac{1}{\phi(s)}ds$. Then, ψ is a Boyd function, and there is a representation $a = \int_0^\infty b(s)\frac{ds}{s}$ with convergence in $\Sigma(\mathbf{A})$, such that

$$\psi(t)J(t,b(t)) \leq C\frac{1}{\phi(t)}K(\xi(1/4\phi(t)),a)$$

for all $t > 0$, where $\xi = (1/\phi)^{-1}$.

Proof. It is clear that $\eta \in \mathcal{B}$ since $\bar{\eta} \leq \bar{\phi}$, and therefore $\psi \in \mathcal{B}$. The remainder of the proof follows from the calculations in Lemma 9.5 of [29]. ☐

Proposition 4.16. *Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$. Suppose that ϕ is a non-decreasing bijection. Then,*

$$K_\infty^{\phi}(\mathbf{A}) = J_\infty^{1/\psi}(\mathbf{A})$$

and

$$K_\infty^{0,\phi}(\mathbf{A}) = J_\infty^{0,\phi}(\mathbf{A}),$$

where, for all $t > 0$, $\psi(t) = -\frac{\eta^2(t)}{t\eta'(t)}$, with $\eta(t) = \frac{1}{t}\int_0^t \frac{1}{\phi(s)}ds$.

Remark 4.17. The previous theorems can be adapted for the case $\underline{b}(\phi) > 0$ and $\overline{b}(\phi) \leq 1$, i.e., the other limiting case.

5. Applications

5.1. **Continuous interpolation of weighted l^q spaces.**

Let $q \in [1,\infty]$ and $\phi \in \mathcal{B}$. For a Banach space X, the space $\ell_\phi^q(X)$ consists of all sequences $(a_j)_j$ in X such that

$$(\phi(2^j)\|a_j\|_X)_j \in \ell^q.$$

This space is equipped with the norm

$$\|(a_j)_j\|_{\ell_\phi^q(X)} := \|(\phi(2^j)\|a_j\|_X)_j\|_{\ell^q}.$$

If $\phi(t) = t^s$ with $s \in \mathbb{R}$, we denote $\ell^q_s(X)$ instead of $\ell^q_\phi(X)$. Additionally, we write $\ell^q_\phi := \ell^q_\phi(\mathbb{C})$.

Lemma 5.1. *Let X be a Banach space, $\phi \in \mathcal{B}$ and $q_0, q_1, q \in [1, \infty]$. If $s_0, s_1 \in \mathbb{R}$ are such that $s_1 \le \underline{b}(\phi)$ and $\overline{b}(\phi) \le s_0$, and if $\gamma \in \mathcal{B}$ is defined by*

$$\gamma(t) = \frac{t^{s_0/(s_0-s_1)}}{\phi(t^{1/(s_0-s_1)})}$$

for $t > 0$, then we have

$$K^\gamma_q(\ell^{q_0}_{s_0}(X), \ell^{q_1}_{s_1}(X)) = \ell^q_\phi(X).$$

Theorem 5.2. *Let $q_0, q_1, q \in [1, \infty]$, and $\gamma, \phi_0, \phi_1 \in \mathcal{B}$. Define $f = \phi_0/\phi_1$ and $\psi = \phi_0/(\gamma \circ f)$. If $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ then*

$$K^\gamma_q(\ell^{q_0}_{\phi_0}(X), \ell^{q_1}_{\phi_1}(X)) = \ell^q_\psi(X).$$

This result can be reformulated in the context of $K^{0,\phi}_\infty$:

Theorem 5.3. *Let $q_0, q_1 \in [1, \infty]$, $\gamma, \phi_0, \phi_1 \in \mathcal{B}$ and define $f = \phi_0/\phi_1$. If $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, and if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then*

$$K^{0,\gamma}_\infty(\ell^{q_0}_{\phi_0}(X), \ell^{q_1}_{\phi_1}(X)) = c_{0,\psi}(X),$$

where $\psi = \phi_0/(\gamma \circ f)$, and $c_{0,\psi}(X)$ is the subspace of $\ell^\infty_\psi(X)$ consisting of the sequences $(a_j)_j$ such that

$$\lim_j \psi(2^j)\|a_j\|_X = 0.$$

Proof. We know that

$$K^\gamma_\infty(\ell^\infty_{\phi_0}(X), \ell^\infty_{\phi_1}(X)) = \ell^\infty_\psi(X).$$

Assuming $\underline{b}(f) > 0$, we can write

$$\ell^\infty_{\phi_0}(X) \hookrightarrow \ell^\infty_{\phi_1}(X).$$

Let $a = a^{(0,j)} + a^{(1,j)}$ be a discrete representation of a in $K^{0,\gamma}_\infty(\ell^\infty_{\phi_0}(X), \ell^\infty_{\phi_1}(X))$. We have

$$\psi(2^j)\|a_j\|_X \le \frac{\|a^{(0,j)}\|_{l^\infty_{\phi_0}(X)}}{\gamma(f(2^j))} + f(2^j)\frac{\|a^{(1,j)}\|_{l^\infty_{\phi_1}(X)}}{\gamma(f(2^j))}$$

and the right-hand side tends to 0 as $j \to \infty$.

Now, let $a \in c_{0,\psi}(X) \hookrightarrow \ell^\infty_\psi(X)$, set $a^{(0,k)} = 0$ and $a^{(1,k)} = a_k e_k$, which provides a discrete representation of a in $K^\gamma_\infty(\ell^\infty_{\phi_0}(X), \ell^\infty_{\phi_1}(X))$ such that

$$\frac{f(2^j)}{\gamma(f(2^j))}\|a^{(1,j)}\|_{l^\infty_{\phi_1}(X)} \le \psi(2^j)\|a_j\|_X.$$

Thus, a belongs to $K^{0,\gamma}_\infty(\ell^\infty_{\phi_0}(X), \ell^\infty_{\phi_1}(X))$. □

Remark 5.4. If $\phi_0/(\gamma \circ f) = \mathrm{id}$, the usual space $c_0(X)$ can be expressed as

$$c_0(X) = K^{0,\gamma}_\infty(\ell^\infty_{\phi_0}(X), \ell^\infty_{\phi_1}(X)).$$

5.2. **Continuous interpolation of Hölder spaces.** Let $C_b(\mathbb{R}^d)$ denot the space of bounded continuous functions in $\mathbb{R}^d$, equipped with the L^∞ norm. Similarly, let $C_b^1(\mathbb{R}^d)$ denote the subspace of continuously differentiable functions with bounded derivatives, equipped with the norm $\|a\|_\infty + \sum_{k=1}^d \|D_k f\|_\infty$. For $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$, $C_b^\phi(\mathbb{R}^d)$ consists of bounded and uniformly ϕ-Hölder continuous functions [16], with the norm

$$\|f\|_\infty + |f|_{C^\phi} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\phi(|x-y|)},$$

The next result extends a classical interpolation identity to the framework of Boyd functions.

Theorem 5.5. *Let $\phi \in \mathcal{B}$ with $0 < \underline{b}(\phi)$, $\overline{b}(\phi) < 1$. Then,*

$$K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) = C_b^\phi(\mathbb{R}^d).$$

Proof. The proof of Example 1.8 in [25] for the standard case can be readily adapted to the setting of Boyd functions (see also [20]). □

In the framework of $K_\infty^{0,\phi}$, we obtain a weighted form of the so-called little Hölder space [25].

Theorem 5.6. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, we have*

$$K_\infty^{0,\phi}(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) = h^\phi(\mathbb{R}^d),$$

where $h^\phi(\mathbb{R}^d)$ is the space of bounded functions a satisfying

$$\lim_{h \to 0} \sup_{x \in \mathbb{R}^d} \frac{|a(x+h) - a(x)|}{\phi(|h|)} = 0.$$

Proof. Let $a = a_0 + a_1$ belong to $K_\infty^{0,\phi}(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d)) \hookrightarrow K_\infty^\phi(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))$. We have

$$\begin{aligned}\frac{|a(x+h) - a(x)|}{\phi(|h|)} &\leq \frac{|a_0(x+h) - a_0(x)|}{\phi(|h|)} + \frac{|a_1(x+h) - a_1(x)|}{\phi(|h|)} \\ &\leq \frac{2\|a_0\|_\infty}{\phi(|h|)} + \frac{\|a_1\|_{C_b^1}|h|}{\phi(|h|)}.\end{aligned}$$

This implies

$$0 \leq \sup_{x \in \mathbb{R}^d} \frac{|a(x+h) - a(x)|}{\phi(|h|)} \leq C \frac{1}{\phi(|h|)} K(|h|, a),$$

showing that a belongs to $h^\phi(\mathbb{R}^d)$.

Now, for a in $h^\phi(\mathbb{R}^d) \hookrightarrow C_b^\phi(\mathbb{R}^d)$, as shown in Theorem 5.5, define

$$a_0^{(t)}(x) = \frac{1}{t^d} \int (a(x) - a(x-y)) \varphi(\frac{y}{t})\, dy \quad \text{and} \quad a_1^{(t)}(x) = \frac{1}{t^d} \int a(y) \varphi(\frac{x-y}{t})\, dy,$$

where φ is a non-negative test function with support contained in the unit ball and satisfying $\int \varphi\, dx = 1$. It follows that

$$\frac{1}{\phi(t)} K(t, a) \leq \frac{1}{\phi(t)} (\|a_0^{(t)}\|_\infty + t\|a_1^{(t)}\|_{C_b^1(\mathbb{R}^d)}).$$

We also observe

$$\frac{1}{\phi(t)}\|a_0^{(t)}\|_\infty \leq \int \sup_{x\in\mathbb{R}^d} \frac{|a(x)-a(x-tu)|}{\phi(t)}\,\varphi(u)\,du,$$

$$\frac{t}{\phi(t)}\|a_1^{(t)}\|_\infty \leq \frac{t}{\phi(t)}\|a\|_\infty,$$

and, for $k \in \{1,\dots,d\}$,

$$\frac{t}{\phi(t)}\|D_k a_1^{(t)}\|_\infty \leq \int \sup_{x\in\mathbb{R}^d} \frac{|a(x-tu)-a(x)|}{\phi(t)}|D_k\varphi(u)|\,du,$$

which implies that $K(t,a)/\phi(t)$ tends to 0 as t tends to 0^+.

To check the other limit, we use the decomposition $a = a + 0$ to directly obtain

$$\frac{1}{\phi(t)}K(t,a) \leq \frac{1}{\phi(t)}\|a\|_\infty,$$

which tends to 0 as t tends to infinity. □

Using Proposition 3.5, the previous result yields to a generalization of Theorem 5.3 in [35]:

Theorem 5.7. *For $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi)$ and $\overline{b}(\phi) < 1$, the closure of $C_b^1(\mathbb{R}^d)$ in $C_b^\phi(\mathbb{R}^d)$ is $h^\phi(\mathbb{R}^d)$.*

5.3. Continuous Interpolation of Sobolev spaces.

Let φ be a function in $\mathcal{S}$ with support in $\{x \in \mathbb{R}^d : 2^{-1} \leq |x| \leq 2\}$, such that $\varphi(x) > 0$ for x satisfying $2^{-1} \leq |x| \leq 2$ and $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1$ for $x \neq 0$. Choose functions ψ_0 and Φ_j ($j \in \mathbb{Z}$) in $\mathcal{S}$ such that

$$\mathcal{F}\Phi_j(x) = \varphi(2^{-j}x) \quad \text{and} \quad \mathcal{F}\psi_0(x) = 1 - \sum_{j=1}^{+\infty} \varphi(2^{-j}x),$$

where $\mathcal{F}$ denotes the Fourier transform. Define $\varphi_0 = \psi_0$ and $\varphi_j = \Phi_j$ for all $j \in \mathbb{N}_0$; $(\varphi_j)_j$ is called a system of test functions. Let $\phi \in \mathcal{B}$ and $(\varphi_j)_j$ be a system of test functions. Following the approach of [24], we define the corresponding generalized Besov space for $p, q \in [1,\infty]$ as

$$B^\phi_{p,q} := \{f \in \mathcal{S}' : (\varphi_j * f)_j \in \ell^q_\phi(L^p)\}.$$

This space is equipped with the norm

$$\|f\|_{B^\phi_{p,q}} ;= \|(\varphi_j * f)_j\|_{\ell^q_\phi(L^p)} = \|(\|\varphi_j * f\|_{L^p})_j\|_{\ell^q_\phi}.$$

Let $\mathcal{B}''$ denote the set of functions $\phi \in \mathcal{B}$ that are C^∞ on $[1,\infty)$ and satisfy $x^m|\phi^{(m)}(x)| \leq C_m\,\phi(x)$ for all $m \in \mathbb{N}$ and $x \in [1,\infty)$. Given $\phi \in \mathcal{B}''$, the generalized Bessel operator $\mathcal{J}^\phi$ is defined on $\mathcal{S}'$ by

$$\mathcal{J}^\phi f = \mathcal{F}^{-1}(\phi(\sqrt{1+|\cdot|^2})\mathcal{F}f).$$

It is straightforward to verify that $\mathcal{J}^\phi$ is a linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$, with $(\mathcal{J}^\phi)^{-1} = \mathcal{J}^{1/\phi}$ and $\mathcal{J}^\phi(\mathcal{S}) = \mathcal{S}$. The generalized (fractional) Sobolev space H^ϕ_p is then defined as

$$H^\phi_p = \{f \in \mathcal{S}' : \|\mathcal{J}^\phi f\|_{L^p} < \infty\}.$$

Proofs of the following results can be found in [19].

Proposition 5.8. *Let $f \in \mathcal{S}'$, $\phi \in \mathcal{B}''$, $p \in [1,\infty]$ and $(\varphi_j)_j$ be a system of test functions. If $\varphi_j * f$ belongs to L^p for all $j \in \mathbb{N}_0$, then*

$$\|\mathcal{J}^\phi \varphi_j * f\|_{L^p} \leq C\,\phi(2^j)\,\|\varphi_j * f\|_{L^p},$$

for all $j \in \mathbb{N}_0$.

Proposition 5.9. *For $\phi \in \mathcal{B}''$ with $\underline{b}(\phi) > 0$, the operator $\mathcal{J}^{1/\phi}$ maps L^p into L^p continuously.*

Theorem 5.10. *Let $\gamma \in \mathcal{B}$, $\phi_0, \phi_1 \in \mathcal{B}''$, $f = \phi_0/\phi_1$, $\psi = \phi_0/(\gamma \circ f)$, and $p \in [1,\infty]$. If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K^\gamma_\infty(H^{\phi_0}_p, H^{\phi_1}_p) = B^\psi_{p,\infty}.$$

In the context of $K^{0,\phi}_\infty$, we get:

Theorem 5.11. *Let $\gamma \in \mathcal{B}$, $\phi_0, \phi_1 \in \mathcal{B}''$ and $f = \phi_0/\phi_1$. For $p \in [1,\infty]$, if $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$ and $0 < \underline{b}(\gamma)$, $\overline{b}(\gamma) < 1$, then*

$$K^{0,\gamma}_\infty(H^{\phi_0}_p, H^{\phi_1}_p) = b^\psi_{p,\infty},$$

for $\psi = \phi_0/(\gamma \circ f)$, where $b^\psi_{p,\infty}$ is the subspace of elements $a \in B^\psi_{p,\infty}$ (equipped with the induced norm) such that

$$\lim_j \psi(2^j)\|\varphi_j * a\|_{L^p} = 0.$$

Proof. For $a = a_0 + a_1$ in $K^{0,\gamma}_\infty(H^{\phi_0}_p, H^{\phi_1}_p)$, we have

$$\psi(2^j)\|\varphi_j * a\|_{L^p} \leq C\psi(2^j)\Big(\frac{1}{\phi_0(2^j)}\|\mathcal{J}^{\phi_0}a_0\|_{L^p} + \frac{1}{\phi_1(2^j)}\|\mathcal{J}^{\phi_1}a_1\|_{L^p}\Big),$$

which implies

$$\psi(2^j)\|\varphi_j * a\|_{L^p} \leq C\frac{1}{\gamma(f(2^j))}K(f(2^j), a).$$

As j tends to infinity, the right-hand side tends to 0 under the given condition

Now, let a belongs to $b^\psi_{p,\infty}$. If $\underline{b}(f) > 0$, then $\Sigma(H^{\phi_0}_p, H^{\phi_1}_p) = H^{\phi_1}_p$. For $a_j = \varphi_j * a$ ($j \in \mathbb{N}_0$), using Hölder inequality, we get that $\sum_j \|a_j\|_{H^{\phi_1}_p}$ is finite. As a consequence, $\sum_{j\in\mathbb{N}} a_j$ converges to a in $\Sigma(H^{\phi_0}_p, H^{\phi_1}_p)$. For $l \in \mathbb{Z}$ and $t \in [f(2^l), f(2^{l+1}))$, define

$$b(t) = \frac{f(2^l)}{\log f(2^l)}a_l,$$

where $a_l = 0$ if $l \in -\mathbb{N}$. Since for $k \in \{0,1\}$, we have

$$\|\varphi_j * a\|_{H^{\phi_k}_p} \leq C\phi_k(2^j)\|\varphi_j * a\|_{L^p},$$

for all $j \in \mathbb{N}$, we get

$$\frac{1}{\gamma(t)}J(t, b(t)) \leq C\frac{J(f(2^j), a_j)}{\gamma(f(2^j))} \leq \frac{\phi_0(2^j)}{\gamma(f(2^j))}\|\varphi_j * a\|_{L^p},$$

hence the conclusion. □

6. Hölder spaces and semigroups

6.1. **Interpolation and semigroups.** We recall here some results concerning generalized interpolation and semigroup theory [27].

Let A be a Banach space and $(T_t)_{t\geq 0}$ a strongly continuous semigroup on A, that is,

- $T_0 = \mathrm{id}$,
- $T_{t+s} = T_t T_s$,
- $T_t a$ tends to a as t tends to 0^+.

The infinitesimal generator G of the semigroup is the linear operator defined by

$$Ga := \lim_{t\to 0^+} \frac{T_t a - a}{t},$$

for all a in the domain

$$D_G := \{a \in A : \lim_{t\to 0^+} \frac{T_t a - a}{t} \text{ exists in } A\}.$$

More generally, for an integer $m \geq 1$, we consider G^m and the corresponding Banach space D_{G^m} endowed with the graph norm

$$\|a\|_{D_{G^m}} := \|a\|_A + \|G^m a\|_A.$$

We shall consider an intermediate spaces $D^q_{G^m}(\phi)$ between D_{G^m} and A. In applications to parabolic differential and integrodifferential equations, G is typically an elliptic operator regarded as the generator of an analytic semigroup on A. The spaces $D^q_G(\phi)$ then provide a natural framework for the study of integrodifferential equations, extending the approach developed in [34, 35].

Definition 6.1. Given $m \in \mathbb{N}_0$ and $q \in [1,\infty]$, for $\phi \in \mathcal{B}$ satisfying $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$, we denote by $D^q_{G^m}(\phi)$ the set of all elements $a \in A$ such that

$$\|a\|_{D^q_{G^m}(\phi)} := \|a\|_A + \|\frac{(T_t - \mathrm{id})^m a}{\phi(t^m)}\|_{L^q_*([0,1])} < \infty.$$

Theorem 16 in [27] can be viewed as a generalization of Theorem 4.10 in [35].

Theorem 6.2. *Let $m \in \mathbb{N}_0$ and $q \in [1,\infty]$, and let $\phi \in \mathcal{B}$. Assume that*

- $0 < \underline{b}(\phi)$ *and* $\overline{b}(\phi) < 1$ *if* $q < \infty$,
- $\sup_{t>0} \min(1, t^{-1})\bar{\phi}(t) < \infty$ *if* $q = \infty$.

Then

$$D^q_{G^m}(\phi) = K^\phi_q(A, D_{G^m}).$$

The following result extends Theorem 2.10 of [34]. It may be regarded as a reduction theorem, since it shows that the interpolation between A and D_{G^m} can be reduced to an interpolation between D_{G^k} and $D_{G^{k+1}}$ for a suitable integer k.

Theorem 6.3. *Let $m \in \mathbb{N}_0$, $q \in [1,\infty]$ and $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$. Then $D^q_{G^m}(\phi) = K^\psi_q(D_{G^k}, D_{G^{k+1}})$, where k is a non-negative integer satisfying $0 \leq k < m\,\underline{b}(\phi) \leq m\,\overline{b}(\phi) < k+1$ and $\psi \in \mathcal{B}$ is defined by the relation $\phi(t^m) = t^k\psi(t)$.*

Proof. We have

$$K^{\phi_k}_1(A, D_{G^{k+1}}) \hookrightarrow D_{G^k} \hookrightarrow K^{\phi_k}_\infty(A, D_{G^{k+1}}),$$

where $\phi_k(t) = t^{k/k+1}$. Since $0 < \underline{b}(\psi)$ and $\overline{b}(\psi) < 1$, Theorem 2 of [27] yields

$$K_q^{\psi}(D_{G^k}, D_{G^{k+1}}) = K_q^{\phi}(A, D_{G^k}),$$

hence the claim follows. $\square$

Theorem 6.4. *Let $m \in \mathbb{N}_0$, $q \in [1, \infty]$ and $\phi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$. Then $D^q_{G^m}(\phi)$ consists of all elements $a \in D_{G^k}$ such that*

$$\|a\|_{D_{G^k}} + \|t^k \frac{(T_t - \mathrm{id})G^k a}{\phi(t^m)}\|_{L^q_*([0,1])} < \infty,$$

where k is a non-negative integer satisfying $0 \leq k < m\,\underline{b}(\phi) \leq m\,\overline{b}(\phi) < k + 1$.

Proof. The assertion follows directly from the proof of Theorem 17 in [27] (see also [37]). $\square$

6.2. **Application to Hölder spaces.** We now apply the results given in the previous section to the Hölder spaces introduced in Section 5.2, thereby extending the approach proposed in [35].

Consider the second derivative as an operator G acting on the space of continuous functions on $[0, 1]$, with suitable domain

$$A = \{a \in C^0([0, 1]) : a(0) = a(1) = 0\},$$

$\|a\|_A = \|a\|_\infty$,

$$D_G = \{a \in C^2([0, 1]) : a(0) = a(1) = a''(0) = a''(1) = 0\},$$

$Ga = a''$. We obtain the following characterization of $D^\infty_G(\phi)$.

Proposition 6.5. *Let $\phi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1/2$ and set $\psi(t) = \phi(t^2)$. Then $D^\infty_G(\phi)$ is equivalent to the Banach space*

$$h^\psi_0 := \{a \in h^\psi([0, 1]) : a(0) = a(1) = 0\},$$

endowedd with the norm of $C^\psi_b([0, 1])$.

Proof. Using Theorems 6.2 and 6.3, the proof of Lemma 7.2 in [34] for the classical setting can be directly adapted to the framework of Boyd functions. $\square$

As a consequence, Theorem 5.6 of [35] remains valid in this generalized setting. To state it, we introduce the following notation : if X is a Banach space, we denote by $C([0, \delta]; X)$ the space of continuous functions on $[0, \delta]$ taking values in X.

Theorem 6.6. *Let $\phi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1/2$, and set $\psi(t) = \phi(t^2)$. Consider the operator $G : D_G \to h^\psi_0$ defined by*

$$D_G = \{a \in C^2([0, 1]) : a'' \in h^\psi([0, 1]),\, a(0) = a(1) = a''(0) = a''(1) = 0\},$$

$Ga = a''$ ($a \in D_G$). Then G is the generator of a bounded analytic semigroup in h^ψ_0 with domain D_G and the graph norm of D_G is equivalent to

$$\|a\|_\infty + \|a''\|_{C^\psi_b([0,1])}.$$

Moreover, G satisfies the maximal regularity property in h^ψ_0: for $\delta > 0$ and every $\varphi \in C([0, \delta]; h^\psi_0)$, the function

$$T * \varphi(t) := \int_0^t T_{t-s}\varphi(s)\, ds \quad (0 \leq t \leq \delta)$$

is the unique $a \in C([0,\delta]; D_G) \cap C^1([0,\delta]; H^\psi)$ *satisfying the equations*

$$a'(t) = Ga(t) + \varphi(t) \quad (0 \leq t \leq \delta)$$

and $a(0) = 0$. *In addition, there exists a constant* $c_\delta > 0$ *such that*

$$\|T * \varphi\|_{C([0,\delta];D_G)} \leq c_\delta \|\varphi\|_{C([0,\delta;H^\psi)},$$

for all $\varphi \in C([0,\delta]; H^\psi)$.

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