Université de Liège
Faculté des Sciences
Unité de Recherche *Mathematics*

Linear numeration systems without a dominant root, alternate base numeration systems, and their links.

Savinien Kreczman

Dissertation présentée
en vue de l'obtention du grade académique de
Docteur en Sciences
Année académique 2025 – 2026

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Jury members

Olivier Carton, Paris Cité University (Reviewer)
Émilie Charlier, University of Liège (Promoter)
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Julien Leroy, University of Liège (Secretary)
Michel Rigo, University of Liège (President)
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Abstract

Numeration systems are ways of representing numbers using words. Many different systems exist, built for different purposes. Connections have been found between some of these, for instance between Rényi numeration systems and positional numeration systems with a dominant root. In this document, we investigate three families of numeration systems that generalize known families by introducing *alternation*. Those are alternate base numeration systems, generalizing Rényi numeration systems, positional numeration systems, no longer requiring a dominant root, and Dumont–Thomas numeration systems, no longer requiring to be built around a fixed point. We introduce these three families, explain how they are linked, then present five chapters, corresponding to five articles, where we study generalizations to the alternate context of some properties of those numeration systems.

2020 Mathematics Subject Classification: 11A67, 11A63, 68R15, 68Q45, 11K16, 11R06

Keywords: numeration systems, positional numeration system, Rényi numeration system, Dumont-Thomas numeration system, dominant root, alternate base, Cantor base, combinatorics on words, Pisot numbers, optimal representations, confluence, spectrum, β-integers, normalization, regular languages, positionality.

Remerciements

Il est de coutume de ne pas dire de choses trop évidentes dans des documents mathématiques. Je souhaite tout de même remercier Emilie Charlier. Son soutien constant, sa guidance éclairée, sa patience, son accompagnement, ses conseils étaient une condition nécessaire à la réalisation de cette thèse. De tout coeur, merci de m'avoir choisi pour cette aventure.

Il est tout aussi évident que mes parents méritent les plus grands remerciements. Ces quatre années n'auraient pas été possibles sans les vingt deux qui les ont précédées. Merci pour votre soutien et votre amour, merci de m'avoir appris à toujours me donner à fond. Surtout, merci de m'avoir laissé choisir les études que je désirais. Je sais que ce n'est pas toujours acquis.

Je remercie aussi ma tante, ma marraine et mon parrain d'adoption. Je sais que vous êtes derrière moi.

La première récréation en secondaire a un effet démesuré sur la vie d'un homme. On y noue les amitiés les plus durables. A cette loterie, j'ai tiré le gros lot. Je remercie Théo, Sébastien, Ludwig, Johann, Eurydice et Mélina pour leur compagnie continue et indéfectible, et pour être des êtres humains de première qualité.

Ma chance a continué a l'université. Benjamin, Guillaume, Antoine, même si nos chemins se sont quelque peu séparés depuis, merci pour le morceau passé ensemble.

Je remercie ma tante de thèse Manon pour son soutien dans des domaines si variés qu'il est impossible à décrire brièvement, et mon oncle de thèse Antoine, qui m'auront permis d'associer le voyage physique au voyage intellectuel. Merci, takk, arigatou. Merci aussi à ma grande soeur de thèse Célia, qui m'a aidé à faire de la recherche avec des roulettes pour me lancer. C'est comme le vélo, ça ne s'oublie pas.

Merci à la grande équipe de mathématiques discrètes de Liège. Je ne sais pas si j'aurais pu trouver meilleur cadre de travail. Emilie, Michel, Julien, Manon, Célia, France, Pierre, Pierre, Antoine, Antoine, Mai-Linh, Léo, Annika, Medhi, Tore, Lucas, merci.

Je remercie Emilie Charlier, Michel Rigo et Sébastien Labbé de m'avoir accompagné tout au long de la thèse. Je remercie également Julien Leroy et Olivier Carton d'avoir accepté de nous rejoindre pour le bouquet final en formant la partie francophone de mon jury. Je suis honoré de vous compter parmi ses membres. Merci particulièrement à Olivier Carton, qui a accepté la lourde tâche de rapporteur.

Merci au F.R.S.-FNRS, dont le soutien financier est certes moins émotionnellement chargé mais certainement pas négligeable. Merci à Thérèse Dupont et au groupe de la retraite à la PhD House de Liège, qui auront fourni un coup d'accélérateur considérable à la rédaction finale de cette thèse.

Je remercie la communauté de numération et de combinatoire des mots que j'ai eu l'occasion d'apprendre à connaître en conférences. Je ne sais pas si l'ambiance est meilleure ailleurs, mais je sais qu'elle est très bien ici.

Je remercie l'Echiquier Mosan de m'avoir appris à perdre avec élégance. Je remercie le Game of Kot de m'avoir appris à faire perdre avec élégance. Je remercie le CO ULiège de m'avoir appris à me perdre avec élégance.

Merci à Martin, toujours si unique que je ne sais pas comment le classer avec d'autres.

J'aimerais terminer par quelques remerciements plus spéciaux mais chers à mon coeur. Merci Mamy. J'aurais dû mieux te connaître. Merci à Jean-Paul ainsi qu'à l'équipe de Wépion de m'avoir montré que les maths pouvaient être plus intéressantes que ce qu'on fait en secondaire. Merci Benjamin de m'avoir donné une raison de me tourner vers le futur quand j'en avais désespérément besoin.

Enfin, merci aux milliers de papillons anonymes ou oubliés dont les battements d'ailes ont construit ma vie. Il me semble qu'ils n'ont pas fait ça trop mal.

Personal acknowledgments

In addition to a great mathematical family, I could count on a terrific trio of mathematical fairy godmothers. Their support and availability made my visit to Prague a wonderful experience. Edito, Lubko, Zuzano, stokrát děkuji.

I must gratefully thank Ľubomíra Dvořáková, Charlene Kalle, and Jörg Thuswaldner for joining us for the last stretch of this thesis and forming the English-speaking branch of my jury. I am honored to count them in. I must especially thank Jörg Thuswaldner, who took on the arduous task of reviewing this manuscript.

I would like to thank the people I had the pleasure of meeting in Prague beyond Trojanova. Alexander, bedankt dat je mijn slechte Nederlands hebt verdragen. Anders, thank you for trusting me enough to go walk in the woods alone with me. Finally, thank you to the Thursday regulars at the Naiad for their welcome.

I'd like to thank the members of Mr.Blind's 3CB server for giving me things to do during long winter evenings. I may not show it often, but the long distance contact is appreciated. May your calcs be correct and your reads on point.

Finally, I'd like to thank the thousands of unseen or forgotten butterflies whose flapping of wings shaped my life. They did an alright job, I reckon.

Scientific acknowledgments

Entire thesis I was supported by the F.R.S.-FNRS Research Fellow grant 1.A.789.23F. I thank Émilie Charlier and Anne-Marie Longrée for their proofreading and helpful suggestions.

Chapters 1 and 9 I thank Manon Stipulanti for her proofreading and helpful suggestions.

Chapter 3 In addition to the FNRS Research Fellow grant, I was supported by a Fédération Wallonie-Bruxelles Travel Grant and by a WBI Excellence WORLD grant. Avec le soutien de Wallonie-Bruxelles International.

Chapter 4 We thank the referee for helpful suggestions. Émilie Charlier is supported by the FNRS grant J.0034.22. Célia Cisternino is supported by the FNRS Research Fellow grant 1.A.564.19F.

Chapter 5 In addition to the FNRS Research Fellow grant, I was supported by a Fédération Wallonie-Bruxelles Travel Grant and by a WBI Excellence WORLD grant. Avec le soutien de Wallonie-Bruxelles International.

Chapter 6 We thank Célia Cisternino for many valuable discussions in the early days of this project.

Chapter 7 Sébastien Labbé is supported by France's Agence Nationale de la Recherche (ANR) project IZES (ANR-22-CE40-0011). Manon Stipulanti is supported by the FNRS Research grant 1.C.104.24F. The authors want to thank Émilie Charlier for useful discussions.

La lutte elle-même vers les sommets suffit à remplir un coeur d'homme.

Albert CAMUS*, Le Mythe de Sisyphe*

It's hard to believe that it's over, isn't it?

Madeline THORSON, *Celeste*

Contents

Introduction **v**

Chapter 1: An overview of numeration systems **1**

1.1 Numeration systems . . . 2

1.1.1 Introduction . . . 2

1.1.2 Combinatorics on words . . . 3

1.1.3 Numeration systems . . . 10

1.2 Positional numeration systems . . . 12

1.3 Rényi numeration systems . . . 18

1.4 Dumont–Thomas numeration systems . . . 26

1.5 Interplay between families of numeration systems . . . 31

Chapter 2: Alternation in numeration systems **41**

2.1 Alternate base numeration systems . . . 42

2.2 Back to positional numeration systems . . . 51

2.3 Extensions of Dumont–Thomas numeration systems . . . 59

2.3.1 A glimpse at a bigger picture . . . 67

Chapter 3: Existence of alternate bases with given expansions of 1 **73**

3.1 Introduction . . . 74

3.2 Preliminaries . . . 76

3.3 The Parry conditions characterize the greedy and the quasi-greedy expansions of 1 . . . 79

3.4 Uniqueness of the alternate base with prescribed greedy or quasi-greedy expansions of 1 . . . 81

3.5 Consequences of a theorem of Furstenberg . . . 83

3.6 Existence of an alternate base with given representations of 1 88

3.7 A family of sequences faithfully coding $\mathcal{B}$-integers . . . 94

3.7.1 Arnoux-Rauzy sequences . . . 97
3.7.2 Sequences associated with N-continued fractions . . . 99

Chapter 4: Ultimately periodic expansions in alternate bases 101
4.1 Introduction . . . 102
4.2 Preliminaries and statements of main results . . . 103
4.3 Preparatory results . . . 105
4.3.1 On $\mathcal{B}$-periodicity and δ-periodicity p by p steps . . . 105
4.3.2 $\mathcal{B}$-expansions of rational numbers . . . 107
4.3.3 Field isomorphism on infinite sums . . . 110
4.4 Proof of Theorem 4.1 . . . 110
4.5 Proof of Theorem 4.3 . . . 113
4.6 Reduction to the case of real base expansions . . . 115

Chapter 5: Alternate bases with the maximal digit property 119
5.1 Introduction . . . 120
5.2 Preliminaries . . . 123
5.3 $\mathcal{B}$-integers, spectrum and optimality . . . 127
5.4 Rewriting system: confluence and normalization . . . 134
5.5 Additional comments . . . 141
5.5.1 Confluence for infinite words . . . 141
5.5.2 A comment on a result of Frougny . . . 154
5.5.3 Other short comments . . . 158

Chapter 6: Regularity of languages associated with U-systems 161
6.1 Introduction . . . 162
6.2 A reminder on notation and useful properties . . . 165
6.3 Elementary necessary conditions for regularity . . . 170
6.4 Our strategy . . . 178
6.5 Vertices with no outgoing edge . . . 181
6.6 Vertices leading to a vertex with no outgoing edge . . . 186
6.7 Vertices in a cycle . . . 192
6.8 Vertices leading to a cycle . . . 207
6.9 Decision procedure . . . 215
6.10 Comments on Hollander's original conjecture . . . 219
6.11 Why obtaining criteria for regularity relying only on recurrence relations satisfied by U is not achievable . . . 221

Chapter 7: Positionality of Dumont–Thomas numeration systems **225**
7.1 Introduction . . . 226
7.2 Reminder on Dumont–Thomas numeration systems . . . 227
7.3 Sketch of the argument . . . 233
7.4 Main result . . . 238
7.5 Particular cases . . . 245

Chapter 8: Additional results **257**
8.1 On the Kepler limit for sequences without a dominant root . 258
8.1.1 Introduction and statement of main result . . . 258
8.1.2 Proof of Theorem 8.1 . . . 260
8.2 Initial conditions where L_U is regular when U has a dominant root . . . 266
8.2.1 Introduction . . . 266
8.2.2 A reminder on Hollander's and our results . . . 267
8.2.3 Main results . . . 270
8.2.4 Examples . . . 273

Chapter 9: Perspectives **279**
9.1 Bertrand numeration systems in the case without a dominant root . . . 279
9.2 More work on Dumont–Thomas numeration systems . . . 283
9.3 The language $\mathrm{Max}(L_U)$ when U has a dominant root . . . 285
9.4 Study of $\mathcal{B}$-integers . . . 291
9.5 Other perspectives . . . 292

Bibliography **295**

Index **309**

Index of notation **313**

Introduction

Let us ask an unusual question. What *are* numbers? Tools for counting? For measuring distances, ratios, volumes,...? Equivalence classes for the relation of being in bijection over finite sets? Purely abstract concepts to be manipulated according to a list of axioms?

Most non-mathematicians do not interact with numbers directly, but rather with *representations* of them. The representation `2026` is understood to refer to the number that is the sum of twice the cube of ten, twice ten, and six times one. In this way, we can work with integers, and even with real numbers, and we can easily compare, add or multiply them, directly by using the representations, without needing to think about the abstract concepts behind the numbers. This system is currently agreed upon in our society, but it is not the only possible one. The number above could have been represented as `MMXXVI` in another time, although we eventually realized that our current system makes computations more tractable. Computers use a different system based on powers of 2 rather than powers of 10, and other civilizations have used systems based on powers of 20 or 60. All these systems save for roman numerals have one thing in common: they are built on an *integer base*. Numbers are written as a sum of powers of the base, and the representation tracks how many times each power is used in the sum. Every position is given a *weight* that multiplies the digit present, and each weight is an element in the sequence of powers of the base. Said in another way, the ratio between the weights of consecutive positions is equal to the base. Such a system is very regular, offers good algorithms that work directly on representations, and is foolproof (as long as the fool doesn't look too closely at the representation $\mathtt{0.9999}\cdots$).

Not all systems commonly in use rely on a single base. Expressions like "Two days, four hours and thirty minutes" or the ever-exotic "five feet eleven inches" reflect *mixed-base* systems where a different multiple might be used when going from one position to the next. Mathematically, these systems were first studied by Cantor in [Can69].

Other correspondences between numbers and representations can arise naturally as a mathematical question is investigated. Maybe representations being easily addable is well and good but we need the addition to be done *in parallel*? Avizienis provides a three-digit binary system that does just that [Avi61]. Maybe we are confronted with a set of numbers that just happens to be easily identifiable, but only if we represent numbers in a particular way? See the use of the Zeckendorf system when solving Wythoff's game in [Fra82]. Maybe the structure and regularity exhibited by integer base systems is a downside, because we are studying objects that are not *quite* perfectly regular? See the introduction of β-integers as models of quasicrystals in [Gaz97]. Maybe we would like numbers to have multiple representations, so as to be able to correct mistakes made when encoding them? In the context of analog-to-digital conversion, β-expansions fix this precise problem [DDGV06]. Maybe we want to define fractal functions where the values on $\mathbb{N}$ are given by partial sums of a specific sequence? See the birth of Dumont–Thomas systems in [DT89].

The field of *numeration systems* stems from this double realization that numbers and representations are distinct, and that the process of representation is of interest in itself. We can now study the set of representations of all numbers, or just of a subset of numbers. We can examine properties of the representation map itself. We can compare representation maps in terms of what properties they bring with them, or how they can be transported from one framework to another.

This doctoral dissertation examines three particular families of numeration systems, all linked together in a beautiful way. *Positional* numeration systems generalize integer bases and replace the sequence of weights, formerly a geometric sequence, by an arbitrary sequence of integers [Zec72, Fra85, Sha94]. *Rényi* numeration systems generalize the representation of real numbers by integer bases, simply by allowing the base to not be an integer [Rén57, Par60, IT74]. Finally, *Dumont–Thomas* numeration systems shift our perspective, and rely on a factorization (into words) of the fixed point of a substitution rather than on a decomposition into sums of given numbers [DT89, LL24a]. Works by Bertrand-Mathis [BM89], Fabre [Fab95], Hollander [Hol98], and more link these three families together, establishing a correspondence between the lexicographically maximal representations found in positional and Rényi systems, and explaining how to deduce from a Rényi system a substitution that will be the base for a Dumont–Thomas system.

However, the correspondence between these three families is limited by some conditions. Namely, the only positional numeration systems that fit in this framework are the ones that verify the so-called *dominant root condition*, and where the quotient of consecutive weights tends to some limit $\beta > 1$ (which will be the base of the corresponding Rényi numeration system). On the side of Dumont–Thomas systems, it is required that the substitution at play have a fixed point.

If we want to study the positional numeration systems that have a regular language, we naturally come to the study of systems where the base sequence $(U_n)_n$ is a linear recurrence sequence, where the ratio of consecutive weights $\frac{U_{n+1}}{U_n}$ tends to a different limit based on the congruence of n modulo some number p that depends on the eigenvalues of the sequence U. It is for this reason that Charlier and Cisternino introduced *alternate bases* in [CC21]. In these systems that generalize Rényi numeration systems, we alternate between p bases $\beta_{p-1}, \ldots, \beta_0$ rather than always using one base β. This allows us to establish a correspondence similar to the one identified by Hollander. The base β_i corresponds to the value of the limit $\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}}$, and a positional numeration system is associated with an alternate base, relying on p bases rather than one like in the Rényi case. On the side of Dumont–Thomas numeration systems, substitutions that have a periodic point of period p naturally fit into this framework, rather than just substitutions with a fixed point. It should be noted that alternate bases can also be seen as a particular case of more general mixed base systems, the *Cantor real base systems* also introduced in [CC21] that generalize Cantor's mixed integer bases. Nevertheless, the connection to regular positional numeration systems and to Dumont–Thomas systems without a fixed point justifies that particular attention be given to the setting of alternate bases.

Thus there is a more general equivalent of the correspondence alluded to above, with a parameter p that corresponds to the number of different limits of $\frac{U_{n+1}}{U_n}$ for positional systems, to the number of bases for alternate bases, and to the period of the periodic point for Dumont–Thomas systems. When p is equal to 1, we recover the setting mentioned above. The aim of this dissertation is to study this correspondence and explore the new families of numeration systems that belong to it, attempting to lift results from the case $p = 1$ or to identify new behavior.

This dissertation is structured as follows.

In Chapter 1, we offer an overview of numeration systems. We define the notion more formally and recall elements of combinatorics on words. We then define and give back elementary properties of the three families mentioned above (positional, Rényi, and Dumont–Thomas numeration systems). To close the chapter, we make more precise the correspondence between these three families, recalling the results of Bertrand-Mathis, Fabre, and Hollander.

In Chapter 2, we lift everything to the general case, where we work with the parameter p that was just mentioned. We introduce alternate base numeration systems and give their elementary properties. We explain why they form such a useful tool to study positional numeration systems with a regular language, detailing the equivalent of Hollander's link in this more general setting. Finally, we explain how Dumont–Thomas systems can be extended to substitutions without a fixed point, and we present an example that justifies the link to positional and alternate base systems.

With the notions introduced, Chapters 3 through 7 form the core of this dissertation. They adapt five articles (two published, one submitted, two in preparation) that explore the new families of numeration systems, lift results from the case where $p = 1$, study new behavior... The articles in question are respectively [CKMP26b, CCK24, CKMP26a, CK25, KLS25b]. Although the flow of the narration is built for the order that they are presented in, they do not strongly depend on one another and can be read independently.

In Chapter 3, we spend the whole chapter lifting a result that is easy in the case where $p = 1$. For Rényi bases, it is not hard to see that any infinite word lexicographically larger than 10^ω has value 1 in exactly one base $\beta > 1$. A celebrated result of Parry [Par60] can then decide if the word is the greedy expansion of 1 or not. In this chapter, we attempt to generalize this result to the case of alternate bases. Given p infinite words $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ that are our candidate representations of 1, does there exist a base $(\beta_{p-1}, \ldots, \beta_0)$ such that $\mathbf{a}_{p-1}$ has value 1 in the base $(\beta_{p-1}, \ldots, \beta_0)$, $\mathbf{a}_{p-2}$ has value 1 in the base $(\beta_{p-2}, \ldots, \beta_0, \beta_{p-1})$, ..., and $\mathbf{a}_0$ has value 1 in the base $(\beta_0, \beta_{p-1}, \ldots, \beta_1)$? If that is so, is this base unique? Under what conditions are the words $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ the greedy expansions of 1? We will introduce a result of Furstenberg [Fur60] on limits of the form

$$\bigcap_{n \geq 0} \mathbb{R}_+^k A_{n-1} \cdots A_1 A_0$$

where $(A_n)_n$ is a suitable sequence of matrices and explain how to construct such a sequence of matrices from a given alternate base. We will detail how Furstenberg's result gives us the existence of an appropriate alternate base in enough cases for us to deduce it for the remaining ones. Finally, we partially solve the problem of uniqueness and give conditions for the candidate words to be the greedy expansions of 1 once the existence of a suitable alternate base is guaranteed.

In Chapter 4, we lift a result of Schmidt [Sch80] on the set of numbers that have an ultimately periodic β-expansion. When β is an integer, this set is clearly the set of rational numbers. Schmidt proved that if β is a Pisot number, we get a similar result: the set of numbers with ultimately periodic expansions is exactly $\mathbb{Q}(\beta)$. The converse is only partial: if all rational numbers have ultimately periodic expansions, the base is a Pisot *or Salem* number. In this chapter, we investigate ultimately periodic expansions in the context of alternate bases. We obtain a generalization of Schmidt's results. Let $(\beta_{p-1}, \ldots, \beta_0)$ be an alternate base with $\delta = \prod_{i=0}^{p-1} \beta_i$. If δ is a Pisot number and $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$, then the numbers with ultimately periodic expansions are the elements of $\mathbb{Q}(\delta)$. For the partial converse, if every rational number has an ultimately periodic expansion in each of the p bases $(\beta_{p-1}, \ldots, \beta_0)$, $(\beta_{p-2}, \ldots, \beta_0, \beta_{p-1})$, ..., $(\beta_0, \beta_{p-1}, \ldots, \beta_1)$, then δ is either a Pisot or a Salem number and $\beta_{p-1}, \ldots, \beta_0$ are all in $\mathbb{Q}(\delta)$. As in the previous chapter, the easiest part when $p = 1$ is the hardest in the general case: we must first prove that the product of bases δ is an algebraic integer. For this, we modify a result from [CCMP23]. After that hurdle, our main trick is to study the numeration system with grouped digits, where we merge the information of p digits into a single digit on a real alphabet. We also use the spectrum, and adapt "elementary" (in the sense of not needing much theoretical background) results of Schmidt to reach our conclusion.

In Chapter 5, we single out a class of particularly well-behaved alternate bases, with the titular *maximal digit property*. We explain what this property is, then we establish a link to several desirable properties that an alternate base can have. The *spectrum* of an alternate base over a given alphabet is the set of values taken by representations with no fractional part written over this alphabet, while the set of $\mathcal{B}$-integers is the set of numbers whose greedy expansion has no fractional part. Those two sets are not always equal. With the correct choice of alphabet for the spectrum, the equality occurs exactly when the base has the maximal digit property. *Optimality*, studied in the context of Rényi systems in [DdVKL12] and defined by us for alternate bases, is another desirable property that intuitively states that the greedy algorithm, which is defined for a local sense of "greedy", is also greedy in a more global sense. Once again, this property occurs exactly for bases with the maximal digit property. *Rewriting systems* associated with a given base were used in [Fro92a, FS92, MPS25]. When the base has suitable properties, they allow *normalization* by rewriting non-admissible factors until a greedy expansion is reached. We study the *confluence* of rewriting systems associated with an alternate base and find out, once again, that this confluence occurs only when the base has the maximal digit property.

Chapter 6 is the seed from which this entire project sprouted. We study the regularity of the *language* of a positional numeration system, that may have a dominant root or not, as opposed to the previous study [Hol98] where this condition was assumed. We show that we can restrict ourselves to positional numeration systems associated with an alternate base and establish a link between maximal words in the positional numeration system and representations of 1 in the alternate base. We then use this link to introduce auxiliary sequences that allow us to control the flow of the greedy algorithm. With this, we can provide characterizations for the regularity of sublanguages of the numeration system, which can be combined into a characterization for the entire language. We end the chapter by discussing the effectiveness of this characterization and comparing it to the case where $p = 1$, answering in the negative a conjecture of Hollander. Although this chapter does not use much theoretical background, the proofs are quite technical.

The final of the article adaptations, Chapter 7 deals with the positionality of Dumont–Thomas numeration systems. In general, a Dumont–Thomas numeration system is not defined using a positional framework, but might end up being positional if the substitution is sufficiently well-behaved. A question asked in Lepšová's thesis [Lep24] is to identify which Dumont–Thomas systems fall in the positional framework. In this chapter, we answer this question by providing a criterion for the positionality of Dumont-Thomas systems. In some cases, we can particularize this criterion into a much simpler one. We end the chapter by relating those simple positional Dumont–Thomas numeration to Bertrand numeration systems, a family of positional numeration systems of particular interest.

Chapter 8 introduces two "rawètes", two additional results that were obtained as we were working on the regularity of languages of positional numeration systems. Those results are too small to deserve an entire chapter, but they are also separate enough of the main study in Chapter 6 that we chose not to include them there. We present them as two sections of one chapter here. The first of those results is a standalone theorem on linear recurrence sequences, that investigates the behavior of the quantity $\frac{U_{i+1}}{U_i}$ when U is a linear recurrence sequence. Fiorenza and Vincenzi in [FV11] provided a necessary and sufficient condition for the sequence to have a dominant root. They call *Kepler limit* the value of this root. In our case, we generalize this result by giving a necessary and sufficient condition for the limits $\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}}$ to exist for $i = 0, \dots, p-1$. Naturally, this mirrors our desires for the generalization of Hollander's results. When $p = 1$, we will find again the result of [FV11]. The second result is a study of the initial conditions that give rise to a numeration with a regular language, in the case where the system *has* a dominant root but Hollander's results do not allow us to decide regularity. Indeed, to decide regularity, Hollander provides some necessary and some sufficient conditions on the polynomial of the recurrence relation satisfied by U, but there are polynomials where regularity depends on the initial conditions. Hollander gives no specific result in this case. We show that the initial conditions corresponding to regular systems form a translate of a polyhedral cone.

Finally, Chapter 9 concludes the dissertation by exposing some avenues for future research. Some of these avenues are more "paved" than others. In some cases, we give precise conjectures that we could observe experimentally, while others contain more general indications of direction.

As a final comment, the five above-mentioned articles do not represent the entirety of my production during my years as a doctoral student. I was able to write on the state-complexity of periodic sequences in [KPRS23] and on a problem of combinatorics on words in [DKP26]. However, the papers presented here form a coherent whole in which these two articles do not fit, justifying their exclusion.

Chapter 1

An overview of numeration systems

The aim of this chapter is to introduce numeration systems as an object of study. We will define this notion and provide several examples, namely positional, Rényi and Dumont–Thomas numeration systems. These will be at the heart of our study throughout the rest of the text, although some chapters will involve some of these families of numeration systems more than others. We will set notation, give directing examples for each relevant family of numeration systems as well as provide bibliographical context. Finally, we will end the chapter by hinting at the interplay that exists between those three families, which we will develop later in Chapter 2 and which will be central to this document's cohesion.

Contents

1.1 Numeration systems **2**

  1.1.1 Introduction 2

  1.1.2 Combinatorics on words 3

  1.1.3 Numeration systems 10

1.2 Positional numeration systems **12**

1.3 Rényi numeration systems **18**

1.4 Dumont–Thomas numeration systems **26**

1.5 Interplay between families of numeration systems **31**

1.1 Numeration systems

1.1.1 Introduction

Chapter 2 in [BR10] opens with the sentence "*Numbers* do exist – independently of the way we represent them, of the way we write them". The number that is the product of the first, second, and fourth prime numbers is commonly written as 42, but this is just a representation that is useful to humans. Every other integer has a similar representation. This eases communication by providing a short form for numbers, and facilitates mathematical computations by enabling algorithms that rely on specific representations of numbers, such as long addition.

The choice of a specific representation for every number is not random, of course, but is made with those two goals in mind. Our system for representing numbers should make it easy to convert between numbers and representations, and should provide us with tractable representation-based algorithms for the mathematical operations that we care about, such as addition or multiplication. We as a society have settled on a base 10 numeration system. A representation of a number is composed of a sequence of digits that can each go from 0 to 9, and each digit carries a value depending on its position in the sequence. To find the number from its representation, we can simply add the values from each position, finding for instance that the representation 42 represents the number $4 \cdot 10 + 2 \cdot 1$.

The human base 10 system is not the only way to represent numbers. Computers are uniquely suited to dealing with sequences that take only two values, and as such prefer to represent numbers using only the digits 0 and 1. For them, the same number would be represented as 101010. Here the value of this word can be computed as $1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 0 \cdot 1$. This is also a system based on positions, but the value of each position is different from that in the base 10 system.

Depending on the problem at hand, we might want to get more creative with our representations of numbers. The need for a fast parallel addition algorithm might lead us to design a three-digit system based on powers of two [Avi61], although this comes at the cost of every number no longer having a unique representation. A study of Wythoff's game might lead us to design a system based on more complex recurrence relations as representations in this system make it easy to identify losing positions [Fra82], where the decimal system does not. Trying to navigate a pentagonal grid in a hyperbolic plane could lead us to elaborate on both previous examples [Knu11, Section 7.1.3]. We might even want to change paradigm entirely for a variety of reasons [Rén57, Fra85, DT89, LR01].

The common denominator between all those paradigms is that we establish a relation between numbers and some representations of them, where a representation is a sequence of digits, each chosen among finitely many. This is what will define a numeration system later in Section 1.1.3. Before that however, we have to define what formalism we will use to deal with the representations of numbers. Representations being finitely-valued discrete sequences make them particularly suited to an interpretation in the field of *combinatorics on words*.

1.1.2 Combinatorics on words

Combinatorics on words is the branch of discrete mathematics that studies sequences of symbols. These sequences are called *words*, while the symbols are called *letters*. This field of study dates back to the early twentieth century and the works of Axel Thue [Thu06, Thu12] and has many applications and links to other domains of mathematics: semigroup theory through the Burnside problem [dLV99], other areas of combinatorics [BF02], or DNA sequencing and compression through tools like the Burrows–Wheeler transform [BW94]. See [BP07] for historical notes. In this dissertation, combinatorics on words will be used as a tool more than as a central object of study, so we simply recall the most central notions and let the reader consult [Lot97] if interested. These few paragraphs are based on the introduction in [KLS25b, Section 2].

General combinatorics on words An *alphabet* A is a finite set and its elements are called *letters*. A (*finite* or *infinite*) *word* over A is a (finite or infinite) sequence of letters belonging to A. For a finite word w over A, we let $|w|$ denote its *length*, i.e., the number of letters it is made of. The *empty word* is denoted by ε. Words are endowed with the operation of *concatenation*. This operation is used with product notation, with the concatenation of w and x being noted wx or $w \cdot x$. For instance, we have $\texttt{0011} \cdot \texttt{001011} = \texttt{0011001011}$. This operation is associative and the empty word is a neutral element, making the set of finite words a monoid. We let A^* denote the set of all finite words over the alphabet A and A^+ the set of all nonempty words over A (note that $A^+ = A^* \setminus \{\varepsilon\}$).

For a word w over A, a *factor* of w is a word y such that there exist words

x, z with $w = xyz$. A *prefix* (resp. *suffix*) is a factor y such that x (resp. z) is empty. The set of factors (resp. prefixes, suffixes) of w is noted $\mathrm{Fac}(w)$ (resp. $\mathrm{Pref}(w)$, $\mathrm{Suff}(w)$). If $w = ps$ for some words p, s, then we write $p^{-1}w = s$ and $ws^{-1} = p$. We let $\mathrm{Pref}_i(w)$ be the prefix of length i of w (for instance, $\mathrm{Pref}_2(\mathtt{0011}) = \mathtt{00}$) and $\mathrm{Suff}_i(w)$ be the suffix of length i of w.

Finite words will be indexed with an initial segment of the natural numbers, but the direction of indexing will depend on the context. When dealing with words for their own sake, or when considering words that naturally extend to the right such as expansions of real numbers, a finite word might be written $w = w_1 w_2 \cdots w_\ell$ or $w = w_0 w_1 \cdots w_{\ell-1}$. When considering words that naturally extend to the left, such as expansions of integers in usual numeration systems, a finite word might be indexed $w = w_{\ell-1} w_{\ell-2} \cdots w_0$. In general, we will use the latter if it makes more sense to align words to the right (think of writing the integers `12`, `123`, and `1234` in a column) and the former if it makes more sense to align words to the left (think of writing the real numbers `0.12`, `0.123`, and `0.1234` in a column). No matter the convention, ℓ is the length of the word and w_i is a letter of w with a given index. We use the notation of intervals to indicate portions of words: if I is an interval of integers, we let w_I denote the factor $(w_i)_{i \in I}$ of w. (Recall the difference between parentheses and square brackets to denote intervals.)

In addition to $|w|$ for the length of a finite word w, we let $|w|_a$ be the number of occurrences of the letter a in w (for instance, $|01101|_0 = 2$ and $|01101|_1 = 3$). The *Parikh vector* of w, noted $\psi(w)$, is the vector with components indexed by A and with $(\psi(w))_a = |w|_a$ for all $a \in A$.

For $\mathbb{D} \in \{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Z}_{<0}\}$, we let $A^{\mathbb{D}}$ the set of words indexed over $\mathbb{D}$. These *infinite* words will often be noted using bold letters. A *two-sided* infinite word, or *biinfinite* word, is a word indexed by $\mathbb{Z}$, whereas a *one-sided* infinite word is one indexed by $\mathbb{N}, \mathbb{N}_0$ or $\mathbb{Z}_{<0}$. We distinguish *left-infinite* and *right-infinite* words. Depending on the context, we will use different indexings for words. In the context of morphisms or Dumont–Thomas numeration systems, it will be more convenient to index right-infinite words by $\mathbb{N}$, as in $\mathbf{u} = u_0 u_1 u_2 \cdots$, left-infinite words by $\mathbb{Z}_{<0}$, and two-sided infinite words by $\mathbb{Z}$, as in $\mathbf{u} = \cdots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \cdots$. The vertical bar will be used to separate the -1-th and 0-th elements of such a word. In the context of Rényi numeration systems, however, it makes more sense to index right-infinite words by $\mathbb{N}_0$ or $\mathbb{Z}_{<0}$, for instance $\mathbf{u} = u_1 u_2 \cdots$, and two-sided infinite words by $\mathbb{Z}$ *with nonnegative indices to the left*, as in $\mathbf{u} = \cdots u_2 u_1 u_0 \bullet u_{-1} u_{-2} \cdots$, mirroring the usual way of writing numbers, with the integer part to the left of the fractional point and the fractional part to the right. This discrepancy stems from the fact that Dumont–Thomas numeration systems use the vocabulary of substitutions, where images are read from left to right, whereas Rényi numeration systems use the vocabulary of the more standard numeration systems, which position the most significant digits to the left.

We note w^ω the right-infinite word composed of repetitions of w, and similarly ${}^\omega w$ the left-infinite such word.

Order on words If the alphabet A is totally ordered by $<$, we define the *lexicographic order* on finite words and on right-infinite words as follows. For two words $u, u' \in A^*$, we write $u <_{\text{lex}} u'$ if u is a proper prefix of u' or if there exist $p, s, s' \in A^*$ and $a, a' \in A$ such that $u = pas$, $u' = pa's'$, and $a < a'$. For two right-infinite words $\mathbf{u}, \mathbf{u}' \in A^{\mathbb{N}}$, we write $\mathbf{u} <_{\text{lex}} \mathbf{u}'$ if there exist $p \in A^*$, $\mathbf{x}, \mathbf{x}' \in A^{\mathbb{N}}$, and $a, a' \in A$ such that $\mathbf{u} = pa\mathbf{x}$, $\mathbf{u}' = pa'\mathbf{x}'$, and $a < a'$. This definition is easily adapted to the comparison of a finite and a right-infinite word.

We may also define the lexicographic order on left-infinite or biinfinite words provided that these words are known to have an infinitely long common prefix. In practice, we will work with words that have an infinitely long left tail of zeros (that is, words $\mathbf{w} = \cdots w_1 w_0 \bullet w_{-1} w_{-2} \cdots$ such that $w_n = 0$ for all n large enough). If $\mathbf{u}$ and $\mathbf{u}'$ are two distinct such words, there uniquely exist $\mathbf{p}$ left-infinite, $a \neq a' \in A$ and $\mathbf{x}, \mathbf{x}'$ right-infinite such that $\mathbf{u} = \mathbf{p}a\mathbf{x}$ and $\mathbf{u}' = \mathbf{p}a'\mathbf{x}'$. in this case, we set $\mathbf{u} <_{\text{lex}} \mathbf{u}'$ if $a < a'$ and $\mathbf{u} >_{\text{lex}} \mathbf{u}'$ if $a > a'$. For finite or infinite words u, u', we write $u \leq_{\text{lex}} u'$ if $u <_{\text{lex}} u'$ or $u = u'$. The lexicographic order can be thought of as follows. First, we left-align the words, padding them to the right with a letter less than all letters in the alphabet if they have different lengths. Then, the lexicographic order between the two words is inherited from the order of the two letters at the leftmost position where they differ.

Another order we can put on finite words is the *radix order* $<_{\text{rad}}$, sometimes called *genealogical order*. If u and u' are finite words, we set $u <_{\text{rad}} u'$ if either $|u| < |u'|$, or $|u| = |u'|$ and $u <_{\text{lex}} v$. Note that the radix order can be seen as induced by the lexicographic order on left-infinite words, if we pad all finite words to the left by a letter less than all letters of A. The radix order is a *well-order* (every set of finite words has a minimal element) and it induces a bijection between A^* and $\mathbb{N}$ for any finite alphabet A.

Languages, automata theory, and regularity A *language* is a set of finite words. A language L is *prefix-closed* if every prefix of a word of L is itself a word of L, and similarly *suffix-closed* and *factorial* for suffixes and factors respectively. A language is *right-extendable* if any word of L can be extended to the right by at least one letter while staying in L: $w \in L \Rightarrow \exists a \in A : wa \in L$.

A *deterministic finite automaton* (DFA) $\mathcal{A}$ can be thought of as a machine that reads words and either accepts or rejects them, thus sorting A^* into the language of accepted words $L_{\mathcal{A}}$ and the language of rejected words. In the case of deterministic finite automata, the machine in question is quite simple as we will see. Languages that are accepted by a DFA are therefore "simple" in some theoretical sense. We provide two more formal points of view on DFAs. We also refer the reader to [Rig14b, Chapter 1] for a much more thorough introduction to the subject, or [Eil74, HU79, Sak09] for complete treatises rather than an introduction.

A DFA can be seen as an oriented labeled graph, with one distinguished *initial state* and a subset of distinguished *final states* (a state can be both initial and final). Edges are called *transitions* and are labeled by letters of some alphabet A. No two outgoing edges from a single vertex can share a label. A word w is *accepted* by $\mathcal{A}$ if there exists a path from the initial state to some final state labeled by w.

Alternatively, a DFA is given by the data (Q, q_0, F, A, δ), where Q is the finite set of states, $q_0 \in Q$ is the initial state, $F \subset Q$ is the set of final states, A is the finite alphabet, and $\delta \colon Q \times A \to Q$ is the transition function, that updates the state of the automaton when reading a letter. We may extend the transition function to words by defining inductively $\delta(q, wa) = \delta(\delta(q, w), a)$ and $\delta(q, \varepsilon) = q$ for all q in Q. We say that the automaton goes from state q to state $\delta(q, w)$ when *reading* w. Then, a word w is accepted by the automaton if $\delta(q_0, w) \in F$.

Nondeterministic finite automata (NFAs) are a "syntactic sugar" extension of DFAs, in the sense that they make constructions easier without permitting more of them. In the graph point of view, multiple states can now be initial, the labels of transitions can now be any finite words, and the condition on outgoing edges from the same origin is dropped. A word is accepted when there is a path from an initial state to a final state labeled by this word. In the other framework, we replace q_0 by a set I of initial states and the transition function by a transition *relation* $\Delta \subset Q \times A^* \times Q$. We extend this relation by saying that $(q_0, w_1, q_1) \in \Delta$ and $(q_1, w_2, q_2) \in \Delta$

implies $(q_0, w_1w_2, q_2) \in \Delta$. Then, we say that w is accepted by $\mathcal{A}$ if there exist $i \in I, f \in F$ such that $(i, w, f) \in \Delta$. Intuitively, nondeterminism offers the automaton to make different choices when reading the word, trying all of them and accepting the word if there is at least one sequence of choices that leads to the correct result.

A language L is said to be *regular* if there is an automaton $\mathcal{A}$ such that $L = L_{\mathcal{A}}$. The key result is that while NFAs look more powerful than DFAs, they are not, and the definition of regular languages can be done equivalently with DFAs or with NFAs ([Rig14b, Algorithm 1.6], originally [Myh57]). This allows us more flexibility when designing an automaton for a given language.

Regular languages can be seen from another point of view: they are the smallest family of languages that contains finite languages and is stable for the three *regular operations*:

- The *union* of two languages L and M is defined as $L \cup M = \{w : w \in L \text{ or } w \in M\}$.
- The *concatenation* of two languages L and M is defined as $LM = \{w : w = xy \text{ for some } x \in L,\ y \in M\}$. In other words, w is in LM if it is the concatenation of a word of L and a word of M.
- The *Kleene star* of a language L is defined as

  $$L^* = \{w : w = w_1w_2 \cdots w_n \text{ for some } n \in \mathbb{N}, w_1, \ldots, w_n \in L\}.$$

  In other words, w is in L^* if it is the concatenation of some number of words of L.

Since regular languages are stable for those three operations, one can show that a language is regular by expressing it as a combination of known-to-be-regular languages by these operations. Using automata, we can additionally show that regular languages are stable for intersection and complementation. We will be working with regular languages in this fashion in Chapter 6.

Note that the Kleene star of the alphabet A, seen as the finite language containing only the words of length 1, indeed corresponds to the set of finite words, justifying the notation A^*.

Example 1.1. Consider the language of words over $\{a, b\}$ that are of length 2 or contain an even number of letters a. Let us build a *regular expression*

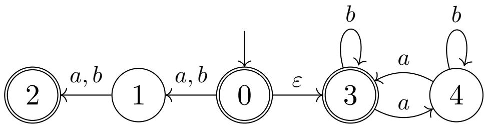


Figure 1.1: Automaton accepting the language $\{aa, ab, ba, bb\} \cup b^*(ab^*ab^*)^*$. Doubly-circle states are accepting. The starting state is state 0.

that generates this language, i.e., an expression starting from finite languages and using regular operations to construct our target language. A block of b's can be written as $\{b\}^*$, or b^* for short. A word contains an even number of as if and only if it can be factorized as a block of b's, then some number of factors of the form ab^*ab^*. Our regular expression for the language is then $\{aa, ab, ba, bb\} \cup b^*(ab^*ab^*)^*$.

Alternatively, we can see that the automaton of Figure 1.1 accepts exactly the words in our language. This automaton is nondeterministic as there are two choices from the starting node. This automaton can still be transformed into a DFA, but this DFA will have more states.

Topology and dynamics on infinite words The set of infinite words is endowed with a topology that we define now. We will consider right-infinite words over a finite alphabet A. Again, we only give a short introduction to the notions and refer the reader to [Rig14a, Subsection 1.2.1] for a more complete explanation. There are three ways to define the topology on words.

We may define a distance on infinite words by noting $\Lambda(\mathbf{x}, \mathbf{y})$ the length of the longest common prefix of $\mathbf{x}$ and $\mathbf{y}$ and setting $\mathrm{d}(\mathbf{x}, \mathbf{y}) = 2^{-\Lambda(\mathbf{x},\mathbf{y})}$. Then, d is an ultrametric distance, which induces a topology. Alternatively, we may define *cylinders*: the cylinder of w is the set $[w] = \{\mathbf{x} : \mathrm{Pref}_{|w|}(\mathbf{x}) = w\}$. We may then define the topology on words by stating that cylinders should form a base of this topology. Balls associated with the distance d in fact coincide with cylinders, so this topology is the same as the previous one. Finally, we may define the discrete topology on the alphabet A, where every set is open. The topology on words is then the product topology on $A^{\mathbb{N}}$.

In any case, $A^{\mathbb{N}}$ endowed with this topology is a compact and complete space. A sequence of words $(\mathbf{w}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{w}$ if $\mathbf{w}_n$ and $\mathbf{w}$ eventually share arbitrarily long prefixes. It is with this in mind that we use the notation lim when talking of words. This distance and topology can be extended to finite words by identifying the word w with the infinite word $w\#^\omega$, where $\#$ is a character that does not belong to the alphabet. Similarly, it can be extended to biinfinite words by redefining $\Lambda(\mathbf{x}, \mathbf{y})$ as $\max\{n : \mathbf{x}_{[-n,n)} = \mathbf{y}_{[-n,n)}\}$, asking cylinders to be centered at the positions -1 and 0, or taking the product topology on $A^{\mathbb{Z}}$ instead. A sequence of (finite or infinite) words $(\mathbf{w}_m)$ then converges to some infinite word $\mathbf{w}$ if $(\mathbf{w}_m)_{[-n,n)}$ is eventually equal to $\mathbf{w}_{[-n,n)}$ for all n.

We may give further structure to the set of infinite words and consider it as a dynamical system. See [Rig14a, Subsection 1.3.3] for a longer introduction, or [Que10, Section 4.2] for a classical reference. We define the *shift operator* σ on right-infinite words and on biinfinite words by

$$\sigma(a_0 a_1 a_2 \cdots) = a_1 a_2 \cdots$$

and

$$\sigma(\cdots a_1 a_0 \bullet a_{-1} a_{-2} \cdots) = \cdots a_0 a_{-1} \bullet a_{-2} a_{-3} \cdots$$

The map σ moves every digit one position to the left, possibly forgetting the leftmost digit if there is one. It is bijective on $A^{\mathbb{Z}}$ but not on $A^{\mathbb{N}}$. This map is continuous, and $(A^{\mathbb{N}}, \sigma)$ and $(A^{\mathbb{Z}}, \sigma)$ are therefore topological dynamical systems, both called the *full shift*. We will mention *subshifts*, which are subsystems of these dynamical systems. A subshift is a set of words that is closed under the operation σ and closed in the topological sense. An example is the set $\overline{\{\sigma^n(\mathbf{w}) : n \in \mathbb{N}\}}$ where $\mathbf{w}$ is a right-infinite word. A subshift can be defined by specifying a set F of *forbidden factors*, letting the shift S be equal to $\{\mathbf{w} : \mathrm{Fac}(\mathbf{w}) \cap F = \emptyset\}$ which is indeed σ-closed and topologically closed. We say that the shift is *of finite type* when it can be defined in this way for some finite F, and *sofic* if it can be defined in this way for some regular language F.

Morphisms and substitutions Given alphabets A, B, a *morphism* is a map $\mu\colon A^* \to B^*$ such that $\mu(uv) = \mu(u)\mu(v)$ for all words $u, v \in A^*$. A morphism is entirely determined by the images of the letters of A. A *substitution* is a morphism $\mu\colon A^* \to A^*$ such that the image $\mu(a)$ is nonempty for every letter $a \in A$ and there exists a *growing* letter $a \in A$, i.e., $\lim_{n\to+\infty} |\mu^n(a)| = +\infty$. A morphism $\mu\colon A^* \to A^*$ is *primitive* if there exists an integer $k \in \mathbb{N}$ such that for all $a, b \in A$, the letter a appears in $\mu^k(b)$.

The *adjacency matrix* of a morphism $\mu\colon A^* \to A^*$, is the matrix M_μ with rows and columns indexed by A and with entry a, b given by $|\mu(b)|_a$. We then have $\psi(\mu(w)) = M_\mu \psi(w)$ where ψ is the Parikh vector. Note that, if we consider the classical adjacency matrix M of $\mu\colon A^* \to A^*$, μ is primitive if and only if M is primitive, which justifies the choice of word. A short argument lets us deduce that if a morphism is primitive, then there exists an integer $k \in \mathbb{N}$ such that for all $a, b \in A$ and for all $\ell \geq k$, the letter a appears in $\mu^\ell(b)$.

Substitutions can naturally be applied to two-sided words by setting

$$\mu(\cdots u_{-3}u_{-2}u_{-1}|u_0u_1u_2\cdots) = \cdots \mu(u_{-3})\mu(u_{-2})\mu(u_{-1})|\mu(u_0)\mu(u_1)\mu(u_2)\cdots$$

Let $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Z}_{<0}\}$ and consider a substitution μ over A. A word $\mathbf{u} \in A^{\mathbb{D}}$ is a *periodic point of* μ if there exists an integer $p \geq 1$ such that $\mu^p(\mathbf{u}) = \mathbf{u}$. In this case, p is called a *period* of the periodic point $\mathbf{u}$. The smallest such integer is called the *period of* $\mathbf{u}$. A periodic point of μ with period $p = 1$ is called a *fixed point* of μ. We let $\mathrm{Per}_{\mathbb{D}}(\mu) = \{\mathbf{u} \in A^{\mathbb{D}} : \mu^p(\mathbf{u}) = \mathbf{u} \text{ for some } p \geq 1\}$ denote the set of periodic points of μ. If $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$, then the *seed* of $\mathbf{u}$ is the pair of letters $u_{-1}|u_0$; see [BG13, §4.1]. If both letters of the seed of a two-sided periodic point are growing, then the periodic point is defined entirely by its seed. More precisely, we have $\mathbf{u} = \lim_{n\to+\infty} \mu^{np}(u_{-1}|u_0)$, where p is a period of $\mathbf{u}$.

1.1.3 Numeration systems

As explained in Section 1.1.1, and with the vocabulary of Section 1.1.2, a numeration system is simply a pair of maps between a set of numbers (mostly $\mathbb{N}$, $[0, 1]$, or $[0, +\infty)$) and a set of words (A^*, $A^{\mathbb{N}}$ or $A^{\mathbb{N}_0}$, and $A^{\mathbb{Z}}$ respectively). These maps are the *evaluation map* val, which maps words to numbers, and the *representation map* rep, which maps numbers to words. Keeping in mind reasonable constraints, such as the representation of a number evaluating to that number, every number being able to be represented, etc, we reach the following definition.

Definition 1.2. Given a set of numbers N (e.g. $\mathbb{N}, [0, 1], [0, +\infty), \ldots$), and a set of words W (e.g. $A^*, A^{\mathbb{N}}, A^{\mathbb{Z}}, \ldots$), a *numeration system* between N and W is a pair of maps, the *representation map*

$$\mathrm{rep}\colon N \to W : n \mapsto \mathrm{rep}(n)$$

and the *evaluation map*

$$\mathrm{val}\colon L \to N : w \mapsto \mathrm{val}(w),$$

where $\mathrm{val} \circ \mathrm{rep} = \mathrm{id}_N$ and $\mathrm{rep}(N) \subset L \subset W$.

In general, we will simply say "numeration system" without mentioning N and W as they will be clear from context. If the numeration system is named U, we will sometimes use rep_U and val_U instead of rep and val to emphasize the dependence on U.

Remark 1.3. The purpose of the inclusions $\mathrm{rep}(N) \subset L \subset W$ is to allow nonstandard representations. In the usual decimal numeration system, it is commonly accepted that `42` and `042` represent the same number, but in a default setting a person who wishes to write this number would write it `42` and not `042`. This is because the words `42` and `042` evaluate to the same number, but rep(42) is the word `42` and not the word `042`. We say that the word `42` is *the representation* or *the expansion* of the number 42 (note the definite article), whereas `042` is *a representation.* In general, L is the set of words that have a value, but may also include words that are not the expansion of their value, whereas $\mathrm{rep}(N)$ only includes those words.

Note also that when dealing with sequences of digits that can be interpreted as both numbers and words, we sometimes use typewriter font to indicate that the sequence of digits is to be interpreted as a word rather than as a number (as an element of W rather than an element of N). This will be the case mainly for Dumont–Thomas systems, but it is not a strict convention.

In the rest of this section, we will introduce the particular numeration systems that will interest us for the remainder of this document. Those are

- *Positional numeration systems* that go between $\mathbb{N}$ and A^*.
- *Rényi numeration systems* and later *alternate bases* and *Cantor bases* that can be seen as either going between $[0, 1]$ and $A^{\mathbb{N}_0}$ or $A^{\mathbb{Z}<0}$, or going between $[0, +\infty]$ and $A^{\mathbb{Z}}$.
- *Dumont–Thomas numeration systems* that also go between $\mathbb{N}$ and A^*, but with different representation and evaluation maps in general.

1.2 Positional numeration systems

Positional numeration systems are perhaps the most direct generalization of our usual (decimal, binary or based on some other integer) numeration systems. It is hard to give a precise start date to their study, with precursor articles including [Ost22, Lek52, Zec72]. The 80's marked an upward trend in interest, with notable articles being the aptly-titled [Fra85] as well as [BM89]. The exposition in this section mimics that of [CK25], which will be Chapter 6 in this thesis.

Definition 1.4. A *positional numeration system* is given by an increasing sequence $U = (U_n)_{n\geq 0}$ of integers such that $U_0 = 1$ and the quotients $\frac{U_{n+1}}{U_n}$ are uniformly bounded. Words are indexed over (an initial segment of) $\mathbb{N}$, increasing from right to left. T he evaluation maps a word $w_{\ell-1}\cdots w_0$ over $\mathbb{N}$ to

$$\mathrm{val}_U(w_{\ell-1}\cdots w_0) = \sum_{n=0}^{\ell-1} w_n U_n.$$

For a given $x \in \mathbb{N}$, any word w with letters in $\mathbb{N}$ such that $\mathrm{val}_U(w) = x$ is said to be a *U-representation* of x. Such a representation need not be unique.

Among all possible U-representations of x, we will consider the one obtained using the greedy algorithm. First, we let ℓ be the least integer such that $x < U_\ell$ and we let $r_\ell = x$. Then for every $n = \ell-1,\dots,0$, we set $a_n = \lfloor \frac{r_{n+1}}{U_n} \rfloor$ and $r_n = r_{n+1} - a_n U_n$. The produced U-representation is

$$\mathrm{rep}_U(x) = a_{\ell-1}\cdots a_0,$$

which is called the *greedy* U-representation of x.

We abuse notation and also note U the numeration system generated by the sequence U. Positional numeration systems will also be called *U-systems* for short. Positional numeration systems form a large and interesting family, and it is valuable to know if a system defined in some other fashion belongs to it.

Definition 1.5. A numeration system over $\mathbb{N}$ (defined differently from above) is said to be *positional* if the representations are words are on an alphabet of numbers and there exists a sequence $(U_n)_{n\in\mathbb{N}}$ such that $\mathrm{val}(w_{\ell-1}\cdots w_0) = \sum_{i=0}^{\ell-1} w_i U_i$.

Definition 1.6. The language of all greedy U-representations, possibly preceded by zeros, i.e., the language

$$L_U = 0^* \mathrm{rep}_U(\mathbb{N}),$$

is called the *numeration language.* It is written over the alphabet

$$A_U = \left\{ 0, \ldots, \sup_{n \geq 0} \left\lceil \frac{U_{n+1}}{U_n} \right\rceil - 1 \right\},$$

called the *numeration alphabet.*

Allowing leading zeros in this fashion will often be more convenient for our future developments. In particular, a word $w_{\ell-1} \cdots w_0 \in A_U^*$ belongs to L_U if and only if

$$\mathrm{val}_U(w_{n-1} \cdots w_0) < U_n \tag{1.1}$$

for all $n \in \{0, \ldots, \ell\}$. This also implies that the language L_U is suffix-closed.

Example 1.7. We introduce two examples that will serve to illustrate various concepts throughout this chapter. The most famous positional numeration system that is not based on an integer base is the Zeckendorf numeration system, based on the Fibonacci sequence [Zec72]. Set $U_0 = 1, U_1 = 2$ then $U_{n+2} = U_{n+1} + U_n$ for all n. This sequence satisfies the conditions mentioned above. We have for instance that the words $\mathtt{1001}, \mathtt{200}, \mathtt{30}$, and $\mathtt{111}$ all have value 6. Among all those, $\mathtt{1001}$ is the one obtained by the greedy algorithm, and it is therefore the expansion of 6, $\mathrm{rep}_U(6)$. Anticipating on the remainder of this section, it is possible to show that the numeration language is the set of words that do not contain the factor $\mathtt{11}$. On the one hand, words that contain this factor clearly cannot be obtained from a greedy algorithm. On the other hand, the number of words of length n that do not contain it matches the number of integers that have a representation of length n or less, which must be U_n. Thus the given language is indeed equal to L_U. The alphabet of the numeration is $\{\mathtt{0}, \mathtt{1}\}$ as digits above $\mathtt{2}$ are never required.

A variation on this is the Fina numeration system, named by Berthé and coauthors in [BFRS20]. The base sequence is the Fibonacci sequence where every other entry has been removed, starting with $1, 3, 8, 21, \cdots$. This is a linear recurrence sequence (see just below) satisfying the relation $U_{n+2} = 3U_{n+1} - U_n$. It can be seen, for instance, that the alphabet of the numeration is $\{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. The expansion of 20 is $\mathtt{211}$, and the expansion of 21 is $\mathtt{1000}$.

As can already be felt with this example, the sequence $(U_n)_{n\in\mathbb{N}}$ that is the base of the positional numeration system will often be a *linear recurrence sequence*. Since this will continue, especially in Chapter 6, we present a primer on the topic.

A primer on linear recurrence sequences Let K be $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$. In general, K can be a principal ideal domain, i.e., a commutative ring with no nontrivial zero divisors where every ideal can be generated by one element. The results of this primer will still be true, but we will only consider the three above cases in this document. We only give the necessary background for our purpose. We refer the reader to [BR11] for more details on linear recurrence sequences, and rational series in general.

A sequence $(U_n)_{n\geq 0}$ of elements in K is said to be a *linear recurrence sequence over* K if there exist $a_0, \ldots, a_{k-1} \in K$ with $k \geq 1$ such that

$$U_{n+k} = a_{k-1}U_{n+k-1} + \cdots + a_1 U_{n+1} + a_0 U_n \tag{1.2}$$

for all $n \geq 0$. We refer to (1.2) as the *linear recurrence relation* satisfied by the sequence $(U_n)_{n\geq 0}$ and to k as the *order* of this linear recurrence relation. The polynomial

$$X^k - a_{k-1}X^{k-1} - \cdots - a_1 X - a_0$$

is called the *characteristic polynomial* of the linear recurrence relation (1.2). The numbers $(U_0, \ldots, U_{k-1})$ are the *initial conditions* of the recurrence sequence. Giving a characteristic polynomial together with a set of initial conditions of the same order completely characterizes a linear recurrence sequence.

If a sequence $(U_n)_{n\geq 0}$ over K satisfies a linear recurrence relation of characteristic polynomial $P \in K[X]$, then it also satisfies all linear recurrence relations whose characteristic polynomials are multiples of P. Given a sequence $(U_n)_{n\geq 0}$, let I_U be the set of all polynomials

$$a_k X^k + a_{k-1}X^{k-1} + \cdots + a_1 X + a_0$$

such that

$$a_k U_{n+k} + a_{k-1}U_{n+k-1} + \cdots + a_1 U_{n+1} + a_0 U_n = 0$$

for all $n \geq 0$. This set I_U is an ideal of the ring of polynomials $K[X]$. The ideal I_U contains a monic polynomial if and only if the sequence $(U_n)_{n\geq 0}$ is a linear recurrence sequence over K. In this case, the unique monic generator of I_U is called the *minimal polynomial* of the sequence $(U_n)_{n\geq 0}$. It is the characteristic polynomial of the linear recurrence relation of smallest order satisfied by $(U_n)_{n\geq 0}$. A celebrated result of Fatou implies that if a sequence of rational numbers $(U_n)_{n\geq 0}$ is a linear recurrence sequence over $\mathbb{Z}$, i.e., if it satisfies a linear recurrence relation with a characteristic polynomial having integer coefficients, then the minimal polynomial of $(U_n)_{n\geq 0}$ also has integer coefficients [BR11, Fat04].

The *eigenvalues* of a linear recurrence sequence over K are the roots of its minimal polynomial. These eigenvalues thus belong to the algebraic closure $\overline{K}$ of the field of fractions of K. The *multiplicity* of an eigenvalue of a linear recurrence sequence is its multiplicity as a root of its minimal polynomial. Any linear recurrence sequence that is not ultimately zero has at least one nonzero eigenvalue. A linear recurrence sequence over K is said to be *strict* if it has only nonzero eigenvalues. In the case where K is an algebraically closed field of characteristic zero, the eigenvalues give rise to a closed formula for the general term of a linear recurrence sequence.

Theorem 1.8. *Let $(U_n)_{n\geq 0}$ be a linear recurrence sequence over an algebraically closed field K of characteristic zero. Let $\alpha_1,\ldots,\alpha_e \in K$ be the nonzero eigenvalues of $(U_n)_{n\geq 0}$ with multiplicities $m_1,\ldots,m_e$ respectively. There exist polynomials $P_1,\ldots,P_e \in K[X]$ of degrees equal to $m_1-1,\ldots,m_e-1$ respectively such that*

$$U_n = \sum_{j=1}^{e} P_j(n)\alpha_j^n$$

for all $n \geq N$, where N is the multiplicity of the eigenvalue 0.

Back to the study of positional numeration systems A particular sublanguage of L_U will prove particularly useful to us: the language of *maximal words*, which collects the lexicographically greatest word of each length within L_U.

The language of maximal words of a language L over a totally ordered alphabet is the language

$$\mathrm{Max}(L) = \{u \in L : \text{for all } v \in L,\ |v| = |u| \implies v \leq_{\mathrm{lex}} u\},$$

where $\leq_{\mathrm{lex}}$ is the lexicographic order. That is, $\mathrm{Max}(L)$ is the language obtained by extracting the lexicographically greatest word of each length present in L. In our case, we may present an alternative characterization of this language.

First, let us recall the following well-known property of greedy U-representations; see for example [Lot02, Chapter 7]. Recall that the *radix order* $<_{\text{rad}}$ sorts words by length first, then lexicographically.

Lemma 1.9. *Let U be a positional numeration system, let $x \in \mathbb{N}$ and let $w \in A_U^* \setminus 0A_U^*$. If $\text{rep}_U(x) <_{\text{rad}} w$ then $x < \text{val}_U(w)$.*

This allows us to obtain the following description of the maximal words in L_U.

Lemma 1.10. *Let U be a positional numeration system. Then*

$$\text{Max}(L_U) = \{\text{rep}_U(U_n - 1) : n \in \mathbb{N}\}.$$

Proof. Clearly, the word $\text{rep}_U(U_n - 1)$ belongs to L_U for all n. From the greedy algorithm, we know that $|\text{rep}_U(U_n - 1)| = n$ and that every word w in L_U of length n is such that $\text{val}_U(w) < U_n$. If w starts with 0, then $w <_{\text{lex}} \text{rep}_U(U_n - 1)$. Otherwise, we get $w \leq_{\text{lex}} \text{rep}_U(U_n - 1)$ by Lemma 1.9. □

We first provide the following lemma. Already present in [Hol98], it gives a characterization of the words in L_U in terms of a lexicographic condition of their suffixes.

Lemma 1.11. *Let U be a positional numeration system. A word $w_{\ell-1} \cdots w_0$ in A_U^* belongs to L_U if and only if*

$$w_{n-1} \cdots w_0 \leq_{\text{lex}} \text{rep}_U(U_n - 1)$$

for all $n \in \{0, \ldots, \ell\}$.

Proof. The necessary condition follows from the fact that L_U is suffix-closed and Lemma 1.10. The sufficient condition is obtained by using (1.1) and an induction on the length of the words. □

In addition, giving a candidate for $\text{Max}(L_U)$ is enough to characterize U fully.

Lemma 1.12. *Let M be a language over a (finite) alphabet included in $\mathbb{N}$. There exists a positional numeration system U such that $M = \mathrm{Max}(L_U)$ if and only if the language M satisfies the following properties:*

- *M has exactly one word of each length.*
- *No word in M starts with the digit 0.*
- *For all words $u_{n-1} \cdots u_0$ and $v_{\ell-1} \cdots v_0$ in M with $n < \ell$, we have $v_{n-1} \cdots v_0 \leq_{\mathrm{lex}} u_{n-1} \cdots u_0$.*

Furthermore, if such a positional numeration system U exists, then it is unique.

Proof. The necessary condition and the uniqueness of the system are straightforward. Let us prove the sufficient condition. So, we suppose that the language M satisfies the three properties given in the statement and we show how to build a positional numeration system U such that $M = \mathrm{Max}(L_U)$.

For each $n \geq 0$, let w_n denote the unique word of length n in M and let $U_n = \mathrm{val}_U(w_n) + 1$. In particular, w_0 is the empty word and $U_0 = 1$. Since the computation of $\mathrm{val}_U(w_n)$ only requires the knowledge of $U_0, \ldots, U_{n-1}$, the terms of the sequence $U = (U_n)_{n \geq 0}$ are obtained recursively. As we have assumed that no word in M starts with 0, this sequence is increasing. We have to show that for all $n \geq 0$, the word w_n belongs to L_U, which is equivalent to showing that $w_n = \mathrm{rep}_U(U_n - 1)$. In particular, we will also obtain that the quotients $\frac{U_{n+1}}{U_n}$ are uniformly bounded by $C + 1$ where C is the maximal element of the alphabet of M.

We proceed by induction on n. For $n = 0$, this is clear. Suppose that $n \geq 1$ and that we have $w_\ell = \mathrm{rep}_U(U_\ell - 1)$ for all $\ell < n$. Write $w_n = w_{n,n-1} \cdots w_{n,0}$ where the $w_{n,\ell}$'s are letters. By Lemma 1.11, we have to show that $w_{n,\ell-1} \cdots w_{n,0} \leq_{\mathrm{lex}} \mathrm{rep}_U(U_\ell - 1)$ for all $\ell \leq n$. For $\ell < n$, by using the properties of the language M and the induction hypothesis, we get that $w_{n,\ell-1} \cdots w_{n,0} \leq_{\mathrm{lex}} w_\ell = \mathrm{rep}_U(U_\ell - 1)$. For $\ell = n$, since $\mathrm{val}_U(w_n) = U_n - 1$ by definition of U and w_n does not start with 0, we get that $w_n \leq_{\mathrm{lex}} \mathrm{rep}_U(U_n - 1)$ from Lemma 1.9. □

Example 1.13. We continue Example 1.7. Through the properties of the Fibonacci sequence, one can see that $\mathrm{rep}(U_n - 1)$ is $(\mathtt{10})^{\frac{n}{2}}$ if n is even and

$\texttt{(10)}^{\frac{n-1}{2}}\texttt{1}$ if n is odd. These indeed are the lexicographically greatest words of their respective length. Thus $\mathrm{Max}(L_U) = \mathrm{Pref}((\texttt{10})^\omega)$. We can note that the language L_U, previously described as the set of words that do not contain the factor $\texttt{11}$, is also adequately described as the set of words whose every suffix is lexicographically less than $(\texttt{10})^\omega$, which highlights Lemma 1.11.

The language $\texttt{21}^*$ satisfies the conditions of Lemma 1.12. Thus there exists a numeration system that has this language as its set of maximal words. In fact, it is the Fina numeration system.

1.3 Rényi numeration systems

Introduced by Rényi in the context of ergodic theory [Rén57], Rényi numeration systems represent real numbers by infinite words. In Rényi's original article, the representation map takes the form of an iterative algorithm. The number to be represented is noted r_0, then a function φ is applied. The integer part of the result is the first digit a_1 of the expansion, and the fractional part is the next remainder r_1. The function φ is then applied to r_1 to get a_2 and r_2, and so on. An infinite word $a_1a_2\cdots$ is obtained, which is such that $r_0 = \varphi^{-1}(a_1 + \varphi^{-1}(a_2 + \cdots))$ if the function φ is sufficiently well-behaved. Rényi's original framework is therefore much more general than the modern acceptation of "Rényi numeration system", as it also includes continued fractions, for instance, which correspond to the function $\varphi(x) = 1/x$. In the modern sense of the term, *Rényi numeration systems* refer specifically to those using a function of the form $\varphi(x) = \beta x$ for some $\beta > 1$.

Rényi numeration systems have garnered a large interest since the original article (over 500 citations), with links to dynamical systems [IT74, Sch97], tilings [Sol97, BS05], transcendence properties [AB07], and a wealth of problems within the field of numeration systems itself [Par60, Sch80, Fro92b, dVK09]. To make clear the links between all of these is beyond the scope of this chapter, but we will explain those that will be relevant to this thesis. In this section, our exposition is guided by a blend of sources, including [Lot02, Section 7.2], and some folklore results. We start by giving a formal definition of those systems.

Definition 1.14. Let β be a real number greater than 1, called the *base*.

The evaluation map of the Rényi numeration system with base β is given by

$$\mathrm{val}_\beta(a_1a_2\cdots) = \sum_{j=1}^{\infty} \frac{a_j}{\beta^j}$$

provided that this series converges.

The series in Definition 1.14 always converges if the word $a_1a_2\cdots$ is a word over a finite alphabet. The numeration system with base β is called *β-numeration* for short, and we will omit the subscript in val_β if the context is clear.

Example 1.15. The usual decimal numeration system is often used to represent real numbers, say $\pi = 3.14159\cdots$. The evaluation map corresponds to choosing $\beta = 10$, as expected. We find for instance that

$$\mathrm{val}(0.499\cdots) = \frac{4}{10} + \frac{9}{100}\sum_{j=0}^{\infty}\frac{1}{10^j} = \frac{4}{10} + \frac{9}{100}\frac{1}{1-\frac{1}{10}} = \frac{5}{10} = \mathrm{val}(0.500\cdots).$$

As evidenced by the previous example, multiple words may have the same value. Among all of them, one is distinguished by using a greedy algorithm, which corresponds to the iterated function algorithm used by Rényi.

Definition 1.16. Let β be a real number greater than 1 and $x \in [0,1]$. The *greedy algorithm* is defined as follows.

Let $r_0 = x$. Then, if r_i has been defined, define

$$a_{i+1} = \lfloor \beta r_i \rfloor \quad \text{and} \quad r_{i+1} = \beta r_i - a_{i+1}.$$

The word $a_1a_2\cdots$ is the *β-expansion* of x, and is noted $d_\beta(x)$.

Note that the β-expansion of any number in $[0,1)$ is a word on the alphabet $A_\beta = \{0,\ldots,\lceil\beta\rceil - 1\}$. The expansion of 1 might contain the digit $\lfloor\beta\rfloor$ if β is an integer.

Example 1.17. In the above example, the 10-expansion of $\frac{1}{2}$ is $0.500\cdots$ rather than $0.499\cdots$.

If we take β to be the golden ratio φ and $x = 1$, we find $a_1 = 1, r_1 = \varphi - 1$, then $a_2 = 1$ and $r_2 = \varphi^2 - \varphi - 1 = 0$, after which the algorithm only produces zeros. Thus $d_\varphi(1) = 110^\omega$. Like in the decimal numeration system, the same number may be represented in multiple ways. For instance, the number $\frac{2}{\varphi^2}$ admits the representations 020^ω and $100(01)^\omega$.

If we now take β to be the square of the golden ratio and $x = 1$, we get $a_1 = 2, r_1 = \varphi^2 - 2 = \varphi - 1$, then $a_2 = 1$ and $r_2 = \varphi - 1$ again. The algorithm therefore loops and we obtain $d_\beta(1) = 21^\omega$.

The greedy algorithm has a number of elementary but important properties. First and most important, it does output a representation of the input.

Proposition 1.18. *For all $\beta > 1$ and $x \in [0, 1]$, if $d_\beta(x) = a_1 a_2 \cdots$, then* $\mathrm{val}(a_1 a_2 \cdots) = x$.

Proof. The equality

$$x = \frac{a_1}{\beta^1} + \frac{a_2}{\beta^2} + \ldots + \frac{a_j}{\beta^j} + \frac{r_j}{\beta^j}$$

can be shown by induction on j. The result follows from taking the limit for j going to infinity and noting that $\frac{r_j}{\beta^j}$ goes to 0 as $r_j \in [0, 1)$ for all j. □

We provide alternative characterizations of the greedy expansion among all representations.

Proposition 1.19 ([Lot02, Lemma 7.2.2, Proposition 7.2.3])**.** *For all $\beta > 1$ and $x \in [0, 1]$, if* $\mathrm{val}(a_1 a_2 \cdots) = x$, *then the following conditions are equivalent.*

(a) The word $a_1 a_2 \cdots$ is the β-expansion of x, $d_\beta(x) = a_1 a_2 \cdots$.

(b) For all $n \in \mathbb{N}_0$, $\sum_{j=n+1}^{\infty} \frac{a_j}{\beta^j} < \frac{1}{\beta^n}$.

(c) The representation $a_1 a_2 \cdots$ is the greatest representation of x in lexicographic order.

Proof. First note that only one representation of x satisfies condition (a) or (c). We now show that only one representation of x satisfies (b). Assume on the contrary that $\mathbf{a} = a_1 a_2 \cdots$ and $\mathbf{b} = b_1 b_2 \cdots$ both have value x and satisfy (b), with $a_1 = b_1, \ldots, a_{n-1} = b_{n-1}$ and $a_n < b_n$. Then

$$\sum_{j=1}^{\infty} \frac{a_j}{\beta^j} < \sum_{j=1}^{n-1} \frac{a_j}{\beta^j} + \frac{a_n}{\beta^n} + \frac{1}{\beta^n} \leq \sum_{j=1}^{n-1} \frac{b_j}{\beta^j} + \frac{b_n}{\beta^n} \leq \sum_{j=1}^{\infty} \frac{b_j}{\beta^j},$$

a contradiction since **a** and **b** have the same value.

It is clear from the greedy algorithm that the greedy expansion satisfies conditions (b) and (c), establishing the desired equivalence since only one representation of x satisfies those conditions. □

The greedy representation is increasing and is right-continuous.

Proposition 1.20 ([Lot02, Proposition 7.2.4])**.** *For $\beta > 1$ and $x, y \in [0, 1]$, we have $x < y$ if and only if $d_\beta(x) <_{\text{lex}} d_\beta(y)$.*

Proof. If $x < y$, then we may obtain a representation $\mathbf{w}$ of y by adding $d_\beta(x)$ and $d_\beta(y - x)$ pointwise. It is clear that $d_\beta(x) <_{\text{lex}} \mathbf{w}$, and $\mathbf{w} \leq_{\text{lex}} d_\beta(y)$ by Proposition 1.19. A similar result holds true for nonstrict inequalities, thus the converse implication holds by contraposition.

□

Proposition 1.21. *For all $\beta > 1$, the function $x \mapsto d_\beta(x)$ from $[0, 1)$ to $A^{\mathbb{N}_0}$ is right-continuous for the Euclidean topology on real numbers and the product topology on infinite words.*

Proof. Consider some $x \in [0, 1)$ and let $d_\beta(x) = a_1 a_2 \cdots$. For all n in $\{1, \ldots, N\}$, we have

$$x - \sum_{j=1}^{n} \frac{a_j}{\beta^j} = \sum_{j=n+1}^{\infty} \frac{a_j}{\beta^j} < \frac{1}{\beta^n}$$

For all y greater than x such that the inequalities $y - \sum_{j=1}^{n} \frac{a_j}{\beta^j} < \frac{1}{\beta^n}$ still hold for $n \in \{1, \ldots, N\}$, $d_\beta(y)$ also starts with $a_1 \cdots a_N$. Since these y form an open set of $[x, +\infty)$ for all N, the conclusion follows. □

Note that we clearly do not have left-continuity, as evidenced by the example of $0.500\cdots$ in decimal: the expansion of any y less than $0.500\cdots$ and close to it starts with 0.4. In fact, we have $\lim_{y \to \frac{1}{2}^-} d_\beta(y) = 0.499\cdots$.

Seeing that the same number can be represented by multiple words, one can ask if there are any tractable characterizations of the words which are expansions of their value and, further, if there are any algorithms to convert a word $\mathbf{w}$ into the expansion of its value $\text{rep}(\text{val}(\mathbf{w}))$, a process called *normalization*. We will now see that identifying which words are *normalized*, i.e., equal to the expansion of their value, can be done elegantly, with a characterization that uses representations of 1. To this end, we must first introduce those representations.

Let $\beta > 1$ be our base, and assume that $d_\beta(1)$ is finite, that is, it ends with an infinite tail of zeros, $d_\beta(1) = t_1 t_2 \cdots t_\ell 0^\omega$ with $t_\ell \neq 0$. It follows that $\frac{1}{\beta^\ell} = \sum_{j=1}^{\ell} \frac{t_j}{\beta^{\ell+j}}$ and therefore $t_1 \cdots t_{\ell-1}(t_\ell - 1)t_1 \cdots t_\ell 0^\omega$ is also a representation of 1. Iterating, we obtain that $(t_1 \cdots t_{\ell-1}(t_\ell - 1))^\omega$ is itself a representation of 1, but is not finite.

Definition 1.22. We say that $d_\beta(1)$ is *finite* if it ends in 0^ω, and infinite otherwise.

Let $\beta > 1$ and $\mathbf{t} = d_\beta(1)$. The *quasi-greedy expansion* of 1 in base β is given by

$$\begin{cases} \mathbf{t}, & \text{if } \mathbf{t} \text{ is infinite;} \\ (t_1 \cdots t_{\ell-1}(t_\ell - 1))^\omega, & \text{if } \mathbf{t} \text{ is finite equal to } t_1 \cdots t_\ell 0^\omega \text{ with } t_\ell \neq 0. \end{cases}$$

It is noted $d^*_\beta(1) = d_1 d_2 \cdots$.

If $\mathbf{t}$ is finite equal to $t_1 t_2 \cdots t_\ell 0^\omega$ with $t_\ell \neq 0$, the *intermediate representations* of 1 are

$$\mathbf{w}_j = (t_1 \cdots t_{\ell-1}(t_\ell - 1))^j t_1 \cdots t_\ell 0^\omega \quad (j \in \mathbb{N}).$$

We will frequently drop the tail of zeros when $d_\beta(1)$ is finite and just say that $d_\beta(1) = t_1 \cdots t_\ell$. We implicitly assume that $t_\ell \neq 0$ when using this notation.

Note that if we modify the greedy algorithm and replace $a_{i+1} = \lfloor \beta r_i \rfloor$ by $a_{i+1} = \lceil \beta r_i \rceil - 1$, this has the effect that $r_{i+1} \in (0, 1]$ for all i, rather than $[0, 1)$. With this modified algorithm, the expansion of 1 becomes $d^*_\beta(1)$. It follows that if $d^*_\beta(1) = d_1 d_2 \cdots$, we have $\sum_{j=n+1}^{\infty} \frac{d_j}{\beta^j} \leq \frac{1}{\beta^n}$, mirroring item (b) of Proposition 1.19.

The quasi-greedy expansion of 1 has many properties.

Proposition 1.23. *Let $\beta > 1$. Then $d^*_\beta(1)$ is the lexicographically largest infinite representation of* 1. *Moreover, $d^*_\beta(1) = \lim_{x \to 1^-} d_\beta(x)$.*

Compare to $d_\beta(1)$ being the largest of all representations of 1.

Proof. First, consider $\mathbf{y} = y_1y_2\cdots$ an infinite word not ending in 0^ω and lexicographically greater than $d^*_\beta(1)$, say $y_1 = d_1, \ldots, y_{n-1} = d_{n-1}$ and $y_n > d_n$. We have

$$1 = \sum_{j=1}^{\infty} \frac{d_j}{\beta^j} \leq \sum_{j=1}^{n-1} \frac{d_j}{\beta^j} + \frac{d_n}{\beta^n} + \frac{1}{\beta^n} \leq \sum_{j=1}^{n} \frac{y_n}{\beta^n} < \sum_{j=1}^{\infty} \frac{y_n}{\beta^n}$$

where the last inequality is due to $\mathbf{y}$ not ending in 0^ω. We have shown that all infinite words lexicographically larger than $d^*_\beta(1)$ have a value greater than 1, thus $d^*_\beta(1)$ must be the lexicographically largest infinite representation of 1.

Moving to the second part of the statement, $\mathbf{z} = \lim_{x\to 1^-} d_\beta(x)$ exists due to Proposition 1.20. It cannot end in 0^ω seeing that for all $x \in [0,1)$ such that $d_\beta(x)$ is finite, there exists y in $(1,x)$ with $d_\beta(y)$ infinite. Since we must have $\mathrm{val}(\mathbf{z}) \leq 1$, it follows from the first part of the statement that $\mathbf{z} \leq_{\mathrm{lex}} d^*_\beta(1)$. It then suffices to notice that any number x satisfying

$$\sum_{j=1}^{n} \frac{d_j}{\beta^j} \leq x < 1 = \sum_{j=1}^{\infty} \frac{d_j}{\beta^j}$$

must have an expansion starting in $d_1\cdots d_n$ (it cannot be lexicographically greater due to the first part of the proof). □

With this representation introduced, we may now explain how this allows us to characterize which words are normalized. This result and others of similar nature are referred to as *Parry conditions*, after Parry who introduced one of them for the first time in [Par60].

Theorem 1.24. *Let $\beta > 1$. The word $\mathbf{a} = a_1a_2\cdots$ is the expansion of some x in $[0,1)$ if, and only if,*

$$\forall n \in \mathbb{N},\ \sigma^n(\mathbf{a}) <_{\mathrm{lex}} d^*_\beta(1).$$

We do not prove this result, but rather refer the reader to [Lot02, Theorem 7.2.9].

A similar result informs us on which infinite words can be expansions of 1 in the first place.

Theorem 1.25 ([Lot02, Corollary 7.2.10]). *Consider the infinite word* $\mathbf{a} = a_1a_2\cdots$, *such that* $a_1 \geq 1$ *and* $0 \leq a_n \leq a_1$ *for all* $n \geq 1$, *and* $\mathbf{a} \neq 10^\omega$. *There exists a unique* $\beta > 1$ *such that* $\mathbf{a}$ *is a* β*-representation of* 1. *Furthermore,* $\mathbf{a}$ *is the* β*-expansion of* 1 *if and only if* $\sigma^n(\mathbf{a}) <_{\text{lex}} \mathbf{a}$ *for all* $n \geq 1$.

Example 1.26. We continue Example 1.17. Since the φ-expansion of 1 is 110^ω, which is finite, we obtain $d^*_\varphi(1) = (10)^\omega$. The intermediate expansions of 1 are $\mathbf{w}_j = (10)^j 110^\omega$, which converge to $d^*_\beta(1)$ as j goes to infinity. Due to Theorem 1.24, a word is the expansion of a number less than 1 if and only if all of its suffixes are lexicographically less than $(10)^\omega$. The two words 020^ω and $100(01)^\omega$ that were given as representations of $\frac{2}{\varphi^2}$ both fail this condition. The true expansion of this number is 10010^ω.

Theorem 1.25 tells us that there is a unique base β where the expansion of 1 is 21^ω. Indeed, we know that this number is φ^2. Expansions in this base never contain a factor of the form 21^n2, nor do they contain the word 21^ω as a suffix. By contrast, the word 1^ω is never $d_\beta(1)$ for any β, as $\sigma^n(1^\omega) = 1^\omega$ for any n. In fact, this is indicative of a much more general property: due to Theorem 1.25, the word $d_\beta(1)$ can never be purely periodic.

We now introduce some of the objects connecting β-expansions to dynamical systems.

Definition 1.27. The transformation T_β is defined by

$$T_\beta \colon [0,1) \to [0,1) : x \mapsto \beta x - \lfloor \beta x \rfloor.$$

The expansion of a number x can be interpreted in terms of the transformation T_β: if $d_\beta(x) = a_1a_2\cdots$, we have $a_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$.

Definition 1.28. The β*-shift* S_β is defined as the topological closure $\overline{D_\beta}$ of the set

$$D_\beta = \{d_\beta(x) : x \in [0,1)\}.$$

The β-shift can be seen as a combinatorial point of view on the dynamical system $([0,1), T_\beta)$, as we have $\sigma \circ \text{rep}_\beta = \text{rep}_\beta \circ T_\beta$. See e.g. [IT74].

Theorem 1.24 above can be reinterpreted in terms of these spaces, as we have

$$\mathbf{a} \in D_\beta \Leftrightarrow \forall n \in \mathbb{N},\, \sigma^n(\mathbf{a}) <_{\text{lex}} d^*_\beta(1)$$

and

$$\mathbf{a} \in S_\beta \Leftrightarrow \forall n \in \mathbb{N},\ \sigma^n(\mathbf{a}) \leq_{\text{lex}} d^*_\beta(1).$$

A similar shift space, called the *noncanonical β-shift*, was introduced more recently ([CCS22]) for reasons that will be exposed in Section 1.5. It is based on the greedy representation of 1 instead of the quasi-greedy one.

Definition 1.29. The *noncanonical β-shift S'_β* is defined as

$$S'_\beta = \{\mathbf{a} \in \{0, \dots, \lfloor\beta\rfloor\}^{\mathbb{N}} : \sigma^n(\mathbf{a}) \leq_{\text{lex}} d_\beta(1)\ \forall n \in \mathbb{N}\}.$$

Example 1.30. In the case of the decimal numeration system ($\beta = 10$), the β-shift is simply the full shift on the alphabet $\{0, \dots, 9\}$. Words ending with a suffix 9^ω are not in D_β but are obtained when taking the topological closure.

When β is the golden ratio, words in the β-shift are those whose every suffix is less than or equal to $(10)^\omega$, which are exactly the words avoiding the factor 11. This shift is therefore of finite type. By contrast, the noncanonical β-shift tolerates words containing the factor 11 provided that it is followed by 0^ω. The shift is therefore described as the set of words avoiding the factors 110^n1 for any n. It is sofic, but is not of finite type.

When β is the square of the golden ratio, the greedy and quasi-greedy expansions are equal since they are infinite. There is only one β-shift, which is the shift of all words whose suffixes are all less than or equal to 21^ω. Alternatively, this corresponds to avoiding the factors 21^n2.

Remark 1.31. We take a moment to mention the case of $\beta = 1$. This is a degenerate example but it will be relevant to Chapter 6. The only numbers that can be represented are 0 and 1. The greedy algorithm applied to $x = 1$ gives the expansion $d_\beta(1) = 10^\omega$, which was specifically excluded from Theorem 1.25. We can still define the words $\mathbf{w}_j = 0^j10^\omega$ which are representations of 1, but their limit is 0^ω which is not. Nevertheless, some results still yield valid interpretations when considering $d^*_\beta(1) = 0^\omega$. We will bring attention to this sort of matter only when it is relevant.

To close this section, it remains to explain how Rényi numeration systems can be used to represent numbers greater than 1. Such numbers will be represented by biinfinite words with a left tail of zeros, indexed with nonnegative indices to the left of the fractional point, mimicking the decimal system. We will often omit this tail of zeros when writing the words.

Definition 1.32. The evaluation map is extended to biinfinite words with a left tail of zeros by setting

$$\mathrm{val}(a_N \cdots a_0 \bullet a_{-1}a_{-2}\cdots) = \sum_{j=-N}^{\infty} \frac{a_{-j}}{\beta^j} = a_N\beta^N + \cdots + a_1\beta + a_0 + \frac{a_{-1}}{\beta} + \cdots$$

The representation map extended to numbers greater than 1 is noted $\langle\cdot\rangle_\beta$ and is defined by

$$\langle x\rangle_\beta = \begin{cases} d_\beta(x), & \text{if } x < 1;\\ \sigma^N({}^\omega 0 \bullet d_\beta(\frac{x}{\beta^N})), & \text{if } \beta^{N-1} \leq x < \beta^N \text{ for some } N \geq 1.\end{cases}$$

That is, we divide the number by a power of β until it is $[0,1)$, use the representation map defined on $[0,1)$, then shift the representation back to undo the division. Note that $\mathrm{val}(\sigma(\mathbf{a})) = \beta\mathrm{val}(\mathbf{a})$ on biinfinite words.

With this, the number 1 now has two greedy representations depending on whether we work on right-infinite or biinfinite words. This is normal as $d_\beta(1)$ is the lexicographically largest representation of 1 that does not use positions to the left of the fractional point, but if those positions are allowed, then $\langle 1\rangle_\beta = 1.0^\omega$ is of course lexicographically greater.

The properties outlined above remain true after small adaptations. In particular, the $\langle x\rangle_\beta$ for $x \geq 0$ are shifts of the $d_\beta(x)$ for $x \in [0,1)$, so the set of factors of the β-shift remains unchanged.

Example 1.33. In the case where β is the golden ratio, the representation of the number 2 is obtained by first dividing by φ^2, so that $\frac{2}{\varphi^2}$ is in $[0,1)$. We know that the expansion of this number is 10010^ω, so the expansion of 2 is obtained by shifting twice: $\langle 2\rangle_\beta = 10.010^\omega$. We find that $\varphi + \frac{1}{\varphi^2}$ is indeed equal to 2.

1.4 Dumont–Thomas numeration systems

Dumont–Thomas numeration systems were introduced in [DT89] by Dumont and Thomas, in the context of defining fractal functions. Since then, these

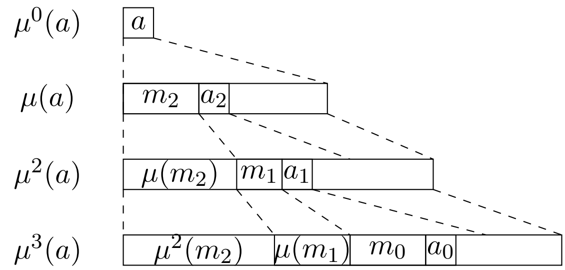


Figure 1.2: An illustration of what it means for a sequence $((m_i, a_i))_{0 \leq i \leq 2}$ to be a-admissible with respect to a substitution μ.

systems have been generalized in different ways, in [Sur20] and [MRST23], and most notably for our purposes in [LL24a] then [KLS25a]. These last extensions will be discussed later in Section 2.3. They have been studied in relation to the Thue-Morse sequence [MM24] and in the context of string attractors [GRS23, GRS24]. The exposition in this subsection is based on that of [KLS25a].

Dumont–Thomas numeration systems represent nonnegative integers with finite words. Unlike positional numeration systems, the idea is not to greedily decompose a number in a sum of terms of a base sequence. Rather, there is an underlying base *substitution* μ with a fixed point $\mathbf{u} = u_0 u_1 \cdots$. To find the representation of the number n, we decompose $\mathbf{u}_{[0,n)}$ as a product of elements $\mu^j(u_0)$, using a greedy algorithm from left to right. The representation of n is then deduced from the number of times that each $\mu^j(u_0)$ is used in this development. We first recall the original definition of Dumont and Thomas.

Definition 1.34. Let μ be a substitution on the alphabet A, with fixed point $\mathbf{u} = u_0 u_1 \cdots$. The sequence $(m_i, a_i)_{i \in \{0,\ldots,k\}} \in (A^* \times A)^{k+1}$ is *admissible* with respect to μ if $m_{i-1} a_{i-1} \in \mathrm{Pref}(\mu(a_i))$ for all i in $\{1, \ldots, k\}$. It is *a-admissible* with respect to μ if furthermore $m_k a_k$ is a prefix of $\mu(a)$.

If the context is clear, we will omit the reference to μ and simply speak of admissible or a-admissible sequences. See Figure 1.2 for an illustration.

The following theorem justifies the definition of a Dumont–Thomas numeration system. It can be found with proof in [DT89] in French, and without proof but with exposition in English in [DT91, Rig14b].

Theorem 1.35. *Let $a \in A$ and let $\mu : A^* \to A^*$ be a substitution. Let $\mathbf{u}$ be a right-infinite fixed point of μ with growing seed $u_0 = a$. For every integer $n \geq 1$, there exist a unique integer $k = k(n)$ and a unique sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that the sequence is a-admissible, $m_{k-1} \neq \varepsilon$, and $u_0 u_1 \cdots u_{n-1} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.*

Definition 1.36. Let $\mu\colon A^* \to A^*$ be a substitution, a be a growing letter, and $\mathbf{u}$ be a right-infinite fixed point of μ with seed $u_0 = a$. Set $\mathtt{c} = \max_{b \in A} |\mu(b)| - 1$ and define the set $D = \{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{c}\}$. We define the map $\operatorname{rep}_{\mu,a}\colon \mathbb{N} \to D^* : n \mapsto \operatorname{rep}_{\mu,a}(n)$ by

$$\operatorname{rep}_{\mu,a}(n) = \begin{cases} \varepsilon, & \text{if } n = 0; \\ |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \geq 1; \end{cases}$$

where $k = k(n)$ is the unique integer and $((m_i, a_i))_{i=0,\ldots,k-1}$ is the unique sequence from Theorem 1.35. This numeration system is called the *Dumont–Thomas numeration system associated with μ and $\mathbf{u}$*.

We provide an alternative, more visual interpretation of these definitions, which we borrow from [KLS25a].

With a morphism $\mu\colon A^* \to A^*$, we can associate a directed graph in the following way: the nodes are the letters of A and for every $a \in A$, if we write $\mu(a) = c_0 \cdots c_\ell$ with $c_i \in A$ for every $i \in \{0, \ldots, \ell\}$, then we draw an arrow labeled by i from a to c_i for every $i \in \{0, \ldots, \ell\}$. This directed graph encodes the images of letters under μ. Similarly, we may unfold the directed graph into a directed tree starting with some fixed vertex that is then called the *root* of the tree. Given $a \in A$, we define the tree $\mathcal{T}_{\mu,a}$ as follows: its root is labeled by a, and if a node of the tree is labeled by x and $\mu(x) = y_0 \cdots y_\ell$ then that node has $\ell + 1$ children labeled from y_0 to y_ℓ, with the edge from x to y_i being labeled by i. Observe that the k-th level of $\mathcal{T}_{\mu,a}$ stores the k-th iteration of μ on the letter a. We say that a node of the tree is *in column n* if there are n nodes of the same level to its left.

The link between the tree $\mathcal{T}_{\mu,a}$ and admissible sequences is presented below.

Proposition 1.37. *An a-admissible sequence $(m_i, a_i)_{i \in \{0,\ldots,k-1\}}$ corresponds to a path in the tree $\mathcal{T}_{\mu,a}$ of length k starting at the root, with nodes labeled*

$a, a_{k-1}, \ldots, a_0$ and edges labeled $|m_{k-1}|, \ldots, |m_0|$. This correspondence is bijective.

Proof. Given a path of length k, define $a_k = a, a_{k-1}, \ldots, a_0$ to be the visited nodes or their labels, and let m_i be the word formed by concatenating the labels of the nodes of children of a_{i+1} that stand to the left of a_i in $\mathcal{T}_{\mu,a}$. It is clear that $m_i a_i$ is a prefix of $\mu(a_{i+1})$ and the sequence is thus a-admissible. The labeling of the edges in $\mathcal{T}_{\mu,a}$ ensures that the edge between a_{i+1} and a_i is labeled $|m_i|$.

Similarly, given an a-admissible sequence, starting at the root and leaving $|m_{k-1}|$ nodes to our left, we use an edge labeled $|m_{k-1}|$ to reach a node labeled a_{k-1}. Iterating this process, we construct the desired path.

The bijectivity is clear. □

Using this property and seeing that paths labeled w and $\mathtt{0}w$ in $\mathcal{T}_{\mu,a}$ lead to the same column, we provide a characterization of Dumont–Thomas expansions.

Proposition 1.38. *Let μ be a substitution with a fixed point $\mathbf{u}$ starting with a. The expansion of n in the associated Dumont–Thomas numeration system is the label of a shortest path from the root to column n in $\mathcal{T}_{\mu,a}$.*

Example 1.39. The *Fibonacci substitution* ϕ is defined by

$$\phi\colon \begin{cases} a & \mapsto ab \\ b & \mapsto a \end{cases}$$

Figure 1.3 displays the directed graph associated with the substitution ϕ and the initial segment of the associated tree. We see for instance that the sequence defined by

$$(m_0, a_0) = (\varepsilon, a),\ (m_1, a_1) = (\varepsilon, a),\ (m_2, a_2) = (a, b)$$

is a-admissible. We then have

$$\mu^2(m_2)\mu^1(m_1)\mu^0(m_0) = \mu^2(a) = aba$$

which is equal to $u_0u_1u_2$. It follows that the expansion of 3 in the associated Dumont–Thomas numeration system is $|m_2| \cdot |m_1| \cdot |m_0| = \mathtt{100}$. Note that if we append $(m_3, a_3) = (\varepsilon, a)$ to the sequence, it is still a-admissible but is not suitable for the Dumont–Thomas numeration system due to the condition $m_{k-1} \neq \varepsilon$ in Theorem 1.35.

This can be verified within the context of Proposition 1.38. The paths labeled by `100` and `0100` both end in column 3, but the former is the shortest one. We see similarly that $\mathbf{u}_{[0,6)}$ is factorized as $\mu^3(a)\mu^2(\varepsilon)\mu^1(\varepsilon)\mu^0(a)$, leading to the representation `1001`. Indeed, the path labeled by `1001` is the shortest that goes from the root to column 6.

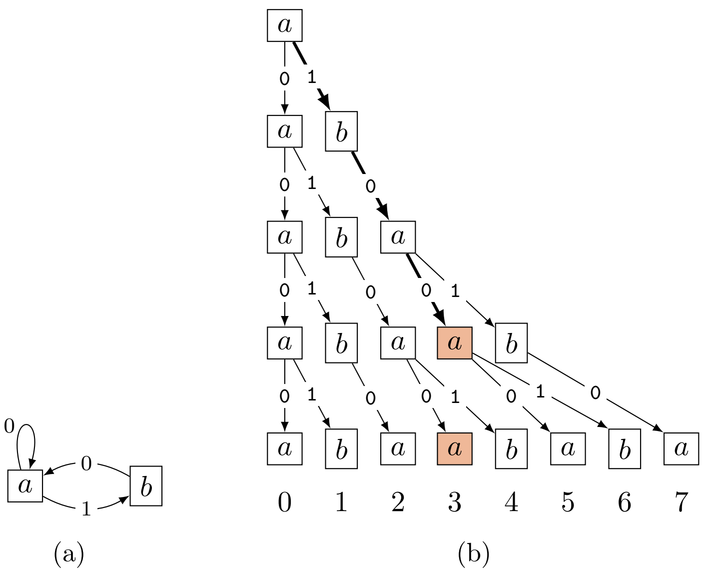


Figure 1.3: On the left, the directed graph associated with the Fibonacci substitution $\phi\colon a \mapsto ab, b \mapsto a$. On the right, the tree $\mathcal{T}_{\phi,a}$ displays all a-admissible sequences of length at most 4 for the Fibonacci substitution ϕ and growing letter a.

The square of the Fibonacci substitution is $\phi^2\colon a \mapsto aba,\ b \mapsto ab$. An initial segment of the corresponding tree is located on Figure 1.4. We can note that $\mathrm{rep}(3) =$ `10` and $\mathrm{rep}(5) =$ `20`. An important consequence is that this system is not a positional system like those of Section 1.2, as the value of `20` would be twice that of `10` in this case.

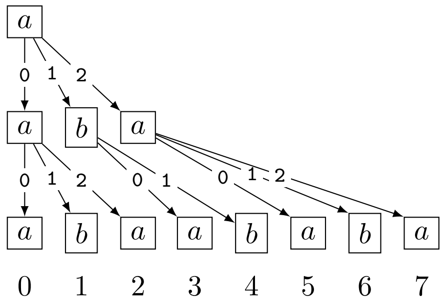


Figure 1.4: The first levels of the tree associated with the substitution ϕ^2.

1.5 Interplay between families of numeration systems

In this section we hint at the larger picture linking the three numeration systems mentioned above. The reader will have noticed that there are similarities between results on positional and Rényi numeration systems (for instance, Lemma 1.11 and Theorem 1.24). This is due to both those systems using a greedy algorithm, but the connections do not stop there.

We will only consider a fraction of all positional numeration systems.

Definition 1.40. The positional numeration system based on the sequence $(U_n)_{n\in\mathbb{N}}$ is said to have a *dominant root* if the limit

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n}$$

exists. In this case, the value of this limit is called the dominant root of the system.

The following critical result by Hollander links positional numeration systems with a dominant root and Rényi numeration systems.

Theorem 1.41 ([Hol98, Section 4])**.** *Let the numeration system U have dominant root β and call $\mathbf{w}_j$ ($j \in \mathbb{N}$ if $d_\beta(1)$ is finite, otherwise $j = 0$) the intermediate representations of 1 in base β. Also let K be a nonzero natural number and let L be a target length ($L \in \mathbb{N}$). For all n larger than some bound $N(K, L)$ and for all $1 \leq k \leq K$, the prefix of length L of $\mathrm{rep}_U(U_n - k)$ coincides with the prefix of length L of some $\mathbf{w}_j$ (where j may depend on n).*

Example 1.42. The Zeckendorf numeration system of Example 1.7 has the golden ratio as a dominant root. We have seen in Example 1.13 that the representation of $U_n - 1$ in this system is $\mathrm{Pref}_n((10)^\omega)$. Finally, in Example 1.26, we have seen that $(10)^\omega$ was precisely the word $d^*_\varphi(1)$, which we know is $\lim_{j\to\infty} \mathbf{w}_j$. In fact, since $\mathbf{w}_j = (10)^j 110^\omega$, we can choose $N(1, L) = N$ and $j = \lceil \frac{n}{2} \rceil$ and we will have the desired result. For $k \geq 1$, we no longer have exact matches to some $\mathbf{w}_j$, but it is not too hard to see that the common prefix can be taken arbitrarily large.

The convergence is not always as impressive as in the previous example. If we consider the positional numeration system given by the linear recurrence sequence

$$U_{n+4} = 2U_{n+3} + U_{n+2} - 2U_{n+1} - U_n$$

and the initial conditions $(1, 2, 3, 6)$, this system also admits the golden ratio as a dominant root. We see that the prefix of $\mathrm{rep}_U(U_n - 1)$ matches a prefix of $\mathbf{w}_0 = 110^\omega$, but the convergence is quite slow. For instance, to see 10 correct digits, we must already go to the expansion of $U_{172} - 1$.

We can see with a variant of the Zeckendorf system that the choice of intermediate expansion is sometimes more fluctuating. If we consider the system given by

$$U_n = U_{n-1} + 2U_{n-2} - U_{n-3} - U_{n-4}$$

with the initial conditions $(1, 2, 4, 6)$, we can see that $\mathrm{rep}_U(U_n - 1)$ is equal to $\mathrm{Pref}_n(110^\omega)$ when n is even, and to $\mathrm{Pref}_n((10)^\omega)$ when n is odd.

The Fina numeration system admits the square of the golden ratio as a dominant root. In this case, the expansion $d_{\varphi^2}(1)$ is infinite, so there are no intermediate expansions. Therefore, the words $\mathrm{rep}_U(U_n - 1)$ must share arbitrarily long prefixes with (and thus converge to) $d_{\varphi^2}(1) = 21^\omega$, which we see is the case.

The prefixes of the maximal words in the U-system partly match the prefixes of some representations of 1 in the β-system. Note that multiple U-systems correspond to the same β-system, due to a wide variety of choices for which representations match, for how long they match, and what happens after the two diverge.

A particular case of notable interest is when the representations of $U_n - 1$ coincide exactly and entirely with prefixes of $d_\beta(1)$, or with prefixes of $d^*_\beta(1)$.

Due to Lemma 1.12 (choosing $M = \mathrm{Pref}(d_\beta(1))$, resp. $M = \mathrm{Pref}(d^*_\beta(1))$, both of which evidently satisfy the three required conditions), only one such system exists for each of the two words. In a sense, the systems obtained (one when $d_\beta(1)$ is infinite, two when it is finite) are the U-systems that most closely correspond to the specific β-system among all U-systems with dominant root β. They are called *Bertrand numeration systems*, and we will now detail some of their properties. They bear the name of Bertrand-Mathis who was the first to study them extensively [BM89], with a later article by Charlier, Cisternino, and Stipulanti [CCS22] completing Bertrand-Mathis's result. This latter article guides our presentation.

Lemma 1.43 ([CCS22, Lemma 3])**.** *Consider a numeration system U. The two following properties are equivalent.*

- *For all words $w \in A_U^*$, $w \in L_U \Leftrightarrow w0 \in L_U$.*
- *There exists an infinite word $a \in A_U^\omega$ such that* $\mathrm{rep}_U(U_n - 1) = \mathrm{Pref}_n(\mathbf{a})$ *for all $n \in \mathbb{N}$.*

In this case, the word $\mathbf{a}$ *satisfies* $\sigma^n(\mathbf{a}) \leq_{\mathrm{lex}} \mathbf{a}$ *for all n.*

Definition 1.44. A positional numeration system is called *Bertrand* if it satisfies one of the above two conditions.

Do note that this definition matches the motivating statement given in the above paragraph. The set $\cup_{\beta>1}\{d_\beta(1), d^*_\beta(1)\}$ is exactly the set of infinite words $\mathbf{a}$ such that $\sigma^n(\mathbf{a}) \leq_{\mathrm{lex}} \mathbf{a}$ for all n, minus the two exceptional words 10^ω and 0^ω([CCS22, Lemma 5]). Thus, up to the exceptional system based on the sequence $U_i = i + 1$, corresponding to $\mathbf{a} = 10^\omega$, the systems that were announced as Bertrand above are exactly those that are Bertrand according to this definition.

Theorem 1.45 ([CCS22, Theorem 2])**.** *A positional numeration system U is Bertrand if and only if one of the three following possibilities occur:*

- *The system is given by $U_i = i + 1$ for all i in $\mathbb{N}$.*
- *The system is given by $U_i = 1 + \sum_{j=1}^{i} a_j U_{i-j}$ for all $i \in \mathbb{N}$, with $a_1 a_2 \cdots = d^*_\beta(1)$. In this case, the system has the dominant root β,*

$\mathrm{Max}(L_U) = \mathrm{Pref}(d^*_\beta(1))$ *and* $L_U = \mathrm{Fac}(S_\beta)$ *where* S_β *is the* β*-shift from Definition 1.28.*

- *The system is given by* $U_i = 1 + \sum_{j=1}^{i} a_j U_{i-j}$ *for all* $i \in \mathbb{N}$*, with* $a_1 a_2 \cdots = d_\beta(1)$*. In this case, the system has the dominant root* β*,* $\mathrm{Max}(L_U) = \mathrm{Pref}(d_\beta(1))$ *and* $L_U = \mathrm{Fac}(S'_\beta)$ *where* S'_β *is the noncanonical* β*-shift from Definition 1.29.*

Numeration systems of the second family are named *canonical* Bertrand numeration systems, whereas those of the third family are named *noncanonical* Bertrand numeration systems.

Example 1.46. The Zeckendorf and Fina numeration systems are the canonical Bertrand numeration systems associated with φ and φ^2 respectively. The latter number does not have a corresponding noncanonical Bertrand numeration system since its expansion of 1 is infinite, but the former does. The noncanonical Bertrand numeration system associated with the golden ratio is the one that is based on the recurrence sequence

$$U_{n+2} = U_{n+1} + U_n + 1 \text{ with initial conditions } (1, 2).$$

For this system, we have $\mathrm{rep}_U(U_n - 1) = \mathrm{Pref}_n(110^\omega)$.

Thus all U-systems with a dominant root β share a light connection to the β-system, and one or two among them share a much stronger connection. To now highlight the link to Dumont–Thomas numeration systems, we must make a small detour through automata theory and restrict ourselves to a large family of well-behaved numbers.

Definition 1.47. A real number $\beta > 1$ is a *Parry number* if $d_\beta(1)$ is finite or eventually periodic. In the first case, β is called a *simple Parry number.*

Remark 1.48. We make an aside on terminology here. In the context of expansions of 1, we will use *eventually periodic* to *exclude* finite expansions (that end in 0^ω). Thus, a Parry number β is simple if and only if $d_\beta(1)$ is finite and nonsimple if and only if $d_\beta(1)$ is eventually periodic. By contrast, we will use *ultimately periodic* to include sequences that are ultimately zero as well.

If β is a Parry number with $d^*_\beta(1) = (d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega$, the set of factors of the β-shift, and therefore the set L_U where U is a canonical Bertrand numeration system, is recognized by a finite automaton that we reproduce on Figure 1.5. Similarly, if β is a simple Parry number with $d_\beta(1) = t_1 \cdots t_\ell 0^\omega$, the set of factors of the noncanonical β-shift, and the language L_U where U is the noncanonical Bertrand numeration system associated with β, is also recognized by a finite automaton (Figure 1.6). Those automata can be found in passing in many places in the literature ([Fab95, Section 2.1], [Bas02, Figure 1], [CCS22, Figures 1 and 2]). We give a detailed construction for reference.

Definition 1.49. Assume that β is a Parry number with the corresponding expansion $d^*_\beta(1) = (d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega$. Define the automaton $\mathcal{A}_\beta$ as follows. The states are $1, \ldots, q+m$. From state $i < q+m$, there is a transition to state $i+1$ with label d_i and to state 1 with labels 0 up to $d_i - 1$. From state $q+m$, there is a transition to state $q+1$ with label d_{q+m} and to state 1 with labels 0 up to $d_{q+m} - 1$. All states are final and the state 1 is initial. Transitions that are not mentioned are implicitly assumed to go to a nonfinal sink state.

Assume now that β is a simple Parry number with $d_\beta(1) = t_1 \cdots t_\ell 0^\omega$. We define the automaton $\mathcal{A}'_\beta$ in a similar fashion, with states $1, \ldots, \ell+1$. From state $i \leq \ell$, there is a transition to state $i+1$ with label t_i and to state 1 with labels 0 up to $t_i - 1$. From state $\ell+1$, there is a self-loop with label 0.

Note that the construction of $\mathcal{A}_\beta$ can be seen as associated with the infinite word $(d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega$ rather than to β. The automaton $\mathcal{A}'_\beta$ can then be seen as associated with the word $(t_1 \cdots t_\ell)(0)^\omega$.

Proposition 1.50. *The language accepted by the automaton $\mathcal{A}_\beta$ is exactly the set of factors of S_β.*

The language accepted by the automaton $\mathcal{A}'_\beta$ is exactly the set of factors of S'_β.

Proof. Recall that S_β is the set $\{\mathbf{a} : \sigma^n(\mathbf{a}) \leq_{\text{lex}} d^*_\beta(1)\ \forall n \in \mathbb{N}\}$ and thus $\text{Fac}(S_\beta) = \{w : \sigma^n(w0^\omega) \leq_{\text{lex}} d^*_\beta(1)\ \forall n \in \mathbb{N}\}$. Recall also that $\sigma^n(d^*_\beta(1)) \leq_{\text{lex}} d^*_\beta(1)$ for all n (from [Par60] or [Lot02, Corollary 7.2.10]).

Now, if w is not accepted by $\mathcal{A}_\beta$, then the factor ranging from the last

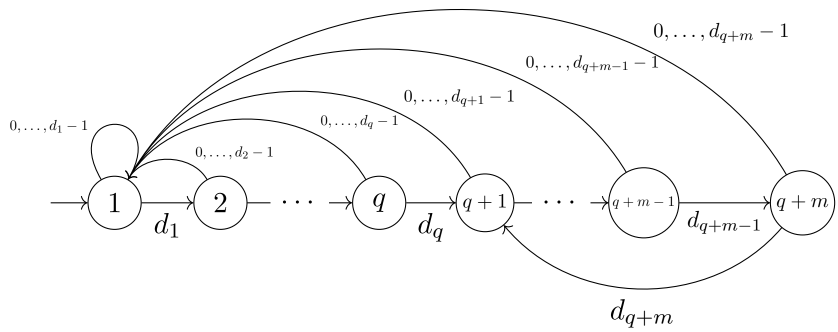


Figure 1.5: Automaton accepting all factors of words in the β-shift S_β when β is a Parry number, with $d^*_\beta(1) = (d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega$

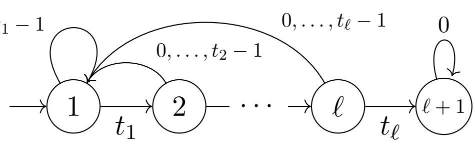


Figure 1.6: Automaton accepting all factors of words in the noncanonical β-shift S'_β when β is a simple Parry number, with $d_\beta(1) = t_1 t_2 \cdots t_\ell 0^\omega$

visit to state 1 to the end of the computation is clearly lexicographically greater than $d^*_\beta(1)$. Conversely, if w contains a factor u greater than $d^*_\beta(1)$ and $\mathcal{A}_\beta$ starts reading this factor in state i, then we have $u >_{\text{lex}} d^*_\beta(1) \geq_{\text{lex}} \sigma^{i-1}(d^*_\beta(1))$, and thus w cannot be accepted by $\mathcal{A}_\beta$ since $\sigma^{i-1}(d^*_\beta(1))$ is the lexicographical supremum of words accepted by $\mathcal{A}_\beta$ when started on state i.

The same proof works for the second part of the statement. □

It is known that automata reading base-k representations of numbers are linked to fixed points of k-uniform morphisms [Cob72]. This results extends to more general numeration systems [Sha88, Rig00]. Thus, from the above-mentioned automata we can deduce some associated substitutions.

Definition 1.51. If β is a Parry number with

$$d^*_\beta(1) = (d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega,$$

define μ_β by

$$\mu_\beta \colon \begin{cases} 1 & \mapsto 1^{d_1} 2 \\ 2 & \mapsto 1^{d_2} 3 \\ & \vdots \\ q+m-1 & \mapsto 1^{d_{q+m-1}}(q+m) \\ q+m & \mapsto 1^{d_{q+m}}(q+1). \end{cases}$$

If β is a simple Parry number with $d_\beta(1) = t_1 \cdots t_\ell 0^\omega$, define μ'_β by

$$\mu'_\beta \colon \begin{cases} 1 & \mapsto 1^{t_1} 2 \\ 2 & \mapsto 1^{t_2} 3 \\ & \vdots \\ \ell & \mapsto 1^{t_\ell}(\ell+1) \\ \ell+1 & \mapsto (\ell+1). \end{cases}$$

The substitution μ_β corresponds to the substitutions introduced by Fabre [Fab95], with the case where β is a simple Parry number introduced slightly differently. The substitution μ'_β was mentioned in [KLS25b], which will be Chapter 7 in this thesis. From a historical perspective, Fabre focused his work on the Bertrand numeration system that was known at the time, which is the canonical Bertrand numeration system, introduced in [BM89]. The noncanonical Bertrand numeration systems were introduced much later in [CCS22] and while the extension of Fabre's results to them is not difficult, Fabre himself of course could not do it.

Finally, we can link Bertrand and Dumont–Thomas numeration systems. We say that two numeration systems are *equal* if every number has the same expansion in both systems.

Theorem 1.52 ([Fab95, Theorem 2], [KLS25b, Propositions 3.19 and 3.21])**.** *The Dumont–Thomas numeration system associated with the substitution* μ_β *and its fixed point starting with* 1 *is equal to the canonical Bertrand numeration system associated with* β*.*

The Dumont–Thomas numeration system associated with the substitution μ'_β *and its fixed point starting with* 1 *is equal to the noncanonical Bertrand numeration system associated with* β*.*

Example 1.53. When β is the golden ratio, the automaton $\mathcal{A}_\beta$ is the one depicted on Figure 1.3a, with a as its starting state. The substitution μ_β is the Fibonacci substitution ϕ that was introduced in Example 1.39 (up to a renaming of letters) and the associated Dumont–Thomas numeration system is therefore the canonical Bertrand numeration associated with β, which we have seen is the Zeckendorf numeration system.

The substitution μ'_β is given by

$$\mu'_\beta \colon \begin{cases} 1 & \mapsto 12 \\ 2 & \mapsto 13 \\ 3 & \mapsto 3 \end{cases}$$

and the associated Dumont-Thomas numeration system is the noncanonical Bertrand numeration system introduced in Example 1.46.

When β is the square of the golden ratio, the substitution μ_β is *not* the substitution ϕ^2. Indeed, we have seen in Example 1.39 that the associated Dumont–Thomas numeration system was not positional. Rather, the substitution μ_{φ^2} is given by $1 \mapsto 112$ and $2 \mapsto 12$.

To conclude this section, we can mention another object related to β and its links to the above-mentioned concepts.

Definition 1.54. If β is a simple Parry number with $d_\beta(1) = t_1 \cdots t_\ell 0^\omega$ and $t_\ell \neq 0$, the *canonical β-polynomial* is the polynomial $P(X) = X^\ell - \sum_{k=1}^{\ell} t_k X^{\ell-k}$. If β is a nonsimple Parry number with the expansion $d^*_\beta(1) = (d_1 d_2 \cdots d_q)(d_{q+1} \cdots d_{q+m})^\omega$, we can define the *extended β-polynomial*

$$P_{q,m} = \left(X^{q+m} - \sum_{i=1}^{q+m} t_i X^{q+m-i} \right) - \left(X^q - \sum_{i=1}^{q} t_i X^{q-i} \right).$$

The term *extended* in the nonsimple case is due to the fact that the preperiod and period can be artificially increased. The *canonical* β-polynomial in this case corresponds to the minimal choices for the preperiod and period, and it can be seen that all the extended β-polynomials are multiples of the canonical one in ways that relate to the choice of the preperiod and period ([Hol98, Section 7]). Indeed, if the minimal preperiod and period are q_0 and m_0 respectively, then the possible other choices are to choose $q \geq q_0$ for the preperiod and $m = km_0$ for the period. We then have $P_{q,m} = X^{q-q_0}(1 + X^{m_0} + \ldots + X^{(k-1)m_0}) P_{q_0,m_0}$. By analogy, we may define an extended β-polynomial in the simple Parry case as any product of the canonical one by some polynomial of the form $X^m(1 + X^\ell + \ldots + X^{(k-1)\ell})$ for some $m \geq 0$, $k \geq 1$. This family of polynomials is linked to the rest of the previously mentioned objects.

Proposition 1.55 ([Hol98, Lemma 7.1])**.** *Let P be an extended or canonical β-polynomial. Then β is a simple root of P and a dominant root of P (in the sense of having the maximal modulus among roots of P and the maximal multiplicity among roots of this modulus). Moreover, if U is a linear recurrence sequence whose minimal polynomial is P, then U has the dominant root β.*

Proposition 1.56. *The automaton $\mathcal{A}_\beta$ and the substitution μ_β have the same adjacency matrix. The characteristic polynomial of this matrix is the corresponding β-polynomial (up to a possible multiplication by -1).*

Proof. The first part of the statement is direct. The second part is proved first in the simple Parry case by induction on the dimension of the matrix, then directly in the nonsimple case. □

Example 1.57. The golden ratio has an expansion of 1 finite and equal to 110^ω, so the canonical φ-polynomial is $X^2 - X - 1$, which is indeed the polynomial directing the recurrence underlying the Zeckendorf numeration system. One can verify that this polynomial is the characteristic polynomial of the substitution μ_φ.

The noncanonical Bertrand numeration system associated with φ has the minimal polynomial $(X^2-X-1)(X-1)$, which is an extended φ-polynomial.

Finally, the square of the golden ratio is a nonsimple Parry number with $d_\beta(1) = 21^\omega$, so the canonical φ^2-polynomial is given by

$$(X^2 - 2X - 1) - (X - 2) = X^2 - 3X + 1,$$

which one can verify is the polynomial directing the recurrence of the Fina numeration system.

Chapter 2

Alternation in numeration systems

In Chapter 1, we have introduced three families of numeration systems and explained how they are connected. In particular, Theorem 1.41 links Rényi numeration systems to positional numeration systems with a dominant root. This result was central to Hollander's study of positional numeration systems with a dominant root that have a regular language [Hol98]. However, not all positional numeration systems with a regular language have a dominant root! To completely study regularity in the context of positional numeration systems, we must develop an equivalent to this theorem that goes beyond dominant roots, and thus develop an extension of Rényi numeration systems to accommodate for this. This is the insight that led Charlier and Cisternino to define alternate bases in their article [CC21]. A couple months before, Caalim and Demegillo had defined a very similar system, motivated by the study of rotational β-transformations on $[0, 1) \times [0, 1)$ [CD20].

In this chapter, we will define alternate base expansions and describe how they are the correct generalization of Rényi numeration systems for the purpose of extending Theorem 1.41. We will also explain how Dumont–Thomas numeration systems can be naturally extended to fit into this framework. In doing so, we introduce the objects that will be studied during the rest of this thesis. Many of the properties in this chapter will be extensions of properties found in Chapter 1, and we will detail how the proofs change and what difficulties are raised by the generalization we undertake.

Contents

2.1 Alternate base numeration systems **42**

2.2 Back to positional numeration systems **51**

2.3 Extensions of Dumont–Thomas numeration systems . . . **59**

2.3.1 A glimpse at a bigger picture 67

2.1 Alternate base numeration systems

As mentioned above, alternate base numeration systems are an extension of Rényi numeration systems, introduced in slightly different forms by Caalim and Demegillo [CD20] and by Charlier and Cisternino [CC21]. The generalization can be pushed further, leading to Cantor real bases, which are also a generalization of Cantor bases [Can69], where the elements of the base have to be integers. In what follows, the focus will be on alternate bases due to their connection to U-systems without a dominant root evidenced by Propositions 2.24 and 2.30. Following their inception, alternate bases were studied extensively by Charlier and coauthors, generalizing many results known for Rényi numeration systems and developing new theory, for instance relating to the dynamical behavior of the system [CCD23], the ultimate periodicity of expansions [CCK24], the normalization problem [CCMP23], and the link to substitutions [CCMP25]. See also [Cha23] for a detailed survey. Our exposition is mostly guided by the original article [CC21].

In Rényi numeration systems, the same base β is used at every step of the greedy algorithm or, in the dynamical framework, the same transformation T_β is applied iteratively. Alternate bases extend this by introducing a finite number p of bases and periodically alternating between them. We start by defining the representation map on the interval $[0, 1]$. Numbers greater than 1 can also be represented, which we will explain later.

Definition 2.1. An *alternate base* $\mathcal{B}$ is given by a p-tuple of real numbers greater than or equal to 1, called *bases* or *base elements*. We note $\mathcal{B} = (\beta_{p-1}, \dots, \beta_0)$, then extend the base to a periodic biinfinite sequence by setting $\beta_{n+p} = \beta_n$ for all $n \in \mathbb{Z}$. We let δ be equal to the product $\beta_{p-1} \cdots \beta_0$.

The evaluation map is given by

$$\mathrm{val}\colon a_1a_2\cdots \mapsto \sum_{j=1}^{\infty} \frac{a_j}{\beta_{-1}\beta_{-2}\cdots\beta_{-j}}$$

provided that this sum converges, which is always the case when the word $a_1a_2\cdots$ is a word on a finite alphabet and one of the β_i is strictly greater than 1.

A number x in $[0,1]$ is represented by a greedy algorithm as follows. First, set $r_0 = x$, then recursively set

$$a_{i+1} = \lfloor \beta_{-i-1} r_i \rfloor \quad \text{and} \quad r_{i+1} = \beta_{-i-1} r_i - a_{i+1}.$$

The word $a_1a_2\cdots$ obtained this way is called the *$\mathcal{B}$-representation of x* or *$\mathcal{B}$-expansion of x*.

Using "*the* expansion" or "*the* representation of x" always refers to the one obtained by the greedy algorithm, which is unique and which corresponds to $\mathrm{rep}(x)$ where rep is the map discussed in Section 1.1.3. A word $\mathbf{a}$ which is such that $\mathrm{val}(\mathbf{a}) = x$ will be called *a* representation of x, with an indefinite article.

Remark 2.2. The choice of indexing the bases with negative numbers, which might seem unnatural at first, is guided by links to other numeration systems that will be found below. In particular, the current system can be adapted to represent nonnegative numbers that might be greater than 1 using biinfinite words (Definition 2.19), in which case indexing the bases decreasingly from left to right will make sense. The connection to U-systems (Propositions 2.24 and 2.30), where the greedy algorithm naturally reads the base elements in decreasing order, also supports this choice.

Remark 2.3. In general, when we write $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, we assume that p is the period of the biinfinite word $(\beta_n)_{n\in\mathbb{Z}}$, which corresponds to choosing p minimal in a sense.

Remark 2.4. In the definition, we allow the base elements to be equal to 1, as this will be relevant when establishing a connection to U-systems (see (3) in Example 2.28). In the case where the base itself is the object of study, it is more common to exclude this possibility and ask that the base elements be strictly greater than 1. The case where all the elements of the base are equal to 1 is particularly pathologic. Seeing the previous remark, this case can be seen as a Rényi system rather than an alternate base. The discussion of Remark 1.31 applies, and we will consider $d_{\mathcal{B}}(1) = 10^\omega$ in this case.

Considering the greedy algorithm outlined above and assuming that the base is not the constant sequence 1, we find that

$$x = \frac{a_1}{\beta_{-1}} + \cdots + \frac{a_j}{\beta_{-1}\cdots\beta_{-j}} + \frac{r_j}{\beta_{-1}\cdots\beta_{-j}}$$

for all $j \geq 1$. Since $r_j \in [0,1)$ for all $j \geq 1$ and $\prod_{i=1}^{j} \beta_{-i} \to \infty$ as $j \to \infty$, we conclude that $d_{\mathcal{B}}(x)$ is indeed a representation of x. We also note that a_i is always in the set $\{0, \ldots, \lfloor \beta_{-i} \rfloor\}$. The alphabet of the numeration is thus $\{0, \ldots, \sup_{i\in\{0,\ldots,p-1\}} \lfloor \beta_i \rfloor\}$, although the entire alphabet may not be seen at all positions. This alphabet will be noted $A_{\mathcal{B}}$.

Example 2.5. Consider the alternate base $\mathcal{B}$ given by $p = 2$ and $(\beta_1, \beta_0) = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$. The computation of $d_{\mathcal{B}}(1)$ using the greedy algorithm is summarized in the following table.

i	r_i	$\beta_{-i-1} r_i$	a_{i+1}
0	1	$\frac{1+\sqrt{13}}{2} \cdot 1 \simeq 2.302$	2
1	$\frac{-3+\sqrt{13}}{2}$	$\frac{5+\sqrt{13}}{6} \cdot \frac{-3+\sqrt{13}}{2} = \frac{-1+\sqrt{13}}{6} \simeq 0.434$	0
2	$\frac{-1+\sqrt{13}}{6}$	$\frac{1+\sqrt{13}}{2} \cdot \frac{-1+\sqrt{13}}{6} = 1$	1
3	0	$\frac{5+\sqrt{13}}{6} \cdot 0 = 0$	0

Afterwards, the algorithm only produces zeros, and the $\mathcal{B}$-expansion of 1 is thus $d_{\mathcal{B}}(1) = 2010^\omega$.

Consider now the alternate base $\mathcal{B} = (\varphi, 3, \varphi)$ where φ is the golden ratio. Representing 1, we successively find $a_1 = 1$ and $r_1 = \varphi - 1$, then $a_2 = \lfloor 3\varphi - 3 \rfloor = 1$ and $r_2 = 3\varphi - 4$, then $a_3 = \lfloor 3 - \varphi \rfloor = 1$ and $r_3 = 2 - \varphi$, then finally $a_4 = 0$ and $r_4 = \varphi - 1$. Since $r_4 = r_1$, the algorithm loops with a period of 3 and the representation of 1 is given by $1(110)^\omega$. On the other hand, the representations of 1 in the bases $(3, \varphi, \varphi)$ and $(\varphi, \varphi, 3)$ are 30^ω and 110^ω respectively.

Let us mention a few basic properties of the greedy representation algorithm. Compare to Propositions 1.19 and 1.20 in Section 1.3.

Proposition 2.6 ([CC21, Lemma 9])**.** *Given an alternate base $\mathcal{B}$ and a word $\mathbf{a} \in \mathbb{N}^{\mathbb{N}_0}$, $\mathbf{a}$ is the $\mathcal{B}$-expansion of a number $x \in [0,1]$ if and only if* $\operatorname{val}(\mathbf{a}) = x$ *and*

$$\sum_{j=J+1}^{+\infty} \frac{a_j}{\prod_{i=1}^{j} \beta_{-i}} < \frac{1}{\prod_{i=1}^{J} \beta_{-i}}$$

for all $J \geq 1$.

Proposition 2.7 ([CC21, Proposition 12])**.** *The $\mathcal{B}$-expansion of $x \in [0,1]$ is lexicographically maximal among all $\mathcal{B}$-representations of x.*

Proposition 2.8 ([CC21, Proposition 13])**.** *The representation map*

$$d_{\mathcal{B}} \colon [0,1] \to A_{\mathcal{B}}^{\mathbb{N}_0}$$

is increasing.

The question of the existence and uniqueness of alternate bases with given representations of 1 is trickier than in the Rényi case. There, Theorem 1.25 provides all the information we need. In the alternate case, it is still clear that an alternate base exists where the given word is a representation of 1, as we can simply take an alternate base that is constant, or use [CC21, Proposition 3] which gives an even stronger result. However, the alternate base is no longer unique. For the simplest example, 110^ω is the representation of 1 in the alternate bases (φ, φ), $(3/2, 2)$ or $(\sqrt{2}, \sqrt{2}+1)$. Intuitively, it is clear that one equation is not enough to uniquely identify a p-tuple of real numbers. For this, we need p equations. In our case, this will amount to setting the desired expansion of 1 in the base $\mathcal{B}$, but also in the bases $(\beta_{p-2}, \ldots, \beta_1, \beta_0, \beta_{p-1}), \ldots,$ and $(\beta_0, \beta_{p-1}, \ldots, \beta_1)$. However, it is now unclear whether a base with such expansions of 1 exists. We will come back to this problem in Chapter 3. In the meantime, this question is one motivation for the introduction of *shifts* of the base $\mathcal{B}$. As another such motivation is the generalization of the quasi-greedy algorithm to the alternate case, we introduce both notions in turn.

Definition 2.9. The *shift* map σ can also be applied to alternate bases. If $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, we set

$$\sigma(\mathcal{B}) = (\beta_{p-2}, \ldots, \beta_0, \beta_{p-1}).$$

The shift map can be iterated and inverted, and we of course have $\sigma^i(\mathcal{B}) = (\beta_{p-1-i}, \ldots, \beta_{-i})$, remembering that $\beta_{n+p} = \beta_n$ for all n.

Note that in particular, we have $\mathrm{val}_{\sigma(\mathcal{B})}(\sigma(\mathbf{a})) = \beta_{-1}\mathrm{val}_{\mathcal{B}}(\mathbf{a}) - a_1$.

The quasi-greedy algorithm works just like in the Rényi case, subtracting 1 from the last digit of a finite expansion of 1 and replacing it by a copy of the representation of 1 at this position. Contrary to the Rényi case however, the correct representation of 1 depends on the position (more precisely, on its congruence modulo p). We describe the greedy algorithm formally and seize the opportunity to introduce notation.

Definition 2.10. For all i, we let $\mathbf{d}_i = d_{\sigma^{-i}(\mathcal{B})}(1)$. It is the expansion of 1 in the base $(\beta_{i-1}, \ldots, \beta_{i-p})$. We let the letters of $\mathbf{d}_i$ be given by $\mathbf{d}_i = t_{i,1}t_{i,2}\cdots$. We therefore have $1 = \sum_{j=1}^{\infty} \frac{t_{i,j}}{\beta_{i-1}\cdots\beta_{i-j}}$.

If $\mathbf{d}_i$ is finite, we let ℓ_i be its length (up to its last nonzero digit) and we write $d_i = t_{i,1}t_{i,2}\cdots t_{i,\ell_i}$ its finite prefix (with $t_{i,\ell_i} \neq 0$ and $\mathbf{d}_i = d_i 0^\omega$). We also set $d_i' = t_{i,1}t_{i,2}\cdots t_{i,\ell_i-1}(t_{i,\ell_i} - 1)$. Still in this case, we define $\mu(i) = (i - \ell_i) \bmod p$. We can now state one step of the quasi-greedy algorithm: we have

$$1 = \mathrm{val}_{\sigma^{-i}(\mathcal{B})}(d_i'\mathbf{d}_{i-\ell_i}) = \mathrm{val}_{\sigma^{-i}(\mathcal{B})}(d_i'\mathbf{d}_{\mu(i)}).$$

Definition 2.11. The *quasi-greedy representation of* 1 *in base* $\sigma^{-i}(\mathcal{B})$ is noted $d^*_{\sigma^{-i}(\mathcal{B})}(1)$ or $\mathbf{d}_i^*$ and is defined recursively by

$$\mathbf{d}_i^* = \begin{cases} \mathbf{d}_i & \text{if } \mathbf{d}_i \text{ is infinite;} \\ d_i'\mathbf{d}^*_{\mu(i)} & \text{if } \mathbf{d}_i \text{ is finite.} \end{cases}$$

The letters of this expansion are written $\mathbf{d}_i^* = d_{i,1}d_{i,2}\cdots$.

Note that since μ is a map from $\{0, \ldots, p-1\}$ to itself, it takes at most p recursive calls before the algorithm either reaches an infinite expansion, or reaches an already seen expansion, in which case the expansions seen by the algorithm can be deduced to be ultimately periodic.

We introduce some more definitions to better keep track of the quasi-greedy algorithm. These notions were introduced in [CK25], which will be Chapter 6 in this thesis.

Definition 2.12. We define $k_{i,j} = \ell_i + \ell_{\mu(i)} + \ldots + \ell_{\mu^{j-1}(i)}$ provided that the lengths involved are all finite. It is the cumulative sum of the lengths of the first j expansions seen in the quasi-greedy algorithm when computing $\mathbf{d}_i^*$.

When $k_{i,j}$ is defined, we define $m_{i,j}$ by $m_{i,j}p = \mu^j(i) - i + k_{i,j}$. This auxiliary quantity will be used in Chapter 6.

We define $\mathbf{w}_{i,j} = d'_i d'_{\mu(i)} \cdots d'_{\mu^{j-1}(i)} \mathbf{d}_{\mu^j(i)}$, the j-th *intermediate representation* of 1, provided that at least j finite expansions are seen before an infinite one in the quasi-greedy algorithm.

Finally, we define the graph G to be the oriented graph whose vertices are $\{0, \ldots, p-1\}$, where there is an arc from i to $\mu(i) = (i - \ell_i) \bmod p$ if $\mathbf{d}_i$ is finite and no arc from i otherwise.

Let us make a few remarks regarding these definitions. First, the definition of the graph G allows us to sort the congruence classes modulo p (identified with the vertices of G) into four categories:

1. Vertices i with no successor.
2. Vertices i leading to a vertex with no successor, i.e., such that there exists $r \geq 1$ such that $\mu^r(i)$ has no successor.
3. Vertices i within a cycle, i.e., such that there exists $r \geq 1$ with $\mu^r(i) = i$.
4. Vertices i leading to a cycle, i.e., such that there exists $r \geq 1$ such that $\mu^r(i)$ belongs to a cycle.

This case distinction will be relevant to our work in Chapter 6. In all cases, we have $k_{i,0} = 0$ and $k_{i,1} = \ell_i$. Also note that $\mathbf{w}_{i,0} = \mathbf{d}_i$. In the first two cases, we have $\mathbf{d}_i^* = \mathbf{w}_{i,s_i}$ where s_i is such that $\mu^{s_i}(i)$ exists and has no successor in G. In the last two cases, $\mathbf{w}_{i,j}$ exists for all j and we have $\mathbf{d}_i^* = \lim_{j\to\infty} \mathbf{w}_{i,j}$. We note that we can rewrite $\mathbf{w}_{i,j}$ as $\mathbf{w}_{i,j} = d'_i d'_{i-k_{i,1}} \cdots d'_{i-k_{i,j-1}} \mathbf{d}_{i-k_{i,j}}$. Finally, we can see that $\operatorname{val}_{\sigma^{-i}(\mathcal{B})}(\mathbf{w}_{i,j}) = 1$ for any j such that $\mathbf{w}_{i,j}$ is defined.

Example 2.13. We continue Example 2.5. For the base $\mathcal{B} = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$, we have seen that $\mathbf{d}_0 = d_{\mathcal{B}}(1) = 2010^\omega$. A similar computation shows that $d_{\sigma(\mathcal{B})}(1) = \mathbf{d}_{-1} = \mathbf{d}_1 = 110^\omega$. From this, the quasi-greedy expansions of 1 are defined by

$$\begin{cases} \mathbf{d}_0^* = d'_0 \mathbf{d}_{-3}^* = 200\mathbf{d}_1^* \\ \mathbf{d}_1^* = d'_1 \mathbf{d}_{-1}^* = 10\mathbf{d}_1^*. \end{cases}$$

Thus we find $\mathbf{d}_1^* = (10)^\omega$ and $\mathbf{d}_0^* = 200(10)^\omega$. The graph G in this case has an arc from 0 to 1 and a loop on 1. We have for instance $k_{0,2} = 5 = |200 \cdot 10|$ and $\mathbf{w}_{0,2} = 200 \cdot 10 \cdot 110^\omega$.

In the base $\mathcal{B} = (\varphi, 3, \varphi)$, we have $\mathbf{d}_0 = 1(110)^\omega$, $\mathbf{d}_1 = 110^\omega$ and $\mathbf{d}_2 = 30^\omega$. From this we find $\mathbf{d}_0^* = 1(110)^\omega$, $\mathbf{d}_1^* = (102)^\omega$ and $\mathbf{d}_2^* = (210)^\omega$. In this case, we have $\ell_0 = \infty$, thus $k_{0,j}$ is not defined for $j \geq 2$ and $\mathbf{w}_{0,j}$ is not defined for $j \geq 1$. There are no intermediate representations of 1 in base $\mathcal{B}$, given that the greedy and quasi-greedy representations of 1 are equal. The graph G is represented on Figure 2.1.

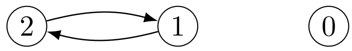


Figure 2.1: The graph G associated with the alternate base numeration system $(\varphi, 3, \varphi)$.

We now turn to proving some properties of the quasi-greedy representations, again with a view to describe which words are expansions of numbers in this numeration system. Compare to Proposition 1.23 and Theorem 1.24.

Proposition 2.14 ([CC21, Proposition 23]). *The expansion $d_{\mathcal{B}}^*(1)$ is lexicographically maximal among all infinite representations of numbers in* $[0, 1]$.

Theorem 2.15 ([CC21, Theorem 26]). *An infinite word* $\mathbf{a}$ *over* $\mathbb{N}$ *is the* $\mathcal{B}$*-expansion of a real number* $x \in [0, 1)$ *if and only if* $\sigma^n(\mathbf{a}) <_{\text{lex}} \mathbf{d}_{-n}^*$ *for all* $n \geq 0$.

Like in the Rényi case, the set $\{d_{\mathcal{B}}(x) : x \in [0, 1)\}$ is called $D_{\mathcal{B}}$.

A result similar to Theorem 1.25 exists, giving necessary conditions for a p-tuple of words to be greedy or quasi-greedy expansions of 1 in a given alternate base. Sufficient conditions are one of the contributions in this thesis and will be considered in Chapter 3.

Theorem 2.16. *In an alternate base* $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$*, the expansions* $\mathbf{d}_{p-1}, \ldots, \mathbf{d}_0$ *and* $\mathbf{d}_{p-1}^*, \ldots, \mathbf{d}_0^*$ *verify*

$$\sigma^j(\mathbf{d}_i) <_{\text{lex}} \mathbf{d}_{i-j} \quad \text{and} \quad \sigma^j(\mathbf{d}_i^*) \leq_{\text{lex}} \mathbf{d}_{i-j}^*$$

for all i *and all* $j \geq 0$.

Proof. We have that $\sigma^j(\mathbf{d}_i)$ is the $\sigma^{j-i}(\mathcal{B})$-expansion of

$$\beta_{i-1}\cdots\beta_{i-j}\left(1-\sum_{k=1}^{j}\frac{t_{i,k}}{\beta_{i-1}\cdots\beta_{i-k}}\right).$$

This quantity is the j-th remainder in the greedy algorithm computing $\mathbf{d}_i$, and is thus in $[0,1)$. Proposition 2.8 applied in $\sigma^{j-i}(\mathcal{B})$ then ensures that $\sigma^j(\mathbf{d}_i) <_{\text{lex}} \mathbf{d}_{i-j}$. A similar proof works for the second part of the statement, using Proposition 2.14 and the fact that $\text{val}_{\sigma^{j-i}(\mathcal{B})}(\sigma^j(\mathbf{d}_i)) \in (0,1]$. □

Corollary 2.17 ([CC21, Proposition 38])**.** *The word $\mathbf{d}_i$ is never purely periodic.*

Proof. If it were purely periodic of period q, we would have $\sigma^{pq}(\mathbf{d}_i) = \mathbf{d}_i = \mathbf{d}_{i-pq}$, contradicting the previous result. □

In what follows, we will be particularly interested in a subset of well-behaved alternate bases.

Definition 2.18. A *Parry alternate base* is an alternate base $\mathcal{B}$ such that $\mathbf{d}_i$ is finite or eventually periodic for every i

Recall that the term *eventually periodic* excludes sequences that end in 0^ω, which are called finite instead, and that *ultimately periodic* by contrast includes those sequences.

Note that an alternate base can be Parry without the elements of the base being Parry numbers, and without the product of all elements of the base being a Parry number. Indeed, consider the alternate base $(3\varphi^2, 2\varphi)$, where φ is the golden ratio. It is easy to check that $\mathbf{d}_0 = 7260^\omega$ and $\mathbf{d}_1 = 31260^\omega$ are both finite, thus this base is Parry alternate. However, it is known [Bas02] that the Parry numbers of degree 2 are exactly the Pisot numbers (see Section 4.2 for a definition of Pisot numbers). Since neither of $3\varphi^2, 2\varphi$ and $6\varphi^3$ are Pisot numbers, they are also not Parry. This fact will be relevant much later, in Section 6.10.

We end this section by introducing two generalizations of alternate bases. The first is Cantor bases. Here, we use a right-infinite sequence of bases $\mathcal{B} = (\beta_{-1}, \beta_{-2}\ldots)$ all greater than 1, rather than a periodic sequence. This generalizes the works of Cantor [Can69], who assumed the bases to be integers. This is the setting of the article [CC21] which was our main guide so far, and therefore many of the properties listed remain true in this context. We highlight the discrepancies. First, we must require $\prod_{i=1}^{\infty} \beta_{-i} = \infty$ in order to ensure that the greedy algorithm converges to a word that indeed represents the correct number. The evaluation map is defined as above, the greedy algorithm then runs as above and the alphabet of the system is $\{0, \ldots, \sup_{i\in\mathbb{N}_0} \lfloor \beta_{-i} \rfloor\}$. Propositions 2.6, 2.7 and 2.8 then remain true. We set $\sigma(\mathcal{B}) = (\beta_{-2}, \beta_{-3}, \ldots)$, which is not invertible in this case. The word $\mathbf{d}_i$ is only defined for $i \leq 0$ and we define $\mu(i) = i - \ell_i$ when $\mathbf{d}_i$ is finite. The map μ is partially defined on $-\mathbb{N}$ rather than on $\{0, \ldots, p-1\}$. Therefore, the quasi-greedy algorithm may run without looping in this setting and the graph G has no cycle, having instead vertices that have infinitely many (indirect) successors. Proposition 2.14 and Theorems 2.15 and 2.16 work as expected. However, Corollary 2.17 fails.

Finally, we explain how to generalize Cantor bases to the representation of positive real numbers, rather than numbers in $[0, 1]$. Alternate bases, which are a particular case of Cantor bases, will also benefit from this generalization. We refer to [CCMP25, Section 3].

Definition 2.19. A *two-way Cantor base* is a sequence $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ of real numbers greater than 1, with $\prod_{i=1}^{\infty} \beta_{-i} = \prod_{i=0}^{\infty} \beta_i = +\infty$.

The evaluation map is defined on all biinfinite sequences $\mathbf{a} = \cdots a_1 a_0 \bullet a_{-1} a_{-2} \cdots \in \mathbb{N}^{\mathbb{Z}}$ with an infinite left tail of zeros (i.e. such that there exists N with $a_n = 0$ for $n \geq N$) by

$$\mathbf{a} \mapsto \sum_{i=0}^{N-1} a_i \beta_{i-1} \cdots \beta_1 \beta_0 + \sum_{i=1}^{\infty} \frac{a_{-i}}{\beta_{-1} \cdots \beta_{-i}}.$$

We recall that the lexicographic order on biinfinite words with a left tail of zeros is defined by setting $\mathbf{a} <_{\text{lex}} \mathbf{b}$ if $a_n < b_n$ where n is the largest index at which $\mathbf{a}$ and $\mathbf{b}$ differ.

The shift operator is defined as above by $\sigma(\mathcal{B}) = (\cdots \beta_0 \beta_{-1} \bullet \beta_{-2} \cdots)$ when $\mathcal{B} = (\cdots \beta_0 \bullet \beta_{-1} \cdots)$. Thus $\text{val}_{\sigma(\mathcal{B})}(\sigma(\mathbf{a})) = \beta_{-1} \text{val}_{\mathcal{B}}(\mathbf{a})$.

The representation map is noted $\langle \cdot \rangle_{\mathcal{B}}$. For numbers $x \in [0, 1)$, we simply set $\langle x \rangle_{\mathcal{B}} = {}^{\omega}0 \bullet d_{\mathcal{B}}(x)$ where $d_{\mathcal{B}}$ is the representation map defined above using the greedy algorithm. For $x \geq 1$, we let $N \geq 1$ be such that $\beta_{N-2} \cdots \beta_0 \leq$

$x < \beta_{N-1} \cdots \beta_0$ (which exists since $\prod_{i=0}^{\infty} \beta_i = +\infty$), then we set

$$\langle x \rangle_{\mathcal{B}} = \sigma^N \left({}^{\omega}0 \bullet d_{\sigma^{-N}(\mathcal{B})} \left(\frac{x}{\beta_{N-1} \cdots \beta_0} \right) \right).$$

That is, to represent a number greater than 1, we divide it by an appropriate product of elements of the base until it is again in $[0, 1)$, then represent it and undo the division by shifting back to the left. Note that $\langle 1 \rangle_{\mathcal{B}} = {}^{\omega}01.0^{\omega}$ for all bases $\mathcal{B}$. In what follows, we will not write left-infinite tails of zeros in representations.

The following basic properties are deduced from Propositions 2.6, 2.7 and 2.8.

Proposition 2.20 ([CCMP25, Lemma 3.1]). *Given a two-way Cantor base $\mathcal{B}$, the word $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ with a left tail of zeros is the expansion of a non-negative number if and only if*

$$\sum_{n=0}^{k} a_n \beta_{n-1} \cdots \beta_0 + \sum_{n=1}^{\infty} \frac{a_{-n}}{\beta_{-1} \cdots \beta_{-n}} < \beta_k \cdots \beta_0 \text{ for all } k \geq 0 \text{ and}$$

$$\sum_{n=k}^{\infty} \frac{a_{-n}}{\beta_{-1} \cdots \beta_{-n}} < \frac{1}{\beta_{-1} \cdots \beta_{-k+1}} \text{ for all } k \geq 1.$$

Proposition 2.21 ([CCMP25, Lemma 3.2]). *Given a two-way Cantor base $\mathcal{B}$, the map $\langle \cdot \rangle_{\mathcal{B}}$ is increasing for the usual order on $\mathbb{R}_{\geq 0}$ and the lexicographic order on biinfinite words with a left tail of zeros. Further, $\langle x \rangle_{\mathcal{B}}$ is lexicographically maximal among all $\mathcal{B}$-representations of x.*

Finally, the following is a consequence of Theorem 2.15.

Proposition 2.22. *A biinfinite word $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ with a left tail of zeros is the $\mathcal{B}$-expansion of a real number in $\mathbb{R}_{\geq 0}$ if and only if $a_{-n-1} a_{-n-2} \cdots <_{\text{lex}} \mathbf{d}^*_{-n}$ for all $n \in \mathbb{Z}$ (where $\mathbf{d}^*_{-n} = d^*_{\sigma^n(\mathcal{B})}(1)$ as above).*

2.2 Back to positional numeration systems

As we have explained in the introduction, the starting point for this research was to investigate positional numeration systems with a regular language.

This question was mostly solved by Hollander in [Hol98] in the case where the numeration system has a dominant root, but this is not the case for all U-systems.

Example 2.23. Consider the U-system based on the sequence defined by the recurrence relation $U_n = 60U_{n-2}$ and the initial conditions $U_0 = 1, U_1 = 10$. Such a system is reminiscent of the Babylonian number system, and it clearly has a regular language: a word is a representation of an integer if and only if its digits at even positions are all in $\{0, \ldots, 9\}$ and its digits at odd positions are all in $\{0, \ldots, 5\}$. However, it doesn't have a dominant root, as the quantity $\frac{U_{n+1}}{U_n}$ alternates between 10 and 6 rather than converging to a limit. By contrast, the numeration language L_U is exactly the set of factors of $D_{\mathcal{B}}$ where $\mathcal{B}$ is the alternate base $(6, 10)$.

The above example both illustrates that the dominant root condition is not enough to understand all U-systems with a regular language, and suggests that alternate bases offer an answer to this. In this section, we introduce two results from the article [CK25], written with Émilie Charlier, that make this connection explicit. The results are presented with different notation from the original article, as we have adapted the notations in this thesis to accommodate more types of systems.

Proposition 2.24 ([CK25, Proposition 10])**.** *Let $U = (U_n)_{n\geq 0}$ be a positional numeration system such that L_U is regular. There exists a positive integer p such that the p limits, for $i \in \{0, \ldots, p-1\}$,*

$$\lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}}$$

exist and can be effectively computed. In particular, the limit

$$\lim_{n\to+\infty} \frac{U_{n+p}}{U_n}$$

exists and is equal to the product of the p above limits.

Proof. We know from [Sha94, Theorem 6] that U must be linear if L_U is regular. Let $\alpha_1, \ldots, \alpha_d$ be the eigenvalues of U with multiplicities $m_1, \ldots, m_d$

respectively. It is classical (see for instance [BR11, Chapter 6]) that

$$U_n = \sum_{j=1}^{d} P_j(n)\alpha_j^n$$

for all sufficiently large n, where the P_j's are polynomials (with coefficients in $\mathbb{C}$) of degree equal to $m_j - 1$.

Since the language L_U is regular, the formal series $\sum_{n\geq 0} U_n X^n$ is $\mathbb{N}$-rational (see for instance [Lot02, Proposition 7.3.7]). The sequence U is not ultimately zero in our framework, hence the series $\sum_{n\geq 0} U_n X^n$ is not a polynomial and we may apply [SS78, Theorem II.10.1]; also see [Ber71, BR11]. This result tells us that the eigenvalues of U of maximum modulus are of the form $\rho\xi$ where $\rho > 0$ and ξ is a root of unity. Moreover, ρ itself is among these eigenvalues and the multiplicity of any such eigenvalue $\rho\xi$ is at most that of ρ.

Let p be the least positive integer such that $\xi^p = 1$ for all ξ as in the previous paragraph. For each $i \in \{0, \ldots, p-1\}$ and for all sufficiently large n, we obtain

$$U_{np+i} = Q_i(n)\rho^{np} + \sum_{\substack{1\leq j\leq d \\ |\alpha_j|<\rho}} \alpha_j^i P_j(np+i)\alpha_j^{np}$$

where

$$Q_i(n) = \sum_{\substack{1\leq j\leq d \\ |\alpha_j|=\rho}} \alpha_j^i P_j(np+i).$$

Since the sequence U is increasing, all the polynomials Q_i share the same degree, for $i \in \{0, \ldots, p-1\}$. This implies that the announced p limits exist: we have

$$\lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}} = \frac{q_{i+1}}{q_i}, \text{ for } i \in \{0, \ldots, p-2\},$$
$$\text{and } \lim_{n\to+\infty} \frac{U_{np+p}}{U_{np+p-1}} = \frac{q_0}{q_{p-1}}\rho^p.$$

where q_i is the leading coefficient of the polynomial Q_i, for every i. Finally, we get that

$$\lim_{n\to+\infty} \frac{U_{np+i+p}}{U_{np+i}} = \rho^p.$$

Since the latter limit does not depend on i, the particular case is also proven. □

Remark 2.25. Without the regularity hypothesis, that is, for an arbitrary linear numeration system $U = (U_n)_{n\geq 0}$, we would only get that the formal series $\sum_{n\geq 0} U_n X^n$ is $\mathbb{Z}$-rational, which is a strictly weaker property than being $\mathbb{N}$-rational. In this case, we would not be able to use the result of Berstel [Ber71] as reported in [SS78, BR11].

Remark 2.26. In general, for an arbitrary linear recurrence sequence $U = (U_n)_{n\geq 0}$ and a given $p \geq 2$, the existence of the limit $\lim_{n\to+\infty} \frac{U_{n+p}}{U_n}$ does not imply the existence of the p limits $\lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}}$, for $i \in \{0, \ldots, p-1\}$. However, this implication is true when the sequence $(U_n)_{n\geq 0}$ is increasing.

See Section 8.1 for the complete result on these limits, which generalize the *Kepler limit* in the case $p = 1$. In this case, the Kepler limit $\lim_{n\to\infty} \frac{U_{n+1}}{U_n}$ of a linear recurrence sequence U is known to exist if and only if U has a single eigenvalue of maximal multiplicity among its eigenvalues of maximal modulus.

Thus, with a positional numeration system having a regular language can be associated an alternate base, whose base elements are given by the p limits of $\frac{U_{np+i+1}}{U_{np+i}}$ as i tends to infinity. Note that nonminimal choices of p only lead us to consider multiples of the minimal possible one, which amounts to considering repetitions of the minimal alternate base. Although this is not forbidden, we will rather assume that p is minimal.

Definition 2.27. We say that a positional numeration system $U = (U_n)_{n\geq 0}$ has an *associated alternate real base* $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ if

$$\beta_i = \lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}}$$

for each $i \in \{0, \ldots, p-1\}$. By considering the minimal possible p for which these limits exist, we may talk about *the* alternate real base associated with U.

Example 2.28. (1) Consider the linear numeration system built on the sequence $U = (U_n)_{n\geq 0}$ defined by $U_{n+4} = 3U_{n+2} + U_n$ for $n \geq 0$ and $(U_0, U_1, U_2, U_3) = (1, 2, 5, 7)$. Then we have

$$\lim_{n\to+\infty} \frac{U_{n+2}}{U_n} = \beta = \frac{3+\sqrt{13}}{2},$$

which is the unique root of maximum modulus of the polynomial $X^2 - 3X - 1$. Further, we have

$$\lim_{n\to+\infty} \frac{U_{2n+2}}{U_{2n+1}} = \beta_1 = \beta - 1 = \frac{1+\sqrt{13}}{2}$$

$$\lim_{n\to+\infty} \frac{U_{2n+1}}{U_{2n}} = \beta_0 = \frac{\beta}{\beta - 1} = \frac{5+\sqrt{13}}{6}.$$

Therefore, this system has no dominant root. The alternate base associated with the numeration system U is given by $\mathcal{B} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ from Example 2.13.

(2) Similarly, the system based on the sequence U given by the relation

$$U_{n+9} = 10U_{n+6} - 18U_{n+3} + 9U_n$$

and the initial conditions

$$(1, 2, 4, 8, 13, 37, 62, 100, 298)$$

is associated with the alternate base $(\varphi, 3, \varphi)$ as we have, for instance, $\frac{U_{3n+2}}{U_{3n+1}} \to 3$ and $\frac{U_{3n+1}}{U_{3n}} \to \varphi$ (this φ being the rightmost one in the triplet $(\varphi, 3, \varphi)$).

(3) Finally, consider the U-system based on the sequence U given by $U_{2n+1} = 2U_{2n}$ and $U_{2n+2} = U_{2n+1} + 1$, together with $U_0 = 1$. The associated language is regular (it is the language of words over $\{0, 1\}$ where a 1 at an odd position is followed by only zeros) and U corresponds to the alternate base $(1, 2)$. This is the very reason why we allowed base elements to be equal to 1 in Definition 2.1.

The next result generalizes Theorem 1.41 to the alternate base case. Before introducing it, we must add one definition, that will allow us to manage the case where an element of the associated alternate base is equal to 1.

Definition 2.29. Let U be a positional numeration system with an associated alternate base $(\beta_{p-1}, \dots, \beta_0)$. Then for integers i, c, n such that $0 \le i < p$, $n \ge 0$ and $1 \le c \le U_{np+i}$, we define

$$\text{rep}_{i,c}(n) = 0^\ell \text{rep}_U(U_{np+i} - c),$$

where $\ell = np + i - |\text{rep}_U(U_{np+i} - c)|$.

The idea is that we pad the greedy representation of $U_{np+i}-c$ with leading zeros in order to obtain a representation of length $np+i$. Most of the time, this consists in doing nothing since if $\beta_i > 1$, then $\mathrm{rep}_{i,c}(n) = \mathrm{rep}_U(U_{np+i}-c)$ for all large enough n. Moreover, we have $\mathrm{rep}_{i,1}(n) = \mathrm{rep}_U(U_{np+i}-1)$ for all $n \geq 0$. However, in the case where $\beta_i = 1$ and $c \geq 2$, these few leading zeros will ensure that we keep working with words of the desired length $np+i$.

Proposition 2.30 ([CK25, Proposition 18]). *Let U be a positional numeration system with an associated alternate base $(\beta_{p-1}, \ldots, \beta_0)$, let $i \in \{0, \ldots, p-1\}$ and let c be a positive integer. For all $L \geq 0$, there exists N such that for all $n \geq N$, there exists $j \in \{0, \ldots, L\}$ such that $\mathrm{rep}_{i,c}(n)$ and $\mathbf{w}_{i,j}$ share a common prefix of length L.*

Proof. For all $L, n \geq 0$, if there exists some $j \in \{0, \ldots, L\}$ such that $\mathrm{rep}_{i,c}(n)$ and $\mathbf{w}_{i,j}$ share a common prefix of length L, we define $j(L,n)$ to be the minimal such j. We show that $j(L,n)$ is well defined for all $L \geq 0$ and all large enough n by induction on L. This holds for $L = 0$ as $j(0,n) = 0$ for all $n \geq 0$. Now, consider a fixed $L \geq 0$ and assume that there exists N such that for all $n \geq N$, the value $j(L,n)$ is well defined. We have to prove the existence of $j(L+1,n)$ for all n large enough.

For each j, we write

$$\mathbf{w}_{i,j} = a_{j,1}a_{j,2}\cdots$$

where we have dropped the dependence on i for the digits since i is fixed in this proof.

Consider $n \geq N$. From the greedy algorithm and the induction hypothesis, we know that the digit indexed by $L+1$ in $\mathrm{rep}_{i,c}(n)$ is equal to $\lfloor b_{n,j(L,n)} \rfloor$ where

$$b_{n,j} = \frac{U_{np+i} - c - a_{j,1}U_{np+i-1} - \cdots - a_{j,L}U_{np+i-L}}{U_{np+i-L-1}}.$$

By hypothesis, for any fixed k, the quotient $\frac{U_{np+i-k}}{U_{np+i-L-1}}$ tends to the product $\beta_{i-k-1}\cdots\beta_{i-L-1}$ as n tends to infinity. Therefore we obtain that, for a fixed j, the quantity $b_{n,j}$ converges to

$$\beta_{i-1}\cdots\beta_{i-L-1}\left(1 - \frac{a_{j,1}}{\beta_{i-1}} - \cdots - \frac{a_{j,L}}{\beta_{i-1}\cdots\beta_{i-L}}\right) \tag{2.1}$$

as n tends to infinity.

Note that by minimality of $j(L,n)$, we have $k_{i,j(L,n)} \leq L$. Indeed, we have $k_{i,0} = 0$ and for $j \geq 1$, the words $\mathbf{w}_{i,j}$ and $\mathbf{w}_{i,j-1}$ share the same prefix of length $k_{i,j} - 1$. Now, we consider some fixed $j \in \{0,\dots,L\}$. For all $n \geq N$ such that $j(L,n) = j$, we study three cases that cover all possible situations.

- If $L \geq k_{i,j+1}$, all nonzero digits of $\mathbf{w}_{i,j}$ are contained in the prefix of length L. This implies that the quantity (2.1) is equal to 0. Since $b_{n,j}$ is nonnegative and goes to 0 as n goes to infinity, its floor must be 0 if n is large enough, hence it coincides with $a_{j,L+1}$.
- If $L = k_{i,j+1} - 1$, then (2.1) is equal to $a_{j,L+1}$, which is positive. In this case, for large enough n, the floor of $b_{n,j}$ is either $a_{j,L+1}$ or $a_{j,L+1} - 1$, which are the digits in position $L+1$ in $\mathbf{w}_{i,j}$ and $\mathbf{w}_{i,j+1}$ respectively.
- Otherwise we have $k_{i,j} \leq L < k_{i,j+1} - 1$ and the quantity (2.1) belongs to the interval $(a_{j,L+1}, a_{j,L+1} + 1)$, which implies that $\lfloor b_{n,j} \rfloor = a_{j,L+1}$ for large enough n.

Since $j(L,n) \leq L$, we get that there exists $N' \geq N$ such that for all $n \geq N'$, the value $j(L+1,n)$ is well defined and is equal to either $j(L,n)$ or $j(L,n)+1$. □

Let us illustrate the breadth of behaviors that might occur within the bounds of this result.

Example 2.31. (1) Consider again example (1) from Example 2.28. Remembering that $\mathbf{d}_0 = 2010^\omega$ and $\mathbf{d}_1 = 110^\omega$, we find $\mathbf{w}_{0,j} = 20(01)^j10^\omega$ and $\mathbf{w}_{1,j} = (10)^j110^\omega$ for $j \geq 0$. The maximal words in the U-system are given by $\mathrm{rep}_U(U_{2n} - 1) = 20(01)^{n-1} = \mathrm{Pref}_{2n}(\mathbf{w}_{0,n-1})$ and $\mathrm{rep}_U(U_{2n+1} - 1) = (10)^n1 = \mathrm{Pref}_{2n+1}(\mathbf{w}_{1,n})$ for $n \geq 1$.

(2) For item (2) from Example 2.28, we have $\mathbf{w}_{0,0} = 1(110)^\omega$, $\mathbf{w}_{1,2j} = (102)^j110^\omega$, $\mathbf{w}_{1,2j+1} = (102)^j1030^\omega$, and similarly $\mathbf{w}_{2,2j} = (210)^j30^\omega$ and $\mathbf{w}_{2,2j+1} = (210)^j2110^\omega$. On the U-system side, we find that

$$\mathrm{rep}_{0,1}(n) = \mathrm{rep}_U(U_{3n} - 1) = 111(011)^{n-1} = \mathrm{Pref}_{3n}(\mathbf{w}_{0,0}),$$

$$\mathrm{rep}_{1,1}(n) = \mathrm{rep}_U(U_{3n+1} - 1) = 110^{3n-2}1 = \mathrm{Pref}_{3n}(\mathbf{w}_{1,0})1,$$

as expected, and $\mathrm{rep}_U(U_{3n+2} - 1)$ shares a long prefix with $\mathbf{w}_{2,2n}$, but has a suffix with different behavior (for instance, $\mathrm{rep}_U(U_{26} - 1) =$

$(210)^7 20111$ whereas $\mathbf{w}_{2,16} = (210)^8 30^\omega$). This is still in accordance with Proposition 2.30, but we see that the behavior past the common prefix might be harder to describe (in fact, in this example, $\{\mathrm{rep}_U(U_{3n+2}-1) : n \in \mathbb{N}\}$ forms a nonregular language, with the nonmatching suffix having unbounded length as n tends to infinity).

(3) Continuing item(3) from Example 2.28 will allow us to justify the introduction of the notation $\mathrm{rep}_{i,c}$. We see that $\mathrm{rep}_U(U_{2n} - 2) = 110^{2n-3}$ for all $n \geq 2$ and that $\mathrm{rep}_U(U_{2n+1} - 2) = 10110^{2n-3}$ for all $n \geq 3$. Observe that both $\mathrm{rep}_U(U_{2n} - 2)$ and $\mathrm{rep}_U(U_{2n-1} - 2)$ have length $2n - 1$. Hence for all $n \geq 3$, the words from Definition 2.29 are given by $\mathrm{rep}_{0,2}(n) = 0110^{2n-3}$ and $\mathrm{rep}_{1,2}(n) = 10110^{2n-3}$. For the alternate base $\mathcal{B} = (1, 2)$, we obtain that

$$\mathbf{w}_{0,2j} = (01)^j 10^\omega, \quad \mathbf{w}_{0,2j+1} = (01)^j 020^\omega,$$
$$\mathbf{w}_{1,2j} = (10)^j 20^\omega, \quad \mathbf{w}_{1,2j+1} = (10)^j 110^\omega$$

for all $j \geq 0$. We thus see that

$$\mathrm{rep}_{0,2}(n) = \mathrm{Pref}_{2n}(\mathbf{w}_{0,2}) \quad \text{and} \quad \mathrm{rep}_{1,2}(n) = \mathrm{Pref}_{2n+1}(\mathbf{w}_{1,3})$$

for all $n \geq 3$, in accordance with Proposition 2.30. Note that in contrast, $\mathrm{rep}_U(U_{2n} - 2)$ starts with 11 whereas none of the words $\mathbf{w}_{0,j}$ do, justifying the switch to $\mathrm{rep}_{i,c}$ in Proposition 2.30.

(4) In the previous examples, the minimal polynomial of U was of the form $f(X^p)$ for some polynomial f. Now, consider $U = (U_n)_{n\geq 0} = (1, 3, 8, 12, 16, 48, \ldots)$ such that $U_{n+3} = 2U_{n+2} - 4U_{n+1} + 8U_n$. This system has no dominant root but we have

$$\beta_3 = \lim_{n\to+\infty} \frac{U_{4+4}}{U_{4n+3}} = \frac{4}{3}, \qquad \beta_2 = \lim_{n\to+\infty} \frac{U_{4n+3}}{U_{4n+2}} = \frac{3}{2},$$
$$\beta_1 = \lim_{n\to+\infty} \frac{U_{4n+2}}{U_{4n+1}} = \frac{8}{3}, \qquad \beta_0 = \lim_{n\to+\infty} \frac{U_{4n+1}}{U_{4n}} = 3.$$

Hence we get $p = 4$ and we can compute

$$\mathbf{w}_{3,0} = 1110^\omega \quad \text{and} \quad \mathbf{w}_{3,j} = 110(1010)^{j-1}10110^\omega \quad \text{for } j \geq 1$$
$$\mathbf{w}_{2,0} = 220^\omega \quad \text{and} \quad \mathbf{w}_{2,j} = 21(1010)^{j-1}10110^\omega \quad \text{for } j \geq 1$$
$$\mathbf{w}_{1,0} = 30^\omega \quad \text{and} \quad \mathbf{w}_{1,j} = 2(1010)^{j-1}10110^\omega \quad \text{for } j \geq 1.$$
$$\mathbf{w}_{0,j} = (1010)^j 10110^\omega \quad \text{for } j \geq 0.$$

The maximal words are given by

$$\begin{aligned}
\mathrm{rep}_U(U_{4n+3}-1) &= 110(1010)^n = \mathrm{Pref}_{4n+3}(\mathbf{w}_{1,n+1})\\
\mathrm{rep}_U(U_{4n+2}-1) &= 21(1010)^n = \mathrm{Pref}_{4n+2}(\mathbf{w}_{2,n+1})\\
\mathrm{rep}_U(U_{4n+1}-1) &= 2(1010)^n = \mathrm{Pref}_{4n+1}(\mathbf{w}_{1,n+1})\\
\mathrm{rep}_U(U_{4n}-1) &= (1010)^n = \mathrm{Pref}_{4n}(\mathbf{w}_{0,n})
\end{aligned}$$

for all $n \geq 0$.

(5) Finally, we consider a system $U = (1, 3, 6, 11, 15, \ldots)$ given by the recurrence relation $U_{n+4} = 2U_{n+2} + 3U_n$. Equivalently, U is given by the relations $U_{2n} = U_{2n-1} + U_{2n-2} - 2(-1)^n$ and $U_{2n+1} = 2U_{2n} + (-1)^n$. In this case we have $p = 2$, with $\beta_1 = 3/2$ and $\beta_0 = 2$. The maximal words are given by

$$\begin{aligned}
\mathrm{rep}_U(U_{4n+3}-1) &= 1110^{4n} = \mathrm{Pref}_{4n+3}(\mathbf{w}_{1,1})\\
\mathrm{rep}_U(U_{4n+2}-1) &= 110^{4n-1}1 = \mathrm{Pref}_{4n+1}(\mathbf{w}_{0,0})1\\
\mathrm{rep}_U(U_{4n+1}-1) &= 20^{4n} = \mathrm{Pref}_{4n+1}(\mathbf{w}_{1,0})\\
\mathrm{rep}_U(U_{4n}-1) &= (10)^{2n} = \mathrm{Pref}_{4n}(\mathbf{w}_{0,2n})
\end{aligned}$$

for all $n \geq 2$. We see that the sequence of maximal words does not admit a limit, even when selecting words of a given parity.

2.3 Extensions of Dumont–Thomas numeration systems

The Dumont–Thomas numeration systems as defined in Section 1.4 were historically introduced for primitive substitutions with a right-infinite fixed point, as that was the most relevant formulation for the problem of fractal functions that Dumont and Thomas were studying. Through [LL21, LL23, LL24a], Labbé and Lepšová uncovered a link between Dumont–Thomas numeration systems and problems related to a tiling of the plane by Wang tiles. The tile to be put at coordinates (m, n) would be decided by an automaton reading the paired representations of m and n in an exotic numeration system, which turned out to be a generalization of a Dumont–Thomas system. However, to tile the entire plane in this fashion rather than the first quadrant, it was of course necessary to define representations for negative numbers as well. In turn, this necessitated another generalization: as the substitutions under study had right-infinite fixed points but no left-infinite fixed points, the authors had to increase the scope of Dumont–Thomas numeration systems to include substitutions with periodic points as well.

Further work was then conducted my coauthors and I in [KLS25a] to investigate the positionality of these new numeration systems, following a question by Lepšová in her thesis ([Lep24, Question 6.5.7]). During this investigation, we simplified the definition of the system by Labbé and Lepšová using a result similar to Proposition 1.38 and discovered evidence of a link to alternate bases, presented here as Example 2.41.

In this section, we present the double generalization of Dumont–Thomas numeration systems that resulted from the above articles and explain the link to alternate bases. Thus the results presented are found in [KLS25a] and its long version [KLS25b] The study of positionality itself will be Chapter 7 in this thesis.

Recall that in the original case, as described in Section 1.4, the Dumont–Thomas representation of n with respect to the substitution μ and the fixed point $\mathbf{u}$ with seed a is obtained in one of two ways. We can define it as $|m_{k-1}|\cdots|m_0|$ where $(m_0, a_0), \ldots, (m_{k-1}, a_{k-1})$ is an a-admissible sequence with $m_{k-1} \neq \varepsilon$, and $u_0 u_1 \cdots u_{n-1} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$. More visually, we can select the label of the shortest path from a to column n in the tree $\mathcal{T}_{\mu,a}$, where this tree is defined by the children of a node labeled x being labeled by $y_0, \ldots, y_\ell$ if $\mu(x) = y_0 \cdots y_\ell$.

By extension of the tree associated with a substitution and a letter as in the previous paragraph, we consider trees associated with substitutions on biinfinite words as follow. For a substitution μ over A and two letters $a, b \in A$, the tree $\mathcal{T}_{\mu,b|a}$ is obtained by setting a start root having two children: the left one is reached with an edge of label `1` and is the root of the tree $\mathcal{T}_{\mu,b}$, and the right one is reached with an edge of label `0` and is the root of the tree $\mathcal{T}_{\mu,a}$. We say that a node is *in column* n if either $n \geq 0$, the node is the right subtree and there are n nodes on the same level to its left in the right subtree, or $n < 0$, the node is in the left subtree and there are $-n-1$ nodes on the same level to its right in the left subtree.

If $\mathbf{u}$ is a fixed point of μ with seed $b|a$, we can once again define the representation of n to be the label of a shortest path from the root to column n in $\mathcal{T}_{b|a}$. We see that the representations of negative numbers will start with `1` whereas the representations of nonnegative numbers will start with `0`. If $\mathbf{u}$ is not a fixed point of μ, but is only a periodic point of period p, it follows that the rows of $\mathcal{T}_{b|a}$ do not converge to an infinite word, but rather that the p subsequences formed by extracting the rows of a given congruence modulo p each converge to a different word. We get around this by choosing only representations of a given length modulo p.

Definition 2.32. Let $\mu\colon A^* \to A^*$ be a substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $b|a$ and period $p \geq 1$. Let $r \in \{0, 1, \ldots, p-1\}$. The *Dumont–Thomas complement numeration system associated with* μ, $\mathbf{u}$ *and* r is defined by letting $\mathrm{rep}_{\mu,\mathbf{u},r}(n)$ be the label of the shortest path of length $r + 1 \bmod p$ from the root to column n in the tree $\mathcal{T}_{b|a}$. Note that when the context is clear, we drop the dependence on μ, $\mathbf{u}$ and r.

The name *complement* numeration system stems from an analogy with the famous two's complement numeration system (see [Knu98, Section 4.1] or Example 7.2). This system was the inspiration for the Fibonacci complement numeration system in [LL21, LL23], which was the motivation for generalizing Dumont–Thomas systems in this fashion.

We also provide the interpretation in terms of admissible sequences. The following two theorems justify the existence and uniqueness of admissible sequences with given properties, which will be the base for our representation map.

Theorem 2.33 ([KLS25b, Theorem 2.11])**.** *Let $\mu : A^* \to A^*$ be a substitution with growing letter $a \in A$. Consider a right-infinite periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{N}}(\mu)$ with $u_0 = a$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \ldots, p-1\}$ modulo p and define $\mathbf{v}_r = \mu^r(\mathbf{u})$. For every integer $n \geq 0$, there exist a unique integer $k = k(n)$ with $k \equiv r \bmod p$ and a unique sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that the sequence is a-admissible,*

$$m_{k-1}m_{k-2}\cdots m_{k-p} \neq \varepsilon \text{ if } k \geq p, \tag{2.2}$$

and $(\mathbf{v}_r)_{[0,n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2})\cdots\mu^0(m_0)$.

The proof of Theorem 2.33 is found in [KLS25b] and mimics that of [LL24a, Theorem 4.1] but we also present it here for the sake of completeness. Some intermediate results are necessary, which we recall too.

Lemma 2.34. [DT89, Lemma 1.1] *Let $\mu\colon A^* \to A^*$ be a substitution and let $k \geq 0$ be an integer. If $((m_i, a_i))_{i=0,\ldots,k}$ in an admissible sequence, then $\sum_{i=0}^{k} \left|\mu^i(m_i)\right| < \left|\mu^k(m_k a_k)\right|$.*

Lemma 2.35. [LL24a, Lemma 3.9] *Let $\mu\colon A^* \to A^*$ be a substitution and let $k \geq 0$ be an integer. Let also $m \in A^*$ and $a \in A$ be such that m is a proper prefix of $\mu^k(a)$. Then there exists a unique a-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that $m = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.*

Proof of Theorem 2.33. As a is a growing letter, the sequence $(\left|\mu^{\ell p+r}(a)\right|)_{\ell\geq 0}$ is (strictly) increasing. Note that, for every $r \in \{0, 1, \ldots, p-1\}$, the word $\mathbf{v}_r$ is also a periodic point of μ with period p. Let $n \geq 0$ be an integer. Setting $|\mu^{-p+r}(a)| = 0$ by convention, there exists a unique integer $\ell = \ell(n) \geq 0$ such that $\left|\mu^{(\ell-1)p+r}(a)\right| \leq n < \left|\mu^{\ell p+r}(a)\right|$. Let $k = k(n) = \ell p + r$ be such that

$$\left|\mu^{k-p}(a)\right| \leq n < \left|\mu^k(a)\right|. \tag{2.3}$$

Now the word $m = v_{r,0}v_{r,1} \cdots v_{r,n-1}$ of length n is a proper prefix of $\mu^k(a)$ (for a visualization, recall that the k-th level of the tree $\mathcal{T}_{\mu,a}$ is a prefix of $\mathbf{v}_r$ since $k \equiv r \bmod p$). By Lemma 2.35, there exists a unique a-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that $m = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)$. It remains to show that $m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon$ if $k \geq p$. Towards a contradiction, we assume that $k \geq p$ and $m_{k-1}m_{k-2} \cdots m_{k-p} = \varepsilon$. By inspecting the definition of admissible sequences with periodic points, it means that $a_{k-p} = a$. Then Lemma 2.34 implies that

$$\begin{aligned} n = |m| = \sum_{j=0}^{k-1} \left|\mu^j(m_j)\right| &= \sum_{j=0}^{k-p-1} \left|\mu^j(m_j)\right| \\ &< \left|\mu^{k-p-1}(m_{k-p-1}a_{k-p-1})\right| \leq \left|\mu^{k-p-1}(\mu(a_{k-p}))\right| = \left|\mu^{k-p}(a)\right|, \end{aligned}$$

which contradicts (2.3). Consequently, $m_{k-1}m_{k-2} \cdots m_{k-p} \neq \varepsilon$, which finishes the proof of existence.

For uniqueness, we already know that the sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ constructed above is the only a-admissible sequence of length k that satisfies the conditions of the statement. It remains to show that other lengths do not lead to more such sequences. First, suppose that $((m'_i, a'_i))_{i=0,\ldots,k'-1}$ is an a-admissible sequence of length k' with $k' < k$ that respects the conditions of the statement. Since $k' \equiv r \bmod p$, we have $k' \leq k - p$. From Lemma 2.34, we deduce that

$$\begin{aligned}\left|\mu^{k'-1}(m'_{k'-1}) \cdots \mu^0(m'_0)\right| &< \left|\mu^{k'-1}(m'_{k'-1} a'_{k'-1})\right| \\ &\leq \left|\mu^{k'}(a)\right| \leq \left|\mu^{k-p}(a)\right| \leq n = |m| \,.\end{aligned}$$

Thus it must be that $\mu^{k'-1}(m'_{k'-1}) \cdots \mu^0(m'_0) \neq m$, which violates the last condition of statement. Now for $k' > k$, we note that the sequence obtained by prefixing

$$\begin{aligned}(m_{k'-1}, a_{k'-1}) &= (\varepsilon, \mu(a)_0), \\ (m_{k'-2}, a_{k'-2}) &= (\varepsilon, \mu^2(a)_0), \ldots, \\ (m_k, a_k) &= (\varepsilon, \mu^{k'-k}(a)_0)\end{aligned}$$

to the sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ produces an a-admissible sequence of length k' which is such that $m = \mu^{k'-1}(m_{k'-1}) \cdots \mu^0(m_0)$. By Lemma 2.35, this is the only such sequence of length k'. But this sequence starts with $m_{k'-1} = \cdots = m_{k'-p} = \varepsilon$, and so it does not satisfy the statement. Thus the uniqueness is shown. □

We now move to left-infinite periodic points.

Theorem 2.36 ([KLS25b, Theorem 2.14])**.** *Let $\mu : A^* \to A^*$ be a substitution with growing letter $b \in A$. Consider a left-infinite periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}_{<0}}(\mu)$ with $u_{-1} = b$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \ldots, p-1\}$ modulo p and define $\mathbf{v}_r = \mu^r(\mathbf{u})$. For every integer $n \leq -1$, there exist a unique integer $k = k(n)$ with $k \equiv r \bmod p$ and a unique sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that the sequence is b-admissible,*

$$\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p}) a_{k-p} \neq \mu^p(b) \text{ if } k \geq p, \tag{2.4}$$

and $(\mathbf{v}_r)_{[-|\mu^k(b)|, n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.

Proof. Note that, for every $r \in \{0, 1, \ldots, p-1\}$, the word $\mathbf{v}_r$ is also a periodic point of μ with period p. Since b is a growing letter, the sequence

$(-\left|\mu^{\ell p+r}(b)\right|)_{\ell\geq 0}$ is (strictly) decreasing. Let $n \leq -1$ be an integer. Setting $|\mu^{-p+r}(b)| = 0$ by convention, there exists a unique integer $\ell = \ell(n)$ such that $-\left|\mu^{\ell p+r}(b)\right| \leq n < -\left|\mu^{(\ell-1)p+r}(b)\right|$. Let $k = k(n) = \ell p + r$ be such that

$$-\left|\mu^{k}(b)\right| \leq n < -\left|\mu^{k-p}(b)\right|. \tag{2.5}$$

Now the word $m = v_{r,-|\mu^k(b)|} \cdots v_{r,n-2}v_{r,n-1}$ of length

$$|m| = \left|\mu^{k}(b)\right| + n < \left|\mu^{k}(b)\right| - \left|\mu^{k-p}(b)\right| \leq \left|\mu^{k}(b)\right| \tag{2.6}$$

is a proper prefix of $\mu^k(b)$. Using Lemma 2.35, there exists a unique b-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ such that

$$m = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2})\cdots\mu^{0}(m_0).$$

We now show that (2.4) holds. Towards a contradiction, we assume that (2.4) is actually an equality. Then we have $a_{k-p} = b$. Writing

$$m = \mu^{k-p}(\mu^{p-1}(m_{k-1})\cdots\mu^{0}(m_{k-p}))\mu^{k-p-1}(m_{k-p-1})\cdots\mu^{0}(m_0),$$

we obtain from (2.4) that

$$|m| = \left|\mu^{k-p}(\mu^{p}(b))\right| - \left|\mu^{k-p}(a_{k-p})\right| + \sum_{j=0}^{k-p-1} \left|\mu^{j}(m_j)\right| \geq \left|\mu^{k}(b)\right| - \left|\mu^{k-p}(b)\right|$$

which contradicts (2.6). Consequently, (2.4) is a non-equality, which finishes the proof of existence.

For uniqueness, we already know that the sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ constructed above is the only b-admissible sequence of length k that satisfies the conditions of the statement. It remains to show that other lengths do not lead to more such sequences. First for $k' > k$, we note that when $k \equiv r \bmod p$, the sequence obtained by prefixing

$$\begin{aligned}
(m_{k'-1}, a_{k'-1}) &= (\mu(b)_{[0,|\mu(b)|-1)}, \mu(b)_{|\mu(b)|-1}),\\
(m_{k'-2}, a_{k'-2}) &= (\mu(a_{k'-1})_{[0,|\mu(a_{k'-1})|-1)}, \mu(a_{k'-1})_{|\mu(a_{k'-1})|-1}),\\
&\ \ \vdots\\
(m_k, a_k) &= (\mu(a_{k+1})_{[0,|\mu(a_{k+1})|-1)}, \mu(a_{k+1})_{|\mu(a_{k+1})|-1})
\end{aligned}$$

to the sequence $((m_i, a_i))_{i=0,\dots,k-1}$ produces a b-admissible sequence of length k' which is such that $v_{[-|\mu^{k'}(b)|,n-1]} = \mu^{k'-1}(m_{k'-1})\cdots\mu^0(m_0)$. By Lemma 2.35, this is the only such sequence of length k'. But this sequence violates (2.4).

For the case $k' < k$, if we had $k \equiv r \bmod p$ and some sequence satisfying the statement existed, we could similarly extend that sequence to a sequence of length k that violates (2.4). But by Lemma 2.35 this would be the only sequence of length k with $\mu^{k-1}(m_{k-1})\cdots\mu^0(m_0) = m$. This is impossible as we have found another such sequence (that does not violate (2.4)) in the proof of existence above. □

We may now use these two results to define a generalization of Dumont–Thomas numeration systems.

Definition 2.37. Let $\mu\colon A^* \to A^*$ be a substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $u_{-1}|u_0$ and period $p \geq 1$. Let $r \in \{0, 1, \dots, p-1\}$ be a residue modulo p. Define $\mathtt{c} = \max_{a\in A}|\mu(a)| - 1$ and the set $D = \{\mathtt{0}, \mathtt{1}, \dots, \mathtt{c}\}$. We define the map $\mathrm{rep}_{\mathbf{u},r}\colon \mathbb{Z} \to \{\mathtt{0},\mathtt{1}\}D^*, n \mapsto \mathrm{rep}_{\mathbf{u},r}(n)$ by

$$\mathrm{rep}_{\mathbf{u},r}(n) = \begin{cases} \mathtt{0}\cdot|m_{k-1}|\cdot|m_{k-2}|\cdots|m_0|, & \text{if } n \geq 0;\\ \mathtt{1}\cdot|m_{k-1}|\cdot|m_{k-2}|\cdots|m_0|, & \text{if } n \leq -1;\end{cases}$$

where $k = k(n)$ is the unique integer congruent to r modulo p and where $((m_i, a_i))_{i=0,\dots,k-1}$ is the unique sequence obtained from Theorem 2.33 (resp. Theorem 2.36) applied on the right-infinite periodic point $\mathbf{u}|_{\mathbb{N}} = u_0u_1\cdots$ (resp. the left-infinite periodic point $\mathbf{u}|_{\mathbb{Z}_{<0}} = \cdots u_{-2}u_{-1}$) with period p.

This numeration system is called the *Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$ and r*. When the context is clear, we drop the dependence on μ, $\mathbf{u}$ and r.

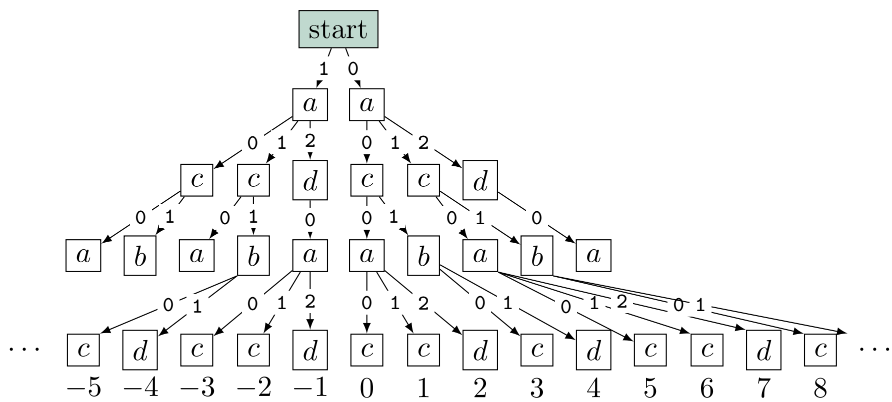
We have now defined two numeration systems both called the Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$ and r, one in Definition 2.32 and one in Definition 2.37. Thankfully, these two systems are one and the same.

Proposition 2.38. *The two numeration systems defined in Definition 2.32 and Definition 2.37 are equal.*

Proof. The link between admissible sequences and paths in $\mathcal{T}_{\mu,a}$ described in Proposition 1.37 can be extended to the tree $\mathcal{T}_{a,b}$. Working with an a-admissible sequence and prefixing 0 to the word $|m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|$ corresponds to paths from the root to the right subtree, while working with a b-admissible sequence and prefixing 1 corresponds to paths going to the left subtree.

For the former, the condition $(\mathbf{v}_r)_{[0,n-1]} = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)$ corresponds to requiring the path to go to column n and (2.2) corresponds to the path being the shortest path of length $r+1 \bmod p$ going to column n. For the latter, $(\mathbf{v}_r)_{[-|\mu^k(b)|,n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$ and (2.4) fill the same role. Thus the two definitions agree on the representation of every n. □

Example 2.39. Consider the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ and its periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{N}}(\mu)$ with growing seed $a|a$ and period $p = 2$. The tree $\mathcal{T}_{\mu,a|a}$ is depicted in Figure 2.2. Depending on the choice of even or odd length for representations, we obtain different numeration systems as illustrated in the table of Figure 2.2.



n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
$\mathrm{rep}_{u,0}(n)$	101	110	111	1	0	001	010	011	020	00100	00101	00110	01000	01001
$\mathrm{rep}_{u,1}(n)$	1111	10	11	12	00	01	02	0010	0011	0100	0101	0102	0110	0111

Figure 2.2: The tree $\mathcal{T}_{\mu,a|a}$ for the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ and the periodic point $\mathbf{u}$ of period $p = 2$ and seed $a|a$. Below, depending on the residue $r \in \{0,1\}$, we obtain a Dumont–Thomas numeration system and we give $(\mathrm{rep}_{\mathbf{u},r}(n))_{-4 \le n \le 9}$ whose lengths are congruent to $r+1 \bmod p$.

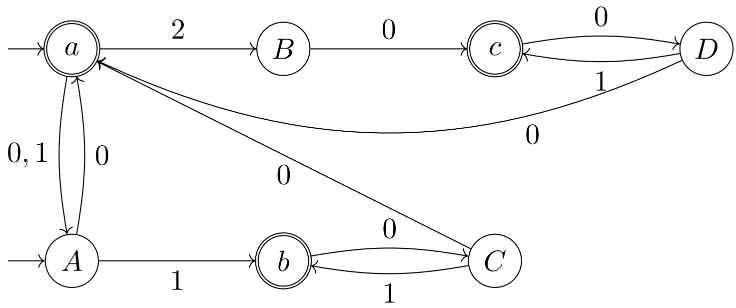


Figure 2.3: Automaton $\mathcal{A}$ built from the words $\lim_{n\to\infty} \mathrm{rep}_U(U_{2n} - 1)$ and $\lim_{n\to\infty} \mathrm{rep}_U(U_{2n+1} - 1)$.

In the special case where we need to represent only nonnegative numbers using a periodic point of a substitution, we slightly adapt the definition to not require the leading sign digit.

Definition 2.40. Let $\mu \colon A^* \to A^*$ be a substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a right-infinite periodic point with growing seed a and period $p \geq 1$. Let $r \in \{0, 1, \ldots, p-1\}$. The *Dumont–Thomas numeration system associated with* μ*,* $\mathbf{u}$ *and* r is defined by letting $\mathrm{rep}_{\mu,\mathbf{u},r}(n)$ be the label of the shortest path of length $r \bmod p$ from the root a to column n in the tree $\mathcal{T}_{b|a}$. Note that when the context is clear, we drop the dependence on μ, $\mathbf{u}$ and r.

2.3.1 A glimpse at a bigger picture

The aim of this subsection is to illustrate that basing Dumont–Thomas numeration systems on a periodic point instead of a fixed point corresponds to using an alternate base instead of a Rényi base, or to removing the dominant root condition from our positional numeration systems.

Example 2.41. We go back to example (1) from Example 2.31 and transfer it to the formalism of Dumont–Thomas systems. In this example, the maximal words were given by $\mathrm{rep}_U(U_{2n} - 1) = 20(01)^{n-1}$ and $\mathrm{rep}_U(U_{2n+1} - 1) = 1(01)^n$.

Let us consider the automaton $\mathcal{A}$ represented on Figure 2.3. This automaton was constructed based on the ultimately periodic words $20(01)^\omega$ and $1(01)^\omega$, in a manner reminiscent of the construction in Definition 1.49. We will see it again later, in the proof of Proposition 6.15 and in Example 6.19.

Let us study the language $L_{\mathcal{A}}$ accepted by this automaton. Since the underlying graph is bipartite between the lowercase and uppercase vertices, $\mathcal{A}$ accepts only words of even length when started on a and of odd length when started on A.

Now, we can show that the words accepted by $\mathcal{A}$ when starting on a lowercase letter are all also accepted when starting on a specifically, and the words accepted when starting on an uppercase letter are also accepted when starting on A specifically. The proof proceeds by induction on the length of the words. From this and from the construction of the automaton, we can deduce that $x \in L_{\mathcal{A}}, |y| = |x|, y <_{\text{lex}} x$ together imply $y \in L_{\mathcal{A}}$.

Since the lexicographical supremum of the words accepted starting from a is $20(01)^\omega$ and that starting from A is $1(01)^\omega$, we conclude that the language accepted by the automaton is

$$L_{\mathcal{A}} = \left\{ x_\ell \cdots x_1 x_0 : \begin{cases} x_{2j-1}x_{2j-2}\cdots x_0 \leq_{\text{lex}} 20(01)^\omega, \\ x_{2j}x_{2j-1}\cdots x_0 \leq_{\text{lex}} 1(01)^\omega \end{cases} \quad \text{for all } j \right\}$$

We notice that this is exactly the language L_U, where U is our target numeration.

Let us now build a Dumont–Thomas numeration system from the automaton $\mathcal{A}$. Much like in Definition 1.51, the underlying graph of $\mathcal{A}$ is associated with a substitution, which we can recover. This substitution is defined by

$$\mu\colon \begin{cases} a \mapsto AAB & A \mapsto ab \\ b \mapsto C & B \mapsto c \\ c \mapsto D & C \mapsto ab \\ & D \mapsto ac \end{cases}$$

which admits a periodic point of period 2 starting with a. Now, the graph of $\mathcal{A}$ can be unfolded into the tree $\mathcal{T}_{\mu,a}$, whose first few levels can be found on Figure 2.4.

The representations of the first few integers in the systems for $r = 0$ (even length) and $r = 1$ (odd length) are given in Table 2.5. Notice that the representations of the first 16 natural numbers in the system with $r = 0$ are consistent with a positional numeration system with the weights sequence starting by $(1, 2, 5, 7, \ldots)$ like our system U from Example 2.28.

We prove that this is not a coincidence, and that our Dumont–Thomas numeration system T associated with $\mu, \mathbf{u}$ and $r = 0$ is in fact equal to the

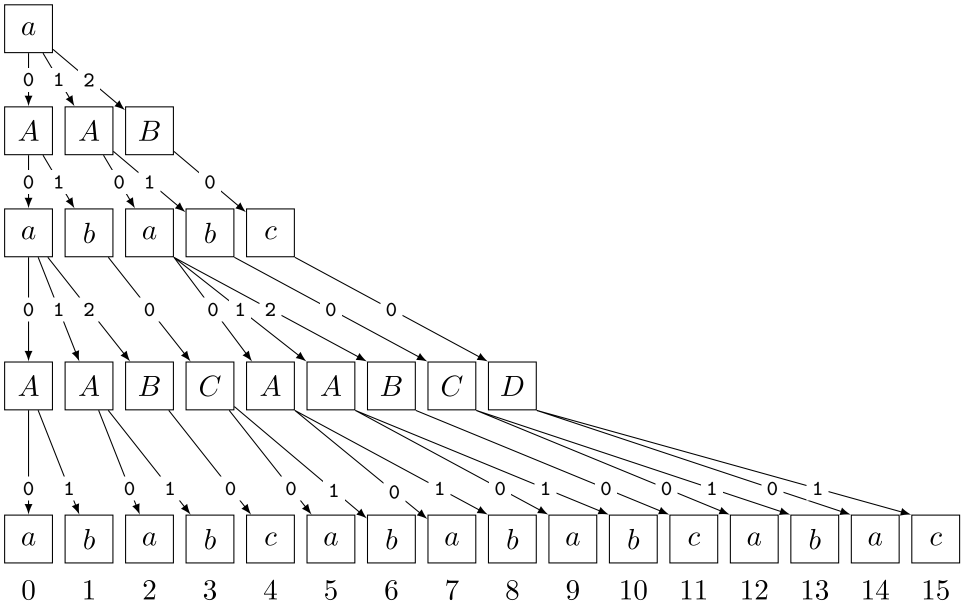


Figure 2.4: First levels of the tree $\mathcal{T}_{\mu,a}$.

n	$\text{rep}_0(n)$	$\text{rep}_1(n)$	n	$\text{rep}_0(n)$	$\text{rep}_1(n)$
0	ε	0	8	1001	200
1	01	1	9	1010	01000
2	10	2	10	1011	01001
3	11	010	11	1020	01002
4	20	100	12	1100	01010
5	0100	101	13	1101	10000
6	0101	102	14	2000	10001
7	1000	110	15	2001	10002

Table 2.5: Even and odd length representations of the first few integers for the system associated with μ and the periodic point starting with a.

system U of item (1) in Example 2.28, up to one modification: adding one leading zero to some representations in the system U so that all expansions have even length.

Indeed, it is clear that rep_T is increasing with the usual order of $\mathbb{N}$ and the radix order in $\{0,1,2\}^*$. Since rep_U also is, it is enough to prove that the sets $\mathrm{rep}_T(\mathbb{N})$ and $\mathrm{rep}_U(\mathbb{N})$ are equal (up to the occasional extra leading zero in T). But $\mathrm{rep}_T(\mathbb{N})$ is the set of shortest paths of even length from the root to columns in $\mathcal{T}_{\mu,a}$. Paths in this tree correspond to words that can be read in the automaton $\mathcal{A}$. Paths of even length from the root of the tree correspond to words of even length that are accepted by $\mathcal{A}$. Taking only shortest paths imposes the condition that the word may not start with 00. Since we have noticed that $L_{\mathcal{A}} = L_U$, we get that $\mathrm{rep}_T(\mathbb{N})$ is the set of words of L_U of even length that do not start with 00, while $\mathrm{rep}_U(\mathbb{N})$ is of course the set of words in L_U that do not start with 0. Thus our result is proved, and our system T (nearly) corresponds to the system U.

Seeing the system U as a Dumont–Thomas system gives us another point of view on its weights. We anticipate a bit on the contents of Chapter 7. From the form of the substitution μ, we can see that the system T will be positional, with the weight of position j being equal to $\left|\mu^j(a)\right|$ if j is even and $\left|\mu^j(A)\right|$ if j is odd.

To compute this, we start by simplifying the automaton $\mathcal{A}$ and its associated substitution. Notice that we can merge states b and c together in $\mathcal{A}$, as we can states A, C, D. If we then convert the minimized automaton back into the substitution

$$\mu' \colon a \mapsto AAB,\ b \mapsto A,\ A \mapsto ab, B \mapsto b,$$

the weights of the system T remain equal to $\left|\mu'^j(a)\right|$ if j is even and $\left|\mu'^j(A)\right|$ if j is odd, as the associated tree has not changed shape.

We notice that $(\mu')^2$ can be seen as a substitution on two letters (either $\{a,b\}$ or $\{A,B\}$), the incidence matrix of which has its characteristic polynomial equal to $X^2 - 3X - 1$. This implies that the sequence of weights in T is indeed a recurrence sequence with recurrence relation $U_{n+4} = 3U_{n+2} + U_n$. The initial conditions $(T_0, \ldots, T_3) = (1, 2, 5, 7)$ can then be computed. As an aside, we can remark that μ' can also be seen as the composition of two substitutions between the two alphabets $\{a,b\}$ and $\{A,B\}$. These two substitutions correspond to the substitutions $\psi(\mathcal{B})$ and $\psi(\sigma(\mathcal{B}))$ that were used in [CCMP25] to S-adically generate the $\mathcal{B}$-integers.

Note that, had we chosen the representations of odd length instead of those of even length, we would also have found a positional numeration system, this time with the weights being $\left|\mu^j(A)\right|$ if j is even and $\left|\mu^j(a)\right|$ if j is odd. As it turns out, this system is then associated to the alternate base $\left(\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2}\right)$, that is, the shift of the alternate base associated to T.

Finally, note that while T is not a Bertrand numeration system, given the structure of the tree $\mathcal{T}_a$ it is clear that for nonempty w, $w \in \mathrm{rep}_T(\mathbb{N})$ if and only if $w00 \in \mathrm{rep}_T(\mathbb{N})$. This sort of property is a good candidate for an extension of Bertrand numeration systems to the alternate case.

Chapter 3

Existence of alternate bases with given expansions of 1

We will now study alternate bases for a few chapters. To produce interesting examples, it will often be wise to specify not the *elements* of the alternate base, but rather the *expansions of* 1 in the alternate base and its shifts.

It is easy to prove in the Rényi case that the word **a** has value 1 in one unique base $\beta > 1$ if and only if **a** is lexicographically greater than 10^ω. Then, the Parry conditions can be used to decide if this word is the expansion of 1 or just a representation. However, in the alternate base case, the corresponding result, i.e., the existence of an alternate base with given representations of 1 is not yet well-understood.

In this chapter, we bridge this gap and establish conditions for given words to be the expansions of 1 in the alternate case. To do so, we use a fixed point theorem on matrices defined from the expansions and obtain the elements of the base from the components of the fixed point. We also obtain a partial result for the uniqueness of such a base. In the last section of the chapter, we use similar techniques to prove the existence of bases with a given sequence of $\mathcal{B}$-integers.

The contents of this chapter are adapted from an article written in collaboration with Émilie Charlier, Zuzana Masáková and Edita Pelantová, prepublished on arXiv [CKMP26b] but not currently published. It should be noted that further work on this topic was prepublished on arXiv by Šťovíček and Pelantová [ŠP26] as this thesis was being reviewed.

Contents

3.1 Introduction **74**

3.2 Preliminaries **76**

3.3 The Parry conditions characterize the greedy and the quasi-greedy expansions of 1 . . . . **79**

3.4 Uniqueness of the alternate base with prescribed greedy or quasi-greedy expansions of 1 . . . . **81**

3.5 Consequences of a theorem of Furstenberg **83**

3.6 Existence of an alternate base with given representations of 1 . . . . **88**

3.7 A family of sequences faithfully coding $\mathcal{B}$-integers **94**

3.7.1 Arnoux-Rauzy sequences 97

3.7.2 Sequences associated with N-continued fractions 99

3.1 Introduction

With the relevant notions introduced, we will now discuss five questions related to our numeration systems in the next five chapters. As we have seen in the previous chapter, the greedy and quasi-greedy expansions of 1, $\mathbf{d}_i$ and $\mathbf{d}_i^*$, are central to the study of alternate base systems and are connected to positional numeration systems as well. As a result, when discussing these families of systems, if we wish to provide examples to illustrate our points, it will be more useful to specify which values these expansions must take, rather than specifying the base elements directly. To anticipate a bit, this is how we proceeded to design Examples 5.7 and 5.8 and even Examples 6.52 and 6.53. We specified what form the expansions of 1 had to take, then found suitable values of $\beta_{p-1}, \dots, \beta_0$ that gave these expansions.

Our method to compute the base elements was inspired by the proof of [CCMP23, Theorem 14]: transform the conditions of having precise expansions of 1 into a matrix form, note that the product δ of the bases must be a root of some polynomial related to the determinant of this matrix, then obtain the values of the bases. However, this method relies on ad-hoc computations and assumptions on the desired expansions of 1. Before the results of this chapter, it was not known whether given certain expansions of 1, there was always a unique alternate base that corresponded to those expansions. As an aside, we note that although this chapter is presented first as the problem is the most fundamental in importance, this research was actually conducted last among all presented chapters, which can explain possible differences in exposition, notably in the examples.

In the Rényi case, it is quite straightforward to show that a sequence $a_1a_2a_3\cdots$ of nonnegative integers not ultimately vanishing serves as the quasi-greedy expansion of 1 for some base $\beta > 1$ if and only if its suffixes are all lexicographic less than or equal to the sequence itself. The proof is done in two steps. First, one realizes that there is a unique solution $X = \beta^{-1} \in (0, 1)$ of the equation

$$1 = \sum_{n\geq 1} a_n X^n$$

by monotonicity and continuity of the map $x \mapsto \sum_{n\geq 1} a_n x^n$ on the interval $(0, 1)$. Second, the lexicographic condition ensures that the β-representation $a_1a_2a_3\cdots$ is lexicographically maximal, i.e., it is the quasi-greedy expansion of 1 for β. See also Theorem 1.25.

As said above however, in the context of alternate bases, or even the two-way Cantor bases of Definition 2.19, the problem is more complicated, and only partial results are known – for instance, [CCMP25, Theorem 6.4] obtains uniqueness but only among some alternate bases. The aim of this article is to close this gap in knowledge in the alternate case. We work at first in the case of alternate bases, but with the notations of the Cantor case as our work relies on associating infinite sequences of matrices with a base.

The first part of the chapter is concerned with the following questions. Given a list of p sequences $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$, does there exist an alternate base $\mathcal{B}$ of length p such that these words are indeed the quasi-greedy expansions with respect to the p shifts of the base? Is such a base unique?

The answer to this problem is formulated in Theorem 3.19, which is our main result. The proof is divided into several steps. First, we show that the lexicographic conditions characterize the greedy and quasi-greedy expansions of 1 in a general Cantor real base (Section 3.3). Then, we address the question of the uniqueness of an alternate base with a given list of quasi-greedy expansions of 1. More precisely, we show that two distinct alternate bases of a given length p cannot share the same list of p quasi-greedy expansions of 1 with respect to the shifted bases $\sigma^i(\mathcal{B})$, for $i \in \{0, \ldots, p-1\}$, provided that these expansions do not start with the digit 1 (Section 3.4). Next, we turn to the existence part of the result. We start by deriving some tools from a fixed point theorem of Furstenberg (Section 3.5). Then we show that given a list of p sequences satisfying some light conditions, there exists an alternate base $\mathcal{B}$ such that these sequences are representations of 1 with respect to the shifted bases $\sigma^i(\mathcal{B})$, for $i \in \{0, \dots, p-1\}$ (Section 3.6). Finally, assuming that the p given sequences satisfy the lexicographic conditions, we derive that they are indeed equal to the quasi-greedy expansions of 1 with respect to the p shifted bases. It should be noted that our result for uniqueness is only partial. Edita Pelantová and Pavel Štovíček have claimed a complete proof of uniqueness in a personal communication to the authors.

In the second part of the paper we use our results in order to show that Cantor real base numeration systems can be used to provide a new interpretation to a large family of infinite words over finite alphabets, including the well-known and extensively studied Arnoux-Rauzy words, or words associated with N-continued fractions as introduced by Langeveld, Rossi and Thuswaldner [LRT23]. Indeed, these infinite words are found among the symbolic codings of sequences of real numbers with no fractional part with respect to a suitably chosen Cantor real base $\mathcal{B}$ (Section 3.7).

3.2 Preliminaries

In this section, we recall the basic notions associated with Cantor real bases that will be relevant throughout this article. We also set some notation. See Definition 2.19

Two-way Cantor real bases [CD20, CC21] offer a generalization of Rényi numeration systems. Rather than being directed by a single base, a whole sequence is given as the base of the numeration system. We let $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ be the base sequence and ask that $\beta_n > 1$ for all n and $\prod_{n=1}^{+\infty} \beta_n = \prod_{n=1}^{+\infty} \beta_{-n} = +\infty$. We consider bi-infinite words $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ over $\mathbb{N}$ with a left tail of zeros, i.e., for which there exists an $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n \geq N$ and $a_{N-1} \neq 0$. We write

$$\mathbf{a} = \begin{cases} a_{N-1} \cdots a_0 \bullet a_{-1} a_{-2} \cdots, & \text{if } N \geq 1; \\ 0 \bullet 0^{-N} a_{N-1} a_{N-2} \cdots, & \text{if } N \leq 0. \end{cases}$$

We define the evaluation of such words as

$$\mathrm{val}_{\mathcal{B}}(\mathbf{a}) = \sum_{n=0}^{+\infty} a_n \beta_{n-1} \cdots \beta_0 + \sum_{n=1}^{+\infty} \frac{a_{-n}}{\beta_{-1} \cdots \beta_{-n}}$$

where the first sum is always finite due to the left tail of zeros. If $\mathrm{val}_{\mathcal{B}}(\mathbf{a}) < +\infty$, then we say that $\mathbf{a}$ is a *$\mathcal{B}$-representation* of the number $\mathrm{val}_{\mathcal{B}}(\mathbf{a})$.

A number might have multiple $\mathcal{B}$-representations. We select the largest representation in the lexicographic order to be the (canonical) *$\mathcal{B}$-expansion* of a given number, where the lexicographic order $<_{\mathrm{lex}}$ is defined by $\mathbf{a} <_{\mathrm{lex}} \mathbf{b}$ if there exists an index $i \in \mathbb{Z}$ such that $a_j = b_j$ for all $j > i$ and $a_i < b_i$. We denote the $\mathcal{B}$-expansion of the number x by $\langle x \rangle_{\mathcal{B}}$. It can be interpreted with a greedy algorithm iteratively computing its digits a_n in decreasing indexing. For $x \geq 1$, let N be the minimal integer such that $x < \beta_{N-1} \cdots \beta_0$ and for $x \in [0, 1)$, let $N = 0$. Set $a_n = 0$ for $n \geq N$. Then set $r_N = \frac{x}{\beta_{N-1} \cdots \beta_0}$ and for all $n \leq N-1$, define $a_n = \lfloor \beta_n r_{n+1} \rfloor$ and $r_n = \beta_n r_{n+1} - a_n$. The following lemma (see [CC21, Corollary 11] and [CCMP25, Lemma 3.1], reproduced here as Proposition 2.20) gives a way to check that the given sequence is a greedy expansion of some real number.

Lemma 3.1. *Let $\mathcal{B} = (\beta_n)_{n \in \mathbb{Z}}$ be a Cantor real base. A sequence $(a_n)_{n \in \mathbb{Z}}$ of nonnegative integers, with $a_n = 0$ for sufficiently large positive n, is the $\mathcal{B}$-expansion of some nonnegative real number if and only if we have*

$$\sum_{n=0}^{k} a_n \beta_{n-1} \cdots \beta_0 + \sum_{n=1}^{\infty} \frac{a_{-n}}{\beta_{-1} \cdots \beta_{-n}} < \beta_k \cdots \beta_0 \quad \text{for all } k \geq 0$$

and

$$\sum_{n=k}^{\infty} \frac{a_{-n}}{\beta_{-1} \cdots \beta_{-n}} < \frac{1}{\beta_{-1} \cdots \beta_{-k+1}} \quad \text{for all } k \geq 1.$$

In what follows, we will be working not only with the base $\mathcal{B}$ itself but also with its shifts. We define $\sigma(\mathcal{B}) = (\beta_{n-1})_{n \in \mathbb{Z}}$. That is, σ maps the base $(\cdots \beta_0 \bullet \beta_{-1} \beta_{-2} \cdots)$ to $(\cdots \beta_0 \beta_{-1} \bullet \beta_{-2} \cdots)$. Iterating the map σ, we get $\sigma^i(\mathcal{B}) = (\beta_{n-i})_{n \in \mathbb{Z}}$ for $i \in \mathbb{Z}$. If $\sigma^p(\mathcal{B}) = \mathcal{B}$ for some $p \geq 1$, meaning that the sequence $\mathcal{B}$ is periodic with period p, we speak of an *alternate base*. In this case, we will abuse notation by simply writing $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, and we call p the *length* of $\mathcal{B}$.

We recall that while the $\mathcal{B}$-expansion of 1 is always given by $\langle 1\rangle_{\mathcal{B}} = 1\bullet 0^\omega$, the one-sided infinite words $\mathbf{d}$ and $\mathbf{d}^*$, corresponding to the greedy and quasi-greedy representations of 1 in the one-way case, will still be of great interest to us.

We define the infinite word $\mathbf{d}_n$ as the lexicographically greatest right-infinite word $\mathbf{t}$ such that $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0\bullet\mathbf{t}) = 1$, whereas $\mathbf{d}_n^*$ is defined as the lexicographically greatest infinite word $\mathbf{d}$ *not ending in a tail of zeros* such that $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0\bullet\mathbf{d}) = 1$. Thus, $\mathbf{d}_n$ is the greedy representation of 1 in the one-way base $(\beta_{n-1}, \beta_{n-2}, \ldots)$, whereas $\mathbf{d}_n^*$ is the quasi-greedy representation of 1 in this same base. By a slight abuse of language, we will talk about $\mathbf{d}_n$ and $\mathbf{d}_n^*$ as the *greedy* and *quasi-greedy* $\sigma^{-n}(\mathcal{B})$-expansions of 1, respectively. Clearly, the words $\mathbf{d}_n$ and $\mathbf{d}_n^*$ may differ only when $\mathbf{d}_n$ ends in a tail of zeros. We let $t_{n,i}$ and $d_{n,i}$, with $i \ge 1$, denote the digits of these words:

$$\mathbf{d}_n = t_{n,1}t_{n,2}t_{n,3}\cdots \quad \text{and} \quad \mathbf{d}_n^* = d_{n,1}d_{n,2}d_{n,3}\cdots$$

Note that we use an increasing indexing for these (one-sided) infinite words. The quasi-greedy expansions of 1 can be recursively obtained from the greedy expansions of 1 as follows:

$$\mathbf{d}_n^* = \begin{cases} t_{n,1}\cdots t_{n,\ell-1}(t_{n,\ell}-1)\mathbf{d}_{n-\ell}^*, & \text{if } \mathbf{d}_n = t_{n,1}\cdots t_{n,\ell}0^\omega \text{ with } t_{n,\ell}\ge 1,\\ \mathbf{d}_n, & \text{otherwise.}\end{cases}$$

Equivalently, the quasi-greedy expansions of 1 are obtained as limits with respect to the product topology on infinite words: for all $n \in \mathbb{Z}$, we have $\lim_{x\to 1^-}\langle x\rangle_{\sigma^{-n}(\mathcal{B})} = 0\bullet\mathbf{d}_n^*$.

The interest of greedy and quasi-greedy expansions of 1 lies in the fact that they can be used to determine when a given $\mathcal{B}$-representation of a number is its $\mathcal{B}$-expansion. See for instance the following result.

Theorem 3.2 ([CCMP25, Corollary 4.4])**.** *A $\mathcal{B}$-representation $a_{N-1}\cdots a_0\bullet a_{-1}a_{-2}\cdots$ of a nonnegative real number is its $\mathcal{B}$-expansion if and only if $a_{n-1}a_{n-2}\cdots <_{\mathrm{lex}} \mathbf{d}_n^*$ for all $n \le N$.*

Focusing on representations of 1, we obtain the following corollary.

Corollary 3.3. *Let $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ be a Cantor real base. For all $n \in \mathbb{Z}$, we have the following properties.*

1. *The sequences* $\mathbf{d}_n$ *and* $\mathbf{d}_n^*$ *start with a nonzero digit.*

2. *The sequence* $\mathbf{d}_n^*$ *does not end in a tail of zeros.*

3. *We have* $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{d}_n) = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{d}_n^*) = 1$.

4. *A sequence* $\mathbf{a}_n = a_{n,1}a_{n,2}\cdots$ *of nonnegative integers with* $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(\mathbf{a}) = 1$ *is equal to* $\mathbf{d}_n$ *if and only if*

$$\mathrm{val}_{\sigma^{-n+j}(\mathcal{B})}(0 \bullet a_{n,j+1}a_{n,j+2}\cdots) < 1$$

for all $j \geq 1$.

5. *(The so-called Parry conditions) We have*

$$t_{n,j+1}t_{n,j+2}\cdots <_{\mathrm{lex}} \mathbf{d}_{n-j}^* \tag{3.1}$$

and

$$d_{n,j+1}d_{n,j+2}\cdots \leq_{\mathrm{lex}} \mathbf{d}_{n-j}^*. \tag{3.2}$$

for all $j \geq 1$.

Proof. The first four properties were shown in [CC21]. The Parry condition (3.1) for the greedy expansions $\mathbf{d}_n$ is a direct consequence of Theorem 3.2. Let now $n \in \mathbb{Z}$ be fixed and let us show the Parry condition for the quasi-greedy expansion $\mathbf{d}_n^*$. Let $(\ell_k)_{k\geq 0}$ be the (finite or infinite) sequence of lengths of the finite greedy-expansions (i.e., not ending in a tail of zeros) that are encountered in the computation of $\mathbf{d}_n^*$. If $j = \ell_1 + \cdots + \ell_k$ for some k, then $d_{n,j+1}d_{n,j+2}\cdots = \mathbf{d}_{n-j}^*$, in which case we are fine. Otherwise, choose k maximal such that $\ell = \ell_1 + \cdots + \ell_k < j$. Then $d_{n,j+1}d_{n,j+2}\cdots \leq_{\mathrm{lex}} t_{n-\ell,j-\ell+1}t_{n-\ell,j-\ell+2}\cdots$, hence we obtain that $d_{n,j+1}d_{n,j+2}\cdots <_{\mathrm{lex}} \mathbf{d}_{n-j}^*$ by using the Parry condition for $\mathbf{d}_{n-\ell}$. □

3.3 The Parry conditions characterize the greedy and the quasi-greedy expansions of 1

First, we recall the following result from [CC21].

Lemma 3.4. *([CC21, Lemma 25]) Let* $\mathcal{B}$ *be a Cantor real base and for all* $n \leq 0$, *let* $\mathbf{a}_n = a_{n,1}a_{n,2}\cdots$ *be a sequence of nonnegative integers such that*

$\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{a}_n) = 1$. *If* $\mathbf{b} = b_1 b_2 \cdots$ *is a sequence of nonnegative integers such that* $b_{n+1}b_{n+2}\cdots <_{\mathrm{lex}} \mathbf{a}_{-n}$ *for all* $n \geq 0$*, then we have* $\mathrm{val}_{\sigma^n(\mathcal{B})}(0 \bullet b_{n+1}b_{n+2}\cdots) < 1$ *for all* $n \geq 0$*, unless there exist* $n \geq 0$ *and* $\ell \geq 1$ *such that*

- $\mathbf{a}_{-n} = a_{-n,1}\cdots a_{-n,\ell}0^\omega$ *where* $a_{-n,\ell} \neq 0$*,*
- $b_{n+1}\cdots b_{n+\ell} = a_{-n,1}\cdots a_{-n,\ell-1}(a_{-n,\ell}-1)$*,*
- $\mathrm{val}_{\sigma^{n+\ell}(\mathcal{B})}(b_{n+\ell+1}b_{n+\ell+2}\cdots) = 1$*,*

in which case we have $\mathrm{val}_{\sigma^n(\mathcal{B})}(0 \bullet b_{n+1}b_{n+2}\cdots) = 1$.

With the next result, we show that the Parry conditions from Corollary 3.3 characterize the greedy and the quasi-greedy expansions of 1.

Proposition 3.5. *Let* $\mathcal{B}$ *be a Cantor real base and for all* $n \in \mathbb{Z}$*, let* $\mathbf{a}_n = a_{n,1}a_{n,2}\cdots$ *be a sequence of nonnegative integers such that* $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{a}_n) = 1$*. Assume moreover that for all* $n \in \mathbb{Z}$*, if* $\mathbf{a}_n$ *ends in* 0^ω *then*

$$a_{n,j+1}a_{n,j+2}\cdots <_{\mathrm{lex}} \mathbf{a}_{n-j} \text{ for all } j \geq 1,$$

and if $\mathbf{a}_n$ *does not end in* 0^ω *then*

$$a_{n,j+1}a_{n,j+2}\cdots \leq_{\mathrm{lex}} \mathbf{a}_{n-j} \text{ for all } j \geq 1.$$

Then we have $\mathbf{a}_n = \mathbf{d}_n$ *for all* $n \in \mathbb{Z}$ *such that* $\mathbf{a}_n$ *ends in a tail of zeros, and we have* $\mathbf{a}_n = \mathbf{d}_n^*$ *for all* $n \in \mathbb{Z}$ *such that* $\mathbf{a}_n$ *does not end in a tail of zeros.*

Proof. Let $n \in \mathbb{Z}$ be fixed. First suppose that $\mathbf{a}_n$ ends in a tail of zeros. By Corollary 3.3, in order to get that $\mathbf{a}_n = \mathbf{d}_n$, it suffices to show that $\mathrm{val}_{\sigma^{-n+j}(\mathcal{B})}(0 \bullet a_{n,j+1}a_{n,j+2}\cdots) < 1$ for all $j \geq 1$. Assume by contradiction that there exists some $j \geq 1$ with $\mathrm{val}_{\sigma^{-n+j}(\mathcal{B})}(0 \bullet a_{n,j+1}a_{n,j+2}\cdots) \geq 1$. Since $\mathbf{a}_n$ ends in 0^ω, there exist only finitely many such j. We pick the greatest such j. Then Lemma 3.4 implies that there exists some $\ell > j$ such that $\mathrm{val}_{\sigma^{-n+\ell}(\mathcal{B})}(0 \bullet a_{n,\ell+1}a_{n,\ell+2}\cdots) = 1$, which obviously contradicts the maximality of j.

Second, suppose that $\mathbf{a}_n$ does not end in a tail of zeros. Let $(x_\ell)_{\ell \geq 0}$ be the sequence of numbers defined by $x_\ell = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet a_{n,1}\cdots a_{n,\ell}0^\omega)$. We have $\lim_{\ell\to+\infty} x_\ell = 1$. Let now fix an $\ell \geq 0$. By Lemma 3.4, we know that $\mathrm{val}_{\sigma^{-n+j}(\mathcal{B})}(0 \bullet a_{n,j+1}\cdots a_{n,\ell}0^\omega) < 1$ for all $j \in \{0,\ldots,\ell-1\}$. Then Lemma 3.1 tells us that $0 \bullet a_{n,1}\cdots a_{n,\ell}0^\omega = \langle x_\ell \rangle_{\sigma^{-n}(\mathcal{B})}$. We obtain that $\mathbf{a}_n = \mathbf{d}_n^*$ by letting ℓ tend to infinity. $\square$

The following two corollaries are immediate.

Corollary 3.6. *Let $\mathcal{B}$ be a Cantor real base and for all $n \in \mathbb{Z}$, let $\mathbf{a}_n = a_{n,1}a_{n,2}\cdots$ be a sequence of nonnegative integers ending in a tail of zeros such that* $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{a}_n) = 1$ *and $a_{n,j+1}a_{n,j+2}\cdots <_{\mathrm{lex}} \mathbf{a}_{n-j}$ for all $j \geq 1$. Then $\mathbf{a}_n = \mathbf{d}_n$ for all $n \in \mathbb{Z}$.*

Corollary 3.7. *Let $\mathcal{B}$ be a Cantor real base and for all $n \in \mathbb{Z}$, let $\mathbf{a}_n = a_{n,1}a_{n,2}\cdots$ be a sequence of nonnegative integers not ending in a tail of zeros such that* $\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet \mathbf{a}_n) = 1$ *and $a_{n,j+1}a_{n,j+2}\cdots \leq_{\mathrm{lex}} \mathbf{a}_{n-j}$ for all $j \geq 1$. Then $\mathbf{a}_n = \mathbf{d}_n^*$ for all $n \in \mathbb{Z}$.*

3.4 Uniqueness of the alternate base with prescribed greedy or quasi-greedy expansions of 1

In [CCMP25], it was shown that there is at most one alternate base with a given list of eventually periodic quasi-greedy expansions of 1. In this section, we partially solve the question on the uniqueness of the base $\mathcal{B}$ in the non-Parry case.

Proposition 3.8. *Let $p \geq 1$ be an integer, let α be the zero of the polynomial $X^p - X^{p-1} - \cdots - 1$ in $[1, 2)$, and let $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ be an alternate base such that $\beta_i > \alpha$ for every i. Then no other alternate base of length p has the same list of greedy (resp. quasi-greedy) expansions of* 1.

Proof. Let $\Gamma = (\gamma_{p-1}, \ldots, \gamma_0)$ be an alternate base such that for all $i \in \{0, \ldots, p-1\}$, the greedy $\sigma^{-i}(\Gamma)$-expansion of 1 coincides with the greedy $\sigma^{-i}(\mathcal{B})$-expansion of 1, which we denote $\mathbf{d}_i = t_{i,1}t_{i,2}t_{i,3}\cdots$.

Let $i \in \{0, \ldots, p-1\}$. The digits $t_{i,n}$ are obtained thanks to the greedy algorithm as follows. We set $r_{i,0} = 1$ and for all $n \geq 1$, set $t_{i,n} = \lfloor \gamma_{i-n} r_{i,n-1} \rfloor$ and $r_{i,n} = \gamma_{i-n} r_{i,n-1} - t_{i,n}$. Thus, we have $r_{i,n} \in [0, 1)$ and $t_{i,n} = \gamma_{i,n} r_{i,n-1} - r_{i,n}$ for all $n \geq 1$. We then have

$$1 = \mathrm{val}_{\sigma^{-i}(\mathcal{B})}(0 \bullet \mathbf{d}_i) = \sum_{n=1}^{+\infty} \frac{t_{i,n}}{\beta_{i-1}\cdots\beta_{i-n}} = \sum_{n=1}^{+\infty} \frac{\gamma_{i-n} r_{i,n-1} - r_{i,n}}{\beta_{i-1}\cdots\beta_{i-n}}.$$

By using that $r_{i,0} = 1$ and by rearranging the terms, we obtain

$$0 = \sum_{n=0}^{+\infty} \left(\frac{\gamma_{i-n-1}}{\beta_{i-n-1}} - 1 \right) \frac{r_{i,n}}{\beta_{i-1} \cdots \beta_{i-n}}.$$

Grouping the terms p by p and by setting $\delta = \beta_{p-1} \cdots \beta_0$, we then get

$$0 = \sum_{j=0}^{p-1} \left(\frac{\gamma_{i-j-1}}{\beta_{i-j-1}} - 1 \right) \frac{1}{\beta_{i-1} \cdots \beta_{i-j}} \sum_{n=0}^{+\infty} \frac{r_{i,pn+j}}{\delta^n}.$$

For all $j \in \{0, \ldots, p-1\}$, by setting

$$E_{i,j} = \frac{1}{\beta_{i-1} \cdots \beta_{i-j}} \sum_{n=0}^{+\infty} \frac{r_{i,pn+j}}{\delta^n}$$

and

$$F_{i,j} = \frac{\gamma_{i-j-1}}{\beta_{i-j-1}} - 1,$$

we can rewrite the latter equality as

$$0 = \sum_{j=0}^{p-1} E_{i,j} F_{i,j}.$$

Observe that for $i \in \{0, \ldots, p-1\}$, the row vectors

$$F_i = (F_{i,0}, \ldots, F_{i,p-1})$$

are circular permutations of one another. In particular, we have

$$F_0 = (F_{i,i}, \ldots, F_{i,i+p-1})$$

where the second indices are seen modulo p. We obtain the matrix equality

$$EF_0^{\intercal} = 0$$

where

$$E = \begin{pmatrix} E_{0,0} & E_{0,1} & \cdots & E_{0,p-2} & E_{0,p-1} \\ E_{p-1,p-1} & E_{p-1,0} & \cdots & E_{p-1,p-3} & E_{p-1,p-2} \\ \vdots & & & & \\ E_{1,1} & E_{1,2} & \cdots & E_{1,p-1} & E_{1,0} \end{pmatrix}.$$

The matrix E has nonnegative entries. Let us show that it is diagonally dominant, i.e., that we have

$$E_{i,0} > \sum_{j=1}^{p-1} E_{i,j} \tag{3.3}$$

for all $i \in \{0, \ldots, p-1\}$. Indeed, we have $E_{i,0} \geq r_{i,0} = 1$ for all $i \in \{0, \ldots, p-1\}$. If $p = 1$, this is already enough. Otherwise, by using the hypothesis that $\beta_i > \alpha$ for all $i \in \{0, \ldots, p-1\}$ and by using that $r_{i,n} \leq 1$ for all $n \geq 0$, we obtain that

$$\sum_{j=1}^{p-1} E_{i,j} < \sum_{j=1}^{p-1} \frac{1}{\alpha^j} \frac{\alpha^p}{\alpha^p - 1} = \frac{\sum_{j=1}^{p-1} \alpha^{p-j}}{\alpha^p - 1} = 1 \leq E_{i,0}$$

as announced.

Hadamard's lemma then tells us that $\det(E) \neq 0$, which implies in turn that F_0 is the zero vector. We then get that $\gamma_i = \beta_i$ for all $i \in \{0, \ldots, p-1\}$, that is, $\Gamma = \mathcal{B}$.

The proof for the quasi-greedy expansions is similar. The digits of the quasi-greedy expansions $\mathbf{d}_n^* = d_{n,1}d_{n,2}\cdots$ are obtained by the following quasi-greedy algorithm. We set $r_{i,0} = 1$ and for all $n \geq 1$, set $d_{i,n} = \lceil \gamma_{i-n} r_{i,n-1} \rceil - 1$ and $r_{i,n} = \gamma_{i-n} r_{i,n-1} - d_{i,n}$. Thus, we have $r_{i,n} \in (0, 1]$ and $d_{i,n} = \gamma_{i-n} r_{i,n-1} - r_{i,n}$ for all $n \geq 1$. The rest of the proof is easily adapted. □

Corollary 3.9. *Let $p \geq 1$ be an integer and let $\mathcal{B}$ be an alternate base of length p such that for all $i \in \{0, 1, \ldots, p-1\}$, the greedy (resp. the quasi-greedy) $\sigma^{-i}(\mathcal{B})$-expansion of 1 starts with a digit that is at least 2. Then no other alternate base of length p has the same list of greedy (resp. quasi-greedy) expansions of 1.*

3.5 Consequences of a theorem of Furstenberg

In this section, we prepare a tool that we will use to prove the existence of Cantor real bases with prescribed representations of 1. The origin of the following theorem can be traced back to Furstenberg [Fur60]; also see [Bir57]. We present this result in a matrix formulation similar to the one often used when dealing with S-adic sequences [BT04, BD14, BMST16]. We note, however, that these articles rely on an implicit primitivity-like assumption which ensures that certain quantities are positive, notably the row vector f of the next statement. As we do not have such an assumption, we must work in general with nonnegative quantities. When required, we will argue that the quantities at hand are positive. See Remark 3.14 for an example where f is only nonnegative.

Theorem 3.10. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence matrices in $\mathbb{N}^{k\times k}$. Assume that there exist a positive matrix P and indices $i_1 < j_1 \le i_2 < j_2 \le \cdots$ such that*

$$\cdots = A_{j_2-1}\cdots A_{i_2} = A_{j_1-1}\cdots A_{i_1} = P.$$

Then there exists a row vector $f \in \mathbb{R}^k_{\ge 0}$ such that

$$\bigcap_{n\ge 0} \mathbb{R}^k_{\ge 0} A_{n-1}\cdots A_0 = \mathbb{R}_{\ge 0} f.$$

Proposition 3.11. *Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of nonnegative integer square matrices such that $(A_n)_{1,1} > 0$ for all $n \in \mathbb{Z}$ and such that $(A_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of Theorem 3.10. Then there exist unique sequences $(\gamma_n)_{n\in\mathbb{Z}}$ of positive real numbers and $(f_n)_{n\in\mathbb{Z}}$ of nonnegative real row vectors with first entry equal to 1 such that*

$$\gamma_n f_{n-1} = f_n A_n \tag{3.4}$$

for all $n \in \mathbb{Z}$. Moreover, if the sequence $(A_n)_{n\in\mathbb{Z}}$ is periodic, then the sequences $(\gamma_n)_{n\in\mathbb{Z}}$ and $(f_n)_{n\in\mathbb{Z}}$ are periodic with the same period.

Proof. For all $n \in \mathbb{Z}$, since the sequence $(A_{n+1+i})_{i\in\mathbb{N}}$ also satisfies the assumptions of Theorem 3.10, there exists a nonnegative real row vector f_n such that

$$\bigcap_{i\ge 0} \mathbb{R}^k_{\ge 0} A_{n+i}\cdots A_{n+1} = \mathbb{R}_{\ge 0} f_n. \tag{3.5}$$

Write $f_n = (f_{n,1}, \ldots, f_{n,k})$. We begin by proving that the row vectors f_{i_r-1} are positive for all $r \ge 1$, where $(i_r)_{r\ge 1}$ is the sequence of indices from Theorem 3.10. Indeed, the intersection

$$\bigcap_{j\ge 0} \mathbb{R}^k_{\ge 0} A_{i_r+j}\cdots A_{i_r} = \bigcap_{j\ge 0} \mathbb{R}^k_{\ge 0} A_{j_r+j}\cdots A_{j_r} P$$

is decreasing and, since $(A_n)_{1,1} > 0$ for all n, each product contains a vector with positive first coordinate. If we consider only vectors of norm 1 in each product, we obtain a decreasing sequence of nonempty compact sets, which must have a nonempty intersection. Thus the intersection contains a nonzero vector. This nonzero vector must belong to $\mathbb{R}^k_{\geq 0}P$, where P is the positive matrix from Theorem 3.10. But this space contains only the zero vector and vectors in $\mathbb{R}^k_{>0}$. Thus f_{i_r-1} is positive, as claimed.

Multiplying now each side of (3.5) by A_n leads to $\mathbb{R}_{\geq 0}f_{n-1} = \mathbb{R}_{\geq 0}f_nA_n$. Now, since $(A_n)_{1,1} > 0$, we have that $f_{n,1} > 0$ implies $f_{n-1,1} > 0$. We deduce that $f_{n,1} > 0$ for all $n \in \mathbb{Z}$. Without loss of generality, we may assume that this first component of f_n is 1 for all $n \in \mathbb{Z}$. Since $f_nA_n \in \mathbb{R}_{\geq 0}f_{n-1}$, we find that there exists $\gamma_n \geq 0$ such that $\gamma_n f_{n-1} = f_nA_n$. Given that the first components of f_n and f_{n-1} are positive, the numbers γ_n must be positive. □

Proposition 3.12. *Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of matrices in $\mathbb{N}^{k\times k}$, with $k \geq 2$, such that $(A_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 3.11. Suppose moreover that there exists $h \in \{1,\ldots,k-1\}$ such that all matrices A_n are of the form*

$$\left(\begin{array}{ccc|c} a_{n,1} & \cdots & a_{n,k-1} & a_{n,k} \\ \hline & & & \\ & \mathbf{I} & & \mathbf{e}_h \\ & & & \end{array}\right)$$

where $a_{n,1} \geq 1$, $\mathbf{I}$ is the identity matrix of size $k-1$ and $\mathbf{e}_h$ is the column matrix of size $k-1$ having 1 in position h and 0 elsewhere. Otherwise stated, we have

$$(A_n)_{i,j} = \begin{cases} a_{n,j}, & \text{if } i = 1; \\ 1, & \text{if } i = j+1; \\ 1, & \text{if } i = h+1 \text{ and } j = k; \\ 0, & \text{otherwise} \end{cases}$$

with the condition $a_{n,1} \geq 1$. Let $(\gamma_n)_{n\in\mathbb{Z}}$ and $(f_n)_{n\in\mathbb{Z}}$ be the sequences from Proposition 3.11, and write $f_n = (f_{n,1},\ldots,f_{n,k})$. Then for all $n \in \mathbb{Z}$, we have

(a) $\gamma_n > 1$

(b) $1 = \displaystyle\sum_{j=1}^{h} \frac{a_{n+j,j}}{\gamma_{n+1}\cdots\gamma_{n+j}} + \frac{f_{n+h,h+1}}{\gamma_{n+1}\cdots\gamma_{n+h}}$

(c) $f_{n+h,h+1} = \sum_{j=h+1}^{k} \frac{a_{n+j,j}}{\gamma_{n+h+1}\cdots\gamma_{n+j}} + \frac{f_{n+k,h+1}}{\gamma_{n+h+1}\cdots\gamma_{n+k}}$.

Proof. We first prove that the row vectors f_n are positive for all $n \in \mathbb{Z}$. As in the proof of Proposition 3.11, if $(i_r)_{r\geq 1}$ is the sequence of indices from Theorem 3.10, then the row vectors f_{i_r-1} are all positive. Since the matrices A_n have a positive entry in each column, any product fA_n where f is a positive row vector is itself a positive row vector. Since $\gamma_n f_{n-1} = f_n A_n$, we see that f_n being positive implies that f_{n-1} also is. So we can conclude that $f_n > 0$ for all n.

Consider now $j \in \{1, \ldots, k\}$. Looking at the j^{th} component of (3.4) and realizing that $f_{n,1} = 1$, we find that

$$\gamma_n f_{n-1,j} = \begin{cases} a_{n,j} + f_{n,j+1}, & \text{if } j < k; \\ a_{n,k} + f_{n,h+1}, & \text{if } j = k \end{cases}$$

for all $n \in \mathbb{Z}$. In particular, for $j = 1$, we get that $\gamma_n = a_{n,1} + f_{n,2} > 1$, which is (a). Let $n \in \mathbb{Z}$. The previous equality taken in $n + j$ instead of n yields

$$\gamma_{n+j} f_{n+j-1,j} = \begin{cases} a_{n+j,j} + f_{n+j,j+1}, & \text{if } j < k; \\ a_{n+j,k} + f_{n+j,h+1}, & \text{if } j = k. \end{cases} \tag{3.6}$$

For $j = 1, \ldots, h$, we divide the equality (3.6) by $\gamma_{n+1}\cdots\gamma_{n+j}$ and sum up. We obtain

$$\sum_{j=1}^{h} \frac{f_{n+j-1,j}}{\gamma_{n+1}\cdots\gamma_{n+j-1}} = \sum_{j=1}^{h} \frac{a_{n+j,j}}{\gamma_{n+1}\cdots\gamma_{n+j}} + \sum_{j=1}^{h} \frac{f_{n+j,j+1}}{\gamma_{n+1}\cdots\gamma_{n+j}},$$

which gives

$$1 + \sum_{j=2}^{h} \frac{f_{n+j-1,j}}{\gamma_{n+1}\cdots\gamma_{n+j-1}} = \sum_{j=1}^{h} \frac{a_{n+j,j}}{\gamma_{n+1}\cdots\gamma_{n+j}} + \sum_{j=2}^{h+1} \frac{f_{n+j-1,j}}{\gamma_{n+1}\cdots\gamma_{n+j-1}}.$$

Deleting identical terms, we obtain (b). For $j = h + 1, \ldots, k$, we divide the equality (3.6) by $\gamma_{n+h+1}\cdots\gamma_{n+j}$ and sum up. We obtain

$$\sum_{j=h+1}^{k} \frac{f_{n+j-1,j}}{\gamma_{n+h+1}\cdots\gamma_{n+j-1}} =$$
$$\sum_{j=h+1}^{k} \frac{a_{n+j,j}}{\gamma_{n+h+1}\cdots\gamma_{n+j}} + \sum_{j=h+1}^{k-1} \frac{f_{n+j,j+1}}{\gamma_{n+h+1}\cdots\gamma_{n+j}} + \frac{f_{n+k,h+1}}{\gamma_{n+h+1}\cdots\gamma_{n+k}}. \tag{3.7}$$

Simplifying as in the previous case, we obtain (c). □

The following result will be used in Section 3.7 in order to show that all S-adic sequences obtained by using some well-identified family of substitutions can be obtained as the faithful coding of the $\mathcal{B}$-integers for some Cantor real base $\mathcal{B}$.

Proposition 3.13. *Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of matrices in $\mathbb{N}^{k\times k}$, with $k \geq 2$, such that $(A_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 3.11. Suppose moreover that the matrices A_n are of the form*

$$\left(\begin{array}{ccc|c} a_{n,1} & \cdots & a_{n,k-1} & a_{n,k} \\ \hline & & & \\ & \mathbf{I} & & \mathbf{0} \\ & & & \end{array}\right)$$

where $a_{n,1} \geq 1$, $\mathbf{I}$ is the identity matrix of size $k-1$ and $\mathbf{0}$ is the zero column matrix of size $k-1$. Otherwise stated, we have

$$(A_n)_{i,j} = \begin{cases} a_{n,j}, & \text{if } i = 1; \\ 1, & \text{if } i = j+1; \\ 0, & \text{otherwise} \end{cases}$$

with the condition $a_{n,1} \geq 1$. Let $(\gamma_n)_{n\in\mathbb{Z}}$ be the sequence from Proposition 3.11. Then for all $n \in \mathbb{Z}$, we have

(a) $\gamma_n \geq 1$

(b) $1 = \sum_{j=1}^{k} \frac{a_{n+j,j}}{\gamma_{n+1}\cdots\gamma_{n+j}}$.

Assuming moreover that $a_{n,k} > 0$ for all $n \in \mathbb{Z}$, then we have $\gamma_n > 1$ for all $n \in \mathbb{Z}$.

Proof. Let $j \in \{1, \ldots, k\}$. Looking at the j^{th} component of the equality (3.4) and realizing that $f_{n,1} = 1$, we see that

$$\gamma_n f_{n-1,j} = \begin{cases} a_{n,j} + f_{n,j+1}, & \text{if } j < k; \\ a_{n,k}, & \text{if } j = k. \end{cases}$$

for all $n \in \mathbb{Z}$. In particular, for $j = 1$, we find $\gamma_n = a_{n,1} + f_{n,2} \geq 1$, which is (a). The latter equality in $n + j$ yields

$$\gamma_{n+j} f_{n+j-1,j} = \begin{cases} a_{n+j,j} + f_{n+j,j+1}, & \text{if } j < k; \\ a_{n+j,k}, & \text{if } j = k. \end{cases} \tag{3.8}$$

For $j = 1, \ldots, k$, dividing (3.8) by $\gamma_{n+1} \cdots \gamma_{n+j}$ and summing up, we find

$$\sum_{j=1}^{k} \frac{f_{n+j-1,j}}{\gamma_{n+1} \cdots \gamma_{n+j-1}} = \sum_{j=1}^{k} \frac{a_{n+j,j}}{\gamma_{n+1} \cdots \gamma_{n+j}} + \sum_{j=1}^{k-1} \frac{f_{n+j,j+1}}{\gamma_{n+1} \cdots \gamma_{n+j}}.$$

Deleting identical terms then gives (b).

If moreover, we have $a_{n,k} > 0$ for all $n \in \mathbb{Z}$, then the matrices A_n all have a positive entry in each column. Arguing as in the beginning of the proof of Proposition 3.12, we obtain that $f_n > 0$ for all n. Then, going back to $\gamma_n = a_{n,1} + f_{n,2}$, we can now deduce that $\gamma_n > 1$ for all n. □

Remark 3.14. It is possible to have $\gamma_n = 1$ if the condition $a_{n,k} > 0$ is not satisfied for all n. Consider the matrices

$$A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and define $A_n = A_{n \bmod 3}$ for all $n \in \mathbb{Z}$. This sequence of matrices satisfies the assumptions of Theorem 3.10 since $A_2 A_1 A_0$ is positive. Nevertheless, using (b) for $n = -1$, we find that

$$1 = \frac{a_{0,1}}{\gamma_0} + \frac{a_{1,2}}{\gamma_0\gamma_1} + \frac{a_{2,3}}{\gamma_0\gamma_1\gamma_2},$$

so $1 = \frac{1}{\gamma_0}$ and $\gamma_0 = 1$.

We can find $\gamma_2 = \frac{1+\sqrt{17}}{2}$ and $\gamma_1 = \frac{3+\sqrt{17}}{4}$ with similar methods. From there, the equalities (3.8) allow us to compute for instance $f_0 = \left(1, 0, \frac{-3+\sqrt{17}}{2}\right)$.

3.6 Existence of an alternate base with given representations of 1

Given p sequences of digits, we aim to show that there exists an alternate base of length p for which these sequences evaluate to 1. We start with a lemma in order to be able to make use of Furstenberg's result.

Lemma 3.15. *Let $p \geq 1$ and for each $i \in \{0, \ldots, p-1\}$, let $\mathbf{a}_i = a_{i,1}a_{i,2}\cdots$ be an eventually periodic sequence of nonnegative integer digits, not starting in* 0 *and not ending in* 0^ω*. Let M, N be such that the p sequences $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ all have preperiod Mp and period Np. Then the sequence $(A_n)_{n\in\mathbb{Z}}$ of square matrices of size $(M+N)p$ defined by*

$$(A_n)_{i,j} = \begin{cases} a_{j-n,j}, & \text{if } i = 1; \\ 1, & \text{if } i = j+1; \\ 1, & \text{if } i = Mp+1 \text{ and } j = (M+N)p; \\ 0, & \text{otherwise,} \end{cases}$$

where the first index in the digits $a_{m,j}$ is considered modulo p, is such that $(A_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 3.11.

Proof. First, observe that for all column vectors $v \in \mathbb{R}_{\geq 0}^{(M+N)p}$ with a positive first entry, for all $\ell \in \{0, \ldots, (M+N)p-1\}$ and all $n_1, \ldots, n_\ell \in \mathbb{Z}$, the $\ell+1$ first entries of the column vector $A_{n_1}\cdots A_{n_\ell}v$ are positive. This can be easily shown by induction on the number ℓ of matrices in the product.

Second, it is easily seen that for all nonzero column vectors $v \in \mathbb{R}_{\geq 0}^{(M+N)p}$ and for all $n_1, \ldots, n_{Mp} \in \mathbb{Z}$, at least one of the last Np entries of the column vector $A_{n_1}\cdots A_{n_{Mp}}v$ is positive.

Third, let us show that for all $n \in \mathbb{Z}$ and all $j \in \{Mp+1, \ldots, (M+N)p\}$, the first entry of the column vector $A_{n+Np-1}\cdots A_n\mathbf{e}_j$ is positive. We have the equalities

$$\left.\begin{aligned} A_n\mathbf{e}_j &= a_{j-n,j}\mathbf{e}_1 + \mathbf{e}_{j+1} \\ A_{n+1}\mathbf{e}_{j+1} &= a_{j-n,j+1}\mathbf{e}_1 + \mathbf{e}_{j+2} \\ &\vdots \\ A_{n+(M+N)p-j-1}\mathbf{e}_{(M+N)p-1} &= a_{j-n,(M+N)p-1}\mathbf{e}_1 + \mathbf{e}_{(M+N)p} \end{aligned}\right\} \text{empty if } j=(M+N)p$$
$$A_{n+(M+N)p-j}\mathbf{e}_{(M+N)p} = a_{j-n,(M+N)p}\mathbf{e}_1 + \mathbf{e}_{Mp+1}$$
$$\left.\begin{aligned} A_{n+(M+N)p-j+1}\mathbf{e}_{Mp+1} &= a_{j-n-Np,Mp+1}\mathbf{e}_1 + \mathbf{e}_{Mp+2} \\ A_{n+(M+N)p-j+2}\mathbf{e}_{Mp+2} &= a_{j-n-Np,Mp+2}\mathbf{e}_1 + \mathbf{e}_{Mp+3} \\ &\vdots \\ A_{n+Np-1}\mathbf{e}_{j-1} &= a_{j-n-Np,j-1}\mathbf{e}_1 + \mathbf{e}_j \end{aligned}\right\} \text{empty if } j = Mp+1$$

Since at least one of the Np digits $a_{j-n,Mp+1},\dots,a_{j-n,(M+N)p}$ is nonzero, we see that the first entry of the column vector $A_{n+Np-1}\cdots A_n\mathbf{e}_j$ is positive.

Putting the previous two observations together, we find that for all $n\in\mathbb{Z}$ and all nonzero column vectors $v\in\mathbb{R}_{\geq 0}^{(M+N)p}$, the first entry of the column vector $A_{n+(M+N)p-1}\cdots A_n v$ is positive. The first observation then gives us that the column vector $A_{n+2(M+N)p-1}\cdots A_n v$ is positive. Since v is arbitrary, the matrix $A_{n+2(M+N)p-1}\cdots A_n$ must be positive. Since the sequence of matrices $(A_n)_{n\in\mathbb{Z}}$ is periodic (with period p), and since $(A_n)_{1,1}$ is clearly positive for all n, we obtain the conclusion. □

Proposition 3.16. *Let $p\geq 1$ and let $\mathbf{a}_{p-1},\dots,\mathbf{a}_0$ be eventually periodic sequences of nonnegative integer digits, not starting in 0 and not ending in 0^ω. Then there exists an alternate base $\mathcal{B}$ of length p such that $\mathrm{val}_{\sigma^{-i}(\mathcal{B})}(0\bullet\mathbf{a}_i)=1$ for all $i\in\{0,\dots,p-1\}$.*

Proof. Without loss of generality, we assume that $\mathbf{a}_{p-1},\dots,\mathbf{a}_0$ all have preperiod Mp and period Np for some $M,N\in\mathbb{N}$. Let $(A_n)_{n\in\mathbb{Z}}$ be the sequence of matrices from Lemma 3.15. Let thus $(\gamma_n)_{n\in\mathbb{Z}}$ and $(f_n)_{n\in\mathbb{Z}}$ be the sequences given by Proposition 3.11, which are periodic with period p. For all $n\in\mathbb{Z}$, set $\beta_n=\gamma_{-n}$ and let $\delta=\beta_0\cdots\beta_{p-1}$. By Proposition 3.12 with $k=(M+N)p$ and $h=Mp$, and keeping in mind the periodicity of f, we obtain that

- $\beta_n>1$
- $1=\sum_{j=1}^{Mp}\frac{a_{n,j}}{\beta_{n-1}\cdots\beta_{n-j}}+\frac{f_{-n,Mp+1}}{\beta_{n-1}\cdots\beta_{n-Mp}}$
- $f_{-n,Mp+1}=\sum_{j=Mp+1}^{(M+N)p}\frac{a_{n,j}}{\beta_{n-Mp-1}\cdots\beta_{n-j}}+\frac{f_{-n,Mp+1}}{\delta^N}$

for all $n\in\mathbb{Z}$. The third item can be rewritten

$$f_{-n,Mp+1}=\frac{\delta^N}{\delta^N-1}\sum_{j=Mp+1}^{(M+N)p}\frac{a_{n,j}}{\beta_{n-Mp-1}\cdots\beta_{n-j}}.$$

Substituting this in the second item yields

$$1=\sum_{j=1}^{Mp}\frac{a_{n,j}}{\beta_{n-1}\cdots\beta_{n-j}}+\frac{\delta^N}{\delta^N-1}\sum_{j=Mp+1}^{(M+N)p}\frac{a_{n,j}}{\beta_{n-1}\cdots\beta_{n-j}}=\mathrm{val}_{\sigma^{-n}(\mathcal{B})}(\mathbf{a}_n)$$

where we have set $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$. □

The following proposition provides lower and upper bounds on the alternate base found in Proposition 3.16.

Proposition 3.17. *Let $p \geq 1$ and for each $i \in \{0,\dots,p-1\}$, let $\mathbf{a}_i = a_{i,1}a_{i,2}\cdots$ be an eventually periodic sequence of nonnegative integer digits, not starting in 0 and not ending in 0^ω. Let $C = Hp+1$ where*

$$H = \max\{a_{i,n} : i \in \{0,\dots,p-1\},\ n \geq 1\}$$

and let

$$L = \min\{\ell \geq 2 : \forall i \in \{0,\dots,p-1\},\ \text{the prefix } a_{i,1}\cdots a_{i,\ell} \text{ of } \mathbf{a}_i \text{ contains at least two nonzero digits}\}.$$

Then the alternate base $\mathcal{B} = (\beta_{p-1},\dots,\beta_0)$ found in Proposition 3.16 is such that

$$\frac{C^L}{C^L-1} < \beta_i \leq C$$

for all $i \in \{0,\dots,p-1\}$.

Proof. Let $i \in \{0,\dots,p-1\}$ and set $\delta = \beta_{p-1}\cdots\beta_0$. As usual, we consider indices reduced modulo p. We have

$$\begin{aligned}
1 &= \mathrm{val}_{\sigma^{-i-1}(\mathcal{B})}(0\bullet\mathbf{a}_{i+1})\\
&\leq \mathrm{val}_{\sigma^{-i-1}(\mathcal{B})}(0\bullet H^\omega)\\
&= H\left(\sum_{j=0}^{p-1}\frac{1}{\beta_i\cdots\beta_{i-j}}\right)\frac{\delta}{\delta-1}\\
&\leq H\cdot\frac{p}{\beta_i}\cdot\frac{\delta}{\delta-1}\\
&\leq \frac{Hp}{\beta_i-1}.
\end{aligned}$$

We get $\beta_i \leq C$. We then have

$$1 = \mathrm{val}_{\sigma^{-i-1}(\mathcal{B})}(0\bullet\mathbf{a}_{i+1}) > \frac{1}{\beta_i} + \frac{1}{\beta_i\cdots\beta_{i-L+1}} \geq \frac{1}{\beta_i} + \frac{1}{C^L}.$$

Therefore, we obtain $\beta_i > \frac{C^L}{C^L-1}$. □

In the next result, given p sequences of digits, we again obtain an alternate base for which these sequences evaluate to 1. Compared to Proposition 3.16, we no longer require each given sequence to be ultimately periodic. The proof uses a reduction to the case of Proposition 3.16.

Proposition 3.18. *Let $p \geq 1$ and for each $i \in \{0, \ldots, p-1\}$, let $\mathbf{a}_i$ be a sequence over a finite alphabet of nonnegative integer digits that is lexicographically greater than 10^ω. Then there exists an alternate base $\mathcal{B}$ of length p such that $\mathrm{val}_{\sigma^{-i}(\mathcal{B})}(0 \bullet \mathbf{a}_i) = 1$ for all $i \in \{0, \ldots, p-1\}$.*

Proof. Write $\mathbf{a}_i = a_{i,1}a_{i,2}\cdots$ for every $i \in \{0, \ldots, p-1\}$. We consider the "quasi-greedy version" of the sequences $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$, which we denote by $\mathbf{b}_{p-1}, \ldots, \mathbf{b}_0$. That is, we define

$$\mathbf{b}_i = \begin{cases} a_{i,1}\cdots a_{i,\ell-1}(a_{i,\ell}-1)\mathbf{b}_{i-\ell}, & \text{if } \mathbf{a}_i = a_{i,1}\cdots a_{i,\ell}0^\omega \text{ with } a_{i,\ell} \geq 1, \\ \mathbf{a}_n, & \text{otherwise} \end{cases}$$

where the first indices are considered modulo p. For each $i \in \{0, \ldots, p-1\}$, the sequence $\mathbf{b}_i = b_{i,1}b_{i,2}\cdots$ does not start with a zero digit and does not end in 0^ω. Let L be the least integer such that for all $i \in \{0, \ldots, p-1\}$, the prefix $b_{i,1}\cdots b_{i,L}$ has at least two nonzero digits. For $i \in \{0, \ldots, p-1\}$ and $N \geq L$, we define

$$\mathbf{b}_i^{(N)} = b_{i,1}\cdots b_{i,N}\, 1^\omega.$$

By Proposition 3.16, there exists an alternate base $\mathcal{B}^{(N)} = (\beta_{p-1}^{(N)}, \ldots, \beta_0^{(N)})$ such that $\mathrm{val}_{\sigma^{-i}(\mathcal{B}^{(N)})}(0 \bullet \mathbf{b}_i^{(N)}) = 1$ for all $i \in \{0, \ldots, p-1\}$. By Proposition 3.17, there exist uniform bounds c and C, with $1 < c < C$, such that all $\beta_i^{(N)}$ belong to the interval $(c, C]$ for N large enough. Then for such N we have

$$1 = \sum_{j=1}^{N} \frac{b_{i,j}}{\beta_{i-1}^{(N)}\cdots\beta_{i-j}^{(N)}} + \sum_{j=N+1}^{+\infty} \frac{1}{\beta_{i-1}^{(N)}\cdots\beta_{i-j}^{(N)}} = \sum_{j=1}^{+\infty} \frac{b_{i,j}}{\beta_{i-1}^{(N)}\cdots\beta_{i-j}^{(N)}} + \mathrm{E}(i,N)$$

with

$$\mathrm{E}(i,N) = \sum_{j=N+1}^{+\infty} \frac{1-b_{i,j}}{\beta_{i-1}^{(N)}\cdots\beta_{i-j}^{(N)}}.$$

Setting $H = \max\{b_{i,n} : i \in \{0, \ldots, p-1\},\ n \geq 1\}$, we obtain that

$$|\mathrm{E}(i,N)| \leq \sum_{j=N+1}^{+\infty} \frac{H}{c^j} = \frac{H}{c^N(c-1)}.$$

As the p-tuples $\mathcal{B}^{(N)}$ all belong to the compact set $[c, C]^p$, the sequence $(\mathcal{B}^{(N)})_{N \geq L}$ has an accumulation point in $[c, C]^p$, which we denote by $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$. Letting N grow while following the subsequence of indices such that $\mathcal{B}^{(N)}$ tends to $\mathcal{B}$, we obtain that $\mathrm{val}_{\sigma^{-i}(\mathcal{B})}(0 \bullet \mathbf{b}_i) = 1$ for all $i \in \{0, \ldots, p-1\}$. This implies that the original sequences $\mathbf{a}_i$ have $\sigma^{-i}(\mathcal{B})$-value equal to 1 as well, i.e., that $\mathrm{val}_{\sigma^{-i}(\mathcal{B})}(\mathbf{a}_i) = 1$ for all $i \in \{0, \ldots, p-1\}$. □

For an alternate base $\mathcal{B}$ of length p, the Parry conditions given in Corollary 3.3 become $t_{i,j+1}t_{i,j+2} \cdots <_{\mathrm{lex}} \mathbf{d}^*_{i-j}$ and $d_{i,j+1}d_{i,j+2} \cdots \leq_{\mathrm{lex}} \mathbf{d}^*_{i-j}$ for all $i \in \{0, \ldots, p-1\}$ and $j \geq 1$, where the index in $\mathbf{d}^*_{i-j}$ is seen modulo p. Our main result is the following one. It shows that given a list of p sequences satisfying the Parry condition, there exists a unique alternate base with these sequences precisely being the corresponding greedy or quasi-greedy expansions of 1.

Theorem 3.19. *Let p be a fixed positive integer and for all $i \in \{0, \ldots, p-1\}$, let $\mathbf{a}_i = a_{i,1}a_{i,2} \cdots$ be a sequence of nonnegative integers lexicographically greater than 10^ω. Assume moreover that for all $i \in \{0, \ldots, p-1\}$, if $\mathbf{a}_i$ ends in 0^ω then*

$$a_{i,j+1}a_{i,j+2} \cdots <_{\mathrm{lex}} \mathbf{a}_{i-j} \text{ for all } j \geq 1, \tag{3.9}$$

and if $\mathbf{a}_i$ does not end in 0^ω then

$$a_{i,j+1}a_{i,j+2} \cdots \leq_{\mathrm{lex}} \mathbf{a}_{i-j} \text{ for all } j \geq 1, \tag{3.10}$$

where the index in $\mathbf{a}_{i-j}$ is considered modulo p. Then there exists an alternate base $\mathcal{B}$ of length p with $\mathbf{d}_i = \mathbf{a}_i$ for all $i \in \{0, \ldots, p-1\}$ such that $\mathbf{a}_i$ ends in 0^ω and $\mathbf{d}^*_i = \mathbf{a}_i$ for all $i \in \{0, \ldots, p-1\}$ such that $\mathbf{a}_i$ does not end in 0^ω. Moreover, the following uniqueness properties hold.

(a) If $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ are all ultimately periodic, then the base $\mathcal{B}$ is unique.

(b) If $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ all start with a digit that is greater than 1, then the base $\mathcal{B}$ is unique.

Proof. The existence part of the result is a straightforward consequence of Propositions 3.8 and 3.18. Now, let us consider the two uniqueness statements. First, suppose that $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ are all ultimately periodic. These p sequences may or may not end in 0^ω. As in the proof of Proposition 3.18, we consider the quasi-greedy version $\mathbf{b}_{p-1}, \ldots, \mathbf{b}_0$ of the sequences $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$.

By [CCMP25, Theorem 6.4], at most one base $\mathcal{B}$ may be such that $\mathbf{d}_i^* = \mathbf{b}_i$ for all $i \in \{0, \dots, p-1\}$. Since we have assumed the lexicographic conditions (3.9) and (3.10), for any such base $\mathcal{B}$, we necessarily have $\mathbf{d}_i = \mathbf{a}_i$ for all $i \in \{0, \dots, p-1\}$ such that $\mathbf{a}_i$ ends in 0^ω and $\mathbf{d}_i^* = \mathbf{a}_i$ for all $i \in \{0, \dots, p-1\}$ such that $\mathbf{a}_i$ does not end in 0^ω. This gives us (a). The second uniqueness statement, i.e., (b), follows from Corollary 3.9. □

Remark 3.20. We presented here only a partial result for the uniqueness. As this thesis was being reviewed, a complete proof of uniqueness, together with an alternative proof of existence, was prepublished on arXiv by Šťovíček and Pelantová [ŠP26]. In this article, the authors prove that for any candidate expansions of 1, there is only one alternate base having these expansions. For the proof of uniqueness, the authors begin by studying in depth a class of matrices linked to alternate base numeration systems. They obtain that all matrices of this class are P-matrices in the sense of Gale and Nikaidô [GN65]. This then forces the map that evaluates p given infinite words in the p bases $\mathcal{B}, \sigma(\mathcal{B}), \dots, \sigma^{p-1}(\mathcal{B})$ to be injective in the arguments $(\beta_{p-1}, \dots, \beta_0)$, concluding the uniqueness. The proof of existence is based on topological arguments.

In the case of nonalternate bases, both the existence and uniqueness proofs are missing. For some special cases of strings $\mathbf{a}_n$, $n \in \mathbb{Z}$, the existence of a base can be deduced in a similar way to the proof of Theorem 3.24.

3.7 A family of sequences faithfully coding $\mathcal{B}$-integers

Let $\mathcal{B}$ be a Cantor real base. A real number is called a *$\mathcal{B}$-integer* if it has no fractional part with respect to $\mathcal{B}$, i.e., if its $\mathcal{B}$-expansion is of the form $\langle x \rangle_{\mathcal{B}} = a_{N-1} \cdots a_0 \bullet 0^\omega$. These numbers form a discrete subset of $\mathbb{R}$, and thus we can encode the distances between consecutive elements of this set by an infinite sequence. If there is precisely one symbol encoding each possible distance, we call this encoding the *faithful* coding of the $\mathcal{B}$-integers (which is clearly unique, up to renaming the letters). It was shown in [CCMP25] that this sequence has an S-adic representation. Let us recall the necessary definitions in order to state this result in a rigorous form.

Definition 3.21. For all $m, n \in \mathbb{N}$, we define

$$\Delta_{m,n} = \mathrm{val}_{\sigma^{-m}(\mathcal{B})}(0 \bullet d_{m+n,n+1} d_{m+n,n+2} \cdots).$$

Then, for all $m \in \mathbb{N}$, we define a map $\pi_m \colon \mathbb{N} \to \mathbb{N}$ by

$$\pi_m(n) = \begin{cases} \pi_m(n'), & \text{if } \Delta_{m,n} = \Delta_{m,n'} \text{ for some } n' < n; \\ n, & \text{otherwise} \end{cases}$$

and an alphabet $A_m = \pi_m(\mathbb{N})$. Finally, for all $m \in \mathbb{N}$, we define a substitution $\varphi_m \colon A_{m+1}^* \to A_m^*$ by

$$\varphi_m(n) = 0^{d_{m+n+1,n+1}} \pi_m(n+1).$$

Theorem 3.22 ([CCMP25, Corollary 5.5])**.** *Let $\mathcal{B}$ be a Cantor real base and $m \in \mathbb{Z}$. The S-adic sequence $\lim_{n\to\infty} \varphi_m \cdots \varphi_{m+n-1}(0)$ is the faithful coding of the $\sigma^{-m}(\mathcal{B})$-integers.*

Our aim in this section is to define a family of substitutions such that any S-adic sequence obtained from this family can be obtained as the faithful coding of the $\mathcal{B}$-integers for some Cantor real base $\mathcal{B}$. We start by defining the family of substitutions.

Definition 3.23. Let $k \geq 2$ be a fixed integer. For a k-tuple $\mathbf{c} = (c_1, \ldots, c_k)$ of positive integers, we define a morphism $\eta_{\mathbf{c}} \colon \{0, \ldots, k-1\}^* \to \{0, \ldots, k-1\}^*$ by

$$\eta_{\mathbf{c}}(j) = \begin{cases} 0^{c_{j+1}}(j+1), & \text{if } j < k-1; \\ 0^{c_k}, & \text{if } j = k-1. \end{cases}$$

Then we define

$$S = \{\eta_{\mathbf{c}} : \mathbf{c} = (c_1, \ldots, c_k),\ c_1 \geq \cdots \geq c_k \geq 1\}. \tag{3.11}$$

Theorem 3.24. *Any S-adic sequence of the form $\lim_{n\to\infty} \psi_0\psi_1 \cdots \psi_{n-1}(0)$, where $\psi_n \in S$ for all $n \geq 0$, is the faithful coding of the $\mathcal{B}$-integers for some Cantor real base $\mathcal{B}$.*

Proof. For all $n \geq 0$, let $\psi_n \in S$ and let $(a_{n+1,1}, \ldots, a_{n+1,k})$ be the corresponding k-tuple of parameters. For $n \leq 0$, we set $(a_{n,1}, \ldots, a_{n,k}) =$

$(1,\dots,1)$. For all $n \in \mathbb{Z}$, we define a square matrix of size k by

$$A_n = \left(\begin{array}{ccc|c} a_{-n+1,1} & \cdots & a_{-n+1,k-1} & a_{-n+1,k} \\ \hline & \mathbf{I} & & \mathbf{0} \end{array}\right)$$

where $\mathbf{I}$ is the identity matrix of size $k-1$ and $\mathbf{0}$ is the zero matrix column of size $k-1$. The product of any k such matrices is positive. Moreover, the product $A_{n+k}\cdots A_{n+1}$ is equal to A_1^k for all $n \geq 0$. Therefore, the sequence $(A_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of Proposition 3.11 and we may apply Propositions 3.11 and 3.13.

Let $(\gamma_n)_{n\in\mathbb{Z}}$ be the sequence given by Proposition 3.11 and for all $n \in \mathbb{Z}$, set $\beta_n = \gamma_{-n}$. By Proposition 3.13, we obtain that $\beta_n > 1$ and

$$1 = \sum_{j=1}^{k} \frac{a_{n-j+1,j}}{\beta_{n-1}\cdots\beta_{n-j}}$$

for all $n \in \mathbb{Z}$. By setting $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$, this can be reexpressed as

$$1 = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet a_{n,1}a_{n-1,2}\cdots a_{n-k+1,k}0^\omega)$$

for all $n \in \mathbb{Z}$. By Corollary 3.6, in order to obtain that the greedy expansions $\mathbf{d}_n$ of 1 are given by

$$\mathbf{d}_n = a_{n,1}a_{n-1,2}\cdots a_{n-k+1,k}0^\omega$$

it suffices to check that these sequences satisfy the Parry condition, i.e., that

$$a_{n-j,j+1}a_{n-j-1,j+2}\cdots a_{n-k+1,k}0^\omega <_{\mathrm{lex}} a_{n-j,1}a_{n-j-1,2}\cdots a_{n-j-k+1,k}0^\omega$$

for all $n \in \mathbb{Z}$ and all $j \geq 1$. These inequalities are indeed satisfied since we have assumed that

$$a_{n,1} \geq a_{n,2} \geq \cdots \geq a_{n,k} \geq 1. \qquad (3.12)$$

The quasi-greedy expansions of 1 are then given by

$$\mathbf{d}_n^* = a_{n,1}a_{n-1,2}\cdots a_{n-k+2,k-1}(a_{n-k+1,k}-1)\mathbf{d}_{n-k}^*.$$

From this, we obtain

$$\begin{aligned}
\Delta_{n,0} &= \Delta_{n,\ell k} = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet a_{n,1}a_{n-1,2}\cdots a_{n-k+1,k}0^\omega) = 1,\\
\Delta_{n,1} &= \Delta_{n,\ell k+1} = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet a_{n,2}a_{n-1,3}\cdots a_{n-k+2,k}0^\omega),\\
&\vdots\\
\Delta_{n,k-1} &= \Delta_{n,\ell k+k-1} = \mathrm{val}_{\sigma^{-n}(\mathcal{B})}(0 \bullet a_{n,k}0^\omega)
\end{aligned}$$

for all $\ell \in \mathbb{N}$. Due to the inequalities (3.12), we derive that

$$\Delta_{n,0} > \Delta_{n,1} > \cdots > \Delta_{n,k-1}.$$

With the notation of Definition 3.21, we have $\pi_n(\mathbb{N}) = \{0, \ldots, k-1\}$ for all $n \in \mathbb{Z}$. (In the context of [CC21], this means that the distances between consecutive $\sigma^{-n}(\mathcal{B})$-integers take precisely k values.)

By Theorem 3.22, in order to finish the proof, it suffices to show that $\varphi_n = \psi_n$ for all $n \in \mathbb{Z}$. By Definition 3.21, for all $n \in \mathbb{N}$ and $j \in \{0, \ldots, k-1\}$, we have $\varphi_n(j) = 0^{d_{n+j+1,j+1}}\pi_n(j+1)$. From what precedes, we know that

$$d_{n+j+1,j+1} = \begin{cases} a_{n+1,j+1}, & \text{if } j < k-1;\\ a_{n+1,k} - 1, & \text{if } j = k-1 \end{cases}$$

and that $\pi_n(j+1) = j+1$ if $j < k-1$ and $\pi_n(k) = 0$. Therefore, for all $j \in \{0, \ldots, k-1\}$, we obtain that

$$\varphi_n(j) = \begin{cases} 0^{a_{n+1,j+1}}(j+1), & \text{if } j < k-1;\\ 0^{a_{n+1,k}}, & \text{if } j = k-1, \end{cases}$$

which is precisely $\psi_n(j)$. □

3.7.1 Arnoux-Rauzy sequences

Arnoux-Rauzy sequences can be viewed as a generalization of Sturmian sequences to multiliteral alphabets. Consider the alphabet $\{0, \ldots, k-1\}$, and let $L_0, \ldots, L_{k-1}$ be the morphisms defined as follows:

$$L_i \colon \{0, 1, \ldots, k-1\}^* \to \{0, 1, \ldots, k-1\}^*,\ j \mapsto \begin{cases} i, & \text{if } j = i;\\ ij, & \text{if } j \neq i \end{cases}$$

for $i \in \{0, \ldots, k-1\}$. It is a known fact that every standard Arnoux-Rauzy sequence $\mathbf{w}$ over the alphabet $\{0, 1, \ldots, k-1\}$ has an S-adic expansion using the k substitutions $L_0, L_1, \dots, L_{k-1}$ [GJ09, Fog02]. More precisely, every standard Arnoux-Rauzy sequence can be written as

$$\mathbf{w} = \lim_{n\to\infty} \psi_0\psi_1\cdots\psi_{n-1}(0)$$

where $\psi_n \in \{L_0, L_1, \dots, L_{k-1}\}$ for all $n \in \mathbb{N}$. In accordance with [Pel24], we say that a standard Arnoux-Rauzy sequence is *regular* if its S-adic expansion can be expressed in the form

$$\psi_0\psi_1\psi_2\cdots = L_{i_1}^{a_1} L_{i_2}^{a_2} L_{i_3}^{a_3}\cdots,$$

for positive integers $a_1, a_2, a_3 \dots$ and where the sequence $(i_n)_{n\ge1}$ is purely periodic with period k and contains every letter in $\{0, \dots, k-1\}$. Note that every standard Sturmian sequence (i.e., a standard Arnoux-Rauzy sequence over a binary alphabet) is regular. Let us realize that up to renaming the letters of the alphabet, we can assume that the S-adic representation of a regular Arnoux-Rauzy sequence is of the form

$$\psi_0\psi_1\psi_2\cdots = L_0^{a_1}L_1^{a_2}L_2^{a_3}\cdots L_{k-1}^{a_k}L_0^{a_{k+1}}L_1^{a_{k+2}}L_2^{a_{k+3}}\cdots \tag{3.13}$$

Example 3.25. The most prominent ternary Arnoux-Rauzy sequence is the so-called Tribonacci sequence $\mathbf{t}$, the fixed point of the substitution $\varphi\colon 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$. On the ternary alphabet $\{0, 1, 2\}$, we have

$$L_0\colon \begin{cases} 0, \mapsto 0; \\ 1, \mapsto 01; \\ 2, \mapsto 02 \end{cases} \qquad L_1\colon \begin{cases} 0, \mapsto 10; \\ 1, \mapsto 1; \\ 2, \mapsto 12 \end{cases} \qquad L_2\colon \begin{cases} 0, \mapsto 20; \\ 1, \mapsto 21; \\ 2, \mapsto 2. \end{cases}$$

As the Tribonacci sequence $\mathbf{t}$ is fixed by φ, it is fixed by φ^3 as well. One can easily check that $\varphi^3 = L_0L_1L_2$. Consequently, $\mathbf{t}$ is a regular Arnoux-Rauzy sequence.

The Tribonacci sequence is an example of a so-called *β-substitution* as introduced by Fabre [Fab95]. Let us define this family of substitutions in the case of *simple Parry numbers*, i.e., real numbers $\beta > 1$ such that the expansion of 1 in the Rényi numeration system with the base β has the form $d_\beta(1) = t_1t_2\dots t_k0^\omega$. In this case, the β-substitution is given by

$$0 \mapsto 0^{t_1}1,\ 1 \mapsto 0^{t_2}2,\ \dots,\ k-2 \mapsto 0^{t_{k-1}}(k-1),\ k-1 \mapsto 0^{t_k}.$$

As shown in [BMP07], the fixed point of such a substitution is an Arnoux-Rauzy sequence if and only if $t_1 = t_2 = \cdots = t_{k-1} = t$ and $t_k = 1$. The fixed point has the S-adic expansion $(L_0^t L_1^t \cdots L_{k-1}^t)^\omega$, hence it is a regular Arnoux-Rauzy sequence.

We obtain as a consequence of Theorem 3.24 that all Arnoux-Rauzy sequences in this family can be expressed as the coding of the B-integers of some Cantor base.

Corollary 3.26. *Every regular k-ary Arnoux-Rauzy sequence is the faithful coding of the $\mathcal{B}$-integers for some Cantor real base $\mathcal{B}$. In particular, it is the case of every standard Sturmian sequence.*

Proof. Let $R\colon \{0,\ldots,k-1\}^* \to \{0,\ldots,k-1\}^*$ be the morphism given by $j \mapsto (j+1) \bmod k$. It is readily seen that for any $i \in \{0,\ldots,k-1\}$, one has $L_i = R^i L_0 R^{-i}$ and therefore $L_i^c = R^i L_0^c R^{-i}$ for all $c \in \mathbb{N}$. Hence, the S-adic representation (3.13) of any regular Arnoux-Rauzy sequence can be expressed as $(L_0^{a_1} R)(L_0^{a_2} R)(L_0^{a_3} R)\cdots$. If for $a \geq 1$, we define $\mathbf{c} = (a, a, \ldots, a, 1) \in \mathbb{N}^k$, then the substitution $\eta_{\mathbf{c}}$ from Definition 3.23 satisfies $L_0^a R = \eta_{\mathbf{c}}$. By Theorem 3.24, there exists a Cantor real base $\mathcal{B}$ such that $\mathbf{w}$ is the faithful coding of the $\mathcal{B}$-integers. □

3.7.2 Sequences associated with N-continued fractions

In [LRT23], Langeveld, Rossi and Thuswaldner study N-continued fractions of numbers, that is, expansions of the form

$$x = \frac{N}{d_1 + \frac{N}{d_2 + \cdots}}.$$

Note that 1-continued fractions coincide with the classical continued fractions. The authors introduce a family of substitutions over a binary alphabet: for $d \geq N$, let

$$\hat{\sigma}_d\colon 0 \mapsto 0^d 1,\ 1 \mapsto 0^N,$$

see [LRT23, Definition 3.1]. To $x \in (0,1) \setminus \mathbb{Q}$ with the N-continued fraction $x = [d_1, d_2, d_3, \ldots]_N$ they assign the so-called *dual NCF sequence*

$$\hat{\omega}(x, N) = \lim_{n\to\infty} \hat{\sigma}_{d_1} \hat{\sigma}_{d_2} \cdots \hat{\sigma}_{d_n}(0).$$

One can easily observe that the substitutions $\hat{\sigma}_{d_n}$ belong to the set of substitutions S defined in Definition 3.23 for $k = 2$. Therefore, Theorem 3.24 has the following corollary.

Corollary 3.27. *Every dual NCF sequence $\hat{\omega}(x, N)$ is the faithful coding of the $\mathcal{B}$-integers for some Cantor real base $\mathcal{B}$.*

Chapter 4

Ultimately periodic expansions in alternate bases

As numeration systems establish a correspondence between words and numbers, it is reasonable to hope for results that link properties of numbers and properties of the words that represent them. In this chapter, we study ultimate periodicity as a property of words. Which numbers have ultimately periodic expansions? For Rényi numeration systems, this question was investigated by Schmidt in [Sch80].

For alternate bases, the question was studied in collaboration with Charlier and Cisternino [CCK24]. This is the article that we present in this chapter. All presented results are taken from this article, with only notational and expositional changes. We generalize Schmidt's results to alternate bases, giving a necessary and a sufficient condition for the set of words with ultimately periodic expansions to be a specific subfield of $\mathbb{R}$ related to the product of all base elements.

It should be noted that some of the results presented here have since been improved by Masáková and Pelantová in [MP24].

Contents

4.1 Introduction **102**

4.2 Preliminaries and statements of main results **103**

4.3 Preparatory results **105**

4.3.1 On $\mathcal{B}$-periodicity and δ-periodicity p by p steps . 105

4.3.2 $\mathcal{B}$-expansions of rational numbers 107

4.3.3 Field isomorphism on infinite sums 110
4.4 Proof of Theorem 4.1 . . . . **110**
4.5 Proof of Theorem 4.3 . . . . **113**
4.6 Reduction to the case of real base expansions **115**

4.1 Introduction

Seeing that a numeration system is nothing more than a correspondence between numbers and words, it makes sense to ask if specific properties of words translate to properties of the numbers they represent. Ultimate periodicity is one of the simpler properties that a word can have, and it is what we will discuss here. In the context of Rényi numeration systems, this property is perhaps more interesting than usual. Given that these systems can be seen as based on an underlying transformation of the unit interval, ultimately periodic representations will correspond to ultimately periodic orbits of the transformation map, which are an interesting feature to know about if we are to study the associated dynamical system.

For the usual integer base numeration systems, it is not too hard to realize (especially with the dynamical interpretation of the greedy algorithm in mind) that a number has an ultimately periodic representation if and only if this number is rational (recall that using the term *ultimately periodic* includes sequences that end in 0^ω). For real base expansions, Schmidt studied in [Sch80] the set $\mathrm{Per}(\beta)$ of numbers that have an ultimately periodic β-expansion, thus corresponding (up to some technicalities) to the points which have a finite orbit under the β-transformation T_β. He proved that if β is a Pisot number, then a number is in $\mathrm{Per}(\beta)$ if and only if this number is in $\mathbb{Q}(\beta)$. A partial converse proved in the same article is that if $\mathbb{Q}(\beta)$ is included in $\mathrm{Per}(\beta)$, then β must be a Pisot or a Salem number.

In this chapter, we generalize the results of Schmidt to alternate base expansions. The chapter is organized as follows. We begin by recalling some notions and notation in Section 4.2 and stating our main results. Section 4.3 contains the preparatory work in order to prove those results (Theorems 4.1 and 4.3). We start by proving two equivalent conditions on $x \in [0, 1)$ in order to belong to $\mathrm{Per}(\mathcal{B})$, allowing us to use the remainders of the greedy algorithms only once out of p steps. Then we obtain a generalization of a result from [CCMP23], which is Theorem 4.6. This generalization represents a crucial argument in the proof of our main result. In Section 4.4, we prove Theorem 4.1 and Corollary 4.2. Section 4.5 is dedicated to the proof of Theorem 4.3. Finally, we consider the restricted case where $p = 1$, i.e., of usual real bases, in Section 4.6. Several arguments used for the general case of alternate bases just vanish in this restricted setting, giving rise to a new proof of Schmidt's result that is simpler than the original one from [Sch80].

4.2 Preliminaries and statements of main results

We start by recalling some notation related to real and alternate bases. For a real base β, the transformation T_β is defined by

$$T_\beta \colon [0,1) \to [0,1) : x \mapsto \beta x - \lfloor \beta x \rfloor.$$

The β-expansion of x can then be defined as $a_1 a_2 \cdots$, where $a_i = \lfloor \beta T^{i-1}(x) \rfloor$.

We will use some elements of Galois theory. Given a number $x \in \mathbb{C}$, we say that x is an *algebraic number* (resp. *algebraic integer*) if there exists a monic polynomial with integer (resp. rational) coefficients that has x as a root. The *minimal polynomial* of x is the unique monic polynomial *of least degree* that has x as a root. The other roots of this polynomial are called the *Galois conjugates* of x. If x is an algebraic integer whose Galois conjugates are all less than 1 in modulus, it is called a *Pisot number*. If x is an algebraic integer whose Galois conjugates are all less than or equal to 1 in modulus, with at least one being equal, then x is called a *Salem number*. We will not need much more than the definition of these sets and the understanding that they are of importance in number theory, but a classical reference on them is [BDGGH$^+$92].

We will note $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ an alternate base, or $(\beta_{-1}, \ldots, \beta_{-p})$ if this is more helpful. In this chapter, we assume that all base elements are greater than 1. We note $\sigma^n(\mathcal{B}) = (\beta_{p-1-n}, \ldots, \beta_{-n})$. The value of the word $\mathbf{a} = a_1 a_2 \cdots$ is given by

$$\operatorname{val}(\mathbf{a}) = \sum_{j=1}^{\infty} \frac{a_j}{\beta_{-1} \cdots \beta_{-j}} = \sum_{m=0}^{\infty} \sum_{i=1}^{p} \frac{a_{mp+i}}{\delta^m \beta_{-1} \cdots \beta_{-i}}$$

where $\delta = \beta_{p-1} \cdots \beta_0$ is the product of all base elements.

As usual, different words may have the same value x. Among those, one is distinguished to be *the expansion* or *the representation* of x using a greedy algorithm: set $r_0 = x$ then, if r_n has been defined, set $a_{n+1} = \lfloor \beta_{-n-1} r_n \rfloor$ and $r_{n+1} = \beta_{-n-1} r_n - a_{n+1}$. Then the *$\mathcal{B}$-expansion* of x is $d_{\mathcal{B}}(x) = a_1 a_2 \cdots$. The digit a_n belongs to $\{0, \ldots, \lceil \beta_{-n} \rceil - 1\}$ and the n-th *remainder* r_n belongs to $[0, 1)$.

We let $\mathrm{Per}(\mathcal{B})$ denote the set of real numbers in $[0, 1)$ having an ultimately periodic greedy $\mathcal{B}$-expansion, that is,

$$\mathrm{Per}(\mathcal{B}) = \{x \in [0, 1) : d_{\mathcal{B}}(x) \text{ is ultimately periodic}\}. \tag{4.1}$$

As in the real base case, the digits of the $\mathcal{B}$-expansion may also be obtained by iterating a well-chosen transformation $T_{\mathcal{B}}$ [CCD23]. The set $\mathrm{Per}(\mathcal{B})$ may then be seen, up to some technicalities, as the set of ultimately periodic points of this map $T_{\mathcal{B}}$. We recall that a base is called *Parry alternate* if the p representations $d_{\mathcal{B}}(1), \ldots, d_{\sigma^{p-1}(\mathcal{B})}(1)$ are all finite or ultimately periodic.

The main goal of this chapter is to prove the following result generalizing Schmidt's theorems [Sch80, Theorems 2.4 and 3.1].

Theorem 4.1. *Let $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ be an alternate base and set $\delta = \prod_{i=0}^{p-1} \beta_i$.*

(a) If $\mathbb{Q} \cap [0, 1) \subseteq \bigcap_{i=0}^{p-1} \mathrm{Per}(\sigma^i(\mathcal{B}))$ then $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ and δ is either a Pisot number or a Salem number.

(b) If δ is a Pisot number and $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ then $\mathrm{Per}(\mathcal{B}) = \mathbb{Q}(\delta) \cap [0, 1)$.

Our proof of Theorem 4.1 is based on algebraic tools such as the alternate base spectrum defined in [CCMP23] as a generalization of the β-spectrum originally introduced by Erdős, Joó and Komornik [EJK90]. A crucial step in our method is a generalization of a result from [CCMP23], which is Theorem 4.6 below. In the reduced case of one real base, we obtain a proof that is much shorter than Schmidt's original one from [Sch80]. Therefore, even though the statement of Theorem 4.1 generalizes Schmidt's results, the proof we give should not be seen as a generalization of Schmidt's original arguments. The new elementary proof of Schmidt's results that we obtain will be provided explicitly in Section 4.6.

Note that the algebraic condition that the bases $\beta_{p-1}, \ldots, \beta_0$ all belong to the extension field $\mathbb{Q}(\delta)$ is trivially satisfied whenever $p = 1$, that is, in the original case of Rényi's expansions. This condition already appeared in [CCMP23]. In that work, the aim was to obtain algebraic descriptions of alternate bases $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ defining a sofic $\mathcal{B}$-*shift*, where for an alternate base $\mathcal{B}$, the $\mathcal{B}$-shift is defined as the topological closure of the set $\bigcup_{i=0}^{p-1}\{d_{\sigma^i(\mathcal{B})}(x) : x \in [0,1)\}$. In [CC21], a combinatorial characterization of the sofic $\mathcal{B}$-shifts was obtained; namely, the $\mathcal{B}$-shift is sofic if and only if $d^*_{\sigma^i(\mathcal{B})}(1)$ is eventually periodic for every $i \in \{0, \ldots, p-1\}$. It was then shown in [CCMP23] that a necessary condition for the $\mathcal{B}$-shift to be sofic is that the product $\delta = \prod_{i=0}^{p-1} \beta_i$ is an algebraic integer and all of the bases $\beta_{p-1}, \ldots, \beta_0$ belong to $\mathbb{Q}(\delta)$. On the other hand, it was also shown that if the product δ is a Pisot number and $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ then the $\mathcal{B}$-shift is sofic.

As a straightforward consequence of Theorem 4.1, we reobtain this result from [CCMP23] generalizing the fact that all Pisot numbers are Parry numbers. Note that in the case $p = 1$, this result was independently obtained by both Bertrand and Schmidt [Ber77, Sch80].

Corollary 4.2. *Let $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ be an alternate base and set $\delta = \prod_{i=0}^{p-1} \beta_i$. If δ is a Pisot number and $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ then $\mathcal{B}$ is a Parry alternate base.*

We also prove the following theorem generalizing [Sch80, Theorem 2.5]. This result is a refinement of the item (a) of Theorem 4.1.

Theorem 4.3. *Let $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ be an alternate base such that the bases $\beta_{p-1}, \ldots, \beta_0$ are all in $\mathbb{Q}(\delta)$, with $\delta = \prod_{i=0}^{p-1} \beta_i$. If δ is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\mathcal{B}) \cap \mathbb{Q}$ *is nowhere dense in* $[0, 1)$.

4.3 Preparatory results

4.3.1 On $\mathcal{B}$-periodicity and δ-periodicity p by p steps

From now on, we fix p and an alternate base $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, and we set $\delta = \prod_{i=0}^{p-1} \beta_i$. We define

$$\begin{aligned} f_{\mathcal{B}} \colon \{0, \ldots, \lceil \beta_{-1} \rceil - 1\} \times \cdots \times \{0, \ldots, \lceil \beta_{-p} \rceil - 1\} &\to \mathbb{R} \\ (a_1, \ldots, a_p) &\mapsto \sum_{i=1}^{p} a_i \beta_{-i-1} \cdots \beta_{-p} \end{aligned}$$

and we consider the image of the map $f_{\mathcal{B}}$ as a set of real digits which we denote by $\mathrm{Dig}(\mathcal{B})$, that is,

$$\mathrm{Dig}(\mathcal{B}) = \left\{ \sum_{i=1}^{p} a_i \beta_{-i-1} \cdots \beta_{-p} : \forall i \in \{1, \ldots, p\},\ a_i \in \{0, \ldots, \lceil \beta_{-i} \rceil - 1\} \right\}. \tag{4.2}$$

These objects were already considered in [CCD23]. For $x \in [0,1)$ and $m \geq 1$, we set

$$\eta_m = f_{\mathcal{B}}(a_{(m-1)p+1}, a_{(m-1)p+2}, \ldots, a_{mp}) \tag{4.3}$$

where $(a_n)_{n \in \mathbb{N}_0} = d_{\mathcal{B}}(x)$. Thus, we have $\eta_m \in \mathrm{Dig}(\mathcal{B})$ and

$$x = \sum_{m=1}^{\infty} \frac{\eta_m}{\delta^m}. \tag{4.4}$$

We will sometimes write $\eta_m(x)$ instead of just η_m when the dependence in x needs to be emphasized. Moreover, for all $m \geq 1$, we have

$$r_{mp} = \delta r_{(m-1)p} - \eta_m \tag{4.5}$$

where r_n is the n-th remainder from the greedy algorithm computing $d_{\mathcal{B}}(x)$.

Proposition 4.4. *Let $x \in [0,1)$. The following assertions are equivalent.*

(a) The sequence $(a_n)_{n \in \mathbb{N}_0}$ is ultimately periodic, i.e., $x \in \mathrm{Per}(\mathcal{B})$.

(b) The sequence $(\eta_m)_{m \in \mathbb{N}_0}$ is ultimately periodic.

(c) There exist distinct $m, m' \in \mathbb{N}$ such that $r_{mp} = r_{m'p}$, i.e., the sequence $(r_{mp})_{m \in \mathbb{N}}$ is not injective.

Proof. If the sequence $(a_n)_{n \in \mathbb{N}_0}$ is ultimately periodic with period t, then so is the sequence $(\eta_m)_{m \in \mathbb{N}_0}$. Hence the implication (a)⇒(b) is verified. The implication (c)⇒(a) follows from the greedy algorithm. Indeed, if there exist $m, m' \in \mathbb{N}$ such that $r_{mp} = r_{m'p}$, then $an_{mp+n} = a_{m'p+n}$ for all $n \in \mathbb{N}_0$. It remains to prove (b)⇒(c). By (4.4) and (4.5), for all $m \in \mathbb{N}$, we have

$$\begin{aligned} r_{mp} &= \delta^m \left(x - \sum_{\ell=1}^{m} \frac{\eta_\ell}{\delta^\ell} \right) \\ &= \delta^m \left(\sum_{\ell=m+1}^{\infty} \frac{\eta_\ell}{\delta^\ell} \right). \end{aligned} \tag{4.6}$$

Suppose that the sequence $(\eta_m)_{m\in\mathbb{N}_0}$ is ultimately periodic, i.e., that there exists $s \in \mathbb{N}$ and $t \in \mathbb{N}_0$ such that $\eta_\ell = \eta_{\ell+t}$ for all $\ell \geq s$. Then

$$r_{sp} = \delta^s \left(\sum_{\ell=s+1}^{\infty} \frac{\eta_\ell}{\delta^\ell} \right) = \delta^{s+t} \left(\sum_{\ell=s+1}^{\infty} \frac{\eta_{\ell+t}}{\delta^{\ell+t}} \right) = \delta^{s+t} \left(\sum_{\ell=s+t+1}^{\infty} \frac{\eta_\ell}{\delta^\ell} \right) = r_{(s+t)p}.$$

□

Remark 4.5. We note that grouping terms p by p in an arbitrary $\mathcal{B}$-representation (not necessarily the greedy one), we do not get such equivalences. More precisely, it might be that, when dealing with other $\mathcal{B}$-representations than the greedy one, the considered representation is not ultimately periodic whereas the corresponding sequence of digits obtain by grouping p by p as in (4.3) is ultimately periodic. For example, for the alternate base $\mathcal{B} = (\varphi, \varphi, \varphi)$ where φ is the golden ratio $\frac{1+\sqrt{5}}{2}$, consider the infinite word $f(t)$ where t is the famous Thue-Morse infinite word $01101001\cdots$ and $f\colon \{0,1\}^* \to \{0,1\}^*$ is the injective coding defined by $f(0) = 100$ and $f(1) = 011$. This word $f(t)$ is a $\mathcal{B}$-representation of $\frac{\varphi}{2}$. Since the Thue-Morse word is not ultimately periodic, the word $f(t)$ is not ultimately periodic either. Since $f_{\mathcal{B}}(0,1,1) = f_{\mathcal{B}}(1,0,0)$, by grouping the terms 3 by 3, we get the purely periodic infinite word $(\varphi^2)^\omega$ over the alphabet $\text{Dig}(\mathcal{B}) = \{0, 1, \varphi, \varphi+1, \varphi^2, \varphi^2 + 1, \varphi^2 + \varphi, \varphi^2 + \varphi + 1\} = \{0, 1, \varphi, \varphi^2, \varphi^2 + 1, \varphi^3, 2\varphi^2\}$.

4.3.2 $\mathcal{B}$-expansions of rational numbers

For our purposes, we need to prove the following theorem, which is a generalization of [CCMP23, Theorems 14 and 19]. Compared with the statements therein, we now authorize alternate representations of numbers of the form $\frac{1}{q}$ for q a nonzero integer instead of alternate representations of 1 only, and the condition that the p sequences all start with a positive digit is now relaxed.

Theorem 4.6. *If for every $i \in \{0, \dots, p-1\}$, there exists a nonzero integer q_i and an ultimately periodic sequence $\mathbf{a}_i = (a_{i,n})_{n\in\mathbb{N}_0}$ of integers such that*

$$\sum_{n=1}^{\infty} \frac{a_{i,n}}{\prod_{k=1}^{n} \beta_{-i-k}} = \frac{1}{q_i}, \tag{4.7}$$

then δ is an algebraic integer. If moreover these p sequences have nonnegative elements and for all $i \in \{0, \dots, p-1\}$, there exists $m_i \in \mathbb{N}$ such that $a_{i,m_ip+1} \geq 1$, then $\beta_{p-1}, \dots, \beta_0 \in \mathbb{Q}(\delta)$.

Note that (4.7) can be rewritten as $\operatorname{val}_{\sigma^i(\mathcal{B})}(\mathbf{a}_i) = \frac{1}{q_i}$. This result can be proven by following the same lines as in the proof of [CCMP23, Theorem 14]. We only give a sketch of the proof underlying the main differences.

Sketch of the proof of Theorem 4.6. We start with the first part of the statement. For every $i \in \{0, \dots, p-1\}$, let q_i be a nonzero integer and let $\mathbf{a}_i = (a_{i,n})_{n\in\mathbb{N}_0}$ be an ultimately periodic sequence of integers such that (4.7) holds. Without loss of generality, we suppose that the p sequences $\mathbf{a}_i$ all have preperiod mp and period kp with $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, for all $i \in \{0, \dots, p-1\}$ and $j \in \{1, \dots, p\}$, we define $g_{i,j}$ as the following polynomial in the indeterminate X:

$$g_{i,j} = q_i \left[\left(\sum_{n=0}^{m+k-1} a_{i,np+j} X^{m+k-1-n} \right) - \left(\sum_{n=0}^{m-1} a_{i,np+j} X^{m-1-n} \right) \right]$$

We can use the periodicity of $\mathbf{a}_i$ to obtain that

$$\delta^m(\delta^k - 1) \left(\sum_{n=0}^{\infty} \frac{a_{i,np+j}}{\delta^n} \right) = \frac{\delta}{q_i} g_{i,j}(\delta)$$

for all i in $\{0, \dots, p-1\}$ and j in $\{1, \dots, p\}$. Then, by expanding the equality $\operatorname{val}_{\sigma^i(\mathcal{B})}(\mathbf{a}_i) = \frac{1}{q_i}$ and splitting the left side in p sums, we can obtain

$$\delta^m(\delta^k - 1) = \sum_{j=1}^{p} g_{i,j}(\delta) \beta_{-i-j-1} \cdots \beta_{-i-p} \tag{4.8}$$

This computation generalizes [CCMP23, Lemmas 16 and 17]. It is also similar to the kind of computation that can be made with the β-polynomials introduced by Hollander in [Hol98], which we will see again in Chapter 6.

Now, we consider the associated polynomial system

$$\begin{cases} X^m(X^k - 1) = \displaystyle\sum_{j=1}^{p} g_{i,j} X_{-i-j-1} \cdots X_{-i-p}, & \text{for } i \in \{0, \dots, p-1\} \\ X = X_{-1} \dots X_{-p} \end{cases}$$

in the $p+1$ indeterminates $X_{-1}, \dots, X_{-p}, X$, with the convention that $X_{n+p} = X_n$ for all integers n. As in the proof of [CCMP23, Theorem 14], by rearranging the equations and by substituting $X_0 \cdots X_{p-1}$ by X, the first p equations of the system can be written in the matrix form

$$(M - X^m(X^k - 1)I_p)\begin{pmatrix} X_{-2}X_{-3}\cdots X_{-p} \\ X_{-3}\cdots X_{-p} \\ \vdots \\ X_{-p} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.9}$$

where

$$M = \begin{pmatrix} g_{1,p} & Xg_{1,1} & \cdots & Xg_{1,p-2} & Xg_{1,p-1} \\ g_{2,p-1} & g_{2,p} & \cdots & Xg_{2,p-3} & Xg_{2,p-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{p-1,2} & g_{p-1,3} & \cdots & g_{p-1,p} & Xg_{p-1,1} \\ g_{0,1} & g_{0,2} & \cdots & g_{0,p-1} & g_{0,p} \end{pmatrix}$$

and I_p is the identity matrix of size p. The fact that the product δ is an algebraic integer is obtained by using the same argument as in the proof of [CCMP23, Theorem 14], namely, that δ is a root of the polynomial in X obtained by computing the determinant of the matrix $M - X^m(X^k - 1)I_p$, which has leading coefficient $(-1)^p$.

Now we turn to the second part of the statement. Thus, from now on, we suppose that the sequences $\mathbf{a}_i$ have nonnegative digits and for all $i \in \{0, \ldots, p-1\}$, let $m_i \in \mathbb{N}$ be such that $a_{i,m_ip+1} \geq 1$. Since we know that

$$g_{i,j}(\delta) = \frac{q_i}{\delta}\delta^m(\delta^k - 1)\sum_{n=0}^{\infty}\frac{a_{i,j+np}}{\delta^n}$$

for all $i \in \{0, \ldots, p-1\}, j \in \{1, \ldots, p\}$, we get that the matrix $M(\delta)$ (that is, the matrix M where the indeterminate X is substituted by δ in every entry) has nonnegative entries. Moreover, for all $i \in \{0, \ldots, p-1\}$, we have $g_{i,1}(\delta) > 0$ since $a_{i,m_ip+1} \geq 1$. Thus the entries $\delta g_{1,1}(\delta)$, $\delta g_{2,1}(\delta), \ldots, \delta g_{p-1,1}(\delta)$, $g_{0,1}(\delta)$ of $M(\delta)$ in respective positions $(1,2)$, $(2,3), \ldots, (p-1,p)$, $(p,1)$ are all positive. Therefore, the matrix $M(\delta)$ is irreducible. Then, in the exact same way as in the last part of the proof of [CCMP23, Theorem 14], we obtain that $\beta_{-1}, \ldots, \beta_{-p} \in \mathbb{Q}(\delta)$ by using the Perron-Frobenius theorem. □

Note that the previous theorem is immediate while considering Rényi expansions. In the alternate base framework, Theorem 4.6 (and [CCMP23, Theorems 14 and 19]) needs much more work.

4.3.3 Field isomorphism on infinite sums

Under the assumption of ultimate periodicity of the coefficients of a power series, and provided that δ and γ are Galois conjugates of moduli greater than one, the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(\delta)$ to $\mathbb{Q}(\gamma)$ mapping δ to γ behaves well with respect to this series.

Lemma 4.7. *Let δ be an algebraic number, let γ be a Galois conjugate of δ with $|\delta|, |\gamma| > 1$, and let ψ be the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(\delta)$ to $\mathbb{Q}(\gamma)$ defined by $\psi(\delta) = \gamma$. For all ultimately periodic sequences $(z_m)_{m\in\mathbb{N}_0}$ over $\mathbb{Q}(\delta)$, we have*

$$\psi\left(\sum_{m=1}^{\infty} \frac{z_m}{\delta^m}\right) = \sum_{m=1}^{\infty} \frac{\psi(z_m)}{\gamma^m}.$$

Proof. Let $(z_m)_{m\in\mathbb{N}_0}$ be an ultimately periodic sequence over $\mathbb{Q}(\delta)$, say of preperiod s and period t. Then

$$\begin{aligned}\psi\left(\sum_{m=1}^{\infty} \frac{z_m}{\delta^m}\right) &= \psi\left(\sum_{m=1}^{s} \frac{z_m}{\delta^m} + \left(\sum_{m=s+1}^{s+t} \frac{z_m}{\delta^m}\right)\frac{\delta^t}{\delta^t - 1}\right)\\ &= \sum_{m=1}^{s} \frac{\psi(z_m)}{\gamma^m} + \left(\sum_{m=s+1}^{s+t} \frac{\psi(z_m)}{\gamma^m}\right)\frac{\gamma^t}{\gamma^t - 1}\\ &= \sum_{m=1}^{\infty} \frac{\psi(z_m)}{\gamma^m}.\end{aligned}$$

□

We note that this lemma will be used for algebraic integers $\delta > 1$ only, but we have chosen to state it in its full generality.

4.4 Proof of Theorem 4.1

We are now ready to prove our main result.

Proof of Theorem 4.1. We start with the first item. Suppose that $\mathbb{Q} \cap [0,1) \subseteq \bigcap_{i=0}^{p-1} \mathrm{Per}(\sigma^i(\mathcal{B}))$. For each $i \in \{0, \ldots, p-1\}$, we can choose a sufficiently large integer m_i such that $\max\{\frac{\beta_{-i-1}\delta^{m_i}}{2}, \delta^{m_i}\} + 1 < \beta_{-i-1}\delta^{m_i}$. This yields that the intervals $(\delta^{m_i}, \beta_{-i-1}\delta^{m_i}]$ and $(\frac{\beta_{-i-1}\delta^{m_i}}{2}, \beta_{-i-1}\delta^{m_i}]$ have lengths greater than one. Therefore, there exists an integer q_i that belongs to both of them.

Thus, for each $i \in \{0, \dots, p-1\}$, we have constructed a rational number of the form $\frac{1}{q_i}$ having a $\sigma^i(\mathcal{B})$-expansion beginning with the prefix $0^{m_i p}1$. Moreover, by hypothesis, these $\sigma^i(\mathcal{B})$-expansions are all ultimately periodic sequences of integers. Therefore, we may apply Theorem 4.6. We get that δ is an algebraic integer and that $\beta_{p-1}, \dots, \beta_0 \in \mathbb{Q}(\delta)$. Now, let γ be a Galois conjugate of δ such that $|\gamma| > 1$ and let ψ be the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(\delta)$ to $\mathbb{Q}(\gamma)$ defined by $\psi(\delta) = \gamma$.

First, we claim that for any $x \in \mathbb{Q} \cap [0, 1)$, we have

$$x = \sum_{m=1}^{\infty} \frac{\psi(\eta_m)}{\gamma^m}. \tag{4.10}$$

Consider $x \in \mathbb{Q} \cap [0, 1)$. By hypothesis, we have $x \in \mathrm{Per}(\mathcal{B})$. Hence, by Proposition 4.4, the sequence $(\eta_m)_{m \in \mathbb{N}}$ is ultimately periodic. Since $x = \psi(x)$ and since $\delta > 1$ and $|\gamma| > 1$, the claim follows by using Lemma 4.7.

Now, for all integers M large enough so that $\frac{1}{\delta} + \frac{1}{\delta^M} < \frac{1}{\beta_{-1} \cdots \beta_{-p+1}}$ and for all real numbers x in the interval $[\frac{1}{\delta}, \frac{1}{\delta} + \frac{1}{\delta^M})$, we have $\eta_1 = f_{\mathcal{B}}(0, \dots, 0, 1) = 1$ and $\eta_2 = \cdots = \eta_M = f_{\mathcal{B}}(0, \dots, 0) = 0$. Any interval of the form $[\frac{1}{\delta}, \frac{1}{\delta} + \frac{1}{\delta^M})$ contains a rational number x, and for each such x, we get from (4.4) and (4.10) together with the previous observation that

$$\frac{1}{\delta} + \sum_{m=M+1}^{\infty} \frac{\eta_m}{\delta^m} = \frac{1}{\gamma} + \sum_{m=M+1}^{\infty} \frac{\psi(\eta_m)}{\gamma^m}.$$

Setting $C = \max\{\eta : \eta \in \mathrm{Dig}(\mathcal{B})\}$ and $D = \max\{|\psi(\eta)| : \eta \in \mathrm{Dig}(\mathcal{B})\}$, we get that

$$\left|\frac{1}{\gamma} - \frac{1}{\delta}\right| \le \sum_{m=M+1}^{\infty} \frac{\eta_m}{\delta^m} + \left|\sum_{m=M+1}^{\infty} \frac{\psi(\eta_m)}{\gamma^m}\right| \le \frac{C}{\delta^M(\delta - 1)} + \frac{D}{|\gamma|^M(|\gamma| - 1)}$$

for all large enough M and all $x \in \mathbb{Q} \cap [\frac{1}{\delta}, \frac{1}{\delta} + \frac{1}{\delta^M})$. Since the last upper bound is independent of the chosen x, by letting M tend to infinity we obtain that $\gamma = \delta$. This proves that δ is either a Pisot number or a Salem number.

We now turn to the second item of the statement. Suppose that δ is a Pisot number and that the bases $\beta_{p-1}, \dots, \beta_0$ belong to $\mathbb{Q}(\delta)$.

First, we prove that $\mathrm{Per}(\mathcal{B}) \subseteq \mathbb{Q}(\delta) \cap [0, 1)$. Let $x \in \mathrm{Per}(\mathcal{B})$. By Proposition 4.4, the sequence $(\eta_m)_{m \in \mathbb{N}_0}$ is ultimately periodic, say with preperiod s

and period t. Then

$$x = \sum_{m=1}^{\infty} \frac{\eta_m}{\delta^m} = \sum_{m=1}^{s} \frac{\eta_m}{\delta^m} + \left(\sum_{m=s+1}^{s+t} \frac{\eta_m}{\delta^m} \right) \frac{\delta^t}{\delta^t - 1}.$$

Since $\mathbb{Q}(\delta)$ is a field and the digits η_m belong to $\mathbb{Q}(\delta)$, we get that $x \in \mathbb{Q}(\delta) \cap [0,1)$.

Second, we turn to the converse inclusion. Let $x \in \mathbb{Q}(\delta) \cap [0,1)$. Proceed by contradiction and suppose that $x \notin \mathrm{Per}(\mathcal{B})$. Then the sequence $(r_{mp})_{m\in\mathbb{N}}$ is injective by Proposition 4.4. By (4.6), we obtain that r_{mp} belongs to the set

$$X^{\Delta}(\delta) = \left\{ \sum_{\ell=0}^{m} a_\ell \delta^{m-\ell} : m \in \mathbb{N},\ a_0, \ldots, a_m \in \Delta \right\}$$

for all $m \in \mathbb{N}$, where $\Delta = -\mathrm{Dig}(\mathcal{B}) \cup \mathrm{Dig}(\mathcal{B}) \cup \{x\}$. The set $X^{\Delta}(\delta)$ is called the δ-*spectrum* over the real digit set Δ. Since the remainders r_{mp} belong to $[0,1)$, the spectrum $X^{\Delta}(\delta)$ has an accumulation point in $\mathbb{R}$. By [CCMP23, Proposition 24], which is indeed valid for all finite digit sets included in $\mathbb{Q}(\delta)$, we get that either δ is not a Pisot number or there exists $i \in \{0, \ldots, p-1\}$ such that $\beta_i \notin \mathbb{Q}(\delta)$. But this is in contradiction with our hypotheses. □

We now show how Corollary 4.2 may be deduced from Theorem 4.1.

Proof of Corollary 4.2. Suppose that δ is a Pisot number and that $\beta_{p-1}, \ldots, \beta_0$ all lie in $\mathbb{Q}(\delta)$. Although Theorem 4.1 deals with real numbers in $[0,1)$, we may easily use it in order to show the ultimate periodicity of $d^*_{\sigma^i(\mathcal{B})}(1)$ for all $i \in \{0, \ldots, p-1\}$. Equivalently, we show that $d_{\sigma^i(\mathcal{B})}(1)$ is ultimately periodic for all $i \in \{0, \ldots, p-1\}$. Since $d_{\sigma^i(\mathcal{B})}(1) = \lfloor \beta_{-i-1} \rfloor d_{\sigma^{i+1}(\mathcal{B})}(\beta_{-i-1} - \lfloor \beta_{-i-1} \rfloor)$, this property is equivalent to the fact that $\beta_{-i-1} - \lfloor \beta_{-i-1} \rfloor \in \mathrm{Per}(\sigma^{i+1}(\mathcal{B}))$ for all $i \in \{0, \ldots, p-1\}$. But from the hypotheses, we know that $\beta_{-i-1} - \lfloor \beta_{-i-1} \rfloor \in \mathbb{Q}(\delta) \cap [0,1)$ for all $i \in \{0, \ldots, p-1\}$. Hence the result follows from Theorem 4.1. □

Remark 4.8. Let us illustrate that the item (b) of Theorem 4.1 cannot be extended to nongreedy representations, i.e., if δ is a Pisot number and $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ then the set of real numbers in $[0,1)$ having ultimately periodic $\mathcal{B}$-representations with respect to an algorithm different from the greedy one need not be equal to $\mathbb{Q}(\delta) \cap [0,1)$. We consider again the example introduced in Remark 4.5. We know that $\frac{\varphi}{2} = \frac{\varphi^3-1}{4} \in \mathbb{Q}(\varphi^3)$. However, the infinite word $f(t)$ is a non ultimately periodic $\mathcal{B}$-representation of $\frac{\varphi}{2}$ although φ^3 is a Pisot number.

Remark 4.9. In the first item of Theorem 4.1, our assumption concerns the p shifts $\sigma^i(\mathcal{B})$ of the alternate base $\mathcal{B}$. However, we need information about the shifts $\sigma^i(\mathcal{B})$ for $i \neq 0$ only when using Theorem 4.6. If we were able to improve Theorem 4.6 by proving that $\mathbb{Q} \cap [0,1) \subseteq \mathrm{Per}(\mathcal{B})$ implies that $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ and δ is an algebraic integer, then we would obtain as a result that $\mathbb{Q} \cap [0,1) \subseteq \mathrm{Per}(\mathcal{B})$ implies that $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$ and δ is either a Pisot number or a Salem number. Otherwise stated, an improvement of Theorem 4.6 gives rise to an improvement of Theorem 4.1. In fact, the improvement in question was later carried out by Masáková and Pelantová in [MP24].

4.5 Proof of Theorem 4.3

Finally, we are able to obtain a much stronger statement than the first item of Theorem 4.1.

Proof of Theorem 4.3. Suppose that δ is an algebraic integer that is neither a Pisot number nor a Salem number. Thus, there exists a Galois conjugate γ of δ such that $\gamma \neq \delta$ and $|\gamma| > 1$. For $x \in [0,1)$, define

$$E(x) = \left\{ n \in \mathbb{N}_0 : \sum_{m=1}^{n} \frac{\eta_m(x)}{\delta^m} \neq \sum_{m=1}^{n} \frac{\psi(\eta_m(x))}{\gamma^m} \right\}$$

where the digits $\eta_m(x)$ are defined as in (4.4) and ψ is the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(\delta)$ to $\mathbb{Q}(\gamma)$ such that $\psi(\delta) = \gamma$.

Let us first show that the set

$$F = \{x \in [0,1) : E(x) \text{ is infinite}\}.$$

is dense in $[0,1)$. Let x be in $[0,1)$ and let ε be a positive real number. We construct an element y of F such that $|x-y| < \varepsilon$. Let $c, d \in \mathbb{N}$ be such that $c < d$, $\delta^d \neq \gamma^d$, $\frac{1}{\delta^d} < \frac{\varepsilon}{2}$ and $\frac{C}{\delta^c(\delta-1)} < \frac{\varepsilon}{2}$ where $C = \max\{\eta : \eta \in \mathrm{Dig}(\mathcal{B})\}$. As a first case, suppose that $c \in E(x)$. In this case, we let

$$y = \sum_{m=1}^{c} \frac{\eta_m(x)}{\delta^m}.$$

Then $\eta_m(y) = \eta_m(x)$ for all $m \in \{1, \ldots, c\}$ and $\eta_m(y) = 0$ for all $m > c$. Therefore, the set $E(y)$ contains all integers $n \geq c$, and hence $y \in F$. Now, suppose that $c \notin E(x)$. Then we let

$$y = \sum_{m=1}^{c} \frac{\eta_m(x)}{\delta^m} + \frac{1}{\delta^d}.$$

In this case, for $m \in \{1, \ldots, c\}$, we have $\eta_m(y) = \eta_m(x)$ and for $m > c$, we have $\eta_m(y) = 0$ if $m \neq d$ and $\eta_d(y) = 1$. We obtain that $n \in E(y)$ for all $n \geq d$, hence $y \in F$ in this case as well. In both cases, we have found the desired element y of F since

$$|x - y| \leq \sum_{m=c+1}^{\infty} \frac{\eta_m(x)}{\delta^m} + \frac{1}{\delta^d} \leq \frac{C}{\delta^c(\delta - 1)} + \frac{1}{\delta^d} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now, since for all $x \in F$ and $\varepsilon > 0$, there exists $n \in E(x)$ such that

$$0 \leq x - \sum_{m=1}^{n} \frac{\eta_m(x)}{\delta^m} < \varepsilon,$$

we get that the set

$$G = \left\{ \sum_{m=1}^{n} \frac{\eta_m(x)}{\delta^m} : x \in [0,1),\ n \in E(x) \right\}$$

is dense in $[0, 1)$ as well. In order to see that $\mathrm{Per}(\mathcal{B}) \cap \mathbb{Q}$ is nowhere dense in $[0, 1)$, it then suffices to show that for all $g \in G$, there exists $N \in \mathbb{N}$ such that

$$\left[g, g + \frac{1}{\delta^N}\right) \cap \mathrm{Per}(\mathcal{B}) \cap \mathbb{Q} \cap [0, 1) = \emptyset.$$

Proceed by contradiction and suppose that there exist $x \in [0, 1)$ and $n \in E(x)$ such that for all $N \in \mathbb{N}$, there exists $y \in \mathrm{Per}(\mathcal{B}) \cap \mathbb{Q} \cap [0, 1)$ such that

$$0 \leq y - \sum_{m=1}^{n} \frac{\eta_m(x)}{\delta^m} < \frac{1}{\delta^N}. \tag{4.11}$$

We consider such x and n. Let now $N \geq n$ and $y \in \mathrm{Per}(\mathcal{B}) \cap \mathbb{Q} \cap [0, 1)$ such that (4.11) holds. Then $\eta_m(y) = \eta_m(x)$ for all $m \in \{1, \ldots, n\}$ and $\eta_m(y) = 0$ for all $m \in \{n + 1, \ldots, N\}$. As in the proof of the item (a) of Theorem 4.1, we obtain that

$$y = \sum_{m=1}^{\infty} \frac{\psi(\eta_m(y))}{\gamma^m}.$$

We then get

$$\left|\sum_{m=1}^{n}\frac{\eta_m(x)}{\delta^m}-\sum_{m=1}^{n}\frac{\psi(\eta_m(x))}{\gamma^m}\right|=\left|\sum_{m=N+1}^{\infty}\frac{\eta_m(y)}{\delta^m}-\sum_{m=N+1}^{\infty}\frac{\psi(\eta_m(y))}{\gamma^m}\right|$$
$$\leq\frac{C}{\delta^N(\delta-1)}+\frac{D}{|\gamma|^N(|\gamma|-1)}$$

where $C=\max\{\eta:\eta\in\mathrm{Dig}(\mathcal{B})\}$ and $D=\max\{|\psi(\eta)|:\eta\in\mathrm{Dig}(\mathcal{B})\}$. Since N can be chosen arbitrarily large, we obtain that

$$\sum_{m=1}^{n}\frac{\eta_m(x)}{\delta^m}=\sum_{m=1}^{n}\frac{\psi(\eta_m(x))}{\gamma^m},$$

contradicting that $n\in E(x)$. □

4.6 Reduction to the case of real base expansions

In this section, we emphasize that our proof of Schmidt's results, i.e., when taking $p=1$ in Theorem 4.1, is in fact simpler that the original one, which relies on several technical lemmas; see [Sch80, Theorems 2.4 and 3.1]. In order to do this, we explicitly give the reduction of our proof to this case.

New proof of Schmidt's theorems. We consider the case where $p=1$ only. So the notation $\mathcal{B}$, β_0 and δ now coincide; so we simply write β for any of them. We start with the first item. Suppose that $\mathbb{Q}\cap[0,1)\subseteq\mathrm{Per}(\beta)$. In particular, there exists an ultimately periodic sequence of integers whose β-value is of the form $\frac{1}{q}$ with $q\in\mathbb{N}_0$. Therefore, β satisfies an equality of the form

$$\frac{1}{q}=\sum_{n=1}^{s}\frac{a_n}{\beta^n}+\left(\sum_{n=s+1}^{s+t}\frac{a_n}{\beta^n}\right)\frac{\beta^t}{\beta^t-1}$$

where $s\in\mathbb{N}$ and $t\in\mathbb{N}_0$. Multiplying the two sides of this equality by $q\beta^s(\beta^t-1)$, we see that β is an algebraic integer. Now, let γ be a Galois conjugate of β such that $|\gamma|>1$ and let ψ be the $\mathbb{Q}$-isomorphism from $\mathbb{Q}(\beta)$ to $\mathbb{Q}(\gamma)$ defined by $\psi(\beta)=\gamma$.

First, we claim that for any $x\in\mathbb{Q}\cap[0,1)$, we have

$$x=\sum_{n=1}^{\infty}\frac{a_n}{\gamma^n}. \tag{4.12}$$

Consider $x \in \mathbb{Q} \cap [0,1)$. By hypothesis, we have $x \in \mathrm{Per}(\mathcal{B})$. Hence the sequence $(a_n)_{n\in\mathbb{N}_0}$ is ultimately periodic. Since $x = \psi(x)$ and since $\beta > 1$ and $|\gamma| > 1$, the claim follows by using Lemma 4.7.

Now, observe that if a real number x belongs to an interval of the form $[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^N})$ with $N \in \mathbb{N}_0$, then $a_1 = 1$ and $a_2 = \cdots = a_N = 0$. For all $N \in \mathbb{N}_0$, the interval $[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^N})$ contains a rational number x, and for each such x, we get from (4.12) and the previous observation that

$$\frac{1}{\beta} + \sum_{n=N+1}^{\infty} \frac{a_n}{\beta^n} = \frac{1}{\gamma} + \sum_{n=N+1}^{\infty} \frac{a_n}{\gamma^n}.$$

We get that

$$\left|\frac{1}{\gamma} - \frac{1}{\beta}\right| \leq \sum_{n=N+1}^{\infty} \frac{a_n}{\beta^n} + \left|\sum_{n=N+1}^{\infty} \frac{a_n}{\gamma^n}\right| \leq \frac{\lceil\beta\rceil - 1}{\beta^N(\beta - 1)} + \frac{\lceil\beta\rceil - 1}{|\gamma|^N(|\gamma| - 1)}$$

for all $N \in \mathbb{N}_0$ and $x \in \mathbb{Q} \cap [\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^N})$. Since the right-most upper bound does not depend on the chosen x, by letting N tend to infinity we obtain that $\gamma = \beta$. This proves that β is either a Pisot number or a Salem number.

We now turn to the second item of the statement. Suppose that β is a Pisot number.

First, we prove that $\mathrm{Per}(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$. Let $x \in \mathrm{Per}(\beta)$. Then $(a_n)_{n\in\mathbb{N}_0}$ is ultimately periodic, say with preperiod s and period t. Then

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} = \sum_{n=1}^{s} \frac{a_n}{\beta^n} + \left(\sum_{n=s+1}^{s+t} \frac{a_n}{\beta^n}\right) \frac{\beta^t}{\beta^t - 1}.$$

Since $\mathbb{Q}(\beta)$ is a field and the digits a_n are integers, we get that $x \in \mathbb{Q}(\beta) \cap [0,1)$.

Second, we turn to the converse inclusion. Let $x \in \mathbb{Q}(\beta) \cap [0,1)$. Proceed by contradiction and suppose that $x \notin \mathrm{Per}(\mathcal{B})$. Then the sequence of remainders $(r_n)_{n\in\mathbb{N}}$ is injective. By (4.6), we obtain that r_n belongs to the spectrum

$$X^{\Delta}(\beta) = \left\{\sum_{\ell=0}^{m} a_\ell \beta^{m-\ell} : m \in \mathbb{N},\ a_0, \ldots, a_{m-1} \in \Delta\right\}$$

for all $n \in \mathbb{N}$, where $\Delta = \{-\lceil\beta\rceil + 1, \ldots, \lceil\beta\rceil - 1\} \cup \{x\}$. Since the remainders r_n belongs to $[0,1)$, the spectrum $X^{\Delta}(\beta)$ has an accumulation point in

$\mathbb{R}$. By [CCMP23, Proposition 24], which is indeed valid for all finite digit sets included in $\mathbb{Q}(\beta)$, we get that β is not a Pisot number. But this is in contradiction with our hypothesis. $\square$

Chapter 5

Alternate bases with the maximal digit property

Not all alternate bases are created equal. Parry alternate bases have more properties than non-Parry ones. We have just seen in Chapter 4 that when the product δ is a Pisot number and all base elements are in $\mathbb{Q}(\delta)$, we get the extra property that numbers with ultimately periodic expansions are those in $\mathbb{Q}(\delta)$. In this chapter, we go further and investigate a restricted class of alternate bases that have a wealth of additional properties.

The titular *maximal digit property* is a combinatorial characterization of these systems, but we will establish links to $\mathcal{B}$-integers, finiteness-type properties, optimality, associated rewriting systems and their confluence, and normalization. We will informally introduce these various properties in the Rényi case, explain how they translate to the alternate base case and how they are all connected to the maximal digit property.

The contents of this chapter are adapted from an article currently written in collaboration with Émilie Charlier, Zuzana Masáková and Edita Pelantová. Portions of the work were presented at conferences, including Journées Montoises d'Informatique Théorique as well as Numeration and Substitution.

Contents

5.1 Introduction **120**

5.2 Preliminaries **123**

5.3 $\mathcal{B}$-integers, spectrum and optimality **127**

5.4 Rewriting system: confluence and normalization **134**

5.5 Additional comments **141**
5.5.1 Confluence for infinite words 141
5.5.2 A comment on a result of Frougny 154
5.5.3 Other short comments 158

5.1 Introduction

Some alternate bases have more properties than others. In fact, that is already the case for real bases. Parry numbers form a good starting point, with the β-shift being sofic [Lot02, Theorem 7.2.13] and a vast number of other properties exclusive to them. We have just seen that Pisot numbers have stronger properties: the set $\mathrm{Per}(\beta)$ of numbers in $[0,1)$ with an ultimately periodic β-expansion is equal to $\mathbb{Q}(\beta) \cap [0,1)$ in this case.

We can ask about numbers that have finite expansions rather than ultimately periodic ones, the set of which is called $\mathrm{Fin}(\beta)$. Such numbers always lie in $\mathbb{Z}[\beta^{-1}] \cap [0,1)$ by definition. The *finiteness property* (F) is satisfied when we have $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap [0,1)$ rather than just one inclusion. Equivalently, we can ask that the sum or difference of numbers that have a finite expansion has a finite expansion itself. This property was introduced in [FS92], where the authors proved that bases β with property (F) must be Pisot numbers (although not all have the property), and that any number β such that $d_\beta(1) = t_1 t_2 \dots t_\ell$ with $t_1 \geq t_2 \geq \dots \geq t_\ell$ is a Pisot number with property (F). The proof of the latter result relies on a rewriting algorithm, slowly changing a representation into the expansion of the same number while preserving finiteness, an idea that we will see again in this chapter. The finiteness property and related ones inspired many over the years, like [Aki98, Tak24] and the articles discussed in the next couple of paragraphs. See also [BR10, Section 2.3.2.2].

Property (F) ensures that the sum of two finite expansions with lengths ℓ_1 and ℓ_2 is itself a finite expansion, but by itself doesn't inform us on the length of this new expansion. The quantity $L_\oplus$, introduced in [AFMP03], measures the number of extra nonzero digits to the right that may be brought on by the process of addition, assuming that this number is finite. Note that $L_\oplus$ itself can still be infinite if the number of extra zeros can be unbounded depending on the terms in the addition. The authors of [AFMP03] proved that $L_\oplus$ is however finite for all Pisot numbers, a result that Bernat improved to all Perron numbers [Ber07]. Note that because we assume that the sum has finite expansion when computing $L_\oplus$, this number being finite does not imply property (F). Indeed, there are Pisot numbers where even the expansion of $\frac{3}{\beta^2}$ is infinite [BR10, Section 2.3.2.2].

The strongest property that we could ask in this style is to have property (F) with $L_\oplus = 0$, indicating that the sum of two numbers with finite expansion is itself a number with a finite expansion that is not longer. When looking at normalization rather than addition, and with the perspective of using a rewriting algorithm to convert representations into expansions, this would correspond to having a rewriting system that never creates a nonzero digit further to the right than the ones that already exist. This can be seen to tie back to an earlier paper of Frougny [Fro92a] on the confluence of linear numeration systems. A consequence of the results of Frougny is that if β is a number with $d_\beta(1) = t_1 t_2 \cdots t_\ell$ with $t_1 = t_2 = \ldots = t_{\ell-1} \geq t_\ell$, such a system exists. Therefore, in this system, words that have a finite representation (on the alphabet $\{0, \ldots, \lceil t_1 \rceil - 1\}$) also have an expansion that is finite and not longer. Numbers with the property are called *confluent Parry numbers*. Unfortunately, this does not guarantee that $L_\oplus = 0$, as the words that are obtained when performing digitwise addition are not written on this alphabet. Nevertheless, this connects confluence to the set of notions that we are discussing, if only through normalization.

The notion of *β-integers* is always in the background when discussing this sort of properties. This notion was introduced by Gazeau [Gaz97], who singled it out as a candidate mathematical model of quasicrystals due to its lattice-like structure. In fact, when β is a Pisot number, the set of β-integers is a Meyer set, a set with strong structural properties [GVG04]. Rényi numeration systems can be extended to all nonnegative real numbers rather than ones in $[0, 1)$, as we have explained in the more general case of Cantor bases (Definition 2.19). The set of β-integers $\mathbb{N}_\beta$ is then the set of all numbers whose expansion has no fractional part. Of course, any number with a finite expansion is obtained by dividing a β-integer by a power of β, linking properties of $\mathbb{N}_\beta$ to properties on finite expansions like property (F).

Underlying multiple of the above-discussed results is a method where representations are normalized (converted into the appropriate expansions) by a system of rewriting rules. Normalization is a staple of the study of numeration systems, but is usually studied through finite state machines ([Fro92b], see also [BR10, Chapter 2]). Frougny's other 1992 paper [Fro92a] links normalization, confluence and rewriting rules to the theory of finiteness-type properties, as we have mentioned above.

A seemingly more distant property is that of *optimality*. The greedy algorithm is such that the digit chosen at a given step is always the one that minimizes the defect between number to be represented and the value of the partial expansion computed so far (mathematically, a_n is chosen so that $x - \sum_{j=1}^{n} \frac{a_j}{\beta^j}$ is positive and minimal). For integer bases, it is also true that greedy representations are *greedy by blocks* in addition to being locally greedy: the choice of $a_1, \ldots, a_n$ ensures that $x - \sum_{j=1}^{n} \frac{a_j}{\beta^j}$ is positive and minimal across all choices for words of length n. This is not always true in real bases! For instance, if β is the positive root of $x^3 - 3x^2 - 2x - 1$ and $x = \frac{3}{\gamma^2} + \frac{3}{\gamma^3} \simeq 0.290$, we have $d_\gamma(x) = \bullet 1002210^\omega$. However, the length-3 prefix of $d_{\gamma(x)}$ is not the best under-approximation of x by a word of length 3, as we have $\mathrm{val}_\gamma(\bullet 100) < \mathrm{val}_\gamma(\bullet 033) \leq x$. A real base is *optimal* if the local optimality of the greedy algorithm generalizes to a more global optimality, as opposed to the previous example. This notion was introduced by Dajani and coauthors in [DdVKL12]. Perhaps surprisingly, the characterization that they found was exactly the one studied twenty years earlier by Frougny for confluent systems, a fact which has seemingly received little attention (it is only mentioned in [MP13, DMV15]).

In summary, in the Rényi case, the notions of confluence, normalization, optimality, β-integers and finiteness-type properties are all intertwined in a tapestry that also includes links to quasicrystal theory. The generalization of these observations to alternate bases was started by Masáková, Pelantová and Studeničová in a recent article [MPS25] where the authors studied property (F) in alternate base systems and found an analogue of Frougny and Solomyak's results. They showed that a condition on the order of digits in the expansions of 1 implies property (F), using a system of rewriting rules as a key argument in the proof.

In this chapter, we go further and uncover more of the tapestry for alternate bases. We introduce the maximal digit property as the analogue of Frougny and Dajani et al.'s condition $t_1 = \ldots = t_{\ell-1} \geq t_\ell$. We show that this condition is necessary and sufficient for optimality, the confluence of an associated rewriting system, and the procurement of a characterization of β-integers.

The article is organized as follows. The basic notions regarding two-way Cantor bases are recalled in Section 5.2 and the maximal digit property is introduced. In Section 5.3, we discuss the equality between spectrum and $\mathcal{B}$-integers as well as optimality. We introduce these notions and relate them to the maximal digit property. Section 5.4 discusses confluence and normalization, introducing them and linking them to the MDP as well. Finally, Section 5.5 provides miscellaneous comments and research perspectives.

5.2 Preliminaries

In this chapter, we will be working with two-way alternate bases. We recall the definitions in the more general setting of Cantor bases (Definition 2.19 onwards in Section 2.1).

Definition 5.1. A *two-way Cantor real base* is given by a sequence $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ of numbers greater than 1 and such that $\prod_{n=0}^{\infty}\beta_n = +\infty$ and $\prod_{n=1}^{+\infty}\beta_{-n} = +\infty$. This sequence is called the *base*. We note it $(\cdots\beta_1\beta_0\bullet\beta_{-1}\cdots)$, with the fractional point $\bullet$ separating the nonnegative and negative indices. The n-th *shift* of this sequence is given by $\sigma^n(\mathcal{B}) = (\beta_{j-n})_j$. That is, applying σ^n brings the fractional point between β_{-n} and β_{-n-1}.

Numbers are represented by biinfinite sequences. To identify the fractional point, the sequence $(a_n)_{n\in\mathbb{Z}}$ is represented by $\cdots a_1a_0\bullet a_{-1}a_{-2}\cdots$. We may omit left or right tails of zeros when writing this. With every sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ with a left tail of zeros, i.e., such that there exists N with $a_n = 0$ for all $n \geq N$, is associated a value given by

$$\mathrm{val}_{\mathcal{B}}(\mathbf{a}) = \sum_{n=0}^{N-1} a_n \prod_{i=0}^{n-1}\beta_i + \sum_{n=1}^{+\infty}\frac{a_{-n}}{\prod_{i=1}^{n}\beta_{-i}}$$

provided that this series converges. If $\mathrm{val}(\mathbf{a}) = x$, we say that $\mathbf{a}$ is a $\mathcal{B}$-representation of x.

A nonnegative real number may have multiple representations. One is distinguished and called the *$\mathcal{B}$-expansion* of x in the following way.

First for x in $[0,1]$, we set $r_0 = x$. For $n \geq 1$, if r_{n-1} has been defined, we set $a_{-n} = \lfloor\beta_{-n}r_{n-1}\rfloor$ and $r_n = \beta_{-n}r_{n-1} - a_{-n}$. We denote the infinite word $(a_{-n})_{n\geq 1}$ by $d_{\mathcal{B}}(x)$.

Let now $N \geq 0$ be the unique N such that $\prod_{i=0}^{N-1}\beta_i \leq x < \prod_{i=0}^{N}\beta_i$, if $x \geq 1$, and $N = -1$ for $x \in (0,1)$. Denote $d_{\sigma^{-N-1}(\mathcal{B})}(x/\prod_{i=0}^{N}\beta_i) =$

$x_N x_{N-1} x_{N-2} \cdots$. Then

$$\langle x \rangle_{\mathcal{B}} \colon = \begin{cases} {}^{\omega}0 \bullet x_{-1} x_{-2} \cdots & \text{if } x \in (0,1), \\ {}^{\omega}0 x_N \cdots x_0 \bullet x_{-1} x_{-2} \cdots & \text{if } x \geq 1. \end{cases}$$

The biinfinite word $\langle x \rangle_{\mathcal{B}}$ is the *$\mathcal{B}$-expansion* of x.

Note that if $x < 1$, the digit a_{-i} in the computation of $d_{\mathcal{B}}(x)$, $i \geq 1$, takes its value in the set $\{0, 1, \ldots, \lceil \beta_i \rceil - 1\}$. As a result, the same is true for biinfinite representations: we have $(\langle x \rangle_{\mathcal{B}})_j \in \{0, \ldots, \lceil \beta_j \rceil - 1\}$ for all $x \geq 0$ and $j \in \mathbb{Z}$. We let the *alphabet* of the numeration be $A = \{0, \ldots, \max_{i=0}^{p-1} \lceil \beta_i \rceil - 1\}$, although some letters may never be admissible at some positions as mentioned.

Note that $\langle x \rangle_{\mathcal{B}} = 0 \bullet d_{\mathcal{B}}(x)$ for $x \in [0, 1)$, but for $x = 1$, we have $\langle 1 \rangle_{\mathcal{B}} = 1 \bullet 0^{\omega}$. The one-way infinite sequence $d_{\mathcal{B}}(1)$ is called the greedy expansion of 1 and it can be easily shown that it is the lexicographically greatest among one-way infinite sequences $\mathbf{a}$ such that $\mathrm{val}_{\mathcal{B}}(0 \bullet \mathbf{a}) = 1$.

We will often omit left- or right-infinite tails of zeros. When the integer part or the fractional part of an expansion is zero, we will sometimes omit it entirely, and write expressions like $\langle x \rangle_{\mathcal{B}} = 11\bullet$.

Definition 5.2. We define $\mathbf{d}_i = d_{\sigma^{-i}(\mathcal{B})}(1)$. We write it $\mathbf{d}_i = t_{i,1} t_{i,2} \cdots$, in such a way that $t_{i,j} \leq \lceil \beta_{i-j} \rceil - 1$ for all i, j. If $\mathbf{d}_i$ is finite (ends in a right-tail of zeros), we let ℓ_i be its length, $d_i = t_{i,1} \cdots t_{i,\ell_i}$ be the corresponding finite word and $d_i' = t_{i,1} \cdots t_{i,\ell_i - 1}(t_{i,\ell_i} - 1)$. If $\mathbf{d}_i$ is infinite, we let its length be $+\infty$.

As usual, the *quasi-greedy representation of* 1 is given recursively by

$$\mathbf{d}_i^* = \begin{cases} \mathbf{d}_i, & \text{if } \mathbf{d}_i \text{ is infinite;} \\ d_i' \mathbf{d}_{i-\ell_i}^*, & \text{if } \mathbf{d}_i \text{ is finite of length } \ell_i. \end{cases}$$

The quasi-greedy representation of 1 is the lexicographically greatest among one-way infinite sequences $\mathbf{a}$ with infinitely many nonzero digits such that $\mathrm{val}_{\mathcal{B}}(0.\mathbf{a}) = 1$, and it can also be defined as a limit. Recall Definition 2.11 and onwards in Section 2.1.

In this paper we mainly work with *alternate bases*, which are Cantor real bases where the base sequence repeats periodically. More precisely, a base is periodic if there exists p (called the *period*) such that $\beta_{n+p} = \beta_n$ for every integer n. We write $\mathcal{B} = (\beta_{p-1}, \dots, \beta_0)$. We also set $\delta = \prod_{i=0}^{p-1} \beta_i$.

There are criteria to decide whether a given infinite word is the $\mathcal{B}$-expansion of some number or not. [CC21, Theorem 26] can be extended using the definitions above to the following statement.

Proposition 5.3. *An infinite word having a left tail of zeros,* $\mathbf{a} = a_{N-1} \cdots a_0 \bullet a_{-1}a_{-2} \cdots$, *is the* $\mathcal{B}$*-expansion of some* $x \geq 0$ *if and only if for all* $n \leq N$,

$$a_{n-1}a_{n-2} \cdots <_{\text{lex}} \mathbf{d}_n^*.$$

Adapting [MPS25, Proposition 2.3], we obtain another criterion that allows us to judge whether a finite word can occur at a given position in the expansion of a real number.

Proposition 5.4. *A finite word* $w_n \cdots w_m$ $(n > m)$ *can occur at positions* n *through* m *of the* $\mathcal{B}$*-expansion of some real number if and only if for all* k *in* $[m, n]$,

$$w_k \cdots w_m <_{\text{lex}} \mathbf{d}_{k+1}.$$

This proposition is only valid for finite words, but it allows for comparison with the greedy expansion of 1 rather than the quasi-greedy one. Words such that the conditions above, called *Parry conditions*, are satisfied are called *admissible.*

Example 5.5. Consider the alternate base given by $\mathcal{B} = (\beta_1, \beta_0) = (3, \frac{4}{3})$. We have $\mathbf{d}_0 = 30^\omega$, $\mathbf{d}_1 = 110^\omega$, and $\mathbf{d}_0^* = 2(10)^\omega$. Note that even though $\mathbf{d}_0$ contains the digit 3, $\langle x \rangle_{\mathcal{B}}$ never does for any x due to the above proposition. This only occurs when one of the base elements is an integer. In other cases, the first digit of $\mathbf{d}_i$ dictates the maximal admissible value at position $i - 1$.

A word $w_n \cdots w_m$ can occur as a factor in some $\mathcal{B}$-expansion if and only if we have $w_{2k+1} \leq 2$, $w_{2k} \leq 1$ and $w_{2k} = 1 \Rightarrow w_{2k-1} = 0$ for all appropriate k. Note the distinction that has to be made according to the value of k mod 2.

We now move on to the central definition of this chapter. It corresponds to the property $d_\beta(1) = t_1t_2 \cdots t_\ell$ with $t_1 = t_2 \ldots = t_{\ell-1}$ that was studied by Frougny [Fro92a] and Dajani et al. [DdVKL12].

Definition 5.6. An alternate base $\mathcal{B}$ of period p has the *maximal digit property* (MDP) if, for every $k \in \mathbb{Z}$, the expansion $\mathbf{d}_k = t_{k,1}t_{k,2}\cdots$ satisfies

$$t_{k,j} = \lceil \beta_{k-j} \rceil - 1 \quad \text{for every } j \in \mathbb{N},\ 1 < j < \ell_k. \tag{5.1}$$

Example 5.7. Consider the alternate base $\mathcal{B}$ of period $p = 3$ defined in the following way. Set δ to be the positive root of $x^3 - 23x^2 + 19x - 4$, then

$$\beta_2 = \frac{\delta^2 - 7\delta + 3}{4\delta - 2},\ \beta_1 = \frac{4\delta - 2}{2\delta} \text{ and } \beta_0 = \frac{2\delta^2}{\delta^2 - 7\delta + 3}.$$

The greedy expansions of 1 in this case are of the form $\mathbf{d}_0 = 31210^\omega$, $\mathbf{d}_2 = 1230^\omega$, $\mathbf{d}_1 = 2310^\omega$. One can check that $\mathcal{B}$ satisfies the MDP.

Example 5.8. On the other hand, the alternate base $\mathcal{B}$ obtained by choosing for δ the greatest root of $x^3 - 115x^2 + 144x - 45$ and

$$\beta_2 = \frac{23\delta^2 - 15\delta}{4\delta^2 - 3\delta},\ \beta_1 = \frac{4\delta^2 - 3\delta}{\delta^2 - 23\delta + 15} \text{ and } \beta_0 = \frac{\delta^2 - 23\delta + 15}{23\delta - 15},$$

does not satisfy the MDP, since the corresponding greedy expansions are of the form $\mathbf{d}_0 = 5330^\omega$, $\mathbf{d}_2 = 4350^\omega$, $\mathbf{d}_1 = 3530^\omega$, and $t_{0,2} = 3 < \lceil \beta_1 \rceil - 1 = 4$.

Remark 5.9. Let us make several comments about the maximal digit property.

1. If $p = 1$, the MDP corresponds to having a real base $\beta > 1$ for which $d_\beta(1) = t^k s$ where $t \geq s \geq 1$, i.e., to confluent Parry numbers, as expected.

2. The maximal digit property is symmetric for all the shifted bases: $\mathcal{B}$ has the MDP if and only if $\sigma^i(\mathcal{B})$ has the MDP.

3. If the length of $\mathbf{d}_i$ is 1 or 2 for every i, then the MDP is satisfied automatically.

4. Condition (5.1) implies that $\mathbf{d}_i$ is finite. Otherwise, (5.1) forces

   $$\mathbf{d}_i = u^\omega \quad \text{where } u = \big(\lceil \beta_{i-1} \rceil - 1\big)\big(\lceil \beta_{i-2} \rceil - 1\big)\cdots\big(\lceil \beta_{i-p} \rceil - 1\big),$$

   but $\mathbf{d}_i$ cannot be purely periodic (Corollary 2.17), a contradiction.

5.3 $\mathcal{B}$-integers, spectrum and optimality

We will investigate some arithmetic properties of bases with the MDP.

Definition 5.10. The set of $\mathcal{B}$-integers is the set of real numbers whose $\mathcal{B}$-expansions have no fractional parts:

$$\mathbb{N}_{\mathcal{B}} = \{x \in \mathbb{R} : \langle x \rangle_{\mathcal{B}} = x_{n-1} \cdots x_0 \bullet 0^{\omega}\}.$$

The notion of *spectrum* was introduced in [EJK90] to answer questions on the uniqueness of expansions. It was elaborated upon in [FP18] where it was linked to the zero automaton, an important concept when dealing with Rényi bases. An alternate base version was then introduced in [CCMP23] which generalized this result.

Intuitively, the spectrum of a given base, with a given alphabet, is the set of values of finite words with no fractional part over this base. Since the spectrum depends on the chosen alphabet in general, we fix one and study the particular associated spectrum. We consider the "plausible" alphabet, given by the simple condition $a_i \leq \lceil \beta_i \rceil - 1$ which was already mentioned above.

Definition 5.11. The spectrum of $\mathcal{B}$ is defined by

$$X_{\mathcal{B}} = \left\{ \sum_{i=0}^{n-1} x_i \beta_{i-1} \dots \beta_0 : n \in \mathbb{N},\ x_i \in \mathbb{N} \text{ and } x_i < \beta_i \text{ for each } i \right\}.$$

The spectrum of $\mathcal{B}$ is the set of values of "plausible" words with no fractional part, whereas the set of $\mathcal{B}$-integers is the set of numbers whose expansion has no fractional part. Obviously, we have $\mathbb{N}_{\mathcal{B}} \subset X_{\mathcal{B}}$ for all bases $\mathcal{B}$. The aim of this section is to investigate when the inclusion is strict.

Example 5.12. We continue Example 5.7. Here if an element x is in $X_{\mathcal{B}}$, it must have a representation $c_{n-1} \cdots c_1 c_0 \bullet$ with no fractional part. If such a representation is nonadmissible, it must be because it contains one of the factors $|312|c$ with $c \geq 1$, $2|31$, or $12|3$, where the vertical bar $|$ is used as a marker of position and always stands between a position congruent to 0 mod 3 and a position congruent to 2 mod 3. No other factors lead to nonadmissibility while satisfying the condition $x_i < \beta_i$ of the spectrum.

If we take the leftmost such nonadmissible factor, the digit to its left must be less than $\beta_i - 1$ for otherwise the selected nonadmissible factor would not be the leftmost. Thus we can subtract $|312|1$, $2|31$ or $12|3$ respectively digitwise from this factor and increment the digit to the left by 1 while keeping a word in the spectrum. For instance, $|212|3$ becomes $|300|0$ and $1|312|2$ becomes $2|000|1$.

Note that performing this operation is always possible when a non-admissible factor exists, preserves the value of the word, decreases the sum of digits of the word and doesn't create nonzero digits any further to the right than they already exist. Thus, iteratively applying this transformation, we necessarily terminate with an admissible word of the same value as the initial word and that doesn't have zeros any further to the right than the initial word. Such a word must therefore be the expansion of x and it must not have a fractional part. Thus x is in $\mathbb{N}_{\mathcal{B}}$, and the spectrum is equal to the set of $\mathcal{B}$-integers.

Example 5.13. We now compare to Example 5.8. We keep the convention that the bar $|$ separates positions 0 and 2 modulo p, and recall that the fractional point $\bullet$ separates positions 0 and -1. Here we have $\mathbf{d}_0 = 533$, with $t_{0,2} = 3$ which is less than the maximum digit $\lceil \beta_1 \rceil - 1 = 4$. Let us therefore consider the word $|540\bullet$ which is not admissible. Its value x is in $X_{\mathcal{B}}$. But x must be equal to $|533\bullet 53$ by substituting β_0 with its value, then to $1|000\bullet 53$. This word is admissible so it must be the expansion of x. Since the expansion of x has a fractional part, x is not in $\mathbb{N}_{\mathcal{B}}$. Thus the non-equality is proved.

This sort of example, where the lack of MDP is used to produce the expansion of a number and a representation of the same number that doesn't go as far to the right, will be recurring in this chapter.

Equality of the $\mathcal{B}$-integers and the spectrum is directly linked to the so-called optimality of $\mathcal{B}$-expansions. This notion is defined to reflect the definition in the Rényi case, where a base is optimal if the greedy algorithm being greedy locally translates to being greedy more globally as discussed in the introduction.

Definition 5.14. Let $\mathcal{B}$ be an alternate base and $x \in [0, 1)$. The expansion $d_{\mathcal{B}}(x) = a_{-1}a_{-2}\cdots$ is *optimal* if, for every $\mathcal{B}$-representation $c_{-1}c_{-2}\cdots$ of x

where $c_i < \beta_i$ for all $i \leq -1$, we have

$$\mathrm{val}_{\mathcal{B}}(\bullet a_{-1} \cdots a_{-k} 0^\omega) \geq \mathrm{val}_{\mathcal{B}}(\bullet c_{-1} \cdots c_{-k} 0^\omega) \text{ for all } k \in \mathbb{N}.$$

We say that the base $\mathcal{B}$ is *optimal* if $d_{\mathcal{B}}(x)$ is optimal for all $x \in [0, 1)$.

Example 5.15. Let us keep considering the base of Example 5.13 and set $\delta = \beta_2\beta_1\beta_0$. Consider the number

$$x = \frac{1}{\delta} + \frac{5}{\delta^2\beta_2} + \frac{3}{\delta^2\beta_2\beta_1}.$$

Its expansion is $\bullet 001|000|53$. As we have seen just above, the word $\bullet 000|540$ has the same value and is thus a representation of x. When considering prefixes of length 6, we have $\mathrm{val}_{\mathcal{B}}(\bullet 001|000) < \mathrm{val}_{\mathcal{B}}(\bullet 000|540)$. Thus this base is not optimal.

Theorem 5.16. *Let $\mathcal{B}$ be an alternate base of period p. Then the following statements are equivalent:*

(a) $X_{\sigma^i(\mathcal{B})} = \mathbb{N}_{\sigma^i(\mathcal{B})}$ for all integers i.

(b) $\mathcal{B}$ is optimal.

(c) $\mathcal{B}$ has the maximal digit property.

The proof of this theorem is divided into three propositions. Proposition 5.17 proves (c)$\Rightarrow$(a), while Proposition 5.19 proves (a)$\Rightarrow$(c). The equivalence (a)$\Leftrightarrow$(b) is proved in Proposition 5.21. Note that if all the base elements are integers, all three above statements are clearly true. Thus, we may assume in these proofs that at least one of the bases is not an integer.

Proposition 5.17. *Let an alternate base $\mathcal{B}$ of period p have the maximal digit property. Then $X_{\sigma^i(\mathcal{B})} = \mathbb{N}_{\sigma^i(\mathcal{B})}$ for all integers i.*

Proof. If the base $\mathcal{B}$ has the maximal digit property, then so does every shifted base $\sigma^i(\mathcal{B})$. Thus we only need to establish that $X_{\mathcal{B}} = \mathbb{N}_{\mathcal{B}}$. As $\mathbb{N}_{\mathcal{B}} \subset X_{\mathcal{B}}$ for every base $\mathcal{B}$, we need to show $X_{\mathcal{B}} \subset \mathbb{N}_{\mathcal{B}}$.

Let $y \in X_{\mathcal{B}}$. Then y has a $\mathcal{B}$-representation of the form $y_n y_{n-1} \cdots y_0 \bullet$ such that $y_j < \beta_j$ for every $j \leq n$. Without loss of generality we assume

$y_n = 0$. By induction on the sum $\sum_{j=0}^{n} y_j$ we show that the $\mathcal{B}$-expansion of y has zero fractional part.

If the word $y_n y_{n-1} \dots y_0$ is $\mathcal{B}$-admissible then obviously y is a $\mathcal{B}$-integer as its expansion must be the word $y_n \cdots y_0\bullet$ itself. This happens in particular in the case when $\sum_{j=0}^{n} y_j = 0$ which means that $y = 0$.

Assume that $y_n y_{n-1} \dots y_0$ is not admissible. Then there exists an index i such that

$$y_{i-1} y_{i-2} \dots y_0 \geq_{\text{lex}} \mathbf{d}_i^* \quad \text{and} \quad y_i < \lceil \beta_i \rceil - 1. \tag{5.2}$$

Indeed, we show that the latter condition is satisfied by the largest index i such that $y_{i-1} \dots y_0 \geq_{\text{lex}} \mathbf{d}_i^*$. We know that $\mathbf{d}_{i+1}^*$ is of the form

$$\mathbf{d}_{i+1}^* = (\lceil \beta_i \rceil - 1) f_2 f_3 f_4 \cdots ,$$

where by the Parry condition, we have $f_2 f_3 f_4 \cdots \leq_{\text{lex}} \mathbf{d}_i^*$. If $y_i = \lceil \beta_i \rceil - 1$ and simultaneously $y_{i-1} \dots y_0 \geq_{\text{lex}} \mathbf{d}_i^*$, then

$$\begin{aligned} y_i y_{i-1} \dots y_0 &\geq_{\text{lex}} (\lceil \beta_i \rceil - 1) \mathbf{d}_i^* \\ &\geq_{\text{lex}} (\lceil \beta_i \rceil - 1) f_2 f_3 f_4 = \mathbf{d}_{i+1}^*, \end{aligned}$$

which contradicts the fact that i is the largest index breaking the Parry condition.

Let $i \leq n$ be an arbitrary index satisfying (5.2) and $k \in \mathbb{N}$ be the smallest index such that $\beta_{i-1}, \dots, \beta_{i-k} \in \mathbb{N}$ and $\beta_{i-k-1} \notin \mathbb{N}$. In particular, if $\beta_{i-1} \notin \mathbb{N}$, then $k = 0$. The choice of k implies $\ell_{i-k-1} > 1$ and $\ell_{i-j} = 1$ if $1 \leq j \leq k$. We will use that for any $j \in \mathbb{Z}$ it holds that

$$\mathbf{d}_j^* = \begin{cases} (\lceil \beta_{j-1} \rceil - 1) \mathbf{d}_{j-1}^*, & \text{if } \beta_{j-1} \in \mathbb{N}; \\ t_{j,1} t_{j,2} \cdots t_{j,\ell_j - 1} (t_{j,\ell_j} - 1) \mathbf{d}_{j-\ell_j}^*, & \text{if } \beta_{j-1} \notin \mathbb{N}. \end{cases}$$

Using the former k times and the latter once yields

$$\mathbf{d}_i^* = u_1 u_2 \cdots u_h \mathbf{d}_{i-h}^*$$

where $h = k + \ell_{i-k}$, $u_j = \lceil \beta_{i-j} \rceil - 1$ for $1 \leq j \leq k+1$ and $u_{k+1} u_{k+2} \cdots u_h = d'_{i-k}$.

Let us stress that

$$\text{val}_{\sigma^{-i}(\mathcal{B})} (0 \bullet u_1 u_2 \dots u_{h-1} (u_h + 1)) = 1. \tag{5.3}$$

Since $\mathbf{d}_i^*(1)$ has infinitely many nonzeros, the inequality $\geq_{\text{lex}}$ in (5.2) must be strict, i.e.,

$$y_{i-1}\cdots y_0 >_{\text{lex}} u_1 u_2 \cdots u_h \mathbf{d}_{i-h}^*. \tag{5.4}$$

Two situations can happen.

Case 1 $y_{i-1}\cdots y_0 >_{\text{lex}} u_1 u_2 \ldots u_h$.

The maximal digit property guarantees that every digit in the list $u_1, u_2, \ldots, u_{h-1}$ is maximal. Therefore we have $i \geq h$, $y_{i-1}\cdots y_{i-h+1} = u_1 u_2 \ldots u_{h-1}$ and $y_{i-h} > u_h = t_{i-k,\ell_{i-k}} - 1$. Necessarily $y_{i-h} = t_{i-k,\ell_{i-k}} + c$ for some $c \in \mathbb{N}$. This fact together with (5.3) allows us to find a new $\mathcal{B}$-representation of y in the form $\tilde{y}_n \tilde{y}_{n-1} \ldots \tilde{y}_0$, where

$$\tilde{y}_j = \begin{cases} y_j + 1, & \text{if } j = i; \\ 0, & \text{if } j = i-1, i-2, \ldots, i-h+1; \\ c, & \text{if } j = i-h; \\ y_j, & \text{otherwise.} \end{cases}$$

Note that $\tilde{y}_i = y_i + 1 < \beta_i$ as $y_i < \lceil \beta_i \rceil - 1$. The sum of digits in the new $\mathcal{B}$-representation of y is strictly smaller than that of the old one and thus by the induction hypothesis $y \in \mathbb{N}_{\mathcal{B}}$.

Case 2 $y_{i-1} \ldots y_{i-h} = u_1 u_2 \ldots u_h$.

Inequality (5.4) implies that $y_{i-h-1} y_{i-h-2} \cdots y_0 >_{\text{lex}} \mathbf{d}_{i-h}^*$. As $y_{i-h} = u_h = t_{i-k,\ell_{i-k}} - 1 < \lceil \beta_{i-h} \rceil - 1$, the index $i-h$ has property (5.2) and we apply the same procedure to the smaller index $i-h$. Since the indices at play are natural numbers, this cannot continue forever and Case 1 must happen at some point.

$\square$

Remark 5.18. In the proof of Proposition 5.17, we have used rewriting rules in order to transform a $\mathcal{B}$-representation of an element of $X_{\mathcal{B}}$ into a lexicographically larger one. This is possible, since in the word $y_n y_{n-1} \ldots y_0$ the digit 1 at position i represents the same value as the word $(\beta_{i-1}-1)(\beta_{i-2}-1)\cdots(\beta_{i-k}-1)\mathbf{d}_{i-k}$ starting at position i. Recall that $k \in \mathbb{N}$ is the smallest such that $\beta_{i-k-1} \notin \mathbb{N}$.

Let us stress that we could start at any position satisfying (5.2).

Proposition 5.19. *Let an alternate base $\mathcal{B}$ of period p satisfy $X_{\sigma^n(\mathcal{B})} = \mathbb{N}_{\sigma^n(\mathcal{B})}$ for all integers n. Then $\mathcal{B}$ has the MDP.*

Proof. Towards a contradiction, assume that there exist $k \in \mathbb{Z}$ and $j, \ell \in \mathbb{N}$, $1 < j < \ell$, such that $t_{k,j} \leq \lceil \beta_{k-j} \rceil - 2$ and $t_{k,\ell} > 0$.

Due to the symmetry of the MDP, without loss of generality, we can assume that $j = k$ and $k \in \mathbb{N}, k \geq 2$. Denote

$$x = \mathrm{val}_{\mathcal{B}}(t_{k,1} \cdots t_{k,k-1}(t_{k,k}+1) \bullet 0^{\omega}).$$

By assumption, $x \in X_{\mathcal{B}}$, and therefore $x \in \mathbb{N}_{\mathcal{B}}$.

Due to the greedy algorithm and as $t_{k,\ell} > 0$,

$$1 = \mathrm{val}_{\mathcal{B}}(1 \bullet 0^{\omega}) > \mathrm{val}_{\mathcal{B}}(0 \bullet t_{k,k+1} t_{k,k+2} \cdots) > 0.$$

Therefore,

$$x > \mathrm{val}_{\mathcal{B}}(t_{k,1} \cdots t_{k,k} \bullet t_{k,k+1} \cdots t_{k,\ell_k} 0^{\omega}) = \mathrm{val}_{\mathcal{B}}(10^k \bullet 0^{\omega}).$$

On the other hand,

$$x < \mathrm{val}_{\mathcal{B}}\left(t_{k,1} \cdots (t_{k,k}+1) \bullet t_{k,k+1} t_{k,k+2} \cdots\right) = \mathrm{val}_{\mathcal{B}}(10^{k-1}1 \bullet 0^{\omega}).$$

If the word $10^{k-1}1 \bullet 0^{\omega}$ is admissible in base $\mathcal{B}$, then we have a contradiction, because there may not exist any $\mathcal{B}$-integer x strictly between the two consecutive $\mathcal{B}$-integers $\mathrm{val}_{\mathcal{B}}(10^k \bullet)$ and $\mathrm{val}_{\mathcal{B}}(10^{k-1}1 \bullet)$.

We will show that under the assumption of the proposition, it cannot happen that the word $10^{k-1}1 \bullet 0^{\omega}$ for $k \geq 2$ is nonadmissible. Take j such that $\mathbf{d}_j$ is lexicographically minimal among all $\mathbf{d}_i$. Among such j, take one for which $\mathbf{d}_{j+1} >_{\mathrm{lex}} \mathbf{d}_j$, if such a j exists. If $\mathbf{d}_j \geq_{\mathrm{lex}} 110^{\omega}$, then $10^{k-1}1$, $k \geq 2$, is admissible in every base $\sigma^i(\mathcal{B})$. Suppose that $\mathbf{d}_j = 10^{\ell}\mathbf{w}$, where $\ell \geq 1$ and $\mathbf{w}$ is a word starting with a nonzero digit.

Consider the number y represented in base $\sigma^{\ell+1-j}(\mathcal{B})$ by the word $10^{\ell-1}1 \bullet 0^{\omega}$. Since $y \in X_{\sigma^{\ell+1-j}(\mathcal{B})}$, by assumption, we have $y \in \mathbb{N}_{\sigma^{\ell+1-j}(\mathcal{B})}$. However, the word $10^{\ell-1}1 \bullet 0^{\omega}$ is not admissible in $\sigma^{\ell+1-j}(\mathcal{B})$, as $10^{\ell-1}1$ is lexicographically larger than $\mathbf{d}_j$. Its value therefore satisfies

$$y > \mathrm{val}_{\sigma^{\ell+1-j}(\mathcal{B})}(10^{\ell+1} \bullet 0^{\omega}).$$

We also have

$$y = \mathrm{val}_{\sigma^{\ell+1-j}(\mathcal{B})}(10^{\ell-1}1 \bullet 0^{\omega}) < \mathrm{val}_{\sigma^{\ell+1-j}(\mathcal{B})}(10^{\ell}1 \bullet 0^{\omega}).$$

Note that $10^\ell 1 \leq_{\text{lex}} \mathbf{d}_j <_{\text{lex}} \mathbf{d}_{j+1}$, and thus $10^\ell 1 \bullet 0^\omega$ is admissible in base $\sigma^{\ell+1-j}(\mathcal{B})$. So we deduce that y lies strictly between two consecutive elements $\text{val}_{\sigma^{\ell+1-j}(\mathcal{B})}(10^{\ell+1} \bullet 0^\omega)$ and $\text{val}_{\sigma^{\ell+1-j}(\mathcal{B})}(10^\ell 1 \bullet 0^\omega)$ in $\mathbb{N}_{\sigma^{\ell+1-j}(\mathcal{B})}$, which is impossible. Therefore, no j exists such that $\mathbf{d}_j$ is lexicographically minimal and $\mathbf{d}_{j+1} >_{\text{lex}} \mathbf{d}_j$. This implies that all the bases have the same greedy expansions of 1, i.e., that we can consider $p = 1$. The case where $d_\beta(1) = 10^\ell \mathbf{w}$ with $\mathbf{w} >_{\text{lex}} 10^\omega$ can be solved just as above. In the case where $d_\beta(1) = 10^\ell 10^\omega$, we can consider y represented by $10^{\ell-1}1\bullet$ and see that y must be strictly between the two consecutive β-integers $\text{val}(10^{\ell+1}\bullet)$ and $\text{val}(10^{\ell+2}\bullet)$, also reaching a contradiction. □

For completing the proof of Theorem 5.16, it remains to prove the equivalence between maximal digit property and optimality of the base. First realize that $\mathcal{B}$ is optimal if and only if $\sigma^i(\mathcal{B})$ is optimal for any i.

Lemma 5.20. *The alternate base $\mathcal{B}$ of period p is optimal if, and only if, $\sigma^i(\mathcal{B})$ is optimal for all $i \in \{0, \ldots, p-1\}$.*

Proof. We first prove that if $\mathcal{B}$ is not optimal, then neither is $\sigma^{-1}(\mathcal{B})$. Indeed, if $\mathcal{B}$ is not optimal, there must exist some $x \in [0, 1)$ whose $\mathcal{B}$-expansion $\langle x \rangle_{\mathcal{B}} = 0 \bullet a_{-1}a_{-2}\cdots$ is not optimal, i.e., there exists some $\mathcal{B}$-representation $0 \bullet c_{-1}c_{-2}\cdots$ of x and some k such that

$$\text{val}_{\mathcal{B}}(0 \bullet a_{-1} \cdots a_{-k}0^\omega) < \text{val}_{\mathcal{B}}(0 \bullet c_{-1} \cdots c_{-k}0^\omega).$$

But then, the number $\frac{x}{\beta_0}$ has the $\sigma^{-1}(\mathcal{B})$-expansion $0 \bullet 0a_{-1}a_{-2}\cdots$ and a $\sigma^{-1}(\mathcal{B})$-representation $0 \bullet 0c_{-1}c_{-2}\cdots$. Clearly,

$$\text{val}_{\sigma^{-1}(\mathcal{B})}(0 \bullet 0a_{-1} \cdots a_{-k}0^\omega) < \text{val}_{\sigma^{-1}(\mathcal{B})}(0 \bullet 0c_{-1} \cdots c_{-k}0^\omega).$$

Thus $\sigma^{-1}(\mathcal{B})$ is not optimal.

Therefore, if $\mathcal{B}$ is optimal then so is $\sigma(\mathcal{B})$, then $\sigma^2(\mathcal{B})$, and so on until $\sigma^p(\mathcal{B})$. Since this is an alternate base, $\sigma^p(\mathcal{B}) = \mathcal{B}$, so all the implications are actually equivalences. Thus we have proved that if one of the bases $\sigma^i(\mathcal{B})$ is optimal, they all are. □

Proposition 5.21. *An alternate base $\mathcal{B}$ is optimal if and only if $X_{\sigma^i(\mathcal{B})} = \mathbb{N}_{\sigma^i(\mathcal{B})}$ for every $i \in \mathbb{Z}$.*

Proof. ($\Leftarrow$) Let $X_{\sigma^i(\mathcal{B})} = \mathbb{N}_{\sigma^i(\mathcal{B})}$ for every $i \in \mathbb{Z}$. Towards a contradiction, assume that there exists $x \in [0,1)$ with an expansion equal to $\langle x\rangle_{\mathcal{B}} = 0 \bullet a_{-1}a_{-2}\cdots$ and a $\mathcal{B}$-representation $0 \bullet c_{-1}c_{-2}\cdots$ such that $c_i < \beta_i$ for all $i \leq -1$ and where for some natural k we have

$$\mathrm{val}_{\mathcal{B}}(0 \bullet a_{-1}\cdots a_{-k}0^\omega) < \mathrm{val}_{\mathcal{B}}(0 \bullet c_{-1}\cdots c_{-k}0^\omega) \leq x.$$

Denote $x' = \mathrm{val}_{\mathcal{B}}(0 \bullet c_{-1}\cdots c_{-k}0^\omega)$ and $y = x' \prod_{j=1}^{k} \beta_{-j} < \prod_{j=1}^{k} \beta_{-j}$. The word $c_{-1}\cdots c_{-k} \bullet 0^\omega$ is a $\sigma^k(\mathcal{B})$-representation of y so y belongs to $X_{\sigma^k(\mathcal{B})}$. So $y \in \mathbb{N}_{\sigma^k(\mathcal{B})}$ and the $\sigma^k(\mathcal{B})$-expansion of y must be of the form $b_{-1}b_{-2}\cdots b_{-k}\bullet 0^\omega$ as $y < \prod_{j=1}^{k}\beta_{-j}$. Equivalently, $\langle x'\rangle_{\mathcal{B}} = 0 \bullet b_{-1}b_{-2}\cdots b_{-k}0^\omega$.

Given that expansion in base $\mathcal{B}$ is increasing between the real numbers with the usual order and words with the lexicographic order, we deduce from the inequalities $\mathrm{val}_{\mathcal{B}}(0.a_{-1}\cdots a_{-k}0^\omega) < x' \leq x$ that

$$a_{-1}\cdots a_{-k}0^\omega <_{\mathrm{lex}} b_{-1}\cdots b_{-k}0^\omega \leq_{\mathrm{lex}} a_{-1}a_{-2}\cdots a_{-k}a_{-k-1}\cdots$$

but this is a contradiction.

($\Rightarrow$) Let $\mathcal{B}$ be optimal. Towards a contradiction, assume that there exists some integer n such that $X_{\sigma^n(\mathcal{B})} \neq \mathbb{N}_{\sigma^n(\mathcal{B})}$. It means that some element x of $X_{\sigma^n(\mathcal{B})}$ does not belong to $\mathbb{N}_{\sigma^n(\mathcal{B})}$. As a result, x has a $\sigma^n(\mathcal{B})$-representation with no fractional part, of the form $c_{\ell-1}\cdots c_0 \bullet 0^\omega$ with $c_i < \beta_i$ for all i, but the $\sigma^n(\mathcal{B})$-expansion of x has a fractional part, and is of the form $a_{\ell-1}\cdots a_0 \bullet a_{-1}\cdots$ (where we have padded to the left by zeros so that the expansions start at the same position).

We show that the number $x' = \frac{x}{\beta_{n+\ell-1}\cdots\beta_{n+1}\beta_n}$ contradicts the optimality of the base $\widetilde{\mathcal{B}} = \sigma^{n-\ell}(\mathcal{B})$. In this base, $0 \bullet a_{\ell-1}a_{\ell-2}\cdots$ is the $\widetilde{\mathcal{B}}$-expansion of x' and $0 \bullet c_{\ell-1}c_{\ell-2}\cdots c_0 0^\omega$ is a $\widetilde{\mathcal{B}}$-representation of x'. But now,

$$\mathrm{val}_{\widetilde{\mathcal{B}}}(0\bullet a_{\ell-1}\cdots a_0 0^\omega) < x' = \mathrm{val}_{\widetilde{\mathcal{B}}}(0\bullet a_{\ell-1}\cdots a_0 a_{-1}\cdots) = \mathrm{val}_{\widetilde{\mathcal{B}}}(0\bullet c_{\ell-1}\cdots c_0 0^\omega)$$

which demonstrates that $\widetilde{\mathcal{B}}$ is not optimal. According to Lemma 5.20, even the base $\mathcal{B}$ is not optimal - a contradiction. □

5.4 Rewriting system: confluence and normalization

Inspired by the method of proof of Proposition 5.17, in accordance with Remark 5.18, we define a rewriting system associated with the expansions

$\mathbf{d}_i$ $(0 \leq i \leq p-1)$. We will study when such a system is confluent, i.e., when different sequences of transformations starting from the same element can always be "merged back" together to obtain the same sequence of digits. Before defining our system, we first recall the definition of rewriting systems and confluence in a general setting.

Definition 5.22. A *rewriting system* is a pair $(E, \to)$ where $\to$ is a binary relation on a set E. Let $\to^*$ be the reflexive transitive closure of $\to$. Then $\to$ is *confluent* if

$$\forall u, x, y \in E, (u \to^* x,\, u \to^* y \Rightarrow \exists v \in E : x \to^* v,\, y \to^* v)\,.$$

From a base $\mathcal{B}$, we define a rewriting system $\rho_{\mathcal{B}}$. We want the rules to represent the operation "remove a nongreedy representation of 1, and add 1 to the digit preceding that representation". To this end, we must keep track of the position of the letters modulo p in some fashion in order to perform only rewritings that preserve the value of the word being rewritten.

To take this into account, our rules will process blocks of p letters that are correctly aligned (starting at position $p-1$ mod p and ending at position 0 mod p). We use a vertical bar $|$ to note a separation between blocks.

We let B be the alphabet of blocks,

$$B = \{x_{p-1}x_{p-2}\cdots x_0 : 0 \leq x_i < \beta_i \text{ for all } i \in \{0, \ldots, p-1\}\}.$$

We further introduce the digitwise addition of words of the same length in $\mathbb{N}^*$ or in B^*. For $x = x_1 \cdots x_\ell$, $y = y_1 \cdots y_\ell$, set

$$x \oplus y = (x_1 + y_1)(x_2 + y_2)\cdots(x_\ell + y_\ell),$$

where addition on the elements of B is itself defined digitwise. Note that using $x \oplus y$ implicitly states that the lengths of x and y are the same.

We now define the rewriting system $\rho_{\mathcal{B}} = (E, \to)$ associated with the alternate base $\mathcal{B}$.

Definition 5.23. Let $\mathcal{B} = (\beta_{p-1}, \beta_{p-2}, \ldots, \beta_0)$ be an alternate base such that $\mathbf{d}_i$ is finite for all $i \in \{0, \ldots, p-1\}$, with $d_i = t_{i,1}t_{i,2}\cdots t_{i,\ell_i}$.

The set E is the set ${}^{\omega}0B^* \bullet B^*0^{\omega}$ of biinfinite words over the alphabet B that have a left-infinite and a right-infinite tail of zeros.

If $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{N}$, set $\to$ to be the empty relation. Otherwise, for every $i \in \{0, 1, \ldots, p-1\}$ we define the *core rule* $U_i \to V_i$ in the following way:

- if $\beta_{i-1} \notin \mathbb{N}$, then

$$U_i = |0^{p-i}d_i0^j \quad \text{and} \quad V_i = |0^{p-i-1}10^h$$

- if $\beta_{i-1} \in \mathbb{N}$ and $k \in \mathbb{N}$ is such that $\beta_{i-1}, \ldots, \beta_{i-k} \in \mathbb{N}$ and $\beta_{i-k-1} \notin \mathbb{N}$, then

$$U_i = |0^{p-i}d'_id'_{i-1}\cdots d'_{i-k+1}d_{i-k}0^j \quad \text{and} \quad V_i = |0^{p-i-1}10^h$$

where in both cases, $j \in \mathbb{N}$ is the smallest integer such that the length of U_i equals 0 mod p and $h \in \mathbb{N}$ is such that lengths of V_i and U_i coincide. In the second case, we recall that $d'_{i-j} = \lceil \beta_{i-1-j} \rceil - 1$ for $0 \leq j \leq k-1$.

We now define the rewriting relation $\to$ by stating that $\mathbf{u} \to \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in E$ if there exist $M > N \in \mathbb{Z}$ and $w \in B^{M-N}$ such that

$$\mathbf{u}_{(-\infty,N]} = \mathbf{v}_{(-\infty,N]} \text{ and } \mathbf{u}_{(M,+\infty)} = \mathbf{v}_{(M,+\infty)}$$
$$\text{and } \mathbf{u}_{(N,M]} = w \oplus U_i \text{ and } \mathbf{v}_{(N,M]} = w \oplus V_i.$$

A *rule* is a rewriting of a finite word, of the form $w \oplus U_i \to w \oplus V_i$.

Intuitively, the core rule $U_i \mapsto V_i$ indicates that a finite expansion of 1, when correctly aligned, can be subtracted provided that we increment the digit immediately to the left so as to preserve the value of the word. The rewriting system then allows us to perform this operation in a context-free manner, replacing a factor of an infinite word with tails of zeros by a factor of the same length.

Example 5.24. Let $\mathbf{d}_1 = 111$ and $\mathbf{d}_0 = 21$, which is obtained by selecting $\beta_1 = \frac{3+\sqrt{5}}{2}$ and $\beta_0 = \frac{1+\sqrt{5}}{2}$. The set of blocks is $B = \{00, 01, 10, 11, 20, 21\}$. We have $|00|21| \to |01|00|$ and $|01|11| \to |10|00|$ as our core rules. The associated rewriting system $\rho_{\mathcal{B}}$ consists of seven rules:

$$|00|21| \to |01|00|, \quad |10|21| \to |11|00|, \quad |20|21| \to |21|00|,$$
$$|01|11| \to |10|00|, \quad |01|21| \to |10|10|,$$
$$|11|11| \to |20|00|, \quad |11|21| \to |20|10|.$$

Example 5.25. Consider the alternate base numeration system $\mathcal{B}$ with period $p = 6$ given by $(\beta_5, \ldots, \beta_0) = (122/23, 230/33, 99/16, 4, 4, 4)$, bringing the six expansions

$$\mathbf{d}_5 = 66, \quad \mathbf{d}_4 = 603, \quad \mathbf{d}_3 = 4, \quad \mathbf{d}_2 = 4, \quad \mathbf{d}_1 = 4, \quad \mathbf{d}_0 = 5203.$$

The core rules are as follows.

$$\begin{array}{lcl}
|066000| & \to & |100000| \\
|006030| & \to & |010000| \\
|000333|520300| & \to & |001000|000000| \\
|000033|520300| & \to & |000100|000000| \\
|000003|520300| & \to & |000010|000000| \\
|000000|520300| & \to & |000001|000000|
\end{array}$$

Example 5.26. Let us continue Example 5.15. We recall that $\mathbf{d}_0 = 533$, $\mathbf{d}_2 = 435$, $\mathbf{d}_1 = 353$ in this system. The core rules are simply

$$\begin{array}{lcl}
|043|500| & \to & |100|000| \\
|003|530| & \to & |010|000| \\
|000|533| & \to & |001|000|.
\end{array}$$

We can see for instance that $000|533|530 \to 001|000|530$ by the core rule $U_0 \to V_0$ and $000|533|530 \to 000|540|000$ by the core rule $U_1 \to V_1$. From there, both words obtained are irreducible in this rewriting system. This now familiar example tells us that the system is not confluent.

We establish a few elementary properties of our system $\rho_{\mathcal{B}}$. These property mostly formalize that the system $\rho_{\mathcal{B}}$ corresponds to the intuitive view that was given above.

Proposition 5.27. *Let $\mathcal{B}$ be an alternate base and $\rho_{\mathcal{B}}$ be the associated rewriting system.*

1. *For each $i \in \{0, \ldots, p-1\}$, if $\mathbf{u} \to \mathbf{v}$ in $\rho_{\mathcal{B}}$, then $\mathrm{val}_{\mathcal{B}}(\mathbf{u}) = \mathrm{val}_{\mathcal{B}}(\mathbf{v})$.*

2. *If $\mathbf{u} \to \mathbf{v}$ in $\rho_{\mathcal{B}}$ and $\mathbf{u}$, when converted back to a word on A indexed with positive numbers to the left, is such that $u_n = 0$ for all $n < N$, then we also have $v_n = 0$ for all $n < N$ in the same conditions. That is, our rewriting rules do not create nonzero digits further to the right than they already exist.*

3. *If a word* $\mathbf{y}$ *is equal to* ${}^{\omega}0|y_{np-1}y_{np-2}\cdots y_0|0^{\omega}$ *with* $y_{np-1}\cdots y_0$ *$\mathcal{B}$-admissible, then no rewriting rule can be applied to the word* $\mathbf{y}$*.*

4. *If $\mathcal{B}$ has the MDP and a word* $\mathbf{y}$ *is written as* ${}^{\omega}0|y_{np-1}y_{np-2}\cdots y_0|0^{\omega}$ *where* $y_{np-1}\cdots y_0$ *is not $\mathcal{B}$-admissible, then there exists some rule in* $\rho_{\mathcal{B}}$ *that can be applied to the word* $\mathbf{y}$*.*

Proof. The first three assertions are easily verified. We focus on the fourth one.

Without loss of generality, let $y = y_{np-1}y_{np-2}\cdots y_0$ be not $\mathcal{B}$-admissible with $y_{np-1} = 0$. There must be some factor $y_{i-1}\cdots y_j$ which is lexicographically greater than or equal to $\mathbf{d}_i^*$. Since $y_{i-1}\cdots y_j$ is a finite word, it must be strictly greater. Among all pairs (i,j) that produce such a nonadmissible factor, choose those that have the largest possible i. As for the discussion around (5.2) in the proof of Proposition 5.17, it then follows that $y_i < \lceil\beta_i\rceil - 1$. Among those pairs, choose the one that has the largest possible j. Now, since $\mathbf{d}_i^*$ starts with

$$(\beta_{i-1}-1)\cdots(\beta_{i-k}-1)t_{i-k,1}\cdots t_{i-k,\ell_{i-k}-1}(t_{i-k,\ell_{i-k}}-1)\mathbf{d}^*_{i-k-\ell_{i-k}}$$

(where k is such that $\beta_{i-1},\ldots,\beta_{i-k}\in\mathbb{N}$ and $\beta_{i-k-1}\notin\mathbb{N}$) and since the first $k+\ell_{i,k}-1$ digits are maximal in the alphabet, the only way to satisfy the optimality conditions that have been imposed on i and j is that

$$y_{i-1}\cdots y_j = (\beta_{i-1}-1)\cdots(\beta_{i-k}-1)t_{i-k,1}\cdots t_{i-k,\ell_{i-k}-1}(t_{i-k,\ell_{i-k}}+c)$$

with $c\geq 0$. As a result, the rule $U_i \to V_i$ from $\rho_{\mathcal{B}}$ may be applied on this factor of $\mathbf{y}$. □

Normalization is a classical problem in numeration systems, which asks whether given a representation $c_nc_{n-1}\cdots$ of a number x, it is possible to construct its expansion $d_{\mathcal{B}}(x) = a_na_{n-1}\cdots$. In our setting, we hope to use the rules in $\rho_{\mathcal{B}}$ to progressively rewrite ${}^{\omega}0c_nc_{n-1}\cdots$ until we reach ${}^{\omega}0a_na_{n-1}\cdots$. Additionally, we would like to apply rules of $\rho_{\mathcal{B}}$ without any strategy and still reach ${}^{\omega}0a_na_{n-1}\cdots$ independently of the order in which the rules are used. We define our target property accordingly.

Definition 5.28. The rewriting system ρ *allows normalization* in base $\mathcal{B}$ on the alphabet B if, for every representation $\mathbf{c}\in E$ of a number x, there exists exactly one irreducible word $\mathbf{u} \in E$ such that $\mathbf{c} \to^* \mathbf{u}$, and this word $\mathbf{u}$ is such that $\mathbf{u} = \langle \mathrm{val}_{\mathcal{B}}(\mathbf{c}) \rangle_{\mathcal{B}}$.

Note that ρ allowing normalization implies that $\langle \mathrm{val}_{\mathcal{B}}(\mathbf{c}) \rangle_{\mathcal{B}}$ is finite when $\mathbf{c}$ is. This property is similar in nature to the property (F) mentioned in the introduction of this chapter. In fact, if the alphabet is large enough it implies the weaker property (PF) where only $\mathbb{N}[\beta^{-1}] \cap [0, 1)$ is included in $\mathrm{Fin}(\beta)$.

Example 5.29. Why consider only finite representations in the above definition? If we set $\mathcal{B} = (\varphi)_{n \in \mathbb{Z}}$ and $\mathbf{c} = {}^{\omega}0.(011)^{\omega}$, then $\mathbf{c}$ cannot be rewritten into the normalized expansion ${}^{\omega}0.(100)^{\omega}$ using finitely many rules (and, indeed, cannot be rewritten into any irreducible word using only finitely many rules).

Having defined all the required objects, we may finally state the main result of this section.

Theorem 5.30. *Let $\mathcal{B}$ be an alternate base such that the $\mathbf{d}_i$ are all finite. The following assertions are equivalent.*

(a) The base $\mathcal{B}$ has the maximal digit property.

(b) The rewriting system $\rho_{\mathcal{B}}$ is confluent over $E = {}^{\omega}0B^* \bullet B^*0^{\omega}$.

(c) The rewriting system $\rho_{\mathcal{B}}$ allows normalization in base $\mathcal{B}$ on the alphabet B.

Proof. First, in the case where $\beta_{p-1}, \ldots, \beta_0$ are all integers, we note that $\mathcal{B}$ has the MDP, $\rho_{\mathcal{B}}$ is confluent as $\to^*$ is the identity, and all representations in E are admissible. Thus, our result holds in this case. Now assume that at least one element of the base is not an integer. We prove three circular implications.

(a)$\Rightarrow$(c). Consider a representation $\mathbf{c} \in E$ of a number x. If $\mathbf{u}$ is an irreducible word such that $\mathbf{c} \to^* \mathbf{u}$, then $\mathrm{val}_{\mathcal{B}}(\mathbf{c}) = \mathrm{val}_{\mathcal{B}}(\mathbf{u})$ by Proposition 5.27. By the same proposition, since $\mathcal{B}$ has the maximal digit property, $\mathbf{u}$ being irreducible implies that it is $\mathcal{B}$-admissible. Therefore, $\mathbf{u}$ must be $\langle \mathrm{val}_{\mathcal{B}}(c) \rangle_{\mathcal{B}}$. Now it remains to show that such a word $\mathbf{u}$ exists. Note that words in E have a finite sum of digits and applying a rule in $\rho_{\mathcal{B}}$ always decreases this sum. Thus, there exists no infinite chain $\mathbf{c}_1 \to \mathbf{c}_2 \to \cdots$. As a result, there must exist an irreducible word $\mathbf{u}$ such that $\mathbf{c} \to^* \mathbf{u}$, as desired.

(c)⇒(b). Assume that $\rho_\mathcal{B}$ allows normalization in base $\mathcal{B}$ on the alphabet B, and select three words $\mathbf{u}, \mathbf{x}, \mathbf{y} \in E$ such that $\mathbf{u} \to^* \mathbf{x}$ and $\mathbf{u} \to^* \mathbf{y}$. Then $\mathbf{x}$ and $\mathbf{y}$ have the same value by Proposition 5.27. Since $\rho_\mathcal{B}$ allows normalization in base $\mathcal{B}$, we have $\mathbf{x} \to^* \langle \mathrm{val}_\mathcal{B}(x)\rangle_\mathcal{B}$ and $\mathbf{y} \to^* \langle \mathrm{val}_\mathcal{B}(y)\rangle_\mathcal{B}$. Both of the right hand sides are the same as $\mathbf{x}$ and $\mathbf{y}$ have the same value. We have therefore shown that $\rho_\mathcal{B}$ is confluent over E.

(b)⇒(a). We proceed by contraposition. Assume that the base $\mathcal{B}$ does not have the MDP, and we will search for $\mathbf{u}, \mathbf{x}, \mathbf{y} \in E$ such that $\mathbf{u} \to^* \mathbf{x}$, $\mathbf{u} \to^* \mathbf{y}$ but there is no $\mathbf{v} \in E$ with $\mathbf{x} \to^* \mathbf{v}$ and $\mathbf{y} \to^* \mathbf{v}$. Since $\mathcal{B}$ does not have the MDP, there exist i and j with $0 \le i \le p-1$ and $1 < j < \ell_i$ such that $t_{i,j} < \lceil \beta_{i-j} \rceil - 1$. We select an i such that an associated j exists and then choose j as large as possible. Thus d_i has as a suffix the word

$$(\lceil \beta_{i-j-1} \rceil - 1)(\lceil \beta_{i-j-2} \rceil - 1)\cdots(\lceil \beta_{i-\ell_i+1} \rceil - 1)t_{i,\ell_i}.$$

Due to the Parry condition, this suffix is lexicographically strictly smaller than $\mathbf{d}^*_{i-j}$, where the strict inequality is due to being a finite word.

Let k be such that $\beta_{i-j-1}, \dots, \beta_{i-j-k} \in \mathbb{N}$ and $\beta_{i-j-k-1} \notin \mathbb{N}$. Consider the word $w = (\lceil \beta_{i-j-1} \rceil - 1)\cdots(\lceil \beta_{i-j-k} \rceil - 1)t_{i-j-k,1}\cdots t_{i-j-k,\ell_{i-j-k}}$. We have $\mathbf{d}^*_{i-j} <_{\mathrm{lex}} w$, thus

$$(\lceil \beta_{i-j-1} \rceil - 1)(\lceil \beta_{i-j-2} \rceil - 1)\cdots(\lceil \beta_{i-\ell_i+1} \rceil - 1)t_{i,\ell_i} <_{\mathrm{lex}} w$$

as well. Due to the maximality of the digits on the left-hand side, it must be that $|w| = k + \ell_{i-j-k} \ge \ell_i - j$,

$$\mathrm{Pref}_{\ell_i-j-1}(w) = (\lceil \beta_{i-j-1} \rceil - 1)(\lceil \beta_{i-j-2} \rceil - 1)\cdots(\lceil \beta_{i-\ell_i+1} \rceil - 1)$$

and $w_{\ell_i-j} \ge t_{i,\ell_i}$. Hence the word

$$\begin{aligned} z &= t_{i,1}\cdots t_{i,j} w \\ &= t_{i,1}\cdots t_{i,j}(\lceil \beta_{i-j-1} \rceil - 1)\cdots(\lceil \beta_{i-j-k} \rceil)t_{i-j-k,1}\cdots t_{i-j-k,\ell_{i-j-k}} \end{aligned}$$

coincides with

$$t_{i,1}t_{i,2}\cdots t_{i,\ell_i-1} w_{\ell_i-j}\cdots w_{k+\ell_{i-j-k}}.$$

Find the smallest $h \in \mathbb{N}$ such that the length of the word $0^{p-i}z0^h$ is a multiple of p, and set $u = 0^{p-i}z0^h$ and $\mathbf{u} = {}^{\omega}0u \bullet 0^{\omega}$. We will use this word to show that the rewriting system associated with the base $\mathcal{B}$ is not confluent.

Consider the following array where the first line is $\mathbf{u}$ and can be rewritten into any of the following two lines using a rule in $\rho_{\mathcal{B}}$. We may go from the first line to the second using the core rule i and from the first to the third using the core rule $i-j \bmod p$.

$$\begin{array}{c|cccccccc}
{}^{\omega}0 & 0^{p-i} & t_{i,1}\cdots t_{i,j-1} & t_{i,j} & w_1\cdots w_{\ell_i-j+1} & w_{\ell_i-j} & w_{\ell_i-j-1}\cdots w_{k+\ell_{i-j-k}} & 0^h & \bullet 0^{\omega} \\
\hline
{}^{\omega}0 & 0^{p-i-1}\,1 & 0\ \cdots\ 0 & 0 & 0\ \cdots\ 0 & (w_{\ell_i-j}-t_{i,\ell_i}) & w_{\ell_i-j-1}\cdots w_{k+\ell_{i-j-k}} & 0^h & \bullet 0^{\omega} \\
{}^{\omega}0 & 0^{p-i} & t_{i,1}\cdots t_{i,j-1} & t_{i,j}+1 & 0\ \cdots\ 0 & 0 & 0\ \cdots\ 0 & 0^h & \bullet 0^{\omega}
\end{array}$$

We now claim that the second and third line in the above array cannot be rewritten into a common word. Note that increasing the digit at a given position can only be done by applying one specific core rule and requires the digit immediately to the right to be nonzero. Note also that after applying some rules to the second line of the array, the position currently containing the leftmost 1 or a position to its left will contain a nonzero digit. If we want the same to happen from the third line, the digit at this position will have to increase, and to do this we will have to use from the third line the same rule that brought us from the first to the second line. However, this would subtract $t_{i,\ell_i} > 0$ from the position that has a w_{ℓ_i-j} in the first line. We have no way of incrementing this position from the third line, as all the positions to the right are zero, and we cannot have a negative digit, so this is not possible. Thus we have found a counterexample to the confluence of the rewriting system, which concludes the proof. □

Observe that for the construction of (b)⇒(a) to hold it was necessary that $t_{i,j}+1$ be an allowable digit, hence why the MDP failing is necessary to use this argument. Note also that this construction is exactly the one used in Example 5.26.

5.5 Additional comments

In this section, we collect miscellaneous comments on related work and possible extensions of the above results.

5.5.1 Confluence for infinite words

Section 5.4 deals only with finite words and bases $\mathcal{B} = (\beta_{p-1}, \beta_{p-2}, \ldots, \beta_0)$ such that $\mathbf{d}_i$ is finite for every $i \in \{0, \ldots, p-1\}$. One can regret this fact as Section 5.3 on the other hand is valid for all bases, including those with infinite expansions of 1. In this section, we offer results regarding confluence in the case where the expansions of 1 may be infinite as well. These results are offered as a comment rather than a main result, due to the overly technical nature of their proofs compared to the improvement that they bring over Theorem 5.30.

We must first define a new system of rewriting rules that is suitable to deal with these new expansions. For historical reasons, we actually define two such systems. The system $\rho''_{\mathcal{B}}$ is the first that we introduced in our research and the one we used in our proofs. When we decided to not include these proofs in the article [CKMP26a], we shifted the presentation in that article to the system $\rho'_{\mathcal{B}}$, which is perhaps simpler to explain and motivate. Nevertheless, these two systems are equivalent.

We start by introducing $\rho'_{\mathcal{B}}$. So far, we have allowed to replace an expansion of 1 by increasing the digit to the left of it by 1. We will keep this idea, but now also allow to replace quasi-greedy expansions of 1 in this fashion.

Definition 5.31. $\mathcal{B} = (\beta_{p-1}, \beta_{p-2}, \ldots, \beta_0)$ be an alternate base. Let E' be the set of biinfinite words over the set of blocks B with a left-infinite tail of zeros. For every i, consider the core rule $U_i \to V_i$ from Definition 5.23 if $\mathbf{d}_i$ is finite and $\beta_{p-1}, \beta_{p-2}, \ldots, \beta_0$ are not all integers. Additionally, consider for every i in $\{0, \ldots, p-1\}$ the core rule $U'_i \to V'_i$ defined by

$$U'_i = |0^{p-i}\mathbf{d}^*_i \quad \text{and} \quad V'_i = |0^{p-i-1}10^\omega.$$

We now define the rewriting relation $\to$ by stating that $\mathbf{u} \to \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in E'$ if either:

- There exist $M > N \in \mathbb{Z}$ and $w \in B^{M-N}$ such that

$$\mathbf{u}_{(-\infty,N]} = \mathbf{v}_{(-\infty,N]} \text{ and } \mathbf{u}_{(M,+\infty)} = \mathbf{v}_{(M,+\infty)}$$
$$\text{and } \mathbf{u}_{(N,M]} = w \oplus U_i \text{ and } \mathbf{v}_{(N,M]} = w \oplus V_i.$$

- There exists $M \in \mathbb{Z}$ and $\mathbf{w}$ such that $w_n = 0$ for all $n > M$, with

$$\mathbf{u}_{(M,+\infty)} = \mathbf{v}_{(M,+\infty)} \text{ and}$$
$$\mathbf{u}_{(-\infty,M]} = \mathbf{w}_{(-\infty,M]} \oplus U'_i \text{ and } \mathbf{v}_{(-\infty,M]} = \mathbf{w}_{(-\infty,M]} \oplus V'_i.$$

  where U'_i and V'_i have been shifted to the correct position so that the notation $\oplus$ makes sense.

For the system $\rho''_{\mathcal{B}}$, the idea is very similar, except that we allow rewritings of all intermediate expansions of 1 (up to and including the quasi-greedy one) rather than just a specific one. Recall that these intermediate expansions of 1 are defined in Definition 2.12, as is the quantity $k_{i,j}$.

Definition 5.32. $\mathcal{B} = (\beta_{p-1}, \beta_{p-2}, \ldots, \beta_0)$ be an alternate base. We adapt Definition 2.12 in the spirit of Definition 2.10. If the word $\mathbf{w}_{i,j}$ ends in a tail of zeros, being equal to $d'_i \cdots d'_{i-k_{i,j-1}} d_{i-k_{i,j}} 0^\omega$, we let $w_{i,j}$ be the finite word $d'_i \cdots d'_{i-k_{i,j-1}} d_{i-k_{i,j}}$ and let $w'_{i,j} = d'_i \cdots d'_{i-k_{i,j-1}} d'_{i-k_{i,j}}$ be the word $w_{i,j}$ with 1 subtracted from the last digit.

Let E' be the set of biinfinite words over the set of blocks B with a left-infinite tail of zeros. For every $i \in \{0, \ldots, p-1\}$ and $j \geq 0$ such that $\beta_{i-k_{i,j}-1} \notin \mathbb{N}$ and $\mathbf{w}_{i,j}$ is defined and ends in a tail of zeros, we define the *core rule* $U_{i,j} \to V_{i,j}$ by

$$U_{i,j} = |0^{p-i} w_{i,j} 0^{h_1} \quad \text{and} \quad V_{i,j} = |0^{p-i-1} 1 0^{h_2}$$

where $h_1 \in \mathbb{N}$ is the smallest integer such that the length of U_i equals 0 mod p and $h_2 \in \mathbb{N}$ is such that lengths of V_i and U_i coincide.

Additionally, we define the core rule $U_{i,\infty} \to V_{i,\infty}$ by

$$U_{i,\infty} = |0^{p-i} \mathbf{d}^*_i \quad \text{and} \quad V_{i,\infty} = |0^{p-i-1} 1 0^\omega.$$

We now define the rewriting relation $\to$ by stating that $\mathbf{u} \to \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in E'$ if either:

- There exist $M, N \in \mathbb{Z}$, $i \in \{0, \ldots, p-1\}$, $j \geq 0$ and $w \in B^{M-N}$ such that

$$\mathbf{u}_{(-\infty,N]} = \mathbf{v}_{(-\infty,N]} \text{ and } \mathbf{u}_{(M,+\infty)} = \mathbf{v}_{(M,+\infty)}$$
$$\text{and } \mathbf{u}_{(N,M]} = w \oplus U_{i,j} \text{ and } \mathbf{v}_{(N,M]} = w \oplus V_{i,j}.$$

- There exists $M \in \mathbb{Z}$ and $\mathbf{w}$ such that $w_n = 0$ for all $n > M$, with

$$\mathbf{u}_{(M,+\infty)} = \mathbf{v}_{(M,+\infty)} \text{ and}$$
$$\mathbf{u}_{(-\infty,M]} = \mathbf{w}_{(-\infty,M]} \oplus U_{i,\infty} \text{ and } \mathbf{v}_{(-\infty,M]} = \mathbf{w}_{(-\infty,M]} \oplus V_{i,\infty}.$$

  where U'_i and V'_i have been shifted to the correct position so that the notation $\oplus$ makes sense.

We say that the rule $\mathbf{u} \to \mathbf{v}$ is *derived* from the core rule (i, j), with $j \geq 0$ or $j = \infty$. When $\mathbf{u} \to \mathbf{v}$ by some rewriting rule, we let the *start* of this rule be the index of the leftmost position that is modified by this rule and its *end* be the index of the rightmost modified position, or ∞ if the rule is infinite. The *width* of a rule is the difference between its start and end. It is a number in $\mathbb{N}_{\geq 2} \cup \{\infty\}$.

For both systems, if $\mathbf{u} \to \mathbf{v}$, then the words $\mathbf{u}$ and $\mathbf{v}$ have the same value in base $\mathcal{B}$.

Example 5.33. Consider the alternate base $\mathcal{B}$ of period 2 where the expansions of 1 are given by $\mathbf{d}_0 = 30^\omega$ and $\mathbf{d}_1 = 1110^\omega$. In the system $\rho'_{\mathcal{B}}$, the core rules are given as follows.

$$\begin{array}{lcl}
|00|21|11| & \to & |01|00|00| \\
|01|11| & \to & |10|00| \\
|00|21|10|21|10|\cdots & \to & |01|00|00|00|\cdots \\
|01|10|21|10|\cdots & \to & |10|00|00|\cdots
\end{array}$$

The first two rules are the same as in $\rho_{\mathcal{B}}$, and the last two are the rules $U'_0 \to V'_0$ and $U'_1 \to V'_1$ that were added.

In the system $\rho''_{\mathcal{B}}$, the core rules are as follows.

$$\begin{array}{lcl}
|00|21|11| & \to & |01|00|00| \\
|00|21|10|21|11| & \to & |01|00|00|00|00| \\
|00|21|10|21|10|21|11| & \to & |01|00|00|00|00|00|00| \\
 & \cdots & \\
|00|21|10|21|10|21|10|\cdots & \to & |01|00|00|00|\cdots \\
\hline
|01|11| & \to & |10|00| \\
|01|10|21|11| & \to & |10|00|00|00| \\
|01|10|21|10|21|11| & \to & |10|00|00|00|00|00| \\
 & \cdots & \\
|01|10|21|10|21|10|\cdots & \to & |10|00|00|\cdots
\end{array}$$

Above the bar, we find the core rules $(0, 2k+1)$ for natural k, and the core rule $(0, \infty)$. Below, we find the core rules $(1, 2k)$ and $(1, \infty)$. Note that some values of i and j are rejected because $\beta_{i-k_{i,j}-1}$ is an integer. For instance, the rule $(0, 0)$ doesn't appear because $\beta_{-1} = 3$. This rule would correspond to $|00|30| \to |01|00|$, but 3 is not a valid digit in the alphabet.

We can note that the rule

$$U_{1,2} \to V_{1,2} : |01|10|21|10|21|11| \to |10|00|00|00|00|00|$$

from $\rho''_{\mathcal{B}}$ can be obtained by combining three rules of $\rho'_{\mathcal{B}}$, those being the rule $|00|21|11| \to |01|00|00|$ twice and $|01|11| \to |10|00|$ once. In fact, this statement is general, as we will see in the next proposition.

We start by proving that the two systems are equivalent, in that $\mathbf{u} \to^*_{\rho'_{\mathcal{B}}} \mathbf{v}$ if and only if $\mathbf{u} \to^*_{\rho''_{\mathcal{B}}} \mathbf{v}$.

Proposition 5.34. *For any two words* $\mathbf{u}, \mathbf{v}$ *in* E', *we have*

$$\mathbf{u} \to^*_{\rho'_{\mathcal{B}}} \mathbf{v} \Leftrightarrow \mathbf{u} \to^*_{\rho''_{\mathcal{B}}} \mathbf{v}.$$

Proof. It is enough to prove that if $\mathbf{u} \to_{\rho'_{\mathcal{B}}} \mathbf{v}$, then $\mathbf{u} \to^*_{\rho''_{\mathcal{B}}} \mathbf{v}$, and that $\mathbf{u} \to_{\rho''_{\mathcal{B}}} \mathbf{v}$ similarly implies $\mathbf{u} \to^*_{\rho'_{\mathcal{B}}} \mathbf{v}$. If all the base elements are integers, only the core rules of infinite width subsist in both systems, which are then clearly equal. Let us assume that at least one base element is noninteger.

The first of these implications is simple. The core rule $U_i \to V_i$ of $\rho'_{\mathcal{B}}$ can be found as the core rule $U_{i,j} \to V_{i,j}$ of $\rho''_{\mathcal{B}}$ by selecting j to be the smallest natural number such that $\beta_{i-j-1} \notin \mathbb{N}$, and the core rule $U'_i \to V'_i$ in $\rho'_{\mathcal{B}}$ corresponds to the core rule $U_{i,\infty} \to V_{i,\infty}$ of $\rho''_{\mathcal{B}}$.

We move to the other implication. Due to the last remark in the previous paragraph, we need only to focus on the rules $U_{i,j} \to V_{i,j}$ with $j < \infty$. Let $w_{i,j}$ be equal to $d'_i d'_{i-k_{i,1}} \cdots d'_{i-k_{i,j-1}} d_{i-k_{i,j}}$. Let $r_1, \ldots, r_s$ be the indices such that $\beta_{i-k_{i,\ell}-1}$ is in $\mathbb{N}$ if and only if ℓ is not among $r_1, \ldots, r_s$. Note that $r_s = j$. We factorize $w_{i,j}$ as

$$w_{i,j} = \left(d'_i d'_{i-1} \cdots d'_{i-k_{i,r_1}}\right)\left(d'_{i-k_{i,r_1+1}} d'_{i-k_{i,r_1+1}-1} \cdots d'_{i-k_{i,r_2}}\right) \cdots \left(d'_{i-k_{i,r_{s-1}+1}} \cdots d_{i-k_{i,r_s}}\right)$$

But now for any $\ell \in \{2, \ldots, s\}$, we have that

$$\begin{aligned} &{}^{\omega}0|0^{p-i}\left(d'_i d'_{i-1} \cdots d'_{i-k_{i,r_1}}\right) \cdots \left(d'_{i-k_{i,r_{\ell-2}+1}} \cdots d'_{i-k_{i,r_{\ell}-1}}\right)\left(d'_{i-k_{i,r_{\ell-1}+1}} \cdots d_{i-k_{i,r_\ell}}\right) 0^{\omega} \\ \to_{\rho'_{\mathcal{B}}} &{}^{\omega}0|0^{p-i}\left(d'_i d'_{i-1} \cdots d'_{i-k_{i,r_1}}\right) \cdots \left(d'_{i-k_{i,r_{\ell-2}+1}} \cdots d_{i-k_{i,r_{\ell-1}}}\right) 0^{\omega} \end{aligned}$$

by core rule $U_{i-k_{i,r_{\ell-1}}+1} \to V_{i-k_{i,r_{\ell-1}}+1}$, and similarly

$${}^{\omega}0|0^{p-i}\left(d'_i d'_{i-1} \cdots d'_{i-k_{i,r_1}}\right) 0^{\omega} \to_{\rho'_{\mathcal{B}}} {}^{\omega}0|0^{p-i-1}10^{\omega}.$$

It follows that $U_{i,j} \to V_{i,j}$ can be obtained as a succession of rules of $\rho'_{\mathcal{B}}$, and the proposition is proved. □

$(i,0\|\infty)$	${}^{\omega}0$	0^{p-i}	$t_{i,1}\cdots t_{i,j-1}$	$t_{i,j}$	$t_{i,j+1}t_{i,j+2}\cdots$
$(i-j,\infty)$	${}^{\omega}0$	0^{p-i}	0^{j-1}	0	$t_{i-j,1}t_{i-j,2}\cdots$
$\mathbf{u}$	${}^{\omega}0$	0^{p-i}	$t_{i,1}\cdots t_{i,j-1}$	$t_{i,j}$	$b_1b_2b_3\cdots$
$\mathbf{x}$	${}^{\omega}0$	$0^{p-i-1}1$	0^{j-1}	0	$(b_1-t_{i,j+1})(b_2-t_{i,j+2})\cdots$
$\mathbf{y}$	${}^{\omega}0$	0^{p-i}	$t_{i,1}\cdots t_{i,j-1}$	$t_{i,j}+1$	$(b_1-t_{i-j,1})(b_2-t_{i-j,2})\cdots$

Table 5.1: Visual help for the construction of the proof of Proposition 5.35

In the rest of this section, we will only work with $\rho''_{\mathcal{B}}$. To alleviate notation, we do not mention it and simply write the rewriting relation as $\to$. We will now prove that this system is confluent if and only if the base $\mathcal{B}$ has the maximal digit property. Since this property implies that all expansions of 1 are finite, we may only have confluence for bases with finite expansions of 1. In fact, switching from $\rho_{\mathcal{B}}$ to $\rho'_{\mathcal{B}}$ or $\rho''_{\mathcal{B}}$ preserves the confluence of the system, and does not introduce new confluent systems. We start by proving that the confluence $\rho''_{\mathcal{B}}$ implies the maximal digit property for $\mathcal{B}$.

Proposition 5.35. *Let $\mathcal{B}$ be an alternate base. If $\rho''_{\mathcal{B}}$ is confluent, then the base $\mathcal{B}$ has the maximal digit property,*

Proof. Let us proceed by contraposition. Assume that $\mathcal{B}$ doesn't have the MDP. Then, there exists some $i \in \{0,\ldots p-1\}$ and $2 \le j < \ell_i$ such that $t_{i,j} < \lceil \beta_{i-j} \rceil - 1$. In particular, note that the β_i are not all integers. We may assume that $t_{i,j+1}$ is nonzero, by iteratively replacing j by $j+1$ it if is. This process must terminate as $\mathbf{d}_i$ cannot contain only zeros from position j onwards.

We will construct words $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$, but there is no $\mathbf{z}$ with $\mathbf{x} \to^* \mathbf{z}$ and $\mathbf{y} \to^* \mathbf{z}$, which will prove that $\rho''_{\mathcal{B}}$ is not confluent. To this end, we consider two core rules: first, the core rule $(i,0)$ if $\mathbf{d}_i$ is finite or the core rule (i,∞) if $\mathbf{d}_i$ is infinite, and second, the core rule $(i-j,\infty)$. See Table 5.1 for a visual help throughout the proof.

Overlap these rules in such a fashion that the first nonzero digit in $U_{i,0}$ is j positions before the one in $U_{i-j,\infty}$ and take the digitwise maximum of the two words. We obtain the word

$$\mathbf{u} = {}^{\omega}0|0^{p-i}t_{i,1}\cdots t_{i,j}b_1b_2b_3\cdots$$

with $b_n = \max(t_{i,j+n}, t_{i-j,n})$ (we deliberately omit the tail of the word to lighten notation). Note that $\mathbf{u}$ is indeed in our alphabet since β_{i-1} is not an integer and due to our choice of core rules. Now, we can apply a rule derived from the core rule $(i, 0)$ or (i, ∞) to obtain

$$\mathbf{x} = {}^{\omega}0|0^{p-i-1}10^{j}(b_1 - t_{i,j+1})(b_2 - t_{i,j+2})\cdots$$

and similarly a rule derived from core rule $(i-j, \infty)$ to obtain

$$\mathbf{y} = {}^{\omega}0|0^{p-i}t_{i,1}\cdots t_{i,j-1}(t_{i,j}+1)(b_1 - t_{i-j,1})(b_2 - t_{i-j,2})\cdots .$$

Note that the digit $(b_1 - t_{i-j,1})$ which is at position $p - i + j + 1$ (in the rest of the proof, we count positions as increasing to the right, with position 1 immediately to the right of the $|$ character) must be 0, as $t_{i,j+1} \leq t_{i-j,1}$ due to the Parry conditions.

We now prove that $\mathbf{x}$ and $\mathbf{y}$ cannot be rewritten into a common word using rules of $\rho''_{\mathcal{B}}$. Indeed, such a common word would need to have a nonzero digit at some position less than $p - i$, since that is the case for all words reachable from $\mathbf{x}$. But for $\mathbf{y}$ to be rewritten into such a word, it would first need to be rewritten into a word that has a nonzero digit at position $p - i$ (just like $\mathbf{x}$). To do so, it is necessary to either increase the digit at position $p - i + j + 1$ (since $t_{i,j+1}$ was chosen greater than 0) or to use a core rule (i, k) with $k > 0$ and to be in the case where $t_{i,1}\cdots t_{i,j+1} = d'_i$. In both cases, it is necessary to have

$$\mathrm{val}_{\sigma^{j+1-i}(\mathcal{B})}\left((b_2 - t_{i-j,2})(b_3 - t_{i-j,3})\cdots\right) \geq 1.$$

However, note that

$$\mathrm{val}_{\sigma^{j+1-i}(\mathcal{B})}\left((b_2 - t_{i-j,2})(b_3 - t_{i-j,3})\cdots\right) \leq \mathrm{val}_{\sigma^{j+1-i}(\mathcal{B})}\left(t_{i,j+2}t_{i,j+3}\cdots\right)$$

since the digits on the left-hand side are digitwise less than or equal to those on the right-hand side. However, any proper suffix of $\mathbf{d}_i$ has value less than 1 in the appropriate base. In particular,

$$\mathrm{val}_{\sigma^{j+1-i}(\mathcal{B})}\left(t_{i,j+2}t_{i,j+3}\cdots\right) < 1,$$

and a contradiction is reached. Thus, $\mathbf{x}$ and $\mathbf{y}$ cannot be rewritten into a common word despite being obtained from a common word, which shows that $\rho''_{\mathcal{B}}$ is not confluent, as desired. □

We turn our attention to the other direction of the proof: if $\mathcal{B}$ has the MDP, then $\rho''_{\mathcal{B}}$ is confluent.

We begin by a study of local confluence: if $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$, what can we say of $\mathbf{x}$ and $\mathbf{y}$? To prepare the rest of the proof, we will have to provide results on the widths of the rules used. Recall that the width of a rule is the length of the interval of positions that it can modify. All rewritings are done in the system $\rho''_{\mathcal{B}}$; we will not specify this to lighten notation.

Lemma 5.36. *Let $\mathcal{B}$ be an alternate base with the maximal digit property. Let $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ be such that $\mathbf{x} \neq \mathbf{y}$, $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$. We have one of the following cases:*

(1) $\exists \mathbf{z} : \mathbf{x} \to \mathbf{z}, \mathbf{y} \to \mathbf{z}$, *and furthermore* $|\mathbf{x} \to \mathbf{z}| \leq |\mathbf{u} \to \mathbf{y}|$ *and* $|\mathbf{y} \to \mathbf{z}| \leq |\mathbf{u} \to \mathbf{x}|$.

(2a) $\mathbf{x} \to \mathbf{y}$. *Furthermore, we have* $|\mathbf{u} \to \mathbf{y}| \geq |\mathbf{u} \to \mathbf{x}|$ *and* $|\mathbf{x} \to \mathbf{y}| = |\mathbf{u} \to \mathbf{y}| - |\mathbf{u} \to \mathbf{x}|$ *if both widths on the right-hand side are finite.*

(2b) $\mathbf{y} \to \mathbf{x}$. *Furthermore, we have* $|\mathbf{u} \to \mathbf{x}| \geq |\mathbf{u} \to \mathbf{y}|$ *and* $|\mathbf{y} \to \mathbf{x}| = |\mathbf{u} \to \mathbf{x}| - |\mathbf{u} \to \mathbf{y}|$ *if both widths on the right-hand side are finite.*

(3a) $\exists \mathbf{z} : \mathbf{x} \to \mathbf{z}, \mathbf{z} \to \mathbf{y}$. *Furthermore, we have* $|\mathbf{u} \to \mathbf{y}| \geq |\mathbf{u} \to \mathbf{x}|$, $|\mathbf{u} \to \mathbf{x}| < \infty$, $|\mathbf{u} \to \mathbf{x}| + |\mathbf{x} \to \mathbf{z}| + |\mathbf{z} \to \mathbf{y}| = |\mathbf{u} \to \mathbf{y}|$ *if the width on the right-hand side is finite, and we may ask that* $|\mathbf{z} \to \mathbf{y}| < \infty$.

(3b) $\exists \mathbf{z} : \mathbf{y} \to \mathbf{z}, \mathbf{z} \to \mathbf{x}$. *Furthermore, we have* $|\mathbf{u} \to \mathbf{x}| \geq |\mathbf{u} \to \mathbf{y}|$, $|\mathbf{u} \to \mathbf{y}| < \infty$, $|\mathbf{u} \to \mathbf{y}| + |\mathbf{y} \to \mathbf{z}| + |\mathbf{z} \to \mathbf{x}| = |\mathbf{u} \to \mathbf{x}|$ *if the width on the right-hand side is finite, and we may ask that* $|\mathbf{z} \to \mathbf{x}| < \infty$.

Proof. Before starting the proof, let us introduce Figure 5.2, which can be used as a visual help throught this proof.

Consider $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$ such that $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$. Let s_x and e_x be the start and end of the rule $\mathbf{u} \to \mathbf{x}$ and let s_y and e_y be those for $\mathbf{u} \to \mathbf{y}$. We set $e_x = \infty$ (resp. $e_y = \infty$) if the relevant rule is infinite. Positions in this proof are counted increasingly from right to left. If the two rules do not overlap (that is, $e_x > s_y$ or $e_y > s_x$), we can apply both of them in any order, and therefore we are clearly in case (1).

If we have $s_x = s_y$, then the two rules must be derived from core rules with the same i. Let $\mathbf{u} \to \mathbf{x}$ be derived from core rule (i, j) and $\mathbf{u} \to \mathbf{y}$

be derived from core rule (i, k). If $j = k$, then $\mathbf{x} = \mathbf{y}$. We assume without loss of generality that $j < k$. See Figure 5.2a. Now, note that $w_{i,k}$ is equal to $w'_{i,j} w_{i-k_{i,j+1},k-j-1}$. As a result, we may apply a rule derived from core rule $(i - k_{i,j+1}, k - j - 1)$ on the digits that were not modified by $\mathbf{u} \to \mathbf{x}$ (and the rightmost digit that was modified) to obtain $\mathbf{y}$ from $\mathbf{x}$. Since $k_{i-k_{i,j+1},k-j} + k_{i,j+1} = k_{i,k+1}$, we are in case $(2a)$ and all the (in)equalities relating to widths do hold. Of course, the assumption $j > k$ leads to case $(2b)$.

Now assume that the two rules do not start at the same position. Without loss of generality we may assume that $s_x > s_y$. (That is, the leftmost digit modified by $\mathbf{u} \to \mathbf{x}$ is to the left of that for $\mathbf{u} \to \mathbf{y}$). Note that the digit at position s_y is increased by 1 when applying $\mathbf{u} \to \mathbf{y}$, and it is thus not the maximal digit of the alphabet in $\mathbf{u}$. From this and the maximal digit property of $\mathcal{B}$, we conclude that it must be the rightmost digit of some block d_i used to define the $w_{i,j}$. Now we discuss some cases based on the relative positions of e_x, s_y, e_y. We know that s_x is bigger than all of these, and we must have $s_y \geq e_x$ since these rules overlap. Of course, $s_y > e_y$. Thus we simply compare e_x and e_y.

Consider the case where $e_x > e_y$ and assume that $\mathbf{u} \to \mathbf{x}$ is derived from core rule (i, j). Since s_y is at the end of a block, we must have $s_y - s_x = k_{i,m}$ for some $m \leq j+1$. Assume $\mathbf{u} \to \mathbf{y}$ is derived from core rule $(i - k_{i,m}, n)$. See Figure 5.2b. Due to the hypotheses, we must have $k_{i,m} + k_{i-k_{i,m},n+1} > k_{i,j+1}$ and as a result $m + n > j$. In this case, up to subtracting some string to $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ digitwise, we have that

$$\begin{array}{rl|lccc}
\mathbf{u} = & {}^{\omega}0 & 0^{p-i} & d'_i d'_{i-k_{i,1}} \cdots d_{i-k_{i,j}} & d'_{i-k_{i,j+1}} \cdots d_{i-k_{i,m+n}} & 0^{\omega} \\
\mathbf{x} = & {}^{\omega}0 & 0^{p-i-1}1 & 0 \cdots 0 & d'_{i-k_{i,j+1}} \cdots d_{i-k_{i,m+n}} & 0^{\omega} \\
\mathbf{y} = & {}^{\omega}0 & 0^{p-i} & d'_i d'_{i-k_{i,1}} \cdots d_{i-k_{i,m-1}} 0 \cdots 01 & 0 \cdots 0 & 0^{\omega}
\end{array}$$

where the dots contain either zeros or blocks of the form d', and the lone 1 in $\mathbf{y}$ is at position e_x. As a result, if we let $\mathbf{z} = 10 \cdots 010 \cdots 0$, with digits 1 at positions s_x and e_x, then $\mathbf{x} \to \mathbf{z}$ using a rule derived from core rule $(i - k_{i,j+1}, m + n - j - 1)$ between the positions e_x and e_y and $\mathbf{y} \to \mathbf{z}$ using a rule derived from core rule $(i, m - 1)$ between positions s_x and s_y. Thus we are in case (1), using that $s_y \geq e_x$ to verify the necessary inequalities.

The other two cases are dealt with in similar fashion. In the case $e_x = e_y$, we see that $\mathbf{y} \to \mathbf{x}$ using a rule derived from core rule $(i, m - 1)$ between the positions s_x and s_y (using the same notation as above). Thus we are in case

(2b) (with $|\mathbf{u} \to \mathbf{x}| = s_x - e_x$, $|\mathbf{u} \to \mathbf{y}| = s_y - e_y$ and $|\mathbf{y} \to \mathbf{x}| = s_x - s_y$). In the final case $e_x < e_y$, we now have $m+n < j$ (still with the same definitions: core rule (i, j) for $\mathbf{u} \to \mathbf{x}$, $s_y - s_x = k_{i,m}$ and core rule $(i - k_{i,m}, n)$ for $\mathbf{u} \to \mathbf{y}$). See Figure 5.2c. We now have, up to subtracting a suitable string,

$$\begin{array}{rl|llllllll} \mathbf{u} = & {}^\omega 0 & 0^{p-i} & d'_i \cdots d'_{i-k_{i,m-1}} & d'_{i-k_{i,m}} \cdots d_{i-k_{i,m+n}} & d'_{i-k_{i,m+n+1}} \cdots d_{i-k_{i,j}} & 0^\omega \\ \mathbf{x} = & {}^\omega 0 & 0^{p-i-1}1 & 0 \cdots 0 & 0 \cdots 1 & 0 \cdots 0 & 0^\omega \\ \mathbf{y} = & {}^\omega 0 & 0^{p-i} & d'_i \cdots d_{i-k_{i,m-1}} & 0 \cdots 0 & d'_{i-k_{m+n+1}} \cdots d_{i-k_{i,j}} & 0^\omega. \end{array}$$

with the dots containing either zeros or blocks d'. Now, we may set

$$\mathbf{z} = {}^\omega 0 | 0^{p-i} d'_i d'_{i-k_{i,1}} \cdots d_{i-k_{i,m-1}} 0 \cdots 010 \cdots$$

with a lone 1 at position e_y. Then, $\mathbf{y} \to \mathbf{z}$ by using core rule $(i - k_{m+n+1}, j - m - n - 1)$ between positions e_y and e_x, and $\mathbf{z} \to \mathbf{x}$ by using core rule $(i, m-1)$ between positions s_x and s_y. Then we are in case (3b).

Of course, the case where $s_x < s_y$ corresponds to cases (1), (2a) and (3a) respectively, by swapping x and y. In the end, for each configuration of s_x, e_x, s_y and e_y we have identified the appropriate case among the five listed, and the lemma is proved. □

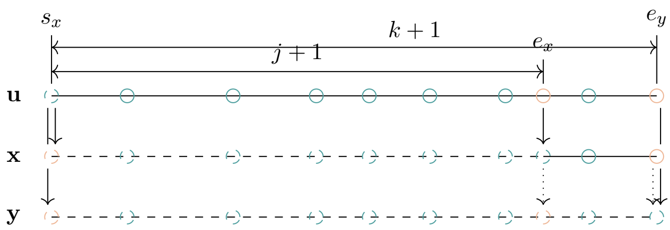
Now that we have a proof of local confluence, we will obtain (global) confluence from it. Unfortunately, Newman's lemma, a common tool to deduce confluence from local confluence [New42, Hue77], doesn't apply here as our system is not *noetherian* (there exist infinite sequences of rewritings). Another property that guarantees confluence is that of *strong confluence*, defined by

$$u \to x, u \to y \Rightarrow \exists v : x \to^* v \text{ and either } y \to v \text{ or } y = v.$$

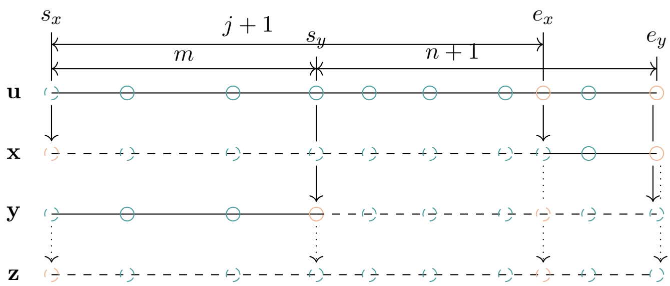
Unfortunately, the system $\rho''_{\mathcal{B}}$ is not strongly confluent either due to case (3b) in the lemma above. In fact, cases (3a) and (3b) together let us build [Hue77, Figure 2], which is the archetypal example of a system with local confluence but not global confluence. To prove the confluence of our system, we will have to resort to more ad-hoc techniques involving monovariants based on the widths of rules. We shall start with the following lemma.

Lemma 5.37. *Assume that ρ is a rewriting system such that every rule can be given a width in $\mathbb{N}_0 \cup \{\infty\}$ and that the system satisfies the conclusion of Lemma 5.36.*

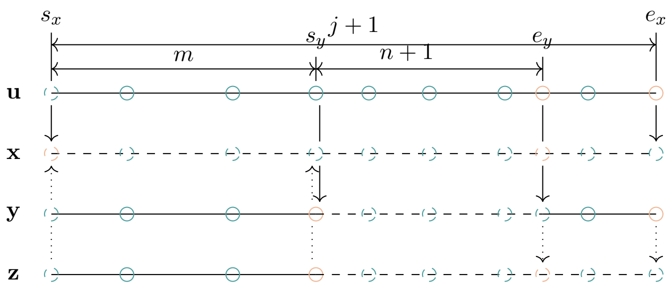
Let u, x, y be such that $u \to^* x$, $u \to^* y$ and some derivation $u \to^* x$ contains only rules of finite width.



(a) Case $s_x = s_y$



(b) Case $s_x > s_y > e_x > e_y$



(c) Case $s_x > s_y > e_y > e_x$

Figure 5.2: Visual representation for the proof of Lemma 5.36. Segments represent a "block" of the form d_i' or d_i. Continuous segments and nodes represent "big" digits, of the form $d_{\cdot,\cdot}$ or $t_{\cdot,\cdot}$. Dashed segments and nodes represent "small" digits, being 0 or 1. Teal nodes correspond to the smaller of two possibilities, red nodes to the bigger of the two. Arrows indicate a rewriting, with $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$ in continuous lines and other rewritings in dotted lines. On top of the picture, some relevant lengths are presented, measured in number of segments.
Notice that applying a core rule of the form $(\cdot, j)$ decreases $j+1$ segments. We know that s_y must be at the end of a segment due to the MDP. In the case of rules of infinite width, the pictures extend infinitely to the right, with only teal nodes.

Then there exists z such that $x \to^* z$, $y \to^* z$ and furthermore the derivation $y \to^* z$ can be chosen to use only rules of finite width, such that the sum of the widths of these rules is less than or equal to that in $u \to^* x$.

Proof. With every pair of derivations $u \to^* x$, $u \to^* y$ where the first derivation uses only rules of finite widths, we associate the pair (α, β) where α is the sum of the widths of all the rules in the first derivation, and β is the number of rules in the second derivation. We prove the result by noetherian induction on (α, β) (where $(\alpha_1, \beta_1) < (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 < \beta_2$).

Note that if one of the derivations has length 0, then z clearly exists (and is x or y depending on which derivation has length 0). This observation covers the case where $\alpha = 0$ or $\beta = 0$. Assume now that this is not the case. Then we can write $u \to x_1 \to^* x$ and $u \to y_1 \to^* y$ for some x_1, y_1. If $x_1 = y_1$, we use the induction hypothesis on $x_1 \to^* x$ and $x_1 \to^* y$, as α has now decreased. Otherwise, we use the conclusion of Lemma 5.36 on u, x_1 and y_1.

In case (1), there exists z_1 such that $x_1 \to z_1$ and $y_1 \to z_1$ with $|y_1 \to z_1| \leq |u \to x_1|$. We may apply the induction hypothesis to $x_1 \to^* x$ and $x_1 \to z_1$ as α has decreased. We obtain some z_2 such that $x \to^* z_2$, $z_1 \to^* z_2$ with only rules of finite width and $|z_1 \to^* z_2| \leq |x_1 \to^* x|$ (where the width of a derivation is the sum of the widths of each rule). Now, we may once again apply the induction hypothesis, this time to $y_1 \to z_1 \to^* z_2$ and $y_1 \to^* y$ (indeed, $|y_1 \to^* z_2| \leq |u \to x_1| + |x_1 \to^* x|$ so α has not increased, but β has decreased by 1). We obtain some z such that $z_1 \to^* z$ and $y \to^* z$ with only rules of finite width, and $|y \to^* z| \leq |y_1 \to^* z_2|$ which is less than or equal to $|u \to x|$. Thus the necessary statement is proved in this case. (The reader may have noticed that this is quite similar to the induction for Newman's lemma).

For case ($2a$), we may apply the induction hypothesis to the derivations $x_1 \to^* x$ and $x_1 \to y_1 \to^* y$ as α has decreased. We find z such that $x \to^* z$ and $y \to^* z$ with only rules of finite width and $|y \to^* z| \leq |x_1 \to^* x| \leq |u \to^* x|$. Therefore z is the one desired by the statement, and the induction is done in this case. For case ($2b$), we apply the induction to $y_1 \to x_1 \to^* x$ and $y_1 \to^* y$. Indeed, we know that

$$|y_1 \to x_1| = |u \to x_1| - |u \to y_1|$$

since $u \to x$ has finite length and that length is greater than $|u \to y|$ (so the latter must also be finite). Since $|u \to y_1| \geq 1$, we have $|y_1 \to x_1| < |u \to x_1|$, so α has decreased. Upon inspection, the z found using the induction hypothesis for those derivations is also the one we need for the derivations starting with u, so this case is done.

For case ($3a$), we apply as in case ($2a$) the induction on $x_1 \to^* x$ and $x_1 \to^2 y_1 \to^* y$, which we can do as α has decreased. The z we find for this new pair of derivations is the one we need. Finally, for case ($3b$), we find z_1 such that $y_1 \to z_1 \to x_1$ and

$$|u \to y_1| + |y_1 \to z_1| + |z_1 \to x_1| = |u \to x_1|$$

as $u \to x_1$ is of finite width. Since $|u \to y_1| \geq 1$, we obtain $|y_1 \to z_1| + |z_1 \to x_1| < |u \to x_1|$. Therefore, we may apply the induction hypothesis on $y_1 \to z_1 \to x_1 \to^* x$ and $y_1 \to^* y$ as α has decreased. The z we find is again the z we need for our original derivations.

In the end, the inductive step is proven in all cases, so the statement holds. □

A slight modification of Lemma 5.37 will give us the tool to work with rules of infinite width.

Lemma 5.38. *Assume that ρ is a rewriting system such that every rule can be given a width in $\mathbb{N}_0 \cup \{\infty\}$ and that the system satisfies the conclusion of Lemma 5.36.*

Let u, x_1, x, y be such that $u \to x_1 \to^* x$, $u \to^* y$ and some derivation of $x_1 \to^* x$ contains only rules of finite width.

Then there exist y_1, z such that $x \to^* z$, $y \to y_1 \to^* z$ and furthermore the derivation $y_1 \to^* z$ can be chosen to use only rules of finite width.

Compared to the previous lemma, now the first rule of the first derivation is allowed to have infinite length, and we lose control over the width of the derivation obtained in the end.

Proof. The proof is similar to the previous one. For two derivations $u \to x_1 \to^* x$ and $u \to^* y$, we let β be the number of rules involved in the derivation $u \to^* y$. First, note that if $\beta = 0$ (i.e. $u = y$), the result is true by setting $z = x$. Otherwise, set $u \to y_1 \to^* y$ and we proceed by induction on β.

If $u \to x_1$, $u \to y_1$ is in case (1), we find z_1 using Lemma 5.36, then apply Lemma 5.37 on $x_1 \to^* x$ and $x_1 \to z_1$ to find some z_2, then apply the induction hypothesis on $y_1 \to z_1 \to^* z_2$ and $y_1 \to^* y$ (β has decreased by 1) to find z.

In case ($2a$), we apply Lemma 5.36 on $x_1 \to^* x$ and $x_1 \to y_1 \to^* y$ to find z.

In case ($2b$) we apply the induction hypothesis to $y_1 \to x_1 \to^* x$ and $y_1 \to^* y$ as β has decreased by 1, and we find z.

In case ($3a$), we apply Lemma 5.36 as in case ($2a$).

Finally, in case ($3b$) we apply the induction hypothesis on $y_1 \to z_1 \to x_1 \to^* x$ and $y_1 \to^* y$ (recall that $z_1 \to x_1$ is of finite width) and we find z as in case ($2b$).

Thus the inductive step is proven in all cases. □

Finally, we may now prove the (global) confluence of the system $\rho''_{\mathcal{B}}$.

Proposition 5.39. *Let $\mathcal{B}$ be an alternate base. If $\mathcal{B}$ has the maximal digit property, then $\rho''_{\mathcal{B}}$ is confluent.*

Proof. Since $\mathcal{B}$ has the maximal digit property, Lemma 5.36 applies to it, and therefore Lemma 5.38 applies to $\rho''_{\mathcal{B}}$. In particular, if $\mathbf{u}, \mathbf{x}, \mathbf{y}$ are such that $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to^* \mathbf{y}$, we can find some $\mathbf{z}$ such that $\mathbf{x} \to^* \mathbf{z}$ and $\mathbf{y} \to^* \mathbf{z}$. But this is statement is known to be equivalent to confluence ([Hue77] states it without even giving a proof, see lemma 3). Indeed, it suffices to decompose a derivation $\mathbf{u} \to^* \mathbf{x}$ into $\mathbf{u} \to \mathbf{x}_1 \ldots \to \mathbf{x}_n = \mathbf{x}$ and use this result iteratively to find $\mathbf{y} = \mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n = \mathbf{z}$ such that $\mathbf{x}_i \to^* \mathbf{z}_i$ and $\mathbf{z}_{i-1} \to^* \mathbf{z}_i$. In the end, $\mathbf{z}$ is such that $\mathbf{x} \to^* \mathbf{z}$ and $\mathbf{y} \to^* \mathbf{z}$. Thus the confluence is proven. □

5.5.2 A comment on a result of Frougny

The statement of Theorem 5.30, particularly the biimplication (a)⇔(b), is inspired by a result of Frougny from [Fro92a]. In this subsection, we restate this result with our notation and offer a few precisions regarding it.

The setting is that of a positional numeration system for natural numbers based on the recurrence relation $U_{j+\ell} = t_1 U_{j+\ell-1} + \ldots + t_\ell U_j$ with $t_i \in \mathbb{N}$ and $t_\ell \neq 0$. In such a system, natural numbers are represented thanks to a greedy algorithm, and $t_1 \cdots t_\ell$ is never a factor of the representation of a natural number. We may introduce the core rule $0 t_1 \cdots t_\ell \to 10^\ell$ and build a rewriting system from this core rule as we have done above, on the alphabet $\{0, \ldots, c\}$ where c is arbitrary but large enough so that every natural number can be represented. We obtain the set of rules

$$\rho_c = \{x_0 x_1 \cdots x_\ell \to (x_0 + 1)(x_1 - t_1) \cdots (x_\ell - t_\ell) :$$

$$0 \leq x_0 < c,\ t_i \leq x_i \leq c \text{ for every } 1 \leq i \leq \ell\}$$

and we extend this to a rewriting system on $E = {}^{\omega}0\{0, \ldots, c\}^* \bullet \{0, \ldots, c\}^* 0^{\omega}$ as above by allowing to rewrite factors independently of context.

If the word $t_1 \cdots t_\ell$ is lexicographically greater than all of its suffixes, then $t_1 \cdots t_\ell 0^{\omega}$ is the β-expansion of 1 for some real number β greater than 1, in which case the rewriting system we have just defined is equal to the system defined by applying Definition 5.23 to the alternate base (β) with $p = 1$. Our work therefore prolongs that of Frougny, but only in the case where the lexicographic condition is respected and only with one specific alphabet. Note that while Frougny addresses the initial conditions of U in her article, they do not impact the rewriting system ρ_c.

Frougny states the following proposition:

Proposition 5.40 ([Fro92a, Proposition 5.6])**.** *The three following conditions are equivalent:*

(a) The rewriting system generated by ρ_c on E is confluent.

(b) Every word on E that contains a factor lexicographically greater than or equal to $t_1 \cdots t_m$ is reducible.

(c) $t_1 = \cdots = t_{\ell-1} = c$.

This statement is not too surprising considering our Theorem 5.30, seeing how condition (c) relates to the maximal digit property.

However, this statement is missing some cases, as the following example shows.

Example 5.41. Consider a system given by the recurrence relation $U_{n+2} = 7U_n$. With it is associated a rewriting system generated by the core rule $007 \to 100$. We claim that this rewriting system is confluent on every alphabet. For alphabets not containing the digit 7, this is vacuously true as there is no rule in the system. For alphabets containing 7, we show the much stronger property that

$$\forall \mathbf{u}, \mathbf{x}, \mathbf{y} \in E : \mathbf{u} \to \mathbf{x},\ \mathbf{u} \to \mathbf{y} \Rightarrow \exists \mathbf{v} : \mathbf{x} \to \mathbf{v},\ \mathbf{y} \to \mathbf{v}.$$

Clearly, if the two factors modified by the rules $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$ do not overlap, the statement is true. Now, notice that in the other cases, the factors where $\mathbf{u}, \mathbf{x}$ and $\mathbf{y}$ disagree have one of the following forms:

$$\mathbf{u}_{[n,n+4)} = w \oplus 0077,\ \mathbf{x}_{[n,n+4)} = w \oplus 1007,\ \mathbf{y}_{[n,n+4)} = w \oplus 0170$$

with w of length 4, or

$$\mathbf{u}_{[n,n+5)} = w \oplus 00707,\ \mathbf{x}_{[n,n+5)} = w \oplus 10007,\ \mathbf{y}_{[n,n+5)} = w \oplus 00800$$

with w of length 5. However, for the first case we may choose $\mathbf{v}_{[n,n+4)} = w \oplus 1100$, and for the second case we may choose $\mathbf{v}_{[n,n+5)} = w \oplus 10100$. Thus the system is confluent. However, the word 010 contains a factor lexicographically greater than 07 yet is not reducible, and we do not have $t_1 = c$ if c is large enough.

We offer the following correction of Frougny's proposition:

Proposition 5.42. *Let* $E = {}^{\omega}0C^* \bullet C^*0^{\omega}$ *with* $C = \{0, \ldots, c\}$ *and* $c \geq \max_{i=1}^{\ell} t_i$*. The following conditions are equivalent:*

(a) The rewriting system generated by ρ_c *on* E *is confluent.*

(b) There exists q *such that* $t_1 \cdots t_\ell = (0^{q-1}c)^k(0^{q-1}b)$ *for some* k *and with* $b \leq c$*.*

Note that such a system is, in a sense, the product of the system based on the rule $0c^kb \to 10^{k+1}$ with itself q times. Our example above now satisfies the second item, with $q = 2$ and $k = 0$. Frougny's result only considers the case $q = 1$. The number q corresponds to the p used for the period of alternate bases. If the sequence U has a dominant root, or if the word $t_1t_2 \cdots t_\ell$ is known to satisfy the Parry conditions, then Frougny's result holds as q must necessarily be 1.

Proof. We prove first that (b)⇒(a). Assume that property (b) holds, and we will show that in this case

$$\forall \mathbf{u}, \mathbf{x}, \mathbf{y} \in E : \mathbf{u} \to \mathbf{x},\ \mathbf{u} \to \mathbf{y} \Rightarrow \exists \mathbf{v} : \mathbf{x} \to \mathbf{v},\ \mathbf{y} \to \mathbf{v}.$$

Note that the application of a rule only modifies letters whose index is of one given class modulo q. Rewritings that concern letters with indices of different classes modulo q can be applied in any order, so we only need to look at pairs of rewritings formed by two applications of the rule that concern the same class modulo q. Similarly, if no letter is modified by both rewritings $\mathbf{u} \to \mathbf{x}$ and $\mathbf{u} \to \mathbf{y}$, these two rewritings can be applied in any order. Thus we may assume that at least one letter is modified by both rewritings. Note that the first letter when applying a rule cannot be c, and all but the last letters where the index is of the chosen class modulo q must be equal to c. Thus $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ must agree outside of some interval $[n, m)$ of length $q(2k+2)+1$ where we have

$$\begin{aligned} \mathbf{u}_{[n,m)} &= w \oplus 0(0^{q-1}c)^k 0^{q-1} b (0^{q-1}c)^k 0^{q-1} b \\ \mathbf{x}_{[n,m)} &= w \oplus 10^{q(k+1)}(0^{q-1}c)^k 0^{q-1} b \\ \mathbf{y}_{[n,m)} &= w \oplus 0(0^{q-1}c)^k 0^{q-1}(b+1)0^{q(k+1)}, \end{aligned}$$

for some $w \in C^{m-n}$. The rule $\mathbf{u} \to \mathbf{x}$ is applied on the first $q(k+1)+1$ positions, the rule $\mathbf{u} \to \mathbf{y}$ on the last $q(k+1)+1$ positions, and they overlap in only one position. But now, we may choose $\mathbf{v}$ to agree with $\mathbf{u}$ outside of $[n, m)$ and such that

$$\mathbf{v}_{[n,m)} = w \oplus 10^{q(k+1)-1}10^{q(k+1)}.$$

The word $\mathbf{v}$ is guaranteed to be in E since $\mathbf{x}$ and $\mathbf{y}$ are. Thus we have proved the claim, and the confluence of our rewriting system is a straightforward consequence of it.

We move on now to proving that (a)⇒(b). For this, we prove the following claim: If ρ_c gives a confluent rewriting system, and if t_i and t_j are both nonzero ($i < j$), then $t_{j-i} = c$.

To this end, set $b_n = \max(t_{j-i+n}, t_n)$ for n in $\{1, \ldots, \ell-j+i\}$ and consider the word ${}^{\omega}00t_1 \cdots t_{j-i} b_1 \cdots b_{\ell-j+i} t_{\ell-j+i+1} \cdots t_\ell 0^{\omega}$, which is indeed a word on E. If t_{j-i} is not c, this word admits the two rewritings

${}^{\omega}0$	0	$t_1 \cdots$	t_{j-i}	b_1	$\cdots$	$b_{\ell-j+i}$	$t_{\ell-j+i+1}$	$\cdots$	t_ℓ	0^{ω}
${}^{\omega}0$	1	$0 \cdots$	0	$b_1 - t_{j-i+1}$	$\cdots$	$b_{\ell-j+i} - t_\ell$	$t_{\ell-j+i+1}$	$\cdots$	t_ℓ	0^{ω}
${}^{\omega}0$	0	$t_1 \cdots$	$t_{j-i}+1$	$b_1 - t_1$	$\cdots$	$t_{\ell-j+i} - t_{\ell-j+i}$	0	$\cdots$	0	0^{ω}

However, these two words cannot be rewritten into a common word. Indeed, to make the last positions equal we need to apply to the first word a rule on its last $\ell+1$ positions, but we cannot do this as that would introduce in column j the letter $\max(t_j, t_i) - t_j - t_i$, which is negative as t_i and t_j are both strictly positive. Thus the rewriting system is not confluent, and the claim is proved.

Back to the proof of (a)⇒(b), if only one of the t_i's is nonzero, it must be t_ℓ and the system satisfies condition (b) with $q = \ell, k = 0$. If now $i < j$ are such that t_i and t_j are both nonzero and the system is confluent, the preceding lemma gives $t_{j-i} = c$, which is nonzero. This process can be iterated along Euclid's algorithm for j and i, until we obtain $t_{\gcd(j,i)} = c$. Therefore, the greatest common divisor of all i such that $t_i \neq 0$ is itself an i such that $t_i \neq 0$. Let us call this quantity q. It must divide ℓ since $t_\ell \neq 0$. But now, we may use the lemma to prove that $t_{\ell-q} = c$, $t_{\ell-2q} = c, \ldots$. This is enough: if i is not multiple of q, then $t_i = 0$ due to the definition of q, if i is multiple of q less than ℓ, then $t_i = c$, and t_ℓ is less than c due to c being the maximum of the t_i's, thus condition (b) is satisfied.

□

5.5.3 Other short comments

In this section, we quickly list some comments too small to deserve an entire subsection and mention some perspectives for further research.

(a) Bases satisfying the maximal digit property also satisfy Property (F), as an easy consequence of [MPS25, Theorem 3.2]. Consequently, the product $\delta = \prod_{i=0}^{p-1} \beta_i$ is a Pisot or Salem number and for all $i \in \mathbb{Z}_p$ we have $\beta_i \in \mathbb{Q}(\delta)$, as seen in Theorem 3.1 of the same article.

(b) Continuing the parallel to [MPS25], we can recall that the authors in Section 5 of that article classify nonadmissible words into two types, Type 1 and Type 2. The authors then explain how to rewrite the nonadmissible words in question to reach expansions and prove their results. We can see in that article that while words of Type 2 can be rewritten without adding zeros to the right, words of Type 1 cannot. For confluence and the associated results, it is thus not too surprising to see that numeration systems where all nonadmissible words are of

Type 2 are precisely those that have the good properties. One can see that the maximal digit property precisely forbids the existence of words of Type 1.

(c) In the text, we considered the specific alphabet A where the letter x_i must be strictly less than β_i. We can ask what would happen for different alphabets. One case that must be treated separately from the rest is the case where some elements of the base are integers and we consider $x_i \leq \beta_i$ rather than $x_i < \beta_i$. In this case, we can adapt the definition of the maximal digit property to

$$t_{k,j} = \lfloor \beta_{k-j} \rfloor \text{ for all } j \in \{1, \dots, \ell_j - 1\}$$

instead of $t_{k,j} = \lceil \beta_{k-j} \rceil - 1$ for $j \geq 2$. We also adapt the definition of the spectrum to reflect the new choice of alphabet and we introduce the core rewriting rule $0\lfloor \beta_i \rfloor \to 10$ for the appropriate congruences modulo p. Then, the results obtained in this chapter stay true , with much the same proofs. If anything, the proofs become slightly easier as there is no need to single out cases where some base is an integer.

(d) However, if we take any larger alphabet, the equivalences obtained here can fail. If there are any i, j such that $j < \ell_i$ and $t_{i,j}$ is less than the maximum allowed digit at this position, we can replicate the construction in the proof of Theorem 5.30 ($(b) \Rightarrow (a)$) to conclude that the associated rewriting system is not confluent. Shifting this construction lets us see that the spectrum and set of $\mathcal{B}$-integers cannot be equal either.

(e) The article [LL24b] generalizes optimality to the framework of *Kakeya sequences* and their associated numerations. A Kakeya sequence is a sequence $(p_n)_{n \in \mathbb{N}_0}$ that tends to zero as n tends to infinity and is such that $p_n \leq \sum_{j=n+1}^{\infty} p_j$ for all n. Every real number smaller than $\sum_{n=1}^{\infty} p_n$ can be represented as a sum of elements p_n, and one canonical expansion can be chosen by using a greedy algorithm.

Kakeya sequences are close to Cantor real numeration systems as a framework. Given a one-way Cantor base $(\beta_n)_{n<0}$ with $\beta_n \in [1, 2)$ for all $n < 0$, we can set $p_n = \frac{1}{\prod_{j=1}^{n} \beta_j}$, which is a decreasing Kakeya sequence. Then, the greedy expansions in the Cantor base and in the Kakeya sequence coincide. A base β_n greater than or equal to 2 can be dealt with by repeating $\frac{1}{\prod_{j=1}^{n}\beta_j}$ in the sequence $\lfloor\beta_n\rfloor$ times (the sequence then becomes nonincreasing rather than decreasing).

A Kakeya sequence is *optimal* if for all x and for all n, the greedy Kakeya expansion $a_1a_2\cdots$ minimizes the defect $x - \sum_{i=1}^{n} a_ip_i$ among all sequences $c_1c_2\cdots$ with $c_i \in \{0,1\}$ such that this defect is nonnegative. This definition corresponds to our definition of optimality in the above-outlined case. Lai and Loreti obtain that for *decreasing* Kakeya sequences, the sequence is optimal if and only if for all n there exists $k_n \geq 1$ with $p_n = \sum_{i=n+1}^{n+k_n} p_i$ ([LL24b, Theorem 1]). The condition of decreasingness has the effect of reducing the correspondence to Cantor bases to the case where all base elements are less than 2. In the case that corresponds to Rényi numeration systems, i.e., where the sequence p_n is geometric, the authors recover that the only real bases in $[1,2)$ with optimal expansions are those where $1 = \sum_{i=1}^{k}\frac{1}{\beta^i}$ for some k, i.e., the ones where $d_\beta(1) = 111\cdots10^\omega$. This corresponds to [DdVKL12].

In the case of alternate base expansions (still with bases in $[1,2)$), Lai and Loreti's condition corresponds to all expansions of 1 being composed only of the digit 1. Due to Parry conditions, this is not possible for greedy expansions, but it is possible if we allow intermediate representations. For instance, the base $(2,3/2)$ admits 111 and 11 as representations of 1. It follows that the Kakeya sequence $(1,1/2,1/3,1/6,1/9,\ldots)$, where $\frac{p_n}{p_{n+1}}$ is 2 if n is odd and $3/2$ if n is even, has optimal expansions.

Seeing that nonincreasing Kakeya sequences correspond to Cantor bases with elements greater than 2, it would be interesting to attempt a generalization of Lai and Loreti's result to this larger framework.

Chapter 6

Regularity of languages associated with a positional numeration system

In this chapter, we tackle the problem of *regularity* of languages associated with positional numeration systems. This problem is of wide theoretical relevance, as it is the key that unlocks the use of automata-theoretic techniques and results on a given numeration system. The most significant contribution on this matter is an article of Hollander [Hol98], that offers some necessary and some sufficient conditions on the recurrence sequence U for the language L_U to be regular. However, Hollander's results are limited to the case of a positional numeration system *without a dominant root*, a class in which not all U-systems with a regular language lie.

Here, we present our results, obtained with Émilie Charlier and prepublished on arXiv ([CK25]). These new results expand the scope of Hollander's work to the case without a dominant root and fill some gaps in his classification. In the end, we obtain a complete characterization of those U-systems that have a regular language. This characterization can be implemented in a semi-decision procedure, which will always detect regularity when it occurs but might not certify nonregularity.

All propositions, theorems and corollaries starting from Section 6.3 are from the article [CK25] unless mentioned, as is most of the exposition. Beyond minor corrections, changes include a rework of Sections 6.1, 6.2 and 6.3 to fit the exposition of the thesis, and a rework of our notation throughout the chapter.

Contents

6.1	Introduction	**162**
6.2	A reminder on notation and useful properties	**165**
6.3	Elementary necessary conditions for regularity	**170**
6.4	Our strategy	**178**
6.5	Vertices with no outgoing edge	**181**
6.6	Vertices leading to a vertex with no outgoing edge	**186**
6.7	Vertices in a cycle	**192**
6.8	Vertices leading to a cycle	**207**
6.9	Decision procedure	**215**
6.10	Comments on Hollander's original conjecture	**219**
6.11	Why obtaining criteria for regularity relying only on recurrence relations satisfied by U is not achievable	**221**

6.1 Introduction

As mentioned in Section 1.1.1, numeration systems are introduced with goals in mind. Depending on the tasks we are interested in performing, we can then try to select the most suitable numeration system. A desirable property of the chosen system is that the operations of interest can be performed by some finite state machine.

In this chapter, we consider U-systems. As explained in the nice chapter by Frougny and Sakarovitch [FS10], perhaps the most fundamental question to be asked is to understand which positional numeration systems support a regular set of representations. That is, does the set of all valid expansions form a regular language in the sense of finite automata? Indeed, the regularity of the set of expansions is a necessary condition to the use of automata-theoretic techniques in relation to numeration systems. We can mention the ability to normalize representations with transducers [Fro92b, FS96], which enables the computation of addition in the numeration system using finite state machines. In turn, this allows us to use tools such as `Walnut`[Mou21, Sha22] to automatically prove results related to the numeration system, following a connection to first-order logic pioneered by Büchi [Büc90].

Most of the commonly used numeration systems, even in the nonstandard numeration systems community, have the property to generate a regular numeration language. This is the case of all Bertrand numeration systems associated with a Parry number (seen in Section 1.5), as we will see shortly (Remark 6.16). All integer base systems as well as the Zeckendorf numeration system [Zec72] based on the Fibonacci sequence are Bertrand numeration systems. Another family of systems with a regular language is that of Pisot numeration systems, which are the positional numeration systems defined by a linear recurrence relation whose characteristic polynomial is the minimal polynomial of a Pisot number [FS96, BH97]. While it is a more general family than Bertrand numeration systems, with more freely given initial conditions, this family is still very special, in particular because of the irreducibility constraint on the characteristic polynomial. These two families share the property of having a dominant root, meaning that the quotient of consecutive terms in the base sequence admits a limit, which in addition is greater than one. On the other hand, even some simple numeration systems, like the one based on the sequence of squares, do not support a regular set of representations [Sha94].

As an attempt to unify the theory of numeration systems with a regular numeration language, Lecomte and Rigo defined the family of abstract numeration systems [LR01, LR10]. Here, the point of view is reversed. One starts with any given regular language, orders the words in the language with respect to the radix order (i.e., by length first and then using the lexicographic order within each length) and declares that a nonnegative integer n is represented by the n-th word in the language. This framework encompasses the previously mentioned families of numeration systems as well as others such as the Dumont-Thomas numeration systems [DT89]. The price of this very general point of view is that we lose the information of the algorithm to directly produce the representation of a number. This raises the question of determining which regular languages can be produced by a positional numeration system. This problem was studied in [KLS25a] for the family of Dumont-Thomas numeration systems, tying this branch back to this document (see Chapter 7).

The question of characterizing positional systems generating a regular numeration language was addressed by Hollander in the case where the numeration system satisfies the dominant root condition [Hol98]. In his study, Hollander provided a detailed, however not full, description of systems with a dominant root and giving rise to a regular set of representations. In particular, he does not entirely solves the issue of the dependence to the initial conditions. Central to Hollander's arguments is a property ([Hol98, Section 4], reproduced here as Theorem 1.41) linking positional numeration systems to Rényi numeration systems. Whenever the dominant root condition is dropped, this link disappears and hence new tools are needed in order to attack the general problem. Part of those tools were mentioned in Section 2.2.

The aim of this chapter is to provide a complete characterization of positional numeration systems that yield a regular numeration language. We solve this problem in its full generality, not adding any particular condition on the given positional numeration system. We will use the tools of Section 2.2 to replace Hollander's lemmas, studying alternate bases of real numbers as introduced in [CC21] as our alternative to Rényi numeration systems. This use case was in fact the original motivation for the study of these new representations of real numbers. In particular, the notion of Parry alternate base developed in [CCMP23] will be central in the present work since we will see that this condition is necessary in order to ensure the regularity of the numeration language of a positional numeration system.

Our main result will be separated into four parts, each handling a different behavior of the greedy algorithm for the representation of 1 via alternate real bases. Together, these results yield a semi-decision procedure for deciding the regularity of the numeration language of a positional numeration system. Testing the ultimate periodicity of the greedy expansions of 1 in an alternate base (in fact, even in a single real base) is a difficult task, that is not known to be decidable in general. But, provided that the associated greedy expansions of 1 are known to be finite or eventually periodic, which we can indeed check in many situations, our results actually provide us with a genuine decision procedure.

The chapter will be structured as follows. In Section 6.2, we recall the basics of positional numeration systems, alternate base numeration systems and their connection, as explained in Chapters 1 and 2. In Section 6.3, we introduce a number of conditions that are necessary for regularity, allowing us to restrict ourselves to the case where these conditions are met. With the preparatory work done, Section 6.4 details our strategy for proving the announced criteria. Sections 6.5, 6.6, 6.7 and 6.8 contain the bulk of the work, carefully analyzing each of the four cases outlined above and obtaining a criterion for each of them. In these four sections, we take care of positioning our results in comparison to those obtained by Hollander, when we restrict our hypotheses to the case of a dominant root: either we recover the previously known results, or we handle a new situation, not occurring in the case of a dominant root. In particular, our results allow the possibility of a base to be equal to 1. In the dominant root case, this corresponds to the polynomial case, which was not treated in [Hol98]. Section 6.9 summarizes our results and comments on their effectiveness. We provide two carefully designed examples in order to illustrate different scenarios. Finally, Sections 6.10 and 6.11 answer Hollander's conjecture in the negative and explain why similar conjectures are unlikely to be true in the non-dominant root case, concluding the chapter.

6.2 A reminder on notation and useful properties

In this section, we quickly recall some notation and useful properties introduced in Chapters 1 and 2.

Definition 6.1. A *positional numeration system* is given by an increasing sequence $U = (U_n)_{n\geq 0}$ of integers such that $U_0 = 1$ and the quotients $\frac{U_{n+1}}{U_n}$ are uniformly bounded.

The *value* of a word $w_{\ell-1}\cdots w_0$ over $\mathbb{N}$ in the system U is written

$$\mathrm{val}_U(w_{\ell-1}\cdots w_0) = \sum_{n=0}^{\ell-1} w_n U_n.$$

The *representation* map is given by the greedy algorithm: to compute the representation of x, we first let ℓ be the least integer such that $x < U_\ell$ and we let $r_\ell = x$. Then for every $n = \ell-1,\dots,0$, we set $a_n = \lfloor \frac{r_{n+1}}{U_n} \rfloor$ and $r_n = r_{n+1} - a_n U_n$. Then the representation of x is $\mathrm{rep}_U(x) = a_{\ell-1}\cdots a_0$.

We abuse notation and note U the numeration system as well.

Definition 6.2. The language of all greedy U-representations, possibly preceded by zeros, i.e., the language

$$L_U = 0^*\mathrm{rep}_U(\mathbb{N}),$$

is called the *numeration language*. It is written over the alphabet

$$A_U = \left\{0,\dots,\sup_{n\geq 0}\left\lceil \frac{U_{n+1}}{U_n}\right\rceil - 1\right\},$$

called the *numeration alphabet*.

As we will see shortly with Proposition 6.15, when studying questions of regularity, we can focus on a particular sublanguage of L_U.

Definition 6.3. The language of maximal words of a language L over a totally ordered alphabet is the language

$$\mathrm{Max}(L) = \{u \in L : \text{for all } v \in L,\ |v| = |u| \Rightarrow v \leq_{\mathrm{lex}} u\},$$

The language $\mathrm{Max}(L_U)$ has nice properties with respect to regularity (see Proposition 6.15 and Lemma 6.18). Even before considering regularity, this language helps with the description of the whole of L_U.

Lemma 6.4 (Lemma 1.10)**.** *Let U be a positional numeration system. Then*

$$\mathrm{Max}(L_U) = \{\mathrm{rep}_U(U_n - 1) : n \in \mathbb{N}\}.$$

Lemma 6.5 (Lemma 1.11)**.** *Let U be a positional numeration system. A word $w_{\ell-1} \cdots w_0 \in A_U^*$ belongs to L_U if and only if*

$$w_{n-1} \cdots w_0 \leq_{\mathrm{lex}} \mathrm{rep}_U(U_n - 1)$$

for all $n \in \{0, \ldots, \ell\}$.

In addition, giving a candidate for $\mathrm{Max}(L_U)$ is enough to characterize U fully.

Lemma 6.6 (Lemma 1.12)**.** *Let M be a language over a (finite) alphabet included in $\mathbb{N}$. There exists a positional numeration system U such that $M = \mathrm{Max}(L_U)$ if and only if the language M satisfies the following properties:*

- *M has exactly one word of each length.*
- *No word in M starts with the digit 0.*
- *For all words $u_{n-1} \cdots u_0$ and $v_{\ell-1} \cdots v_0$ in M with $n < \ell$, we have $v_{n-1} \cdots v_0 \leq_{\mathrm{lex}} u_{n-1} \cdots u_0$.*

Furthermore, if such a positional numeration system U exists, then it is unique.

The introduction of the language $\text{Max}(L_U)$ also allows us to make the connection to alternate bases.

Proposition 6.7 (Proposition 2.24)**.** *Let $U = (U_n)_{n\geq 0}$ be a positional numeration system such that L_U is regular. There exists a positive integer p such that the p limits, for $i \in \{0, \ldots, p-1\}$,*

$$\lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}}$$

exist and can be effectively computed. In particular, the limit

$$\lim_{n\to+\infty} \frac{U_{n+p}}{U_n}$$

exists and is equal to the product of the p above limits.

Thus we may introduce an alternate base of real numbers associated to the U-system.

Definition 6.8 (Definition 2.27)**.** We say that a positional numeration system $U = (U_n)_{n\geq 0}$ has an *associated alternate real base* $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$ if

$$\beta_i = \lim_{n\to+\infty} \frac{U_{np+i+1}}{U_{np+i}}$$

for each $i \in \{0, \ldots, p-1\}$. By considering the minimal possible p for which these limits exist, we may talk about *the* alternate real base associated with U.

We now introduce again the notation of alternate bases, developed in Section 2.1.

Definition 6.9 (Definitions 2.1 and 2.9)**.** An *alternate base* $\mathcal{B}$ is given by a p-tuple of real numbers greater than or equal to 1, called *bases* or *base elements*. We note $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, then extend the base to a periodic biinfinite sequence by setting $\beta_{n+p} = \beta_n$ for all $n \in \mathbb{Z}$.

The evaluation map is given by

$$\mathrm{val}\colon a_1a_2\cdots \mapsto \sum_{j=1}^{\infty} \frac{a_j}{\beta_{-1}\beta_{-2}\cdots\beta_{-j}}$$

provided that this sum converges, which is always the case when the word $a_1a_2\cdots$ is a word on a finite alphabet and one of the β_i is strictly greater than 1.

A number x in $[0,1]$ is represented by a greedy algorithm as follows. First, set $r_0 = x$, then recursively set

$$a_{i+1} = \lfloor \beta_{-i-1} r_i \rfloor \quad \text{and} \quad r_{i+1} = \beta_{-i-1} r_i - a_{i+1}$$

if r_i is defined. The word $a_1a_2\cdots$ obtained this way is called the *$\mathcal{B}$-representation of x* and is noted $d_{\mathcal{B}}(x)$.

The *shift map* σ can also be applied to alternate bases. Given a base $\mathcal{B} = (\beta_{p-1},\ldots,\beta_0)$, we set

$$\sigma(\mathcal{B}) = (\beta_{p-2},\ldots,\beta_0,\beta_{p-1}).$$

The shift map can be iterated and inverted, and we of course have $\sigma^i(\mathcal{B}) = (\beta_{p-1-i},\ldots,\beta_{-i})$, remembering that $\beta_{n+p} = \beta_n$ for all n. We will of course use notation such as $d_{\sigma^i(\mathcal{B})}(x)$ to denote expansions in a shift of the base.

Note that in our case it is important to allow the base elements to be equal to 1, as $\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}}$ may well be equal to 1.

We are particularly interested in representations of 1, as we recall.

Definition 6.10 (Definitions 2.10, 2.11 and 2.12)**.** For all i, we let $\mathbf{d}_i = d_{\sigma^{-i}(\mathcal{B})}(1)$. It is the expansion of 1 in the base $(\beta_{i-1},\ldots,\beta_{i-p})$. We let the letters of $\mathbf{d}_i$ be given by $\mathbf{d}_i = t_{i,1}t_{i,2}\cdots$. We therefore have $1 = \sum_{j=1}^{\infty} \frac{t_{i,j}}{\beta_{i-1}\cdots\beta_{i-j}}$.

If $\mathbf{d}_i$ is finite, we let ℓ_i be its length (up to its last nonzero digit) and we write $d_i = t_{i,1}t_{i,2}\cdots t_{i,\ell_i}$ its finite prefix (with $t_{i,\ell_i} \neq 0$ and $\mathbf{d}_i = d_i0^{\omega}$). We also set $d'_i = t_{i,1}t_{i,2}\cdots(t_{i,\ell_i}-1)$. Still in this case, we define $\mu(i) = (i-\ell_i) \bmod p$.

The *quasi-greedy representation of* 1 *in base* $\sigma^{-i}(\mathcal{B})$ is noted $d^*_{\sigma^{-i}(\mathcal{B})}(1)$ or $\mathbf{d}^*_i$ and is defined recursively by

$$\mathbf{d}^*_i = \begin{cases} \mathbf{d}_i & \text{if } \mathbf{d}_i \text{ is infinite;} \\ d'_i\mathbf{d}^*_{\mu(i)} & \text{if } \mathbf{d}_i \text{ is finite.} \end{cases}$$

The letters of this expansion are written $\mathbf{d}_i^* = d_{i,1}d_{i,2}\cdots$.

We define $k_{i,j} = \ell_i + \ell_{\mu(i)} + \ldots + \ell_{\mu^{j-1}(i)}$ provided that the lengths involved are all finite. It is the cumulative sum of the first j expansions seen in the quasi-greedy algorithm when computing $\mathbf{d}_i^*$.

We define

$$\mathbf{w}_{i,j} = d_i' d_{i-k_{i,1}}' \cdots d_{i-k_{i,j-1}}' \mathbf{d}_{i-k_{i,j}} \tag{6.1}$$

the j-th *intermediate representation* of 1, provided that at least j finite expansions are seen before an infinite one in the quasi-greedy algorithm. This can also be defined as

$$\mathbf{w}_{i,j} = d_i' d_{\mu(i)}' \cdots d_{\mu^{j-1}(i)}' \mathbf{d}_{\mu^j(i)} \tag{6.2}$$

and we will use the most convenient of the two notations during our developments.

We also define the values $m_{i,j}$ by

$$m_{i,j}p = \mu^j(i) - i + k_{i,j} \tag{6.3}$$

Finally, we define the graph G to be the oriented graph whose vertices are $\{0, \ldots, p-1\}$, where there is an arc from i to $\mu(i) = (i - \ell_i) \bmod p$ if $\mathbf{d}_i$ is finite and no arc from i otherwise.

The quantity $m_{i,j}$ is a correcting term counting an offset as we go through the quasi-greedy algorithm. Assume we start computing $\mathbf{d}_i^*$. After j steps of the quasi-greedy algorithm starting we must compute $\mathbf{d}_{i-k_{i,j}}$. We know that the number $i - k_{i,j}$ is congruent to $\mu^j(i)$ modulo p, but what multiple of p is involved if we want the exact value? We now have the answer; it is $-m_{i,j}p + \mu^j(i)$. Of course, in the setting of alternate bases where there is an underlying periodicity of period p, this is not too interesting, but on the side of U-systems, this one new notation will greatly simplify the expressions at play. See Figure 6.1 for a visual help, to be interpreted in the context of U-systems. We can see for instance that we have $m_{i,j+h} = m_{i,j} + m_{\mu^j(i),h}$.

We recall the crucial result regarding the lexicographic conditions on the expansions of 1 in different shifts of the base.

Theorem 6.11 (Theorem 2.16). *In an alternate base $\mathcal{B} = (\beta_{p-1}, \ldots, \beta_0)$, the words $\mathbf{d}_{p-1}, \ldots, \mathbf{d}_0$ and $\mathbf{d}_{p-1}^*, \ldots, \mathbf{d}_0^*$ verify*

$$\sigma^j(\mathbf{d}_i) <_{\text{lex}} \mathbf{d}_{i-j} \quad \text{and} \quad \sigma^j(\mathbf{d}_i^*) \leq_{\text{lex}} \mathbf{d}_{i-j}^*$$

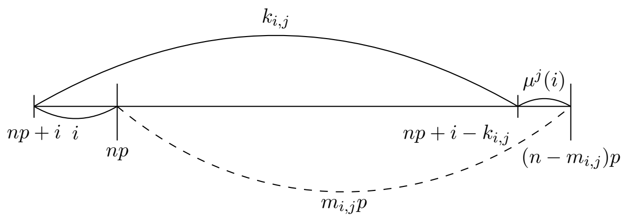


Figure 6.1: Schematic representation of the definition of $m_{i,j}$

for all i and all $j \geq 0$.

The point of all these notations is that, up to a small modification, the language $\operatorname{Max}(L_U)$ and some related ones share a strong connection to the words $\mathbf{w}_{i,j}$ in the associated alternate base.

Definition 6.12 (Definition 2.29)**.** Let U be a positional numeration system with an associated alternate base $(\beta_{p-1}, \ldots, \beta_0)$. Then for integers i, c, n such that $0 \leq i < p$, $n \geq 0$ and $1 \leq c \leq U_{np+i}$, we define

$$\operatorname{rep}_{i,c}(n) = 0^\ell \operatorname{rep}_U(U_{np+i} - c),$$

where $\ell = np + i - |\operatorname{rep}_U(U_{np+i} - c)|$.

Proposition 6.13 (Proposition 2.30)**.** *Let U be a positional numeration system with an associated alternate base $(\beta_{p-1}, \ldots, \beta_0)$, let $i \in \{0, \ldots, p-1\}$ and let c be a positive integer. For all $L \geq 0$, there exists N such that for all $n \geq N$, there exists $j \in \{0, \ldots, L\}$ such that $\operatorname{rep}_{i,c}(n)$ and $\mathbf{w}_{i,j}$ share a common prefix of length L.*

6.3 Elementary necessary conditions for regularity

Since we aim to study which numeration systems have a regular language, we may first describe some necessary conditions for this and then restrict ourselves to the case where these conditions are met (since if the conditions are not met, the language is certainly not regular). We introduce a couple of such conditions in this section.

The following well-known result – see for instance [Sha94, Lor95] – asserts that we can restrict our study to *linear numeration systems*, which are positional numeration systems where the sequence $(U_n)_{n\in\mathbb{N}}$ is a linear recurrence sequence over $\mathbb{Z}$. For such a sequence, there exist integers $c_0, \ldots, c_{m-1}$ such that

$$U_{n+m} = c_{m-1}U_{n+m-1} + \ldots + c_0 U_n \tag{6.4}$$

for all $n \geq 0$. As presented in Section 1.2, the polynomial

$$X^m - c_{m-1}X^{m-1} - c_2X^{m-2} - \cdots - c_0$$

is called the *characteristic polynomial* of the linear recurrence relation (6.4).

Proposition 6.14. *Any positional numeration system with a regular numeration language is linear.*

We will use the notions related to linear recurrence sequences (minimal polynomial, eigenvalues, multiplicity,...) that were defined near the beginning of Section 1.2.

Now, we explain why the language of maximal words is such a powerful tool when studying regularity.

Proposition 6.15.

- *If L is a regular language, then the language* $\mathrm{Max}(L)$ *is regular.*
- *If U is a positional numeration system such that* $\mathrm{Max}(L_U)$ *is regular, then the numeration language L_U is also regular.*

For the sake of completeness, we reproduce Shallit's proof of the first item from [Sha94], and give a new construction of an automaton accepting a numeration language built from the knowledge of the lexicographically maximal words (provided that they form a regular language). Another construction of such an automaton was proposed in [Hol98], but the resulting automaton contains far more states than the one we describe here, since it is obtained as a product of several automata.

Remark 6.16. In particular, Bertrand numeration systems associated with Parry numbers have a regular language since $\mathrm{Max}(L_U) = \mathrm{Pref}(\mathbf{d}_\beta(1))$ or $\mathrm{Pref}(\mathbf{d}^*_\beta(1))$, both of which are clearly regular when β is a Parry number.

Remark 6.17. Also note that for an arbitrary language L, it is not the case that the regularity of the language $\mathrm{Max}(L)$ implies that of L. For example, the Dyck language of well-parenthesized binary words is well known to be nonregular whereas its language of maximal words with the order $0 < 1$ is $(01)^*$, which is regular. Hence, the hypothesis that the language is derived from a numeration system is important, as such languages have some structure with respect to the lexicographic order.

We mention a lemma, proved in [Sha94], that provides a useful decomposition of *slender* regular languages, i.e., containing a bounded number of words of each length. Since $\mathrm{Max}(L_U)$ is slender, this lemma will apply to it, making questions of regularity easier to answer than for L_U in full.

Lemma 6.18. *A language is slender and regular if and only if it is a finite union of languages of the form xy^*z.*

Proof of Proposition 6.15. We reproduce Shallit's simple argument [Sha94] for proving the regularity of the language of lexicographically maximal words of each length when the starting language is regular. Let L be an arbitrary regular language written over a totally ordered alphabet $(A, <)$. In order to show that $\mathrm{Max}(L)$ is also regular, we can equivalently show that $L \backslash \mathrm{Max}(L)$ is regular. Let $\mathcal{A} = (Q, i, F, A, \delta)$ be a deterministic finite automaton accepting L. Consider now the following nondeterministic finite automaton: $\mathcal{B} = (Q \times Q \times \{e, \ell\}, (i, i, e), F \times F \times \{\ell\}, A, R)$, where the transition relation R contains the following transitions:

- $(p, q, e) \xrightarrow{a} (\delta(p, a), \delta(q, a), e)$ for all $p, q \in Q$ and $a \in A$;
- $(p, q, e) \xrightarrow{a} (\delta(p, a), \delta(q, b), \ell)$ for all $p, q \in Q$ and $a, b \in A$ with $a < b$;
- $(p, q, \ell) \xrightarrow{a} (\delta(p, a), \delta(q, b), \ell)$ for all $p, q \in Q$ and $a, b \in A$.

Transitions in the second component of Q nondeterministically guess a word of L that is greater than the word read by $\mathcal{B}$. As a result, a given word is accepted by $\mathcal{B}$ if and only if it belongs to L and there exists a lexicographically greater word in L of the same length: $\mathcal{B}$ accepts exactly the words in $L \setminus \mathrm{Max}(L)$.

Let us now turn to the second item. The proof is constructive, and will be illustrated in Example 6.19 below. Let U be a positional numeration system and suppose that the lexicographically maximal words of each length in L_U form a regular language. Lemma 6.18 combined with elementary arithmetical considerations implies that this language can be decomposed in the following way:

$$\mathrm{Max}(L_U) = F \cup \left(\bigcup_{j=0}^{d-1} x_j y_j^* z_j \right)$$

where the unions are disjoint, F is a finite language, $d \geq 1$, and for all j, we have $x_j, y_j, z_j \in A_U^*$, $x_j \neq \varepsilon$, $|y_j| = d$, $|x_j z_j| \equiv j \pmod{d}$, $|x_{j+1} z_{j+1}| = |x_j z_j| + 1$, and y_j and z_j do not share the same first letter. Let $n = |x_0 z_0|$, so that we have $F = \{\mathrm{rep}_U(U_\ell - 1) : \ell < n\}$ and $|x_j z_j| < n + d$ for all $j \in \{0, \dots, d-1\}$.

We now construct an NFA $\mathcal{A}$ as follows. For each $j \in \{0, \dots, d-1\}$, we consider two finite automata P_j and S_j, each accepting finitely many words:

- P_j accepts the words $w \in L_U$ such that $|w| < n$ and $|w| \equiv j \pmod{d}$.
- S_j accepts the words $w \in L_U$ such that $|w| = |z_j|$ and $w \leq_{\mathrm{lex}} z_j$.

We add the states (j, k) for all $j \in \{0, \dots, d-1\}$ and $k \in \{0, \dots, |x_j y_j| - 1\}$, and the following transitions, with $x_j y_j = a_{j,1} \cdots a_{j,|x_j y_j|}$ where each $a_{j,k}$ is a letter:

- $(j, k) \xrightarrow{a_{j,k+1}} (j, k+1)$ for all $k \in \{0, \dots, |x_j y_j| - 2\}$;
- $(j, |x_j y_j| - 1) \xrightarrow{a_{j,|x_j y_j|}} (j, |x_j|)$;
- $(j, k) \xrightarrow{a} ((j - k - 1) \bmod d, 0)$ for all $k \in \{0, \dots, |x_j y_j| - 1\}$ and $a < a_{j,k+1}$.

Finally, for each $j \in \{0, \dots, d-1\}$, we add

- an ε-transition from $(j, 0)$ to the initial state of the automaton P_j;
- an ε-transition from $(j, |x_j|)$ to the initial state of the automaton S_j.

The initial states are the states $(j, 0)$ and the final states are the final states of P_j and S_j for all $j \in \{0, \dots, d-1\}$.

The idea is that the states (j, k) keep track of what prefix of a maximal element we are reading, with state (j, k) specifically corresponding to having read k letters in some $x_j y_j^\omega$. Notice the similarities with the constructions in Figure 1.5 and Example 2.41.

We claim that

$$L_U = L(\mathcal{A}) \setminus (K_1 \cup K_2)$$

where $L(\mathcal{A})$ denotes the language accepted by the NFA $\mathcal{A}$, and K_1 and K_2 are the following regular languages:

$$K_1 = \bigcup_{\ell=1}^{n+d-1} A_U^*\{s \in A_U^\ell : s >_{\text{lex}} \text{rep}_U(U_\ell - 1)\}$$

$$K_2 = \bigcup_{j=0}^{d-1} A_U^* x_j y_j^* \{s \in A_U^{|z_j|} : s >_{\text{lex}} z_j\}.$$

By the stability properties of regular languages, this will imply that L_U is a regular language.

We start with the inclusion $L_U \subseteq L(\mathcal{A}) \setminus (K_1 \cup K_2)$. Let $w \in L_U$. We show by induction on its length that w is accepted by $\mathcal{A}$ from the state $(j, 0)$ where $j = |w| \bmod d$. Our base case is as follows: if $|w| < n$ then w is accepted by $\mathcal{A}$ by first following the ε-transition from $(j, 0)$ to the initial state of P_j, and then is accepted by P_j by definition. Note that since $n \geq d$, the base case contains all possible lengths modulo d. Now, suppose that $|w| \geq n$ and that all shorter words s of L_U are accepted by $\mathcal{A}$ from the state $(k, 0)$ where $k = |s| \bmod d$. We know that $\text{rep}_U(U_{|w|} - 1) \in x_j y_j^* z_j$. More precisely, we have $\text{rep}_U(U_{|w|} - 1) = x_j y_j^m z_j$ with $m = (|w| - |x_j z_j|)/d$. Lemma 6.5 yields that $w \leq_{\text{lex}} \text{rep}_U(U_{|w|} - 1)$. If $x_j y_j^m$ is a prefix of w, i.e., $w = x_j y_j^m s$, then the suffix s is such that $s \in L_U$, $|s| = |z_j|$ and $s \leq_{\text{lex}} z_j$. In this case, the word w is accepted by first following the path labeled by $x_j y_j^m$ from $(j, 0)$ to $(j, |x_j|)$, then by taking the ε transition from $(j, |x_j|)$ to the initial state of S_j, and then following the unique path labeled by s in S_j. Since S_j accepts s by definition, this shows that w is accepted by $\mathcal{A}$ from $(j, 0)$. Now suppose that $w = pas$, $\text{rep}_U(U_{|w|} - 1) = pbs'$ with $|p| < |x_j y_j^m|$, $a, b \in A_U$, $a < b$. Then there is a path reading pa from $(j, 0)$ to $(k, 0)$ where $k = (|w| - |pa|) \bmod d = |s| \bmod d$. By induction hypothesis, the suffix s is accepted from $(k, 0)$, hence w is accepted by $\mathcal{A}$ from $(j, 0)$. Moreover, Lemma 6.5 also implies that w does not belong to K_1 nor K_2.

We now turn to the converse inclusion. Let $w \in L(\mathcal{A}) \setminus (K_1 \cup K_2)$. Then no suffix of w belongs to $K_1 \cup K_2$ either. Again, we proceed by induction on the length of the words. By construction of the automaton $\mathcal{A}$, the word w must be accepted from the state $(j, 0)$ where $j = |w| \bmod d$. If $|w| < n$, then $w \in L_U$ since $w \notin K_1$. Now, suppose that w is an accepted word of length $|w| \geq n$, and that all shorter words accepted by $\mathcal{A}$ but not belonging to K_1 nor K_2 belong to L_U. Since $|w| \geq n$, the lexicographically greatest word in L_U of length $|w|$ is $\mathrm{rep}_U(U_{|w|} - 1) = x_j y_j^m z_j$ with $j = |w| \bmod d$ and $m = (|w| - |x_j z_j|)/d$. Let us compare w and $\mathrm{rep}_U(U_{|w|} - 1)$.

If these two words first differ in the prefix of length $|x_j y_j^m|$, then by construction of $\mathcal{A}$, the first differing letter in w must be less than the corresponding letter in $\mathrm{rep}_U(U_{|w|} - 1)$, i.e., $w = pas$, $\mathrm{rep}_U(U_{|w|} - 1) = pbs'$, $|p| < |x_j y_j^m|$, $a, b \in A_U$, $a < b$. Our accepting path must start in $(j, 0)$, and after reading the prefix pa, ends in the state $(k, 0)$ where $k = (|w| - |pa|) \bmod d = |s| \bmod d$. Therefore, the suffix s is accepted by $\mathcal{A}$ from this state $(k, 0)$. Since $s \notin K_1 \cup K_2$, we may apply the induction hypothesis. We get that s belongs to L_U. Now, consider a suffix t of w such that $|t| > |s|$. Then $p = p_1 p_2$ and $t = p_2 as <_{\mathrm{lex}} p_2 bs'$. Since $p_2 bs'$ is a suffix of $\mathrm{rep}_U(U_{|w|} - 1)$, it satisfies $p_2 bs' \leq_{\mathrm{lex}} \mathrm{rep}_U(U_{|t|} - 1)$ by Lemma 6.5. Hence $t \leq_{\mathrm{lex}} \mathrm{rep}_U(U_{|t|} - 1)$. By Lemma 6.5 again, we obtain that $w \in L_U$.

We are left with the case where w and $\mathrm{rep}_U(U_{|w|} - 1)$ are either equal or differ in the last $|z_j|$ digits. This means that $w = x_j y_j^m s$ with $|s| = |z_j|$. Since $w \notin K_1 \cup K_2$ and $|s| < n + d$, there are strong restrictions on the suffix s: it must satisfy $s \in L_U$ and $s \leq_{\mathrm{lex}} z_j$. Therefore, as in the previous paragraph, all suffixes t of w satisfy $t \leq_{\mathrm{lex}} \mathrm{rep}_U(U_{|t|} - 1)$, which implies that $w \in L_U$ by Lemma 6.5. □

We illustrate this construction on an example. This example was found by specifying the form $\mathrm{Max}(L_U)$ had to take to be illustrative, then using Lemma 6.6 to construct the sequence U from the language, illustrating the usefulness of this lemma as well.

Example 6.19. Let M be the following regular language, which is intended to provide the candidate maximal words :

$$M = \ 21(11)^*00 \ \cup \ 2101(01)^*1 \ \cup \ \{\varepsilon, 1, 11, 101\}.$$

With the notation of the proof of Proposition 6.15, we have

$$d = 2,\ n = 4,\ x_0 = 21,\ y_0 = 11,\ z_0 = 00,\ x_1 = 2101,\ y_1 = 01,\ z_1 = 1.$$

By Lemma 6.6, there exists a unique positional system $U = (U_n)_{n \geq 0}$ such that $M = \mathrm{Max}(L_U)$. One can check that it is given by the recurrence relation $U_{n+9} = 8U_{n+7} - 10U_{n+5} + 2U_{n+3}$ for $n \geq 0$ and the initial conditions

$(U_0, \ldots, U_8) = (1, 2, 4, 6, 17, 44, 116, 286, 760)$. In Figure 6.2, we have drawn the automaton described in the proof of Proposition 6.15. The finitely many words accepted by the automata P_0, P_1, S_0, S_1 are listed explicitly. The initial states are marked with an incoming arrow. The accepting paths all end in one of the automata P_0, P_1, S_0, S_1. Compare to Figure 2.3, where the auxiliary automata P_0, P_1, S_0, S_1 are not needed, as the short maximal words in that example follow the same pattern as the long ones.

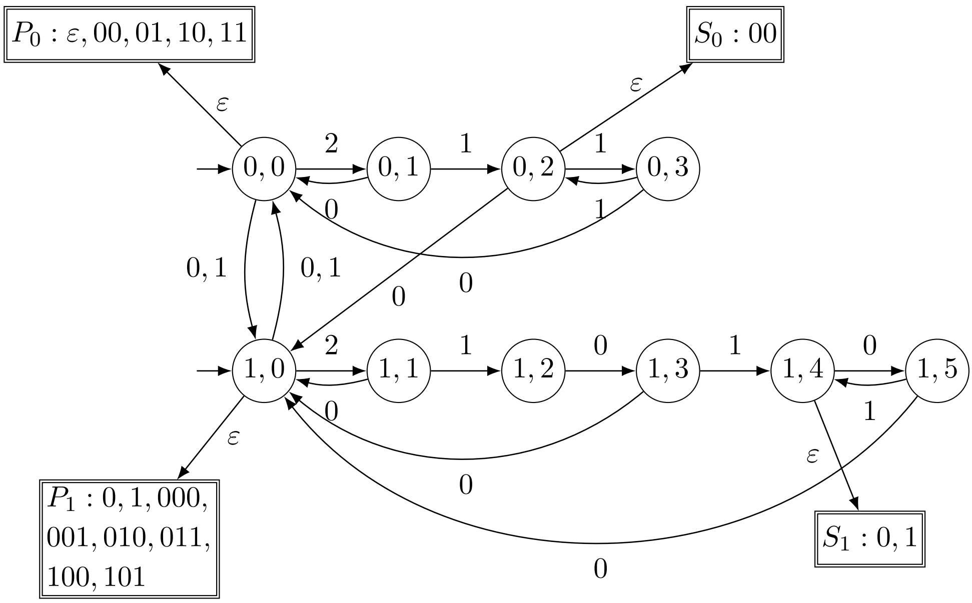


Figure 6.2: The nondeterministic automaton $\mathcal{A}$ built from the set of maximal words $M = 21(11)^*00 \cup 2101(01)^*1 \cup \{\varepsilon, 1, 11, 101\}$.

This example shows that words in $K_1 \cup K_2$ may be accepted by the automaton $\mathcal{A}$ despite the fact that they do not belong to L_U. For example, the words in $21(11)^*01$ are accepted by $\mathcal{A}$ by following a path starting in the state $(0, 0)$ and ending in P_1. However they do not belong to L_U as they are lexicographically greater than the maximal corresponding words, which belong to $21(11)^*00$. Since the suffix 01 is lexicographically greater than $z_0 = 00$, one has $21(11)^*01 \subset K_2$. Now, consider the word 111. It is accepted by $\mathcal{A}$ by following the path from $(1, 0)$ to $(0, 0)$ labeled by 1, and then taking the ε-transition going to P_0, which accepts 11. This word 111 belongs to K_1 as it is of length $3 < n + d = 6$ and is lexicographically greater than 101, the maximal word of length 3.

Studying the regularity of the language $\mathrm{Max}(L_U)$ is easier than that of L_U itself since $\mathrm{Max}(L_U)$ is a *thin* language, that is, it contains at most one word of each length [PS95]. This allows us to use Lemma 6.18.

We introduce one last straightforward lemma, which will allow us to separate the study of the regularity of $\mathrm{Max}(L_U)$ according to the residue class modulo p for some suitable p.

Lemma 6.20. *Let L be a language over a finite alphabet A and let $p, N \in \mathbb{N}$ with $p \geq 1$. The language L is regular if and only if the p languages*

$$\{w \in L : |w| \equiv i \pmod{p},\ |w| \geq N\}$$

are all regular.

Now, we provide an additional necessary condition by showing how the connection to alternate bases can enter play with great effect.

We first extract some useful information from Proposition 6.13 in the case where $\mathbf{d}_i$ is infinite. In terms of the graph G, this means that i is a vertex with no outgoing edge.

Corollary 6.21. *Let U be a positional numeration system with an associated alternate base $(\beta_{p-1}, \dots, \beta_0)$, let $i \in \{0, \dots, p-1\}$ be such that $\mathbf{d}_i$ is infinite, and let c be a positive integer. We have*

$$\lim_{n\to\infty} \mathrm{rep}_{i,c}(n) = \mathbf{d}_i$$

where the limit is taken with respect to the product topology.

Proof. Since $\mathbf{d}_i = \mathbf{d}_i^*$, only $j = 0$ is possible in Proposition 6.13. □

We now prove that when $\mathcal{B}$ is an alternate base associated with a regular numeration system, the expansions of 1 must necessarily be ultimately periodic. Alternatively, we can require that all quasi-greedy expansions of 1 are eventually periodic. This corresponds to the notion of *Parry alternate base*, defined in Definition 2.18. Recall that *eventually periodic* excludes finite sequences, while *ultimately periodic* includes them.

Proposition 6.22. *Let U be a positional numeration system with a regular numeration language. Then the associated alternate base is such that $\mathbf{d}_i$ is finite or eventually periodic for all i.*

Proof. Let p be the length of the alternate base and consider $i \in \{0, \dots, p-1\}$ such that $\mathbf{d}_i$ is infinite. We have to show that $\mathbf{d}_i$ is ultimately periodic. By Corollary 6.21, the sequence of finite words $(\text{rep}_U(U_{np+i} - 1))_{n \geq 0}$ converges to the infinite word $\mathbf{d}_i$. Proposition 6.15 and Lemmas 6.18 and 6.20 then imply that $\mathbf{d}_i$ is ultimately periodic. □

6.4 Our strategy

We consider an arbitrary positional numeration system U and we aim at determining whether the numeration language L_U is regular or not. At this point, we have obtained several necessary conditions for L_U to be regular. We thus put ourselves in the restricted situation where these conditions are satisfied. Namely, we suppose that U is linear and that there is a Parry alternate base $\mathcal{B} = (\beta_{p-1}, \dots, \beta_0)$ associated with U. In view of Proposition 6.15 and Lemmas 6.4 and 6.20, in order to understand when L_U is indeed regular, it suffices to study the regularity of the p languages

$$L_i = \{\text{rep}_U(U_{np+i} - 1) : n \geq 0\}$$

for $i \in \{0, \dots, p-1\}$. Either these p languages are all regular, in which case the numeration language L_U is regular, or one of them is not regular, in which case L_U is not regular.

We will see that the regularities of these p languages are interconnected. Our approach is based on the graph G which is meant to encode the connections between the p shifts of the alternate base when performing the quasi-greedy algorithm (see Definition 6.10). We can separate vertices of the graph, and therefore congruence classes modulo p, into four categories:

1. Vertices i with no successor.
2. Vertices i leading to a vertex with no successor, i.e., such that there exists $r \geq 1$ such that $\mu^r(i)$ has no successor.
3. Vertices i within a cycle, i.e., such that there exists $r \geq 1$ with $\mu^r(i) = i$.
4. Vertices i leading to a cycle, i.e., such that there exists $r \geq 1$ such that $\mu^r(i)$ belongs to a cycle, but where i itself doesn't.

Each of these categories will be treated separately in the proof of our main result.

The following example illustrates the four categories of vertices described above as well as some notations of Definition 6.10.

Example 6.23. Let $U = (U_n)_{n\geq 0}$ be defined by $U_{n+10} = 16U_{n+5} - 9U_n$ for $n \geq 0$ and the following initial conditions:

n	0	1	2	3	4	5	6	7	8	9
U_n	1	2	3	6	10	19	29	48	96	151

For $i \in \{0, \dots, 4\}$, the limits

$$\beta_i = \lim_{n\to+\infty} \frac{U_{5n+i+1}}{U_{5n+i}}$$

exist and can be effectively computed:

i	β_i
4	$\frac{11+3\sqrt{55}}{17}$
3	$\frac{2+\sqrt{55}}{6}$
2	2
1	$\frac{6+3\sqrt{55}}{17}$
0	$\frac{11+3\sqrt{55}}{22}$

The product of these bases is $8 + \sqrt{55}$, which is a root of $X^2 - 16X^2 + 9$ as expected. Set $\mathcal{B} = (\beta_4, \dots, \beta_0)$. We get the following greedy and quasi-greedy expansions of 1 in $\mathcal{B}$ and its shifts. Recall that $\mathbf{d}_i = d_{\sigma^{-i}(\mathcal{B})}(1)$ is the expansion that starts with the base element β_{i-1}, hence the offset of one: $\mathbf{d}_3$ corresponds to starting with β_2.

i	$\mathbf{d}_i$
0	1110^ω
4	$11(00010)^\omega$
3	20^ω
2	110^ω
1	110^ω

i	$\mathbf{d}_i^*$
0	$(11010)^\omega$
4	$11(00010)^\omega$
3	$1(10110)^\omega$
2	$(10110)^\omega$
1	$1011(00010)^\omega$

For example, the intermediate $\mathcal{B}$-expansions and $\sigma^{-3}(\mathcal{B})$-expansions of 1 are given by

$$\mathbf{w}_{0,1} = 110 \cdot 110^\omega \qquad \mathbf{w}_{3,1} = 1 \cdot 110^\omega$$
$$\mathbf{w}_{0,2} = 110 \cdot 10 \cdot 1110^\omega \qquad \mathbf{w}_{3,2} = 1 \cdot 10 \cdot 1110^\omega$$
$$\mathbf{w}_{0,3} = 110 \cdot 10 \cdot 110 \cdot 110^\omega \qquad \mathbf{w}_{3,3} = 1 \cdot 10 \cdot 110 \cdot 110^\omega$$
$$\vdots \qquad \vdots$$

There are no intermediate $\sigma^{-4}(\mathcal{B})$-expansions of 1 since $\mathbf{w}_{4,0} = \mathbf{d}_4$ is infinite. The associated graph G is depicted in Figure 6.3. The four categories outlined above correspond to $\{4\}, \{1\}, \{0, 2\}$ and $\{3\}$, respectively.

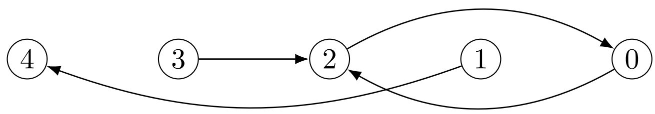


Figure 6.3: The graph G associated with the numeration system U of Example 6.23.

Our strategy will be to consider the four families of vertices separately and, for each, provide a characterization of the regularity of L_i when i is this family. After dealing with these four cases separately, we will combine them in order to describe a decision procedure that will allow us to determine whether or not the numeration language L_U is regular. These results will not be illustrated until Section 6.9, where two appropriate examples will be presented to exhibit a variety of behaviors. We invite the reader to consult Example 6.53 for Sections 6.5 and 6.6, and Example 6.52 for Sections 6.7 and 6.8.

Since the regularities of the languages L_i with $i \in \{0, \ldots, p-1\}$ are linked together, we will be lead to not only consider the maximal words of each length, but also words of the form $\mathrm{rep}_U(U_n - c)$ where c is a constant. Note that Proposition 6.13 was already stated in this spirit. The following definition will be used in the statements of the four main theorems that describe the situation for each category of vertices in the graph G.

Definition 6.24. For $i \in \{0, \ldots, p-1\}$ and positive integers c, we define

$$L_{i,c} = \{\mathrm{rep}_{i,c}(n) : n \geq 0,\ U_{np+i} \geq c\}$$

where the notation $\mathrm{rep}_{i,c}(n)$ has been defined in Definition 6.12.

Again, the condition that $U_{np+i} \geq c$ will not be a true restriction as for any $N \geq 0$, the language $L_{i,c}$ is regular if and only if so is the language $L_{i,c} \cap \{w \in A_U^* : |w| \geq N\}$. Note that we always have $L_i = L_{i,1}$ as $\text{rep}_{i,1}(n) = \text{rep}_U(U_{np+i} - 1)$ for all $n \geq 0$.

6.5 Vertices with no outgoing edge

In this section, we consider a (fixed) vertex i with no outgoing edge in the graph G. This means that the greedy and the quasi-greedy $\sigma^{-i}(\mathcal{B})$-expansions of 1 coincide, i.e., $\mathbf{d}_i = \mathbf{d}_i^*$. Since we are dealing with a Parry alternate base, this infinite word is eventually periodic. We let q_0 be the minimal preperiod of $\mathbf{d}_i$ and we let m_0 be its minimal period that is a multiple of p. Thus, we have

$$\mathbf{d}_i = t_{i,1} \cdots t_{i,q_0} (t_{i,q_0+1} \cdots t_{i,q_0+m_0})^\omega.$$

The following definition generalizes an idea of Hollander [Hol98] to the non-dominant root case.

Definition 6.25. For $i \in \{0, \ldots, p-1\}$ such that $\mathbf{d}_i$ is infinite, integers $q \geq 0$ and $m \geq 1$, and integers n such that $np + i \geq q + m$, we define

$$(\Delta_{i,q,m})_n = \left(U_{np+i} - \sum_{\ell=1}^{q+m} t_{i,\ell} U_{np+i-\ell} \right) - \left(U_{np+i-m} - \sum_{\ell=1}^{q} t_{i,\ell} U_{np+i-m-\ell} \right).$$

In what follows, we will be concerned with properties of the sequence $(\Delta_{i,q,m})_n$ for large values of n only. Therefore, we will not always pay much attention to verify that the condition $np + i \geq q + m$ is satisfied. Otherwise stated, we will only consider these values provided that they are indeed well defined. Also, we will allow ourselves to talk about the sequence $\Delta_{i,q,m}$ in reference to the sequence of values $(\Delta_{i,q,m})_n$ for large enough n.

Lemma 6.26. *Let $i \in \{0, \ldots, p-1\}$ be such that $\mathbf{d}_i$ is infinite. For all $q \geq q_0$, all $k \geq 1$ and all n large enough so that the following expressions are well defined, we have*

$$(\Delta_{i,q,km_0})_n = (\Delta_{i,q_0,km_0})_n = \sum_{\ell=0}^{k-1} (\Delta_{i,q_0,m_0})_{n-\ell\frac{m_0}{p}}.$$

Proof. Since $q \geq q_0$, we have

$$(\Delta_{i,q,km_0})_n - (\Delta_{i,q_0,km_0})_n = -\sum_{\ell=q_0+km_0+1}^{q+km_0} t_{i,\ell} U_{np+i-\ell} + \sum_{\ell=q_0+1}^{q} t_{i,\ell} U_{np+i-km_0-\ell}.$$

Since $t_{i,\ell} = t_{i,\ell-km_0}$ for $\ell > q_0 + km_0$, the two sums cancel and we obtain the first announced equality.

We now turn to the second equality of the statement. We proceed by induction on k. For $k = 1$, the result is immediate. Now, assume that the equality holds for some $k \geq 1$ and let us show it for $k + 1$. Then we have

$$\begin{aligned}(\Delta_{i,q_0,(k+1)m_0})_n &= (\Delta_{i,q_0+m_0,km_0})_n + (\Delta_{i,q_0,m_0})_{n-k\frac{m_0}{p}} \\ &= (\Delta_{i,q_0,km_0})_n + (\Delta_{i,q_0,m_0})_{n-k\frac{m_0}{p}} \\ &= \sum_{\ell=0}^{k-1} (\Delta_{i,q_0,m_0})_{n-\ell\frac{m_0}{p}} + (\Delta_{i,q_0,m_0})_{n-k\frac{m_0}{p}} \\ &= \sum_{\ell=0}^{k} (\Delta_{i,q_0,m_0})_{n-\ell\frac{m_0}{p}}\end{aligned}$$

where we have used the first part of the statement for the second step and the induction hypothesis for the third one. □

We now present the main result of this section.

Theorem 6.27. *Let $i \in \{0, \dots, p-1\}$ be such that $\mathbf{d}_i$ is infinite. The following assertions are equivalent.*

(a) The language L_i is regular.

(b) For all $c \geq 1$, the languages $L_{i,c}$ are regular.

(c) There exists $k \geq 1$ such that the sequence Δ_{i,q_0,km_0} is ultimately zero.

(d) There exists $q \geq q_0$ and $k \geq 1$ such that the sequence Δ_{i,q,km_0} is ultimately zero.

Proof. Clearly, (b) implies (a). By using the first equality of Lemma 6.26, we see that (c) and (d) are equivalent. We show that (a) implies (d). Thus, we suppose that the language L_i is regular. By Lemma 6.18 and Corollary 6.21, we can split this language into a union of the form

$$L_i = F \cup \bigcup_{e=0}^{M-1} xy^* z_e$$

where F is a finite language, M is a positive integer, $|y| = Mp$ and $\mathbf{d}_i = xy^\omega$. We must have that $|x| \geq q_0$ and $|y|$ is a multiple of m_0. Set $q = |x|$. We aim to show that $\Delta_{i,q,Mp}$ is eventually zero.

Let n be large enough so that $\mathrm{rep}_{i,1}(n - M) \notin F$. Then there exist $e \in \{0, \ldots, M-1\}$ and $r \geq 1$ such that $\mathrm{rep}_{i,1}(n) = xy^r z_e$ and $\mathrm{rep}_{i,1}(n-M) = xy^{r-1} z_e$. Thus, we have

$$\begin{aligned} U_{np+i} - U_{(n-M)p+i} &= (U_{np+i} - 1) - (U_{(n-M)p+i} - 1) \\ &= \mathrm{val}_U(xy^r z_e) - \mathrm{val}_U(xy^{r-1} z_e) \\ &= \mathrm{val}_U(xy0^{(r-1)|y|+|z_e|}) - \mathrm{val}_U(x0^{(r-1)|y|+|z_e|}) \\ &= \sum_{\ell=1}^{q+Mp} t_{i,\ell} U_{np+i-\ell} - \sum_{\ell=1}^{q} t_{i,\ell} U_{(n-M)p+i-\ell}. \end{aligned}$$

We have therefore shown that $\Delta_{i,q,Mp}$ is ultimately zero, as expected.

We now show that (d) implies (b). Let $c \geq 1$ be fixed, and assume that there is some $q \geq q_0$ and m multiple of m_0 such that $\Delta_{i,q,m}$ is ultimately zero, i.e., there exists N_1 such that $(\Delta_{i,q,m})_n = 0$ for all $n \geq N_1$. As m_0 is a multiple of p, we can write $m = Mp$. By Corollary 6.21, there exists N_2 such that for all $n \geq N_2$, the prefix of length $q + m$ of the word $\mathrm{rep}_{i,c}(n)$ is $t_{i,1} \cdots t_{i,q+m}$. Let $N = \max\{N_1, N_2\}$ and consider a fixed $n \geq N$. Set z to be the suffix defined as

$$\mathrm{rep}_{i,c}(n) = t_{i,1} \cdots t_{i,q} z.$$

We show by induction that

$$\mathrm{rep}_{i,c}(n + aM) = t_{i,1} \cdots t_{i,q} (t_{i,q+1} \cdots t_{i,q+m})^a z. \tag{6.5}$$

for all $a \geq 0$. The base case $a = 0$ is the definition of z. Now, fix some $a \geq 0$ and suppose that (6.5) holds. Let s be the suffix defined as

$$\mathrm{rep}_{i,c}(n + (a+1)M) = t_{i,1} \cdots t_{i,q+m} s.$$

We have to show that $s = (t_{i,q+1} \cdots t_{i,q+m})^a z$. These two words have the same length $np + am + i - q$. Moreover, they both belong to L_U since they are suffixes of words in L_U. Therefore, in order to see that these words are actually equal, it suffices to show that they have the same value. Since $(\Delta_{i,q,m})_{n+(a+1)M} = 0$, we know that

$$U_{(n+(a+1)M)p+i} - U_{(n+aM)p+i} = \sum_{\ell=1}^{q+m} t_{i,\ell} U_{(n+(a+1)M)p+i-\ell} - \sum_{\ell=1}^{q} t_{i,\ell} U_{(n+aM)p+i-\ell}.$$

Using the induction hypothesis, the latter equality and the definition of s, we successively obtain

$$\begin{aligned} \mathrm{val}_U\big((t_{i,q+1} \cdots t_{i,q+m})^a z\big) &= U_{(n+aM)p+i} - c - \sum_{\ell=1}^{q} t_{i,\ell} U_{(n+aM)p+i-\ell} \\ &= U_{(n+(a+1)M)p+i} - c - \sum_{\ell=1}^{q+m} t_{i,\ell} U_{(n+(a+1)M)p+i-\ell} \\ &= \mathrm{val}_U(s), \end{aligned}$$

as desired. Thus, we have shown that for all $n \geq N$, there exists a finite word z_n such that

$$\{\mathrm{rep}_{i,c}(n + aM) : a \geq 0\} = t_{i,1} \cdots t_{i,q} (t_{i,q+1} \cdots t_{i,q+m})^* z_n.$$

As in Lemma 6.20, this implies that the language $L_{i,c}$ is regular since

$$L_{i,c} \cap \{w \in A_U^* : |w| \geq Np + i\} = \bigcup_{n=N}^{N+M-1} \{\mathrm{rep}_{i,c}(n + aM) : a \geq 0\}.$$

□

The following result shows that Theorem 6.27 can be used in practice to decide whether the language L_i is regular.

Proposition 6.28. *Assume that the eigenvalues of U are known and let $i \in \{0, \ldots, p-1\}$ be such that $\mathbf{d}_i$ is eventually periodic. Then the condition (c) of Theorem 6.27 is effective.*

Proof. Since $\mathbf{d}_i$ is known to be eventually periodic, the values q_0 and m_0 (as defined in the beginning of Section 6.5) can be found by inspecting the remainders in the greedy algorithm. Note that if U satisfies a recurrence relation of characteristic polynomial P, then the same is true for Δ_{i,q_0,m_0}. As a result, the eigenvalues of Δ_{i,q_0,m_0} are among those of U and can be effectively identified. Now, given Lemma 6.26, for any $k \geq 1$, the sequence Δ_{i,q_0,km_0} is ultimately zero if and only if the sequence Δ_{i,q_0,m_0} ultimately satisfies the recurrence relation of characteristic polynomial $1 + X^{\frac{m_0}{p}} + \ldots + X^{(k-1)\frac{m_0}{p}}$. Since

$$\left(1 - X^{\frac{m_0}{p}}\right)\left(1 + X^{\frac{m_0}{p}} + \ldots + X^{(k-1)\frac{m_0}{p}}\right) = 1 - X^{\frac{km_0}{p}},$$

one of the following two cases must happen. If the eigenvalues of Δ_{i,q_0,m_0} are all zero or roots of unity of order not dividing $\frac{m_0}{p}$, then the least common multiple of those orders is the desired k so that (c) holds. Otherwise, there exists no k such that (c) holds. □

In the dominant root case studied by Hollander, Theorem 6.27 corresponds to the case where β is a nonsimple Parry number; see [Hol98, Lemmas 7.3 and 7.4]. In this case, we have $p = 1$ and we only have to consider the sequence $\Delta_{q,m}$ defined by

$$(\Delta_{q,m})_n = \left(U_n - \sum_{\ell=1}^{q+m} t_\ell U_{n-\ell}\right) - \left(U_{n-m} - \sum_{\ell=1}^{q} t_\ell U_{n-m-\ell}\right)$$

where the coefficients t_j are the digits of the greedy expansion of the dominant root, which is a Parry number: $d_\beta(1) = t_1 \cdots t_q (t_{q+1} \cdots t_{q+m})^\omega$. It is easily seen that this sequence is ultimately zero if and only if the base sequence U ultimately satisfies the linear recurrence relation of characteristic polynomial

$$P_{q,m} = \left(X^{q+m} - \sum_{\ell=1}^{q+m} t_\ell X^{q+m-\ell}\right) - \left(X^q - \sum_{\ell=1}^{q} t_\ell X^{q-\ell}\right).$$

Such a polynomial $P_{q,m}$ is called an *extended β-polynomial* by Hollander. In particular, whenever q and m are chosen to be the minimal preperiod q_0 and period m_0 of $d_\beta(1)$, we obtain the so-called *Parry polynomial* P_{q_0,m_0}, also called *the β-polynomial*. As noted by Hollander, all extended β-polynomials can be obtained by multiplying the Parry polynomial P_{q_0,m_0} by some very specific polynomials, as we have

$$P_{q,km_0} = X^{q-q_0}(1 + X^{m_0} + \cdots + X^{(k-1)m_0}) P_{q_0,m_0} \tag{6.6}$$

for any $q \geq q_0$ and $k \geq 1$.

With these comments, we see that the first part of the main result of [Hol98] can be re-obtained as a corollary of Theorem 6.27.

Corollary 6.29 ([Hol98]). *Let U be a positional numeration system with a dominant root $\beta > 1$ such that $d_\beta(1)$ is eventually periodic, i.e., β is a non-simple Parry number. The numeration language L_U is regular if and only if the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial.*

The relationships given in Lemma 6.26 can be viewed as an analogue of (6.6). However, in the general setting considered in this paper, i.e., with possibly $p \geq 2$, there are no such nice polynomials associated with the sequences $\Delta_{i,q,m}$. This makes the situation more complex and forces us to work with the graph G as a whole rather than to independently consider each vertex i of G.

6.6 Vertices leading to a vertex with no outgoing edge

In this section, we investigate the regularity of L_i when there is a nontrivial path from i to a vertex $\mu^s(i)$ with no outgoing edge in the graph G. This means that $\mathbf{d}_i$ is finite but, after $s \geq 1$ steps, the quasi-greedy algorithm sees an infinite expansion $\mathbf{d}_{\mu^s(i)}$ and stops. Note that this situation never occurs whenever $p = 1$, that is, whenever the numeration system U has a dominant root. As in the previous section, our proofs will rely on the introduction of an auxiliary sequence Δ. However, the definition of Δ must be adapted to account for the finiteness of our expansions. This definition will be used in this section as well as the two following ones.

Definition 6.30. For $i \in \{0, \ldots, p-1\}$ such that $\mathbf{d}_i$ is finite and for all integers n such that $np + i \geq \ell_i$, we let

$$(\Delta_i)_n = U_{np+i} - \sum_{\ell=1}^{\ell_i} t_{i,\ell} U_{np+i-\ell}.$$

As before, we are only concerned with such values for large n, so we usually omit to verify the condition $np + i \geq \ell_i$. We will be interested not only in the values of Δ_i for the current i, but also for its successors $\mu^h(i)$ in the graph G, for $h \in \{0, \dots, s-1\}$.

Our main result in this section is the following one. In particular, it describes a previously unseen behavior when only considering systems with a dominant root.

Theorem 6.31. *Let $i \in \{0, \dots, p-1\}$ be such that there exists $s \geq 1$ such that $\mathbf{d}_{\mu^s(i)}$ is infinite but $\mathbf{d}_i, \mathbf{d}_{\mu(i)}, \dots, \mathbf{d}_{\mu^{s-1}(i)}$ are all finite, and assume that the languages $L_{\mu(i)}, \dots, L_{\mu^s(i)}$ are regular. Then the following assertions are equivalent.*

- *(a) The language L_i is regular.*
- *(b) For all $c \geq 1$, the languages $L_{i,c}$ are regular.*
- *(c) The sequence Δ_i is ultimately periodic.*

Proof. The proof proceeds by induction on $s \geq 1$. We assume that the equivalences hold for $\mu(i), \dots, \mu^{s-1}(i)$ and show them for i. The base case $s = 1$ and the induction step will be addressed simultaneously. Note that, for the base case, our only assumption is that $L_{\mu(i)}$ is regular since there are no previous equivalences to check.

Since the languages $L_{\mu(i)}, \dots, L_{\mu^{s-1}(i)}$ are assumed to be regular, we know by the induction hypothesis that the sequences $\Delta_{\mu(i)}, \dots, \Delta_{\mu^{s-1}(i)}$ are ultimately periodic, and also that for all $c \geq 1$, the languages $L_{\mu(i),c}, \dots, L_{\mu^{s-1}(i),c}$ are regular. Since $L_{\mu^s(i)}$ is also assumed to be regular and $\mathbf{d}_{\mu^s(i)}$ is infinite, we can also make use of Theorem 6.27.

Clearly, (b) implies (a). We show that (a) implies (c). Thus, we assume that L_i is regular. By Lemma 6.18 combined with some elementary arithmetical considerations, it can be written as a disjoint union of the form

$$L_i = F \cup \bigcup_{e=0}^{M-1} x_e y_e^* z_e$$

where F is a finite language, $M \geq 1$ can be chosen to be a multiple of the periods of the sequences $\Delta_{\mu(i)}, \dots, \Delta_{\mu^{s-1}(i)}$ and a multiple of $k\frac{m_0}{p}$, where k is such that $(\Delta_{\mu^s(i),q_0,km_0})$ is ultimately zero (with the notation of Theorem 6.27). We can also require that for each e, we have $|y_e| = Mp$, $|x_e z_e| \equiv i \pmod p$, and moreover, $|x_0z_0| = tMp + i$ for some $t \geq 0$, and $|x_{e+1}z_{e+1}| = |x_ez_e| + p$ if $e < M - 1$.

Our aim is to show that

$$(\Delta_i)_n = (\Delta_i)_{n-M} \tag{6.7}$$

for all large n. Consider $n \geq (t+1)M$ and let $e = n \bmod M$. We have $\mathrm{rep}_{i,1}(n) = x_ey_e^rz_e$ and $\mathrm{rep}_{i,1}(n-M) = x_ey_e^{r-1}z_e$ for some $r \geq 1$. By Proposition 6.13, we know that for all L and all large enough n, the prefix of length L of $\mathrm{rep}_{i,1}(n)$ coincides with the prefix of length L of some $\mathbf{w}_{i,j}$. Due to our hypothesis on i, the possible j are $0, 1, \ldots, s$. We consider the cases $j < s$ and $j = s$ separately.

First, suppose that $j < s$. Then

$$\mathbf{w}_{i,j} = d'_i \cdots d'_{\mu^{j-1}(i)} d_{\mu^j(i)} 0^\omega.$$

In this case, we see that y_e can only contain zeros, i.e., we have $\mathbf{w}_{i,j} = x_e 0^\omega$ and $y_e = 0^{Mp}$. Then

$$\begin{aligned}
U_{np+i} - U_{(n-M)p+i} \\
&= (U_{np+i} - 1) - (U_{(n-M)p+i} - 1) \\
&= \mathrm{val}_U(x_e 0^{rMp} z_e) - \mathrm{val}_U(x_e 0^{(r-1)Mp} z_e) \\
&= \left(\sum_{h=0}^{j} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{np+i-k_{i,h}-\ell} - \sum_{h=1}^{j} U_{np+i-k_{i,h}} \right) \\
&\quad - \left(\sum_{h=0}^{j} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{(n-M)p+i-k_{i,h}-\ell} - \sum_{h=1}^{j} U_{(n-M)p+i-k_{i,h}} \right).
\end{aligned}$$

Rearranging the terms, this gives

$$\begin{aligned}
&\left(U_{np+i} - \sum_{\ell=1}^{\ell_i} t_{i,\ell} U_{np+i-\ell} \right) - \left(U_{(n-M)p+i} - \sum_{\ell=1}^{\ell_i} t_{i,\ell} U_{(n-M)p+i-\ell} \right) \\
= \quad & -\sum_{h=1}^{j} \left(U_{np+i-k_{i,h}} - \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{np+i-k_{i,h}-\ell} \right) \\
& + \sum_{h=1}^{j} \left(U_{(n-M)p+i-k_{i,h}} - \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{(n-M)p+i-k_{i,h}-\ell} \right).
\end{aligned}$$

Using Definition 6.30 and (6.3), this can be reexpressed as

$$(\Delta_i)_n - (\Delta_i)_{n-M} = -\sum_{h=1}^{j} \Big((\Delta_{\mu^h(i)})_{n-m_{i,h}} - (\Delta_{\mu^h(i)})_{n-m_{i,h}-M} \Big) \qquad (6.8)$$

Since M is a multiple of all the periods of the sequences $\Delta_{\mu(i)}, \ldots, \Delta_{\mu^{s-1}(i)}$, every term of the sum on the right-hand side is ultimately zero. We thus get (6.7) for all large n such that the corresponding j is less than s. Note that for the base case $s = 1$, this sum is in fact empty, thus the right-hand side is 0, which leads us to the same conclusion.

Second, we consider the case $j = s$. We have

$$\mathbf{w}_{i,s} = d'_i \cdots d'_{\mu^{s-1}(i)} \mathbf{d}_{\mu^s(i)}$$

where $\mathbf{d}_{\mu^s(i)}$ is eventually periodic. Since we know that $\mathbf{d}_{\mu^s(i)}$ is not purely periodic (see [CC21, Proposition 38] reproduced here as Corollary 2.17), we must have

$$x_e = d'_i \cdots d'_{\mu^{s-1}(i)} t_{\mu^s(i),1} \cdots t_{\mu^s(i),q}$$

and

$$y_e = t_{\mu^s(i),q+1} \cdots t_{\mu^s(i),q+Mp}$$

for some q larger than the preperiod of $\mathbf{d}_{\mu^s(i)}$. Similarly to the previous case, we obtain

$$\begin{aligned}
&U_{np+i} - U_{(n-M)p+i} \\
&= \left(\sum_{h=0}^{s-1} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{np+i-k_{i,h}-\ell} - \sum_{h=1}^{s} U_{np+i-k_{i,h}} + \sum_{\ell=1}^{q+Mp} t_{\mu^s(i),\ell} U_{np+i-k_{i,s}-\ell} \right) \\
&\quad - \left(\sum_{h=0}^{s-1} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{(n-M)p+i-k_{i,h}-\ell} - \sum_{h=1}^{s} U_{(n-M)p+i-k_{i,h}} \right. \\
&\qquad \left. + \sum_{\ell=1}^{q} t_{\mu^s(i),\ell} U_{(n-M)p+i-k_{i,s}-\ell} \right).
\end{aligned}$$

Using Definition 6.30, Definition 6.25 and (6.3), we get

$$\begin{aligned}
(\Delta_i)_n-&(\Delta_i)_{n-M}\\
&=-\sum_{h=1}^{s-1}\Big((\Delta_{\mu^h(i)})_{n-m_{i,h}}-(\Delta_{\mu^h(i)})_{n-m_{i,h}-M}\Big)\\
&\quad-\Bigg(U_{np+i-k_{i,s}}-\sum_{\ell=1}^{q+Mp}t_{\mu^s(i),\ell}U_{np+i-k_{i,s}-\ell}\\
&\qquad\quad -U_{(n-M)p+i-k_{i,s}}+\sum_{\ell=1}^{q}t_{\mu^s(i),\ell}U_{(n-M)p+i-k_{i,s}-\ell}\Bigg)\\
&=-\sum_{h=1}^{s-1}\Big((\Delta_{\mu^h(i)})_{n-m_{i,h}}-(\Delta_{\mu^h(i)})_{n-m_{i,h}-M}\Big)+(\Delta_{\mu^s(i),q,Mp})_{n-m_{i,s}}.
\end{aligned}$$

As in the previous case, every term of the sum over h is ultimately zero. Moreover, the additional term $(\Delta_{\mu^s(i),q,Mp})_{n-m_{i,s}}$ is also ultimately equal to zero by using Lemma 6.26 and the link between Theorem 6.27 and the choice of M. Therefore (6.7) also holds for all large n such that the corresponding j is equal to s.

We now move on to the proof that (c) implies (b). Assume that the sequence Δ_i is ultimately periodic, say with period M, and let $c \geq 1$ be fixed. By Lemma 6.20, in order to show that $L_{i,c}$ is regular, it suffices to prove that there exists N such that the M languages $\{\mathrm{rep}_{i,c}(nM+e) : n \geq N\}$ are regular, for $e \in \{0,\ldots,M-1\}$.

Consider some fixed $e \in \{0,\ldots,M-1\}$. By Proposition 6.13, we know that for n large enough, the word $\mathrm{rep}_{i,c}(nM+e)$ starts with $t_{i,1}\cdots t_{i,\ell_i-1}$ and the next digit is either t_{i,ℓ_i} or $t_{i,\ell_i}-1$. This next digit is

$$\begin{aligned}
&\left\lfloor\frac{U_{(nM+e)p+i}-c-\sum_{\ell=1}^{\ell_i-1}t_{i,\ell}U_{(nM+e)p+i-\ell}}{U_{(nM+e)p+i-\ell_i}}\right\rfloor\\
=&\left\lfloor\frac{(\Delta_i)_{nM+e}-c+t_{i,\ell_i}U_{(nM+e)p+i-\ell_i}}{U_{(nM+e)p+i-\ell_i}}\right\rfloor\\
=&t_{i,\ell_i}+\left\lfloor\frac{(\Delta_i)_{nM+e}-c}{U_{(nM+e)p+i-\ell_i}}\right\rfloor.
\end{aligned}$$

By hypothesis, $(\Delta_i)_{nM+e}$ is ultimately a constant. We denote this constant value by $C_{i,e}$. We separate our work based on whether $C_{i,e} \geq c$ or $C_{i,e} < c$.

First, suppose that $C_{i,e} \geq c$. For n large, the next digit of $\text{rep}_{i,c}(nM+e)$ is t_{i,ℓ_i} and

$$\text{rep}_{i,c}(nM+e) \in d_i 0^* \text{rep}_U(C_{i,e}-c).$$

Therefore, there exists N and r such that

$$\{\text{rep}_{i,c}(nM+e) : n \geq N\} = d_i 0^r (0^{Mp})^* \text{rep}_U(C_{i,e}-c),$$

which shows that this language is regular.

Second, suppose that $C_{i,e} < c$. In this case, for large n, the next digit of the word $\text{rep}_{i,c}(nM+e)$ is $t_{i,\ell_i} - 1$ and, using the notation (6.3), we obtain that

$$\text{rep}_{i,c}(nM+e) = d_i' \,\text{rep}_{\mu(i),c-C_{i,e}}(nM+e-m_{i,1}).$$

As a result, there exists N such that

$$\{\text{rep}_{i,c}(nM+e) : n \geq N\} = d_i' L_{\mu(i),c-C_{i,e}} \cap A^r (A^{Mp})^*,$$

where $r = (NM+e)p + i$. By the induction hypothesis if $s \geq 2$, and by Theorem 6.27 for the base case $s = 1$, the equivalence between (a) and (b) holds for $\mu(i)$. Since the language $L_{\mu(i)}$ is regular by hypothesis, we get that the language $L_{\mu(i),c-C_{i,e}}$ is regular. This allows us to conclude that the language $\{\text{rep}_{i,c}(nM+e) : n \geq N\}$ is regular in this case as well. □

Remark 6.32. As in Proposition 6.28, we argue that the condition (c) of Theorem 6.31 can be used effectively to decide the regularity of a given L_i, assuming the eigenvalues of U are known. Since the sequence Δ_i satisfies the same recurrence relations as U, its eigenvalues form a subset of those of U and can be computed effectively. It then suffices to check whether these eigenvalues are all zero or roots of unity, which is equivalent to Δ_i being ultimately periodic.

It should be noted that when $L_{\mu(i)}$ (or a further successor) is not regular, L_i may or may not be regular. Let us illustrate this comment with two examples.

Example 6.33. On the one hand, one can consider the system U given by the relations $U_{2n} = 3U_{2n-1} + 1$ and $U_{2n-1} = 2U_{2n-2} + 2U_{2n-3} - U_{2n-4} - 1$ and the initial conditions $(U_0, \ldots, U_5) = (1, 2, 7, 16, 49, 122)$. This sequence satisfies the linear recurrence relation $U_{n+6} = 9U_{n+4} - 11U_{n+2} + 3U_n$ and its associated alternate base is $(3, \frac{4+\sqrt{13}}{3})$. We have $\mathbf{d}_0 = 30^\omega$ and $\mathbf{d}_1 = 21^\omega$. Thus, $\mu(0) = 1$. The corresponding graph G is depicted in Figure 6.4. Recall that in Section 6.5, we expect the period m_0 of the infinite expansion to be a multiple of p. So, with $i = 1$ and $p = 2$ here, we have $q_0 = 1$ and $m_0 = 2$. We find that $(\Delta_{1,1,2})_n$ is ultimately equal to -1. From Lemma 6.26, we then find that $(\Delta_{1,1,2k})_n$ is ultimately equal to $-k$ for all $k \geq 1$. Therefore, the criterion (c) of Theorem 6.27 tells us that the language L_1 is not regular. However, the language L_0 is given by $30(00)^*$ and thus is regular.

On the other hand, if the system U is defined by the relations $U_{2n} = 3U_{2n-1}$ and $U_{2n-1} = 2U_{2n-2} + 2U_{2n-3} - U_{2n-4} - 1$ and the initial conditions $(1, 2, 6, 14, 42, 105)$, it satisfies the same recurrence relation, is associated with the same alternate base, and we again find that $(\Delta_{1,1,2})_n$ is ultimately equal to -1. Thus, L_1 is again not regular. However, this time we have $L_0 = 2 \cdot L_1$, thus this language is not regular either.

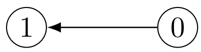


Figure 6.4: The graph G associated with the alternate base $B = (3, \frac{4+\sqrt{13}}{3})$.

6.7 Vertices in a cycle

In this section, we examine the regularity of L_i when i is part of a cycle in the graph G. In this case, there exists some r such that $\mu^r(i) = i$. Consequently, $\mathbf{d}_i, \ldots, \mathbf{d}_{\mu^{r-1}(i)}$ are all finite and $\mathbf{d}_i^* = (d_i' \cdots d_{\mu^{r-1}(i)}')^\omega$. We know that, up to taking large enough n, the words $\text{rep}_U(U_{np+i} - 1)$ share prefixes of arbitrary length with one of the $\mathbf{w}_{i,j}$'s. However, these choices for all vertices of the cycle are not independent of one another, which brings additional information on top of Proposition 6.13. This observation will be made clear shortly as it will be one of the main arguments in what follows.

The next result gives a necessary condition for the regularity of all languages "in the cycle" that will allow us to focus on ultimately periodic sequences.

Proposition 6.34. *Let $i \in \{0, \ldots, p-1\}$ be such that there exists $r \geq 1$ such that $\mu^r(i) = i$. If the languages $L_i, L_{\mu(i)}, \ldots, L_{\mu^{r-1}(i)}$ are all regular, then the sequences $\Delta_i, \Delta_{\mu(i)}, \ldots, \Delta_{\mu^{r-1}(i)}$ are all ultimately periodic.*

Proof. Suppose that the languages $L_i, L_{\mu(i)}, \ldots, L_{\mu^{r-1}(i)}$ are all regular. Then by Lemma 6.18 combined with arithmetic considerations, they can be decomposed in disjoint unions as follows. For every $h \in \{0, \ldots, r-1\}$, we have

$$L_{\mu^h(i)} = F_h \cup \bigcup_{e=0}^{M-1} x_{h,e} y_{h,e}^* z_{h,e}$$

where F_h is a finite language, $M \geq 1$ with $|y_{h,e}| = Mp$, $|x_{h,e} z_{h,e}| \equiv \mu^h(i) \pmod{p}$, $|x_{h,0} z_{h,0}| = tMp + \mu^h(i)$ for some t, and $|x_{h,e+1} z_{h,e+1}| = |x_{h,e} z_{h,e}| + p$ for each e. Without loss of generality, we can ask that Mp is a multiple of $k_{i,r}$, which is the sum of the lengths of the finite expansions $\mathbf{d}_{\mu^h(i)}$ for $h \in \{0, \ldots, r-1\}$. We let M' be this multiple, so that we have

$$Mp = M' k_{i,r} = k_{i,M'r}.$$

See Figure 6.5 for a visual help with the methods of addressing of the integers in this section.

Given the symmetry of the situation, in order to show the result it is enough to prove that Δ_i is ultimately periodic. To this end, we will show that

$$(\Delta_i)_n = (\Delta_i)_{n-M} \tag{6.9}$$

for large n.

Consider $e \in \{0, \ldots, M-1\}$ and $n \geq tM$ (or larger if needed) such that $n \equiv e \pmod{M}$. Thus, we have $\text{rep}_U(U_{np+i} - 1) \in x_{0,e} y_{0,e}^* z_{0,e}$. By Proposition 6.13, we know that for all L and all large enough n, the prefix of length L of $\text{rep}_U(U_{np+i} - 1)$ coincides with the prefix of length L of some $\mathbf{w}_{i,j}$. This implies that either $x_{0,e} y_{0,e}^\omega = \mathbf{w}_{i,j_0}$ for some $j_0 \geq 0$ or $x_{0,e} y_{0,e}^\omega = \mathbf{d}_i^*$. We consider these two cases separately.

First, we suppose that $x_{0,e} y_{0,e}^\omega = \mathbf{w}_{i,j_0}$ for some $j_0 \geq 0$. Then $y_{0,e} = 0^{Mp}$. Now for $h \in \{1, \ldots, j_0\}$ and $e' = (e - m_{i,h}) \bmod M$, we also have that $x_{h,e'} y_{h,e'}^\omega$ is either $\mathbf{w}_{\mu^h(i),j_h}$ for some $j_h \geq 0$ or $\mathbf{d}_{\mu^h(i)}^*$. However, in our case only the former is possible, with $j_h \leq j_0 - h$. Indeed, otherwise the suffix of length $np + i - k_{i,h}$ of $\text{rep}_U(U_{np+i} - 1)$ would be lexicographically greater than $\text{rep}_U(U_{np+i-k_{i,h}} - 1)$ for large n, in contradiction with Lemma 6.5. Similarly

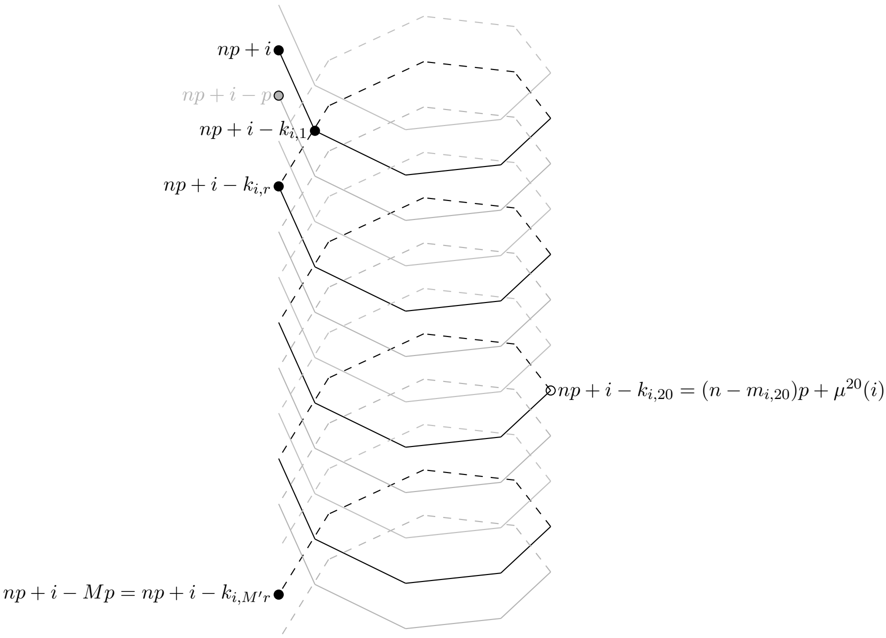


Figure 6.5: A schematic representation (in three dimensions) of the notions at play in the proof of Proposition 6.34. The integers are represented as a coil, with n and $n-p$ being vertically aligned. Smaller integers are lower in the picture. A segment represents following a (finite) greedy representation of 1, and leads from $np+i$ to $np+i-l_i$. Here $r=8$, $k_{i,r}=3p$, $M=12$ and the picture repeats periodically even including the various values of Δ, with exactly one period being represented in this picture. There are $\frac{M}{M'}=\frac{k_{i,r}}{p}$ disjoint spirals, which will become relevant in Theorem 6.39.

to (6.8) and using that $m_{i,h+j} = m_{i,h} + m_{\mu^h(i),j}$, we obtain

$$(\Delta_{\mu^h(i)})_{n-m_{i,h}} - (\Delta_{\mu^h(i)})_{n-m_{i,h}-M} \\ = -\sum_{j=1}^{j_h} \left((\Delta_{\mu^{h+j}(i)})_{n-m_{i,h+j}} - (\Delta_{\mu^{h+j}(i)})_{n-m_{i,h+j}-M} \right). \tag{6.10}$$

We can now prove by descending induction on $h \in \{0, \ldots, j_0\}$ that

$$(\Delta_{\mu^h(i)})_{n-m_{i,h}} - (\Delta_{\mu^h(i)})_{n-m_{i,h}-M}$$

is ultimately equal to 0. If $h = j_0$ then $j_h = 0$ and the right-hand side of (6.10) is trivially zero. Now, let $h \in \{0, \ldots, j_0 - 1\}$ and suppose that the claim holds for $h+1, \ldots, j_0$. Then each term of the sum of the right-hand side of (6.10) is ultimately zero by induction hypothesis since $h+j \in \{h+1 \ldots, j_0\}$ as we have $h + j_h \leq j_0$.

Second, we suppose that $x_{0,e}y_{0,e}^{\omega} = \mathbf{d}_i^*$. Let us argue that, in this case, for every $h \in \{1, \ldots, M'r - 1\}$, the word $\mathrm{rep}_U(U_{np+i-k_{i,h}} - 1)$ has long common prefixes either with some $\mathbf{w}_{\mu^h(i),j_h}$ with $j_h < M'r - h$ or with $\mathbf{d}^*_{\mu^h(i)}$. Indeed, we know that the word $\mathrm{rep}_U(U_{np+i-Mp} - 1)$ also shares long prefixes with $x_{0,e}y_{0,e}^{\omega}$, which we are assuming is $\mathbf{d}_i^*$. Since $k_{i,M'r} = Mp$, we see that $\mathrm{rep}_U(U_{np+i-k_{i,h}} - 1)$ cannot share long common prefixes with some $\mathbf{w}_{\mu^h(i),j_h}$ with $j_h \geq M'r - h$ for otherwise, its suffix of length $np + i - k_{i,M'r}$ would be lexicographically greater than $\mathrm{rep}_U(U_{np+i-Mp} - 1)$, in contradiction with Lemma 6.5.

We define a finite sequence $(h_q)_{1 \leq q \leq Q}$ as follows. Set $h_1 = 1$. Then, if h_q is defined and is less than $M'r$, and if there is some $j_{h_q} < M'r - h_q$ such that $\mathrm{rep}_U(U_{np+i-k_{i,h_q}} - 1)$ has long common prefixes with $\mathbf{w}_{\mu^{h_q}(i),j_{h_q}}$, then we set $h_{q+1} = h_q + j_{h_q} + 1$. Since this sequence is increasing and bounded above by $M'r$, it must end with some h_Q. From the previous paragraph, the word $\mathrm{rep}_U(U_{np+i-k_{i,h_Q}} - 1)$ has long common prefixes with $\mathbf{d}^*_{\mu^{h_Q}(i)}$. See Figure 6.6 for a visual explanation.

We thus have the following factorizations: first,

$$\mathrm{rep}_U(U_{np+i} - 1) = d_i' d_{\mu(i)}' \cdots d_{\mu^{h_Q - 1}(i)}' w z_{0,e}$$

where w is a prefix of $\mathbf{d}^*_{\mu^{h_Q}(i)}$; then, for every $q \in \{1, \ldots, Q-1\}$, there exists $e_q \in \{0, \ldots, M-1\}$ such that

$$\mathrm{rep}_U(U_{np+i-k_{i,h_q}} - 1) \in d'_{\mu^{h_q}(i)} d'_{\mu^{h_q+1}(i)} \cdots d'_{\mu^{h_{q+1}-2}(i)} d_{\mu^{h_{q+1}-1}(i)} 0^* z_{\mu^{h_q}(i),e_q}$$

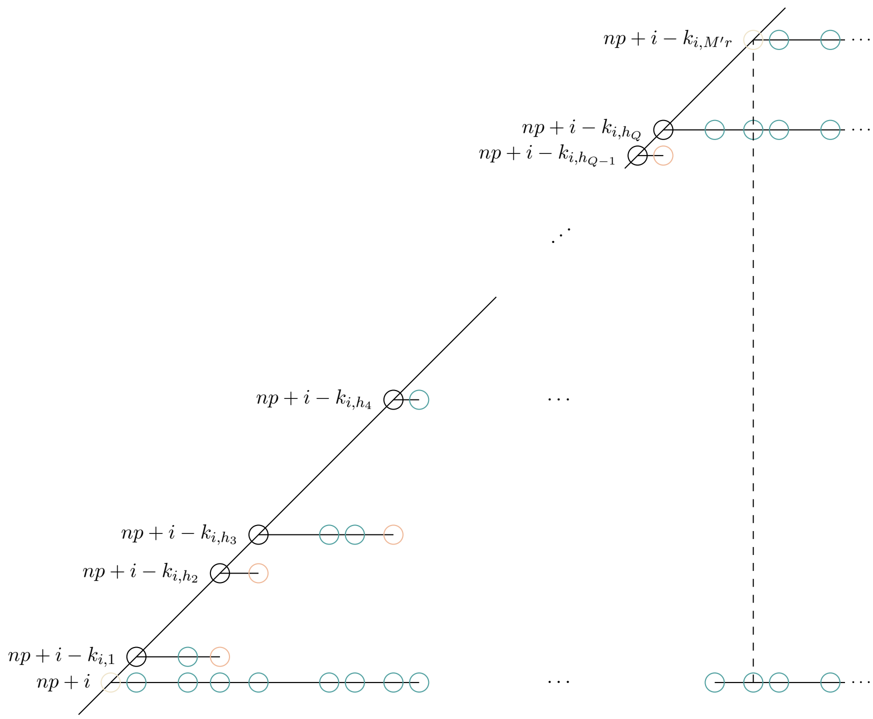


Figure 6.6: Visual explanation for the definition of $(h_q)_{1\le q\le Q}$. The language $\{\mathrm{rep}_U(U_n-1) : n\in\mathbb{N}\}$ can be schematically represented as a triangle, by right-aligning all these representations, with the expansion of U_n-1 having length n and being the n-th slice of the triangle. Here, the label to the left of a line indicates the length of the representation on this line. Segments correspond to following a representation of 1 in the associated alternate base. The teal nodes correspond to following the word d', with the last digit being decreased by 1, whereas red nodes correspond to following the word d, with the last digit having the "correct" value (in representations, this red node is likely followed by zeros to the right according to Lemma 6.36). Note that due to the Parry conditions, there cannot be a teal node above a red node. The two beige nodes are distant of Mp, corresponding to a full period. Since we assumed that $x_{0,e}y_{0,e}^{\omega}=\mathbf{d}_i^*$, all red nodes horizontally between the two beige nodes must be to the left of the vertical dotted line. The sequence $(h_q)_{1\le q\le Q}$ is constructed iteratively, starting just above the leftmost beige node, then going to the right until a red node is found then adding the node vertically above it to the sequence. This sequence must necessarily end in a node that follows an infinite expansion, with no red nodes to its right.

and finally, there exists $e_Q \in \{0, \ldots, M-1\}$ such that

$$\operatorname{rep}_U(U_{np+i-k_{i,h_Q}} - 1) = w' z_{\mu^{h_Q}(i), e_Q}$$

where w' is a prefix of $\mathbf{d}^*_{\mu^{h_Q}(i)}$.

Using these expansions, we can write

$$\begin{aligned}
0 = U_{np+i} - 1 - \sum_{h=0}^{h_Q-1} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{np+i-k_{i,h}-\ell} + \sum_{h=1}^{h_Q} U_{np+i-k_{i,h}} - \operatorname{val}_U(w z_{0,e}) \\
- \sum_{q=1}^{Q-1} \Bigg(U_{np+i-k_{i,h_q}} - 1 - \sum_{h=h_q}^{h_{q+1}-1} \sum_{\ell=1}^{\ell_{\mu^h(i)}} t_{\mu^h(i),\ell} U_{np+i-k_{i,h}-\ell} \\
+ \sum_{h=h_q+1}^{h_{q+1}-1} U_{np+i-k_{i,h}} - \operatorname{val}_U\big(z_{\mu^{h_q}(i), e_q}\big) \Bigg) \\
- \Big(U_{np+i-k_{i,h_Q}} - 1 - \operatorname{val}_U\big(w' z_{\mu^{h_Q}(i), e_Q}\big) \Big).
\end{aligned}$$

Multiple cancellations yield

$$(\Delta_i)_n = \operatorname{val}_U(w z_{0,e}) - \operatorname{val}_U\big(w' z_{\mu^{h_Q}(i), e_Q}\big) - \sum_{q=1}^{Q-1} \big(\operatorname{val}_U\big(z_{\mu^{h_q}(i), e_q}\big) + 1\big).$$

In the right-hand side of this equality, only w and w' depend on n, since the sequence $(h_q)_{1 \le q \le Q}$ and the suffixes $z_{0,e}$ and $z_{\mu^{h_q}(i), e_q}$ only depend on e. Since w and w' are both prefixes of the same infinite word, one must be a prefix of the other. Since moreover

$$|w z_{0,e}| = |w' z_{\mu^{h_Q}(i), e_Q}| = np + i - k_{i,h_Q},$$

we get that

$$\operatorname{val}_U(w z_{0,e}) - \operatorname{val}_U\big(w' z_{\mu^{h_Q}(i), e_Q}\big) = \begin{cases} \operatorname{val}_U(z_{0,e}) - \operatorname{val}_U\big(v z_{\mu^{h_Q}(i), e_Q}\big), & \text{if } w' = wv; \\ \operatorname{val}_U(v z_{0,e}) - \operatorname{val}_U\big(z_{\mu^{h_Q}(i), e_Q}\big), & \text{if } w = w'v. \end{cases} \tag{6.11}$$

Because $|v|$ is either $|z_{0,e}| - |z_{\mu^{h_Q}(i), e_Q}|$ or $|z_{\mu^{h_Q}(i), e_Q}| - |z_{0,e}|$, this value only depends on e. This proves that (6.9) hold for all large n, as desired. □

In order to state the main result of this section, we need a new definition. This definition and the following lemma are stated in the more general case of a vertex that is either in a cycle or leading to a cycle in the graph G, as they will be used in both the current and the next sections.

Definition 6.35. Let $i \in \{0, \ldots, p-1\}$ be such that there exist $s \geq 0$ and $r \geq 1$ with $\mu^{s+r}(i) = \mu^s(i)$. For all integers $j \geq 0$ and all integers n such that $np + i \geq k_{i,j}$, we let

$$(\Delta_i^{(j)})_n = \sum_{h=0}^{j-1} (\Delta_{\mu^h(i)})_{n-m_{i,h}}.$$

That is, $(\Delta_i^{(j)})_n$ is the cumulative sum of j values of the sequences $\Delta_{\mu^h(i)}$, for $h \in \{0, \ldots, j-1\}$. These values are taken along relevant positions with respect to the execution of the greedy algorithm when representing $U_{np+i} - 1$. The following result is what motivates the previous definition.

Lemma 6.36. *Let $i \in \{0, \ldots, p-1\}$ be such that there exist $s \geq 0$ and $r \geq 1$ with $\mu^{s+r}(i) = \mu^s(i)$. For all integers $c \geq 1$ and $j \geq 0$, there exists N such that for all $n \geq N$,*

- *if $(\Delta_i^{(h)})_n < c$ for all $h \in \{1, \ldots, j\}$, then*

$$\text{rep}_{i,c}(n) = d'_i d'_{\mu(i)} \cdots d'_{\mu^{j-1}(i)} \text{rep}_{\mu^j(i), c_n}(n - m_{i,j}),$$

  where $c_n = c - (\Delta_i^{(j)})_n$;

- *if $(\Delta_i^{(h)})_n < c$ for all $h \in \{1, \ldots, j-1\}$ and $(\Delta_i^{(j)})_n \geq c$, then*

$$\text{rep}_{i,c}(n) \in d'_i d'_{\mu(i)} \cdots d'_{\mu^{j-2}(i)} d_{\mu^{j-1}(i)} 0^* \text{rep}_U\big((\Delta_i^{(j)})_n - c\big).$$

Proof. We proceed by induction on j. The result is trivial for $j = 0$. Let thus $j \geq 1$ be such that the result holds for $j-1$, and let $c \geq 1$. By induction hypothesis, there exists N_1 such that for all $n \geq N_1$ with

$$(\Delta_i^{(h)})_n < c \quad \text{for } h \in \{1, \ldots, j-1\}, \tag{6.12}$$

we have

$$\text{rep}_{i,c}(n) = d'_i d'_{\mu(i)} d'_{\mu^{j-2}(i)} \text{rep}_{\mu^{j-1}(i), b_n}(n - m_{i,j-1})$$

where $b_n = c - (\Delta_i^{(j-1)})_n$. Therefore, and by using Proposition 6.13, there exists $N_2 \geq N_1$ such that for all $n \geq N_2$ where the inequalities (6.12) hold, the prefix of length $k_{i,j}$ of $\text{rep}_{i,c}(n)$ coincides with that of either $\mathbf{w}_{i,j-1}$ or $\mathbf{w}_{i,j}$. Thus, for such n, the prefix of length $\ell_{\mu^{j-1}(i)}$ of $\text{rep}_{\mu^{j-1}(i),b_n}(n - m_{i,j-1})$ is either $d'_{\mu^{j-1}(i)}$ or $d_{\mu^{j-1}(i)}$. The last digit of this prefix is given by

$$\left\lfloor \frac{U_{np+i-k_{i,j-1}} - c + (\Delta_i^{(j-1)})_n - \sum_{\ell=1}^{\ell_{\mu^{j-1}(i)}-1} t_{\mu^{j-1}(i),\ell} U_{np+i-k_{i,j-1}-\ell}}{U_{np+i-k_{i,j}}} \right\rfloor .$$

Using Definitions 6.30 and 6.35, this digit can be rewritten as

$$\begin{aligned} &\left\lfloor \frac{(\Delta_i^{(j-1)})_n + (\Delta_{\mu^{j-1}(i)})_{n-m_{i,j-1}} - c}{U_{np+i-k_{i,j}}} \right\rfloor + t_{\mu^{j-1}(i),\ell_{\mu^{j-1}(i)}} \\ &= \left\lfloor \frac{(\Delta_i^{(j)})_n - c}{U_{np+i-k_{i,j}}} \right\rfloor + t_{\mu^{j-1}(i),\ell_{\mu^{j-1}(i)}}. \end{aligned}$$

If $(\Delta_i^{(j)})_n \geq c$, then this digit is $t_{\mu^{j-1}(i),\ell_{\mu^{j-1}(i)}}$ and

$$\text{rep}_{i,c}(n) \in d'_i d'_{\mu(i)} \cdots d'_{\mu^{j-2}(i)} d_{\mu^{j-1}(i)} 0^* \text{rep}_U\big((\Delta_i^{(j)})_n - c\big)$$

as expected. Otherwise, if $(\Delta_i^{(j)})_n < c$, then this digit is $t_{\mu^{j-1}(i),\ell_{\mu^{j-1}(i)}} - 1$ and

$$\text{rep}_{i,c}(n) = d'_i d'_{\mu(i)} \cdots d'_{\mu^{j-1}(i)} \text{rep}_{\mu^j(i),c_n}(n - m_{i,j})$$

where $c_n = c - (\Delta_i^{(j)})_n$, as expected. □

We will also need the following lemma on circular sums.

Lemma 6.37. *Consider a finite sequence $\delta_0, \ldots, \delta_{M-1}$ such that $\sum_{m=0}^{M-1} \delta_m < 0$, and set $\delta_m = \delta_{m \bmod M}$ for $m \geq M$. Then there exists $j \in \{0, \ldots, M-1\}$ such that for every $t \in \{0, \ldots, M-1\}$, we have $\sum_{m=j}^{j+t} \delta_m < 0$.*

Proof. Let $j \in \{0, \ldots, M-1\}$ be such that $\sum_{m=0}^{j-1} \delta_m$ is maximal, and choose j maximal among all possible such values. If $t \in \{0, \ldots, M-1-j\}$, then

$$\sum_{m=j}^{j+t} \delta_m = \sum_{m=0}^{j+t} \delta_m - \sum_{m=0}^{j-1} \delta_m < 0$$

by choice of j. If $t \in \{M-j, \ldots, M-1\}$, then we get that

$$\sum_{m=j}^{j+t} \delta_m = \sum_{m=j}^{M-1} \delta_m + \sum_{m=M}^{j+t} \delta_m = \sum_{m=0}^{M-1} \delta_m - \sum_{m=0}^{j-1} \delta_m + \sum_{m=0}^{j+t-M} \delta_m < 0,$$

since the first sum is negative by assumption, and the third is less or equal than the second by choice of j. □

We introduce one last definition.

Definition 6.38. Let $i \in \{0, \ldots, p-1\}$ be such that there exists $r \geq 1$ such that $\mu^r(i) = i$ and such that the sequences $\Delta_i, \Delta_{\mu(i)}, \ldots, \Delta_{\mu^{r-1}(i)}$ are all ultimately periodic with a common period M. For all integers $j \geq 0$ and all $e \in \{0, \ldots, M-1\}$, we let $\Delta_{i,e,j}$ denote the ultimate constant value of $(\Delta_{\mu^j(i)})_{nM+e-m_{i,j}}$ and we let $\Delta_{i,e}^{(j)}$ denote the ultimate constant value of $(\Delta_i^{(j)})_{nM+e}$, so that we have

$$\Delta_{i,e}^{(j)} = \sum_{h=0}^{j-1} \Delta_{i,e,h}.$$

We are ready to state the main result of this section.

Theorem 6.39. *Let $i \in \{0, \ldots, p-1\}$ be such that there exists $r \geq 1$ such that $\mu^r(i) = i$, in which case $\mathbf{d}_i, \ldots, \mathbf{d}_{\mu^{r-1}(i)}$ are finite and $\mathbf{d}_i^* = (d_i' \cdots d_{\mu^{r-1}(i)}')^\omega$. We assume that the sequences $\Delta_i, \Delta_{\mu(i)}, \ldots, \Delta_{\mu^{r-1}(i)}$ are all ultimately periodic with a common period M such that $Mp = M'k_{i,r}$ with $M' \geq 1$.*

The following assertions are equivalent.

(a) The languages $L_i, L_{\mu(i)} \ldots, L_{\mu^{r-1}(i)}$ are all regular.

(b) For all $c \geq 1$, the languages $L_{i,c}, L_{\mu(i),c} \ldots, L_{\mu^{r-1}(i),c}$ are all regular.

(c) For all $e \in \{0, \ldots, \frac{k_{i,r}}{p} - 1\}$ and $j \in \{0, \ldots, r-1\}$, we have $\Delta_{\mu^j(i),e}^{(M'r)} \geq 0$.

(d) For all $e \in \{0, \ldots, \frac{k_{i,r}}{p} - 1\}$, we have $\Delta_{i,e}^{(M'r)} \geq 0$.

Proof. We start by showing a few properties which hold in the setting of this theorem. For all $j, h \geq 0$, we have $k_{i,j+h} = k_{i,j} + k_{\mu^j(i),h}$ and $m_{i,j+h} =$

$m_{i,j} + m_{\mu^j(i),h}$. Therefore, for all $e \in \{0, \ldots, M-1\}$, we have

$$\Delta_{i,e,j+h} = \Delta_{\mu^j(i),e',h} \tag{6.13}$$

where $e' = (e - m_{i,j}) \bmod M$. In particular, note that e' does not depend on h. From our hypotheses, we have that $k_{\mu^j(i),M'r} = k_{i,M'r} = M'k_{i,r} = Mp$. We obtain that $k_{i,j+M'r} = k_{i,j} + Mp$ and $m_{i,j+M'r} = m_{i,j} + M$. Hence

$$\Delta_{i,e,j+M'r} = \Delta_{i,e,j}. \tag{6.14}$$

We get that, on the one hand,

$$\Delta_{i,e}^{(M'r)} = \Delta_{\mu^j(i),e'}^{(M'r)} \tag{6.15}$$

and, on the other hand,

$$\Delta_{i,e}^{(cM'r)} = c\Delta_{i,e}^{(M'r)}. \tag{6.16}$$

for all $c \geq 1$. In particular, since $\mu^r(i) = i$, note that $k_{i,r} = m_{i,r}p$ and that the relation (6.15) with $j = r$ becomes $\Delta_{i,e}^{(M'r)} = \Delta_{i,e'}^{(M'r)}$ where $e' = (e - \frac{k_{i,r}}{p}) \bmod M$. Therefore, assuming that $\Delta_{\mu^j(i),e}^{(M'r)} \geq 0$ for all $e \in \{0, \ldots, \frac{k_{i,r}}{p} - 1\}$ implies that $\Delta_{\mu^j(i),e}^{(M'r)} \geq 0$ for all $e \in \{0, \ldots, M-1\}$.

We obtain from the previous paragraph that (d) implies (c). The fact that (b) implies (a) is obvious.

In order to show that (c) implies (b), we will show the stronger fact that if for all $e \in \{0, \ldots, \frac{k_{i,r}}{p} - 1\}$, we have $\Delta_{i,e}^{(M'r)} \geq 0$, then for all $c \geq 1$, the language $L_{i,c}$ is regular. Thus, we consider a fixed $c \geq 1$, and we suppose that $\Delta_{i,e}^{(M'r)} \geq 0$ for all $e \in \{0, \ldots, \frac{k_{i,r}}{p} - 1\}$. From Lemma 6.20, to get the regularity of the language $L_{i,c}$, it is enough to prove that the languages

$$L_{i,c,e} = \{\mathrm{rep}_{i,c}(n) : n \geq 0,\ n \equiv e \pmod{M},\ U_{np+i} \geq c\}.$$

are regular for all $e \in \{0, \ldots, M-1\}$. We fix such an e and consider two cases.

First, suppose that there exists $q \in \{1, \ldots, cM'r\}$ such that $\Delta_{i,e}^{(q)} \geq c$. We choose a minimal such q. Thus we have $\Delta_{i,e}^{(q')} < c$ for $q' < q$. By Lemma 6.36, there exists N such that for all $n \geq N$, we have

$$\mathrm{rep}_{i,c}(nM + e) \in d'_i d'_{\mu(i)} \cdots d'_{\mu^{q-2}(i)} d_{\mu^{q-1}(i)} 0^* \mathrm{rep}_U\big(\Delta_{i,e}^{(q)} - c\big).$$

We get that $L_{i,c,e}$ is regular as

$$L_{i,c,e} = F \cup \Big(d'_i d'_{\mu(i)} \cdots d'_{\mu^{q-2}(i)} d_{\mu^{q-1}(i)} 0^* \mathrm{rep}_U\big(\Delta_{i,e}^{(q)} - c\big) \cap \big(A_U^{Mp}\big)^* A_U^{(NM+e)p+i}\Big)$$

where F is a finite language.

Second, suppose that $\Delta_{i,e}^{(q)} < c$ for all $q \in \{1, \dots, cM'r\}$. Combining our assumption that $\Delta_{i,e}^{(M'r)} \geq 0$ with (6.16), we get that in fact $\Delta_{i,e}^{(M'r)} = 0$. By Lemma 6.36 and by using that $m_{i,cM'r} = cM$, there exists N such that for all $n \geq N$, we have

$$\begin{aligned}\mathrm{rep}_{i,c}(nM + e) &= d'_i d'_{\mu(i)} \cdots d'_{\mu^{cM'r-1}(i)} \mathrm{rep}_{i,c}(nM + e - m_{i,cM'r}) \\ &= d'_i d'_{\mu(i)} \cdots d'_{\mu^{cM'r-1}(i)} \mathrm{rep}_{i,c}((n - c)M + e).\end{aligned}$$

We iterate this argument ℓ times until $n - \ell c < N$. We obtain again that $L_{i,c,e}$ is regular as

$$L_{i,c,e} = F \cup \left(d'_i d'_{\mu(i)} \cdots d'_{\mu^{cM'r-1}(i)} \right)^* G$$

where F and G are finite languages, namely,

$$\begin{aligned}F &= \{\mathrm{rep}_{i,c}(nM + e) : 0 \leq n < N - c,\ U_{(nM+e)p+i} \geq c\} \\ G &= \{\mathrm{rep}_{i,c}((N - t)M + e) : 1 \leq t \leq c\}.\end{aligned}$$

We turn our attention to show that (a) implies (d). We proceed by contraposition. Thus, we assume that there is some $e \in \{0, \dots, M - 1\}$ with $\Delta_{i,e}^{(M'r)} < 0$. We have to show that one of the r languages $L_i, L_{\mu(i)} \dots, L_{\mu^{r-1}(i)}$ is not regular. In view of (6.14) and by Lemma 6.37, there exists $j \in \{0, \dots, M'r - 1\}$ such that for all $t \in \{1, \dots, M'r\}$, we have

$$\sum_{h=j}^{j+t-1} \Delta_{i,e,h} < 0.$$

We are going to show that the language $L_{\mu^j(i)}$ is not regular. Using (6.13), the latter sum can be reexpressed as

$$\sum_{h=0}^{t-1} \Delta_{i,e,j+h} = \sum_{h=0}^{t-1} \Delta_{\mu^j(i),e',h} = \Delta_{\mu^j(i),e'}^{(t)}$$

where $e' = (e - m_{i,j}) \bmod M$. In view of (6.15) and (6.16), for all $t \in \{1, \dots, M'r\}$ and all $c \geq 1$, we have

$$\Delta_{\mu^j(i),e'}^{(t+cM'r)} = \Delta_{\mu^j(i),e'}^{(t)} + c\Delta_{\mu^{j+t}(i),e''}^{(M'r)} = \Delta_{\mu^j(i),e'}^{(t)} + c\Delta_{i,e}^{(M'r)} < 0$$

where $e'' = (e - m_{i,j+t}) \bmod M$. We obtain that

$$\Delta^{(t)}_{\mu^j(i),e'} < 0$$

for all $t \geq 1$.

Now, let us fix some $C \geq 1$. Lemma 6.36 ensures that there exists n such that for all $c \in \{0, \dots, C-1\}$, we have

$$\operatorname{rep}_{\mu^j(i),1}((n+c)M + e') = d'_{\mu^j(i)} d'_{\mu^{j+1}(i)} \cdots d'_{\mu^{j+cM'r-1}(i)} \operatorname{rep}_{\mu^j(i),c'}(nM + e')$$

where we have set $c' = 1 - c\Delta^{(M'r)}_{i,e}$, and where we have used (6.15) and (6.16), as well as the equality $m_{\mu^j(i),cM'r} = cM$. In particular, the C suffixes

$$\operatorname{rep}_{\mu^j(i),c'}(nM + e'), \qquad \text{for } c' \in \{1 - c\Delta^{(M'r)}_{i,e} : 0 \leq c < C\},$$

are distinct and all of the same length $(nM + e')p + \mu^j(i)$. This implies that $L_{\mu^j(i)}$ is not regular as slender regular languages have a uniformly bounded number of suffixes of the same length by Lemma 6.18. □

Remark 6.40. As in the previous sections, Theorem 6.39 can be used effectively in order to decide the regularity of all the languages L_i from a cycle. If the eigenvalues of U are known, then those of the sequences $\Delta_{\mu^j(i)}$ can be computed, allowing us to test whether they are all ultimately periodic. If this is indeed the case, then the values of M, M' and $\Delta^{(M'r)}_{i,e}$ can be computed, and the condition (d) of Theorem 6.39 can be tested.

In the dominant root case, this section concerns the case where the dominant root is a simple Parry number, i.e., $d_\beta(1) = t_1 \dots t_\ell 0^\omega$. In this situation, the graph G contains a single vertex with a loop, and Proposition 6.34 and Theorem 6.39 reduce to the following corollary, where the sequence $\Delta = (\Delta_n)_{n \geq \ell}$ is defined by

$$\Delta_n = U_n - \sum_{k=1}^{\ell} t_k U_{n-k}.$$

Corollary 6.41. *Let U be a positional numeration system with a dominant root $\beta \geq 1$ such that $d_\beta(1)$ is finite of length ℓ, i.e., β is a simple Parry number or $\beta = 1$.*

- *If the numeration language L_U is regular, then the sequence Δ is ultimately periodic.*
- *Assume that the sequence Δ is ultimately periodic with a preperiod $N \geq \ell$ and a period $M = M'\ell$ with $M' \geq 1$ (we include in the preperiod the ℓ undefined terms at the start of the sequence). Then the numeration language L_U is regular if and only if*
$$\sum_{j=0}^{M'-1} \Delta_{n-j\ell} \geq 0$$
for all $n \in \{N+M-\ell, \ldots, N+M-1\}$.

Proof. This follows from Proposition 6.34 and Theorem 6.39 where the notation boils down to $p = 1$, $i = 0$, $r = 1$, $k_{i,r} = \ell$. □

In comparison with [Hol98], this result is new. It provides a necessary and sufficient condition for the regularity of L_U in the simple Parry dominant root case, which is precisely the case that Hollander did not solve entirely. In particular, Hollander exhibited an example showing that the regularity of L_U can depend on the initial conditions, and not only on the characteristic polynomial of a linear recurrence satisfied by U. The example is the following. Suppose that U satisfies the linear recurrence of characteristic polynomial $(X-1)(X-3)$. This exactly means that the sequence Δ is constant as this sequence is given by $\Delta_n = U_n - 3U_{n-1}$, thus it satisfies the linear recurrence of characteristic polynomial $X - 1$. As shown by Hollander, by choosing the initial conditions $U_0 = 1$, $U_1 = 4$, we obtain a regular numeration language, whereas by choosing the initial conditions $U_0 = 1$, $U_1 = 2$, we obtain a non-regular numeration language. This situation was not handled by Hollander's result but is covered by Corollary 6.41. The notation drastically reduces since the dominant root of the system is 3 and Δ is constant, hence we get $\ell = N = M = M' = 1$. For the initial conditions $U_0 = 1$, $U_1 = 4$, we get that $U_n = 3U_{n-1} + 1$ and $\Delta_n = 1$ for all $n \geq 1$. Therefore, Corollary 6.41 tells us that L_U is regular. However, for the initial conditions $U_0 = 1$, $U_1 = 2$, we get that $U_n = \frac{3^n+1}{2}$ and $\Delta_n = -1$ for all $n \geq 1$. In this case, Corollary 6.41 yields that L_U is not regular.

The results of this section also include the case where U has 1 as a dominant root. We note that this case was not considered at all in [Hol98]. We discuss this particular case in the next corollary.

Corollary 6.42. *Let U be a positional numeration system with the dominant root 1. Then the numeration language L_U is regular if and only if the sequence $(U_{n+1} - U_n)_{n\geq 0}$ is ultimately periodic.*

In particular, if the sequence U is ultimately a polynomial, i.e., there exists a polynomial $P \in \mathbb{C}[X]$ and an integer $N \geq 0$ such that $U_n = P(n)$ for all $n \geq N$, then the numeration language L_U is regular if and only if this polynomial has integer coefficients and degree 1, i.e., there exists $a, b \in \mathbb{Z}$ with $a > 0$ such that $U_n = an + b$ for all $n \geq N$.

Proof. As for Corollary 6.41, under the dominant root hypothesis, the notation of this section reduces to $p = 1$, $i = 0$, $r = 1$, $k_{i,r} = \ell$. Moreover, since the dominant root is 1, we also have $\ell = 1$ and the sequence Δ is given by $\Delta_n = U_n - U_{n-1}$ for $n \geq 1$. By Corollary 6.41, if L_U is regular then Δ is ultimately periodic. Let us argue that the converse also holds. Suppose that Δ is ultimately periodic with preperiod $N \geq 1$ and period $M \geq 1$. With the notation of Corollary 6.41, we have $M' = M$ and in order to obtain that L_U is regular, it suffices to show that

$$\sum_{j=0}^{M-1} \Delta_{N+j} = \sum_{j=0}^{M-1} (U_{N+j} - U_{N+j-1}) \geq 0.$$

This is clearly the case as U is an increasing sequence of integers.

The particular case is straightforward. □

Example 6.43. The sequence U given by the initial conditions $U_0 = 1, U_1 = 2$ and the relation $U_n = U_{n-2} + 3$ for $n \geq 2$ yields a regular numeration language as $(U_{n+1} - U_n)_{n\geq 1} = (1, 2, 1, 2, \ldots)$. Indeed, it easily checked that $L_U = 10^* \cup 1(00)^*1 \cup \{\varepsilon\}$. On the other hand, Shallit proved in [Sha94] that the numeration system U given by $U_n = (n+1)^2$ for all $n \geq 0$ has a nonregular numeration language. This result can be recovered by the characterization given in Corollary 6.42.

Finally, similarly to the discussion ending Section 6.5, let us explain how the second part of Hollander's main result from [Hol98] can be re-obtained as a consequence of Corollary 6.41.

Corollary 6.44 ([Hol98])**.** *Let U be a numeration system with a dominant root $\beta > 1$ such that $d_\beta(1)$ is finite of length ℓ, i.e., β is a simple Parry* *number.*

- *If the numeration language L_U is regular, then the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial multiplied by $(X^\ell - 1)$.*
- *If the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial, then the numeration language L_U is regular.*

Proof. Suppose that $d_\beta(1) = t_1 \dots t_\ell 0^\omega$ with $t_\ell > 0$. Then $d^*_\beta(1)$ is equal to $(t_1 \dots t_{\ell-1}(t_\ell - 1))^\omega$. The Parry polynomial was introduced at the end of Section 6.5 for nonsimple Parry numbers. Mimicking the definition by using the digits of the quasi-greedy expansion $d^*_\beta(1) = (t_1 \dots t_{\ell-1}(t_\ell - 1))^\omega$, the Parry polynomial is defined as

$$P_{0,\ell} = \left(X^\ell - \sum_{k=1}^{\ell-1} t_k X^{\ell-k} - (t_\ell - 1) \right) - X^0 = X^\ell - \sum_{k=1}^{\ell} t_k X^{\ell-k}$$

and the extended β-polynomials as $P_{N,M\ell} = X^N(1 + X^\ell + \dots + X^{(M-1)\ell})P_{0,\ell}$ for $N \geq 0$ and $M \geq 1$. Thus, the first part of Corollary 6.41 amounts to saying that if the numeration language L_U is regular, then the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is of the form $X^N(X^M - 1)P_{0,\ell}$ for some $N \geq 0$ and $M \geq 1$. Since $X^N(X^M - 1)P_{0,\ell} = (X^\ell - 1)P_{N,M\ell}$, this proves the first item of the result.

For the second item, suppose that the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial $P_{N,M\ell}$. This exactly means that the sequence Δ satisfies a linear recurrence relation whose characteristic polynomial is given by $X^N(1 + X^\ell + \dots + X^{(M-1)\ell})$. This in turn means that

$$\sum_{j=0}^{M-1} \Delta_{n-j\ell} = 0$$

for all $n \geq N + M\ell$. Since $X^{M\ell} - 1 = (X^\ell - 1)(1 + X^\ell + \dots + X^{(M-1)\ell})$, we get that Δ is ultimately periodic with period $M\ell$. Therefore, the second part of Corollary 6.41 tells us that the numeration language L_U is regular. □

6.8 Vertices leading to a cycle

This is the final case in our discussion. We are now focusing on vertices i in the graph G that have a nontrivial path leading to a cycle in G. In this case, there exists $s \geq 1$ and $r \geq 1$ such that $\mu^{s+r}(i) = \mu^s(i)$. The expansions $\mathbf{d}_i, \ldots, \mathbf{d}_{\mu^{s+r-1}(i)}$ are all finite and $\mathbf{d}_i^* = d_i' \cdots d'_{\mu^{s-1}(i)}(d'_{\mu^s(i)} \cdots d'_{\mu^{s+r-1}(i)})^\omega$. In the remainder of this section, we assume that $s \geq 1$ is minimal such that there exists $r \geq 1$ with $\mu^{s+r}(i) = \mu^s(i)$, to avoid duplicating the previous case. As for Section 6.6, we note that this situation does not occur in the dominant root case of [Hol98], i.e., whenever $p = 1$. We begin with a lemma.

Lemma 6.45. *For any sequence $(x_n)_{n \geq 0}$ of complex numbers and any positive integer M, the following conditions are equivalent.*

(a) The sequence $(x_n)_{n \geq 0}$ is a linear recurrence sequence and all its nonzero eigenvalues are M-th roots of unity and have multiplicity at most 2.

(b) The sequence $(x_{n+M} - x_n)_{n \geq 0}$ is ultimately periodic with period M.

Proof. We start by proving that (a) implies (b). We suppose that (a) holds and we let $\lambda_1, \ldots, \lambda_d$ be the nonzero eigenvalues of $(x_n)_{n \geq 0}$. Then there exist $N \geq 0$ and $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{C}$ such that

$$x_n = \sum_{i=1}^{d} (a_i n + b_i) \lambda_i^n$$

for all $n \geq N$. We get

$$x_{n+M} - x_n = \sum_{i=1}^{d} \Big((a_i(n+M) + b_i)\lambda_i^{n+M} - (a_i n + b_i)\lambda_i^n \Big) = \sum_{i=1}^{d} a_i M \lambda_i^n$$

and

$$\begin{aligned} x_{n+2M} - x_{n+M} &= \sum_{i=1}^{d} \Big((a_i(n+2M) + b_i)\lambda_i^{n+2M} - (a_i(n+M) + b_i)\lambda_i^{n+M} \Big) \\ &= \sum_{i=1}^{d} a_i M \lambda_i^n \end{aligned}$$

for all $n \geq N$. Thus, we see that the sequence $(x_{n+M} - x_n)_{n \geq 0}$ is ultimately periodic with period M.

We now prove the converse implication. Suppose that there exists $N \geq 0$ such that for all $n \geq N$, we have $x_{n+2M} - x_{n+M} = x_{n+M} - x_n$, or equivalently, $x_{n+2M} = 2x_{n+M} - x_n$. Then the sequence $(x_n)_{n\geq 0}$ satisfies the recurrence relation of characteristic polynomial $X^N(X^{2M} - 2X^M + 1)$, which is $X^N(X^M - 1)^2$. Therefore, the eigenvalues of $(x_n)_{n\geq 0}$ are either 0 or M-th roots of unity with multiplicity at most 2, as expected. □

Proposition 6.46. *Let $i \in \{0, \ldots, p-1\}$ be such that there exist $s \geq 1$ minimal and $r \geq 1$ with $\mu^{s+r}(i) = \mu^s(i)$. If the languages $L_i, L_{\mu(i)}, \ldots, L_{\mu^{s+r-1}(i)}$ are all regular, then the sequence Δ_i satisfies the two conditions of Lemma 6.45 for some $M \geq 1$.*

Proof. Suppose that the languages $L_i, L_{\mu(i)}, \ldots, L_{\mu^{s+r-1}(i)}$ are all regular. By Lemma 6.18 combined with a few arithmetic considerations, they can be decomposed in disjoint unions as follows. For every $h \in \{0, \ldots, s+r-1\}$, we have

$$L_{\mu^h(i)} = F_h \cup \bigcup_{e=0}^{M-1} x_{h,e} y_{h,e}^* z_{h,e}$$

where F_h is a finite language, $M \geq 1$ with $|y_{h,e}| = Mp$, $|x_{h,e}z_{h,e}| \equiv \mu^h(i) \pmod p$, $|x_{h,0}z_{h,0}| = tMp + \mu^h(i)$ for some t, and $|x_{h,e+1}z_{h,e+1}| = |x_{h,e}z_{h,e}| + p$ for each e. Without loss of generality, we can ask that Mp is a multiple of $k_{\mu^s(i),r}$, which is the sum of the lengths of the finite expansions $\mathbf{d}_{\mu^{(i)}}$ for $h \in \{s, \ldots, s+r-1\}$. We let M' be this multiple, so that we have

$$Mp = M' k_{\mu^s(i),r} = k_{\mu^s(i),M'r}.$$

From the proof of Proposition 6.34, we know that

$$(\Delta_{\mu^h(i)})_n = (\Delta_{\mu^h(i)})_{n-M} \tag{6.17}$$

for all large n and all $h \in \{s, \ldots, s+r-1\}$.

Let us show that for each $e \in \{0, \ldots, M-1\}$, there exists a constant $\Gamma_{i,e}$ such that

$$(\Delta_i)_n - (\Delta_i)_{n-M} = \Gamma_{i,e}$$

holds for all large n with $n \equiv e \pmod M$. We proceed by induction on $s \geq 1$. The base case $s = 1$ and the induction step will be addressed simultaneously. Fix $e \in \{0, \ldots, M-1\}$ and let $n \geq (t+1)M$ be such that $n \equiv e \pmod M$.

Then $\operatorname{rep}_U(U_{np+i}-1) \in x_{0,e}y_{0,e}^*z_{0,e}$ and similarly as in the proof of Proposition 6.34, either $x_{0,e}y_{0,e}^\omega = \mathbf{w}_{i,j}$ for some $j \geq 0$ or $x_{0,e}y_{0,e}^\omega = \mathbf{d}_i^*$, and we consider these two cases separately.

First, we suppose that $x_{0,e}y_{0,e}^\omega = \mathbf{w}_{i,j}$ for some $j \geq 0$. Then $y_{0,e} = 0^{Mp}$. As for (6.8) in the proof of Theorem 6.31, we find

$$(\Delta_i)_n - (\Delta_i)_{n-M} = -\sum_{h=1}^{j}\Big((\Delta_{\mu^h(i)})_{n-m_{i,h}} - (\Delta_{\mu^h(i)})_{n-m_{i,h}-M}\Big).$$

Using the induction hypothesis combined with (6.17) for $h \in \{s,\dots,s+r-1\}$, for large enough n, each term of the sum in the right-hand side is equal to a constant depending only on the residue class e modulo M. (In particular, this constant is 0 for $h \geq s$.)

Second, we suppose that $x_{0,e}y_{0,e}^\omega = \mathbf{d}_i^*$. In this case, we find

$$(\Delta_i)_n - (\Delta_i)_{n-M} = \\ -\left(\sum_{h=1}^{s-1}\Big((\Delta_{\mu^h(i)})_{n-m_{i,h}} - (\Delta_{\mu^h(i)})_{n-m_{i,h}-M}\Big)\right) - \left(\Delta^{(M'r)}_{\mu^s(i)}\right)_{n-m_{i,s}}. \quad (6.18)$$

As for the first case, for large enough n, each term of the sum in the right-hand side is equal to a constant. Moreover, from the proof of Proposition 6.34, we know that the sequences $\Delta_{\mu^s(i)}, \Delta_{\mu^{s+1}(i)} \dots, \Delta_{\mu^{s+r-1}(i)}$ are all ultimately periodic with period M. This implies that for all large n, the additional term has the constant value $\Delta^{(M'r)}_{\mu^s(i),e'}$, where $e' = (e - m_{i,s}) \bmod M$. □

There are indeed cases where i is on a path leading to a cycle in G, is such that the corresponding language L_i is regular and the sequence Δ_i satisfies the conditions of Lemma 6.45 without being periodic. This is illustrated by the following example.

Example 6.47. Consider the sequence U given by $U_0 = 1$ and the relations

$$U_n = \begin{cases} U_{n-1} + U_{n-2} + 1, & \text{if } n \equiv 0 \pmod 2; \\ 3U_{n-1} - \frac{n-1}{2}, & \text{if } n \equiv 1 \pmod{10}; \\ 3U_{n-1}, & \text{otherwise.} \end{cases}$$

This sequence is indeed a linear recurrence sequence, satisfying the linear recurrence relation $U_{n+22} = 4U_{n+20} + 2U_{n+12} - 8U_{n+10} - U_{n+2} + 4U_n$ for

$n \geq 0$. It is associated with the alternate base $\left(\frac{4}{3}, 3\right)$. This base yields $\mathbf{d}_0 = 110^\omega$, $\mathbf{d}_1 = 30^\omega$, $\mathbf{d}_0^* = (10)^\omega$ and $\mathbf{d}_1^* = 2(10)^\omega$. The associated graph G is depicted in Figure 6.7.

Figure 6.7: The graph G associated with the alternate base numeration system $\left(\frac{4}{3}, 3\right)$.

One can see by inspection that the language L_U is regular, as we have

$$\mathrm{rep}_U(U_n - 1) = \begin{cases} 110^{n-2}, & \text{if } n \equiv 0 \pmod 2; \\ 2(10)^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{10}; \\ 2110^{n-3}, & \text{otherwise.} \end{cases}$$

The sequence Δ_1 is given by

$$(\Delta_1)_n = U_{2n+1} - 3U_{2n} = \begin{cases} -n, & \text{if } n \equiv 0 \pmod 5; \\ 0, & \text{otherwise.} \end{cases} \tag{6.19}$$

It is a linear recurrence sequence with minimal polynomial $X^{10} - 2X^5 + 1 = (X^5 - 1)^2$. Thus, we see all fifth roots of unity as eigenvalues of this sequence. Of course, the choice of $M = 5$ was arbitrary here, and this example shows that any root of unity can occur as a double eigenvalue of Δ_i in this case.

Now that we have reduced our study to linear recurrence sequences with eigenvalues that are roots of unity with multiplicity at most 2, we investigate precisely which of these sequences lead to regular languages. This is the point of our next and final main result. We will see that although eigenvalues of multiplicity 2 may occur, these occurrences are heavily constrained. In fact, only one new type of behavior arises.

In order to state this result, we introduce one last definition.

Definition 6.48. Let $i \in \{0, \ldots, p-1\}$ be such that there exist $s \geq 1$ and $r \geq 1$ such that $\mu^{s+r}(i) = \mu^s(i)$, with s minimal, and such that the sequences $\Delta_i, \Delta_i, \ldots, \Delta_{\mu^{s-1}}$ all satisfy the condition (b) of Lemma 6.45 with a common M. For all $j \in \{0, \ldots, s-1\}$ and all $e \in \{0, \ldots, M-1\}$, we let $\Gamma_{i,e,j}$ the ultimate constant value of $(\Delta_{\mu^j(i)})_{nM+e-m_{i,j}} - (\Delta_{\mu^j(i)})_{(n-1)M+e-m_{i,j}}$.

Theorem 6.49. *Let $i \in \{0, \dots, p-1\}$ be such that there exist $s \geq 1$ and $r \geq 1$ such that $\mu^{s+r}(i) = \mu^s(i)$, with s minimal. In this case the greedy expansions $\mathbf{d}_i, \dots, \mathbf{d}_{\mu^{s+r-1}(i)}$ are finite and $\mathbf{d}_i^* = d'_i \cdots d'_{\mu^{s-1}(i)}(d'_{\mu^s(i)} \cdots d'_{\mu^{s+r-1}(i)})^\omega$. We assume that the languages $L_{\mu(i)}, \dots, L_{\mu^{s+r-1}(i)}$ are all regular, that the sequences $\Delta_i, \Delta_{\mu(i)}, \dots, \Delta_{\mu^{s-1}(i)}$ all satisfy the condition (b) of Lemma 6.45 with a common M such that $Mp = M'k_{\mu^s(i),r}$ with $M' \geq 1$, and that the sequences $\Delta_{\mu^s(i)}, \dots, \Delta_{\mu^{s+r-1}(i)}$ are all ultimately periodic with period M.*

The following assertions are equivalent.

(a) The language L_i is regular.

(b) For all $c \geq 1$, the language $L_{i,c}$ is regular.

(c) For all $e \in \{0, \dots, M-1\}$, either $\Gamma_{i,e,0} = 0$, or $\Gamma_{i,e,0} = -\Delta^{(M'r)}_{\mu^s(i),e'} < 0$ where $e' = (e - m_{i,s}) \bmod M$ and $\Gamma_{i,e,j} = 0$ for all $j \in \{1, \dots, s-1\}$.

Proof. We prove this proposition by induction on s. We assume that the equivalences hold for $\mu(i), \dots, \mu^{s-1}(i)$ and show them for i. The base case $s = 1$ and the induction step will be addressed simultaneously. It is clear that (b) implies (a).

Let us show that (c) implies (b). Consider some fixed $c \geq 1$. Seeing Lemma 6.20, it suffices to show that the language

$$L_{i,c,e} = \{\operatorname{rep}_{i,c}(n) : n \geq 0,\ n \equiv e \pmod M,\ U_{np+i} \geq c\}.$$

is regular for each $e \in \{0, \dots, M-1\}$. We fix such an e and consider the two cases given in (c) separately. In what follows, we let n be an integer congruent to e modulo M and since we are interested in ultimate properties only, we always ask n to be sufficiently large so that all the future claims hold.

First, consider the case where $\Gamma_{i,e,0} = 0$. This means that the value $(\Delta_i)_n$ is eventually equal to a constant that we name $\Delta_{i,e,0}$ by analogy with Definition 6.38. If $\Delta_{i,e,0} \geq c$ then we get from Lemma 6.36 that

$$\operatorname{rep}_{i,c}(n) \in d_i 0^* \operatorname{rep}_U(\Delta_{i,e,0} - c),$$

which implies that $L_{i,c,e}$ is indeed regular. Now, suppose that $\Delta_{i,e,0} < c$. Lemma 6.36 yields

$$\operatorname{rep}_{i,c}(n) = d'_i \operatorname{rep}_{\mu(i),c'}(n - m_{i,1}).$$

where $c' = c - \Delta_{i,e,0}$. Letting $e' = (e - m_{i,1}) \bmod M$, we get that the languages $L_{i,c,e}$ and $d'_i L_{\mu(i),c',e'}$ coincide on all long enough words. Since we have assumed $L_{\mu(i)}$ to be a regular language, the language $L_{\mu(i),c',e'}$ is regular by induction hypothesis if $s \geq 2$ and by Theorem 6.39 if $s = 1$. This proves that $L_{i,c,e}$ is regular in this case as well.

Second, suppose that $\Gamma_{i,e,0} = -\Delta^{(M'r)}_{\mu^s(i),e'} < 0$ where $e' = (e - m_{i,s}) \bmod M$ and that

$$\Gamma_{i,e,j} = 0 \quad \text{for } j \in \{1, \ldots, s-1\}. \tag{6.20}$$

Using the ultimate periodicity of the sequences $\Delta_{\mu^s(i)}, \ldots, \Delta_{\mu^{s+r-1}(i)}$ combined with (6.20), we have that

$$\left(\Delta_i^{(j)}\right)_{n+tM} = \left(\Delta_i^{(j)}\right)_n + t\Gamma_{i,e,0} \tag{6.21}$$

for all $j, t \geq 0$. Since $\Gamma_{i,e,0} < 0$, we obtain that $\left(\Delta_i^{(j)}\right)_n < c$ for all $j \in \{1, \ldots, s + M'r\}$ and all large enough n. As a result, Lemma 6.36 ensures that

$$\mathrm{rep}_{i,c}(n) = d'_i \cdots d'_{\mu^{s+M'r-1}(i)} \mathrm{rep}_{\mu^s(i),c_n}(n - M - m_{i,s})$$

where $c_n = c - \left(\Delta_i^{(s)}\right)_n + \Gamma_{i,e,0}$. We have used that $m_{i,s+M'r} = m_{i,s} + M$ and that

$$\left(\Delta_i^{(s+M'r)}\right)_n = (\Delta_i^{(s)})_n + (\Delta_{\mu^s(i)}^{(M'r)})_{n-m_{i,s}} = (\Delta_i^{(s)})_n - \Gamma_{i,e,0}.$$

Similarly, we obtain

$$\mathrm{rep}_{i,c}(n - M) = d'_i \cdots d'_{\mu^{s-1}(i)} \mathrm{rep}_{\mu^s(i),c_n}(n - M - m_{i,s})$$

where we have used (6.21) in order to get that $c - \left(\Delta_i^{(s)}\right)_{n-M} = c - \left(\Delta_i^{(s)}\right)_n + \Gamma_{i,e,0} = c_n$. We see that $\mathrm{rep}_{i,c}(n)$ and $\mathrm{rep}_{i,c}(n-M)$ share the same suffix of length $(n - M)p + i - k_{i,s}$. From there, we conclude that there exists $N \geq 1$ and a suffix w such that for n larger than N and congruent to e modulo M, we have

$$\mathrm{rep}_{i,c}(n) \in d'_i \cdots d'_{\mu^{s-1}(i)} \left(d'_{\mu^s(i)} \cdots d'_{\mu^{s+M'r-1}(i)}\right)^* w.$$

Thus $L_{i,c,e}$ is regular, as expected.

We now prove that (a) implies (c). We will proceed by contraposition and show that if the condition (c) is not met, then we can construct C different suffixes of identical length of words in L_i for arbitrary C, which contradicts the regularity of this language by Lemma 6.18.

We suppose that there exists $e \in \{0, \dots, M-1\}$ such that $\Gamma_{i,e,0} \neq 0$ and, either $\Gamma_{i,e,0} \neq -\Delta^{(M'r)}_{\mu^s(i),e'}$ where $e' = (e - m_{i,s}) \bmod M$, or there exists $j \in \{1, \dots, s-1\}$ such that $\Gamma_{i,e,j} \neq 0$. We pick such an e and we fix $C \geq 1$. As above, we consider large n with $n \equiv e \pmod{M}$.

First, we consider the case where $\Gamma_{i,e,0} > 0$. Since $(\Delta_i)_{n+tM} = (\Delta_i)_n + t\Gamma_{i,e,0}$ for all $t \geq 0$, one has $(\Delta_i)_n > 0$ if n is large enough. Lemma 6.36 gives us

$$\mathrm{rep}_{i,1}(n) \in d_i 0^* \mathrm{rep}_U((\Delta_i)_n - 1).$$

Note that, in the present case, at least one of the bases β_j of the alternate base $\mathcal{B} = (\beta_{p-1}, \dots, \beta_0)$ is greater than 1, which implies that $U_{np+j+1} - U_{np+j}$ tends to infinity as n does. Therefore, up to letting n grow, we can build C distinct words

$$\mathrm{rep}_U((\Delta_i)_n - 1), \mathrm{rep}_U((\Delta_i)_{n+M} - 1), \dots, \mathrm{rep}_U((\Delta_i)_{n+(C-1)M} - 1)$$

of identical length which are suffixes of words in L_i.

Now, we suppose that $\Gamma_{i,e,0} < 0$. Using the ultimate periodicity of the r sequences $\Delta_{\mu^s(i)}, \dots, \Delta_{\mu^{s+r-1}(i)}$, we have

$$\left(\Delta_i^{(j)}\right)_{n+tM} = \left(\Delta_i^{(j)}\right)_n + t \sum_{h=0}^{s-1} \Gamma_{i,e,h} \tag{6.22}$$

for all $j \geq s$ and $t \geq 0$. By induction hypothesis, since the languages $L_{\mu(i)}, \dots, L_{\mu^{s-1}(i)}$ are assumed to be regular, the condition (c) is met for $\mu(i), \dots, \mu^{s-1}(i)$. Thus, for each $h \in \{1, \dots, s-1\}$, since $\Gamma_{i,e,h} = \Gamma_{\mu^h(i),f,0}$ with $f = (e - m_{i,h}) \bmod M$, we obtain that either $\Gamma_{i,e,h} = 0$, or $\Gamma_{i,e,h} = -\Delta^{(M'r)}_{\mu^s(i),e'} < 0$ with $e' = (e - m_{i,s}) \bmod M$ and $\Gamma_{i,e,h'} = 0$ for every $h' \in \{h+1, \dots, s-1\}$, where we have used that $m_{\mu^h(i),s-h} \equiv (m_{i,s} - m_{i,h}) \pmod{M}$. In particular, we have $\Gamma_{i,e,h} \leq 0$ for every $h \in \{1, \dots, s-1\}$. Therefore we have

$$\left(\Delta_i^{(j)}\right)_{n+tM} \leq \left(\Delta_i^{(j)}\right)_n + t\Gamma_{i,e,0}$$

for all $j \geq s$ and $t \geq 0$. Since $\Gamma_{i,e,0} < 0$, we obtain that $\left(\Delta_i^{(j)}\right)_n \leq 0$ for all $j \geq s$ and all large enough n (depending on j). Lemma 6.36 then gives us that for all $c \in \{0, \dots, C-1\}$, we have

$$\mathrm{rep}_{i,1}(n + cM) = d'_i \cdots d'_{\mu^{s-1}(i)} \left(d'_{\mu^s(i)} \cdots d'_{\mu^{s+M'r-1}(i)}\right)^c \mathrm{rep}_{i,c'}(n - m_{i,s})$$

where $c' = 1 - \big(\Delta_i^{(s+cM'r)}\big)_{n+cM}$. Yet, using (6.22) and (6.16), we get

$$\begin{aligned}\big(\Delta_i^{(s+cM'r)}\big)_{n+cM} &= \big(\Delta_i^{(s)}\big)_{n+cM} + \big(\Delta_{\mu^s(i)}^{(cM'r)}\big)_{n+cM-m_{i,s}} \\ &= \big(\Delta_i^{(s)}\big)_n + c\sum_{j=0}^{s-1}\Gamma_{i,e,j} + c\Delta_{\mu^s(i),e'}^{(M'r)}\end{aligned}$$

where $e' = (e - m_{i,s}) \bmod M$. Let us argue that the constant

$$Q = \sum_{j=0}^{s-1}\Gamma_{i,e,j} + \Delta_{\mu^s(i),e'}^{(M'r)}$$

is different from 0. If $\Gamma_{i,e,j} = 0$ for all $j \in \{1, \dots, s-1\}$, then by choice of e, we must have $\Gamma_{i,e,0} \neq -\Delta_{\mu^s(i),e'}^{(M'r)}$, hence $Q \neq 0$. Now, suppose that $\Gamma_{i,e,j} \neq 0$ for some $j \in \{1, \dots, s-1\}$. Then, by the induction hypothesis, as explained at the beginning of this case, we must have $\Gamma_{i,e,j} = -\Delta_{\mu^s(i),e'}^{(M'r)} < 0$ and $\Gamma_{i,e,j'} = 0$ for all $j' \in \{1, \dots, s-1\}$ that are distinct from j. Since $\Gamma_{i,e,0} < 0$, we also get $Q \neq 0$. Thus, up to letting n grow, once again, we see that we can build C distinct words

$$\mathrm{rep}_{i,c'}(n - m_{i,s}), \quad \text{for } c' \in \{1 - \big(\Delta_i^{(s)}\big)_n - cQ : 0 \leq c < C\},$$

of identical length which are suffixes of words in L_i. □

Remark 6.50. Let us argue that the condition (c) of Theorem 6.49 can be used effectively. Once the eigenvalues of U are known, we can find the eigenvalues of the sequences $\Delta_{\mu^j(i)}$, for $j \in \{0, \dots, s-1\}$, and check whether these sequences all satisfy the condition (a) of Lemma 6.45. If this is the case, and assuming that we already know that the sequences $\Delta_{\mu^j(i)}$ with $j \in \{s, \dots, s+r-1\}$ are ultimately periodic, a common M (as in the statement of Theorem 6.49) can be computed. From there, the values $\Gamma_{i,e,j}$ and $\Delta_{\mu^s(i),e'}^{(M'r)}$ can be computed, and Theorem 6.49 can be used to decide the regularity of L_i.

Example 6.51. We resume Example 6.47 to illustrate the notation of Theorem 6.49. We have $p = 2$, $i = 1$, $s = r = 1$, $k_{0,1} = 2$, and $M = M' = 5$. We also have $(\Delta_0)_n = 1$ for all n while the values $(\Delta_1)_n$ were given in (6.19) We get $\Gamma_{1,e,0} = 0$ for $e = 1, 2, 3, 4$. Now, consider the case where $e = 0$. For any n, we have $\Gamma_{1,0,0} = (\Delta_1)_{5n} - (\Delta_1)_{5n-5} = -5n - (-5n + 5) = -5$. Finally, we get $m_{1,1} = 1$, $e' = (e - m_{1,1}) \bmod 5 = 4$ and $\Delta_{0,4}^{(5)} = \sum_{h=0}^{4} \Delta_{0,4,h} = \sum_{h=0}^{4} (\Delta_0)_{5n+4-h} = 5$, in accordance with Theorem 6.49.

6.9 Decision procedure

The results of this work can be used in practice to decide if the language L_U associated with any given positional numeration system U is regular. In this section, we describe the semi-decision procedure induced by our results, namely Propositions 6.22, 6.34 and 6.46 and Theorems 6.27, 6.31, 6.39 and 6.49. Below, we will argue that in the situation where the obtained alternate base is Parry, we indeed get a decision procedure.

(1) Using Proposition 6.14, identify the values of p and $\beta_{p-1}, \ldots, \beta_0$.

(2) Compute the infinite words $\mathbf{d}_0, \ldots, \mathbf{d}_{p-1}$ and the graph G. By Proposition 6.22, if one of the infinite words $\mathbf{d}_0, \ldots, \mathbf{d}_{p-1}$ is aperiodic, then L_U is not regular. In this case, the algorithm stops with "no".

(3) Otherwise, study the regularity of the languages $L_0, \ldots, L_{p-1}$.

   (a) For each vertex i with no outgoing edge, use the item (c) of Theorem 6.27 to check whether the corresponding language L_i is regular or not. As soon as one can find a nonregular such language, the algorithm stops with "no".

   (b) Proceed with the vertices i leading to a vertex with no outgoing edge, starting with the vertices at distance 1, then distance 2, etc. For each such vertex i, use the item (c) of Theorem 6.31 to decide whether the corresponding language L_i is regular. As soon as one can find a nonregular such language, the algorithm stops with "no".

   (c) For each cycle in G, first check whether every sequence Δ_i with i in the cycle is ultimately periodic. If at least one of them is not, then the algorithm stops with "no", following Proposition 6.34. Otherwise, find a common period M of these sequences such that Mp is a multiple of $k_{i,r}$ where r is the length of the cycle and i is any vertex of the cycle. Then use the item (d) of Theorem 6.39 to decide whether all languages L_i, with i in the cycle, are regular. If at least one of them is not regular, the algorithm stops with "no".

(d) Finally, consider the vertices i leading to a cycle, starting with the vertices at distance 1, then distance 2, etc. For each such vertex i, use Proposition 6.46 and the item (c) of Theorem 6.49 to decide whether the corresponding language L_i is regular. As soon as one can find a nonregular such language, the algorithm stops with "no".

(4) If the languages L_i for all i in the graph have been checked to be regular, then L_U itself is regular. The algorithm stops with "yes".

Note that, at step (3), we can run the tests in parallel, for each path to either a vertex with no outgoing edge or a cycle. First, we can consider all the vertices with no outgoing edge and the cycles, and then proceed iteratively with the vertices at distance $d \geq 1$ to such vertices or cycles, for increasing values of d until we have considered all vertices of the graph.

The greedy algorithm provides a semi-algorithm for testing if a given expansion of 1 is ultimately periodic: we may simply generate digits t_i and memorize the remainders r_i that appear in the algorithm (to recall, $t_i = \lfloor \beta_{-i} r_{i-1} \rfloor$ and $r_i = \beta_{-i} r_{i-1} - t_i$). If two remainders are equal, then the algorithm loops and the expansion of 1 is ultimately periodic. In particular, we have a semi-algorithm that tests whether a given alternate base is Parry. However, testing when an alternate base is not Parry is harder. Techniques such as those of Section 6.11 can sometimes be applied, but there is no known general algorithm.

If we assume that the eigenvalues of U are known and that the associated alternate base is known to be Parry, Proposition 6.28 and Remarks 6.32, 6.40 and 6.50 ensure that the above procedure can be carried out effectively.

We now illustrate the above decision procedure with two examples, chosen to present a variety of behaviors.

Example 6.52. Let us consider the system U generated by the recurrence relation

$$U_{n+18} = 23U_{n+15} + 5U_{n+12} - 46U_{n+9} - 7U_{n+6} + 23U_{n+3} + 3U_n$$

and the initial conditions

$$(U_0, \ldots, U_{17}) = (1, 4, 9, 20, 70, 175, 489, 1641, 4015, 11294, 37898, 92748, 261291, 876620, 2145176, 6043562, 20275863, 49617086).$$

The minimal polynomial of U is $P(X^3)$ where

$$P(X) = X^6 - 23X^5 - 5X^4 + 46X^3 + 7X^2 - 23X - 3$$
$$= (X+1)^2(X-1)^2\left(X - \frac{23-\sqrt{541}}{2}\right)\left(X - \frac{23+\sqrt{541}}{2}\right).$$

As in the proof of Proposition 6.7, we find $p = 3$ and we obtain closed formulas for U_{3n+2}, U_{3n+1} and U_{3n} from which we find

$$\beta_2 = \frac{19+\sqrt{541}}{15}, \quad \beta_1 = \frac{11+\sqrt{541}}{14} \quad \text{and} \quad \beta_0 = \frac{17+\sqrt{541}}{12}.$$

Using the semi-algorithm described above, we find that the alternate base $\mathcal{B}$ is Parry as we obtain

$$\mathbf{d}_0 = 220^\omega, \quad \mathbf{d}_2 = 21110^\omega \quad \text{and} \quad \mathbf{d}_1 = 310^\omega.$$

The graph G is depicted on Figure 6.8.

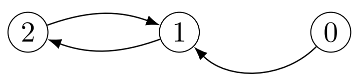


Figure 6.8: The graph G associated with the alternate base B of Example 6.52.

Now we may start studying the regularity of the languages L_i individually. We start with L_1 and L_2. Using Definition 6.30, for $n \geq 2$, we find that

$$(\Delta_1)_n = \begin{cases} -1, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad (\Delta_2)_n = \begin{cases} -1, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

This can be proved by noting that Δ_1 and Δ_2 ultimately satisfy the recurrence relation given by P. We may thus proceed past Proposition 6.34 and verify the criterion (d) in Theorem 6.39 for $i = 1$. Note that this criterion only needs to be checked for one vertex in the cycle. Here we have $M = 2$, $r = 2$, $k_{1,r} = 6$, $M' = 1$, $m_{1,0} = 0$ and $m_{1,1} = 1$. Using Definition 6.38, we can compute the following values, where n is taken large enough to enter the ultimate constant part:

$$\Delta_{1,0}^{(2)} = \Delta_{1,0,0} + \Delta_{1,0,1} = (\Delta_1)_{2n+0-0} + (\Delta_2)_{2n+0-1} = -1 + 2 = 1$$
$$\Delta_{1,1}^{(2)} = \Delta_{1,1,0} + \Delta_{1,1,1} = (\Delta_1)_{2n+1-0} + (\Delta_2)_{2n+1-1} = 1 - 1 = 0.$$

Since $\Delta_{1,e}^{(2)} \geq 0$ for $e \in \{0,1\}$, the languages L_1 and L_2 are both regular.

It remains to check the regularity of the language L_0. Here we find that for $n \geq 1$, we have

$$(\Delta_0)_n = \begin{cases} -1, & \text{if } n \text{ is even;} \\ -6n, & \text{if } n \text{ is odd.} \end{cases}$$

As in the previous paragraph, this can be proved by noting that Δ_0 ultimately satisfies the recurrence relation given by P. So $(\Delta_0)_{n+2} - (\Delta_0)_n$ is ultimately periodic with period 2, and we may proceed past Proposition 6.46 and verify the criterion (c) in Theorem 6.49. Here $M = 2$ satisfies the assumptions of the statement. Using Definition 6.48 and choosing large enough n to be in the constant part, we have

$$\Gamma_{0,0,0} = (\Delta_0)_{2n} - (\Delta_0)_{2n-2} = 0$$
$$\Gamma_{0,1,0} = (\Delta_0)_{2n+1} - (\Delta_0)_{2n-1} = -12.$$

Since $\Gamma_{0,1,0}$ is not zero, it must be equal to $-\Delta_{1,0}^{(2)}$. But this fails to be the case as $\Delta_{1,0}^{(2)} = 1$. Therefore, the language L_0 is not regular, hence the numeration language L_U is not regular either.

Example 6.53. Consider the system U generated by the linear recurrence relation

$$U_{n+13} = -U_{n+12} - U_{n+11} + 22U_{n+10} + 22U_{n+9} + 22U_{n+8} + 13U_{n+7} + 13U_{n+6} + 13U_{n+5} - 10U_{n+4} - 10U_{n+3} - 10U_{n+2}$$

and the initial conditions

$$(U_0, \dots, U_{12}) = (1, 4, 8, 22, 71, 185, 476, 1614, 4179, 10740, 36396, 94271, 242238).$$

The minimal polynomial of U factors as $X^2(X+1)(X^2-X+1)(X^2+X+1)(X^6-23X^3+10)$. Multiplying this polynomial by $X-1$, we see U also satisfies the linear recurrence relation of characteristic polynomial $X^2P(X^3)$, where $P = (X^2-1)(X^2-23X+10)$. In this example, we find $p = 3$ and we obtain closed formulas for U_{3n+2}, U_{3n+1} and U_{3n}, from which the associated alternate base $\mathcal{B} = (\beta_2, \beta_1, \beta_0)$ can be computed. We find

$$\beta_2 = \frac{19+\sqrt{489}}{16}, \quad \beta_1 = \frac{53+\sqrt{489}}{29} \quad \text{and} \quad \beta_0 = \frac{5+\sqrt{489}}{8}.$$

This alternate base is Parry since the corresponding greedy expansions of 1 are

$$\mathbf{d}_0 = 21^\omega, \quad \mathbf{d}_2 = 220^\omega \quad \text{and} \quad \mathbf{d}_1 = 310^\omega.$$

The associated graph is depicted in Figure 6.9.

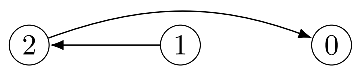


Figure 6.9: The graph G associated with the alternate base of Example 6.53.

We must first decide the regularity of L_0, then L_2, then L_1. For L_0, the expansion $\mathbf{d}_0$ has minimal preperiod $q_0 = 1$ and the minimal period that is a multiple of p is $m_0 = 3$. Using Definition 6.25, we obtain

$$(\Delta_{0,1,3})_n = \begin{cases} -1, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

for all $n \geq 2$. This can be seen from the initial values of this sequence and the fact that it ultimately follows the linear recurrence relation given by P. Lemma 6.26 then gives us $(\Delta_{0,1,6})_n = (\Delta_{0,1,3})_n + (\Delta_{0,1,3})_{n-1} = 0$. The language L_0 is then regular by using the criterion (c) of Theorem 6.27 with $k = 2$.

For L_2 and subsequently L_1, using Definition 6.30, we have

$$(\Delta_2)_n = -1 \qquad \text{and} \qquad (\Delta_1)_n = \begin{cases} 1, & \text{if } n \text{ is even;} \\ -3, & \text{if } n \text{ is odd.} \end{cases}$$

for all $n \geq 2$. Therefore, we may use the criterion (c) of Theorem 6.31 to deduce that L_2, then L_1 are regular. In the end, the numeration language L_U is regular.

6.10 Comments on Hollander's original conjecture

In this section, we go back to Hollander's original conjecture, which was the starting point for this research. We restate it with our notation.

Conjecture 6.54 ([Hol98], Section 8.2)**.** *If L is regular, there exists p such that the limit*

$$\lim_{n\to\infty} \frac{U_{np+i}}{U_{(n-1)p+i}}$$

exists and is independent of i. Furthermore, the minimal polynomial P of the recurrence relation satisfied by U is of the form $P(X) = Q(X^p)$ where Q is the minimal polynomial for a recurrence which gives rise to a regular language.

We should note that in [Hol98] this conjecture is given in the form of a comment rather than a formal statement. Therefore, some parts of the statement are difficult to interpret unambiguously. In particular, it is not clear in Hollander's statement whether P must be the *minimal* polynomial of U, rather than any polynomial associated with a recurrence relation satisfied by U.

The first part of the statement was proven in Proposition 6.7. For the second part of the statement, we give two examples that refute the conjecture as written above and provide a reason to disbelieve the interpretation where P need not be minimal.

Example 6.55. Consider again the system U generated by the recurrence relation $U_{n+3} = 2U_{n+2} - 4U_{n+1} + 8U_n$ and the initial conditions $(U_0, U_1, U_2) = (1, 3, 8)$, which was the fourth item of Example 2.31. It can be seen that L_U is regular in this case, and that $\lim_{n\to\infty} \frac{U_{n+4}}{U_n} = 16$. Nevertheless, the minimal polynomial of U is $X^3 - 2X^2 + 4X - 8$, which is not of the form $Q(X^4)$.

The next example illustrates a more fundamental objection to the conjecture. There exists an alternate base which is Parry but whose product of elements is not Parry (in the nonalternate sense). For such a base, even if its minimal polynomial is of the form $Q(X^p)$, the polynomial Q cannot be the minimal polynomial of a sequence giving rise to a regular language.

Example 6.56. Consider the system U generated by the recurrence relation $U_{n+6} = 9U_{n+3} - 9U_n$ and the initial conditions $(U_0, U_1, U_2, U_3, U_4, U_5) = (1, 2, 3, 9, 15, 24)$. In this case, the numeration language L_U is again regular. However, the associated alternate base is $(3, \varphi, \varphi)$ where φ is the golden ratio, and $3\varphi^2$ is not a Parry number. Therefore, $X^2 - 9X + 9$ is not the minimal polynomial of a recurrence which gives rise to a regular language, independently of the choice of initial conditions. Note that taking multiples of p cannot fix this as no power of $3\varphi^2$ is a Parry number. Indeed, no power of $3\varphi^2$ is a Pisot number and it is known that Parry and Pisot numbers of degree 2 coincide [Bas02].

The latter example does not completely refute Hollander's conjecture in the setting where P is not assumed to be minimal, but for the conjecture to hold there would need to exist a Parry number such that $X^2 - 9X + 9$ is a factor of its Parry polynomial (and similarly for any Parry alternate base whose product of elements is not a Parry number).

6.11 Why obtaining criteria for regularity relying only on recurrence relations satisfied by U is not achievable

The results in Hollander [Hol98] are of a different nature compared to ours. Rather than extracting new sequences from U and deciding the regularity of L_U based on them, Hollander aims to link directly the regularity of L_U with polynomials giving recurrence relations satisfied by U, which is a more tractable criterion. The aim of this section is to present evidence that we cannot replicate this in our setting.

Indeed, in Hollander's case with $p = 1$, the knowledge of the minimal polynomial P of U gives us the value of the dominant root β, from where Hollander's study can take place. In our case however, knowing the minimal polynomial of U only tells us the value of the product of all bases, $\delta = \beta_{p-1} \cdots \beta_0$, but does not inform us on the values of $\beta_{p-1}, \ldots, \beta_0$ themselves. As a result, the behavior of U rarely depends on just the minimal polynomial. We illustrate this by exhibiting a polynomial P and various sets of initial conditions that lead to differing behaviors for U.

Example 6.57. Consider the polynomial $P = X^8 - 2X^6 - 2X^4 - 2$ and the associated recurrence relation $U_{n+8} = 2U_{n+6} + 2U_{n+4} + 2U_n$. The polynomial P has two roots of maximal modulus, which are $-\sqrt{\delta}$ and $\sqrt{\delta}$ where $\delta \simeq 2.80$ is the dominant root of the polynomial $X^4 - 2X^3 - 2X^2 - 2$. Consequently, a generic increasing sequence that satisfies this recurrence relation is associated with an alternate base of length 2.

First, consider the initial conditions $(U_0, \ldots, U_7) = (1, 2, 4, 6, 12, 17, 34, 47)$. We can obtain closed formulas for U_{2n+1} and U_{2n}. From these, we find the alternate base associated with U, which is $(\beta_1, \beta_0) = (2, \delta/2)$. We find $\mathbf{d}_0 = 20^\omega$

and $\mathbf{d}_1 = 10100010^\omega$. In this case, the alternate base is Parry and the associated graph is a cycle of length 2. We find that the numeration language is regular in this case. In fact, the maximal words are exactly the prefixes of the quasi-greedy expansions $\mathbf{d}_0^* = (11010000)^\omega$ and $\mathbf{d}_1^* = (10100001)^\omega$, depending on the parity of their length. To anticipate a bit, this system is 2-Bertrand in the sense of Definition 9.2.

Now, consider the initial conditions $(U_0, \dots, U_7) = (1, 2, 3, 5, 8, 13, 21, 34)$. Similarly, we may obtain values for β_1 and β_0, which are $\beta_1 = \frac{81}{755} + \frac{371}{755}\delta - \frac{28}{755}\delta^2 + \frac{18}{755}\delta^3$ and $\beta_0 = \frac{\delta}{\beta_1} = \frac{226}{119} + \frac{59}{119}\delta + \frac{45}{119}\delta^2 - \frac{25}{119}\delta^3$. We prove that this base is not Parry, as both greedy expansions of 1 are infinite and aperiodic. We prove this by showing that the greedy algorithm started on 1 does not reach the same remainder twice, using an idea of Schmidt ([Sch80, CCK24], see also Chapter 4). We know that $1, \delta, \delta^2$ and δ^3 form a base of $\mathbb{Q}(\delta)$ as a $\mathbb{Q}$-vector space. Considering components in this base, multiplication by any element γ in $\mathbb{Q}(\delta)$ can be represented by a matrix M_γ in $\mathbb{Q}^{4\times 4}$. In particular, we have

$$M_\delta = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

We now consider these matrices M_γ as elements of $\mathbb{C}^{4\times 4}$, so that we can diagonalize them. Let $\delta_1 = \delta, \delta_2, \delta_3, \delta_4$ be the Galois conjugates of δ, with $|\delta_2| > 1$. It is easily seen that for each $k \in \{1, 2, 3, 4\}$, the matrix M_δ admits the eigenvector $v_k = (-2\delta_k - 2\delta_k^2 + \delta_k^3, -2 - 2\delta_k + \delta_k^2, -2 + \delta_k, 1)^T$ with eigenvalue δ_k. For any $\gamma \in \mathbb{Q}(\delta)$, if we decompose $\gamma = a + b\delta + c\delta^2 + d\delta^3$, then $M_\gamma = aI + bM_\delta + cM_\delta^2 + dM_\delta^3$. Therefore, the same vectors v_k are eigenvectors of M_γ, with corresponding eigenvalues $a + b\delta_k + c\delta_k^2 + d\delta_k^3$. Therefore, all matrices M_γ are simultaneously diagonalizable by the matrix $S = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$, hence in particular the matrices M_{β_1} and M_{β_0}.

If we now express the remainders in the greedy algorithm when applied to 1 in the base of the $\mathbb{C}$-vector space $\mathbb{C}^4$ corresponding to the eigenvectors found above, one component corresponds to a Galois conjugate δ_2 of δ which is approximately -1.13. In this component, multiplications by β_1 in the greedy algorithm correspond to a multiplication by $\frac{81}{755} + \frac{371}{755}\delta_2 - \frac{28}{755}\delta_2^2 + \frac{18}{755}\delta_2^3$, which is approximately -0.53, multiplications by β_0 in the greedy algorithm correspond to a multiplication by $\frac{226}{119} + \frac{59}{119}\delta_2 + \frac{45}{119}\delta_2^2 - \frac{25}{119}\delta_2^3$, which is approximately 2.12, and the subtraction of 1 that is sometimes performed

Initial conditions	Behavior
$(1, 2, 3, 5, 9, 15, 25, 40)$	Both expansions are eventually periodic.
$(1, 2, 3, 5, 8, 13, 21, 39)$	One expansion is finite, one is eventually periodic.
$(1, 2, 3, 5, 8, 13, 21, 36)$	One is eventually periodic, one is aperiodic.

Table 6.10: Choices of initial conditions giving rise to differing behaviors in Example 6.57.

between two such multiplications corresponds to adding approximately 0.11. From this, it can be seen that if an absolute value of 3 or more is reached on this component when performing the greedy algorithm, then the value of this component tends to infinity as n does. We can numerically verify that the value of this component after 50 steps of the greedy algorithm for $\mathbf{d}_0$ is about 5.32 and the one for $\mathbf{d}_1$ is about -4.63. As a result, the process as seen in $\mathbb{C}^4$ and in the base of eigenvectors never reaches a periodic point, and neither do the process as seen in the canonical base or the original greedy algorithm. Therefore, both expansions are aperiodic.

Other initial conditions of note are listed in Table 6.10. We have not been able to characterize for which initial conditions the associated alternate base is Parry, which is necessary for regularity.

Chapter 7

Positionality of Dumont–Thomas numeration systems

In Sections 1.4 and 2.3, we have seen how to define a numeration system from a substitution, by factoring a prefix of a periodic point as a product of images of its seed. This kind of numeration system, introduced by Dumont and Thomas and later generalized by Labbé and Lepšová, has some overlap with the positional numeration systems of Chapter 6. We have seen in Example 1.53 how some Dumont–Thomas numeration systems correspond exactly to specific U-system, where others are not even positional. In this chapter, we present a criterion to decide the positionality of a Dumont–Thomas numeration system given by its substitution.

The results in this chapter were obtained in collaboration with Sébastien Labbé and Manon Stipulanti. A short version was published in the proceedings of the 2025 WORDS conference [KLS25a], and a longer version was prepublished on arXiv [KLS25b]. This chapter is an adaptation of these two articles, with the exposition and notation reworked to fit the context of this thesis but with no changes to the results.

Contents

7.1 Introduction **226**

7.2 Reminder on Dumont–Thomas numeration systems . . . **227**

7.3 Sketch of the argument **233**

7.4 Main result **238**
7.5 Particular cases **245**

7.1 Introduction

As we just saw in Chapter 6, regularity is a desirable property and deciding when a numeration system has an associated regular language is not always easy. This has led Lecomte and Rigo to reverse the framework ([LR01], see also [BR10, Chapter 3] for a general presentation). Since positional numeration systems are such that the representation map is increasing for well-chosen orders on $\mathbb{N}$ and A^*, why not define a numeration system directly by specifying the language of the numeration and requiring the representation map to be increasing? Since the radix order is a well-order on A^*, if L is an infinite language there is only one increasing bijection between $(L, <_{\mathrm{rad}})$ and $(\mathbb{N}, <)$. So we promptly define this bijection to be the evaluation map, and its inverse to be the representation map. We have just defined an *abstract numeration system* and, if L is regular, the numeration language of our system must be regular! A simple example is given by the abstract numeration system S built on the language $L = \mathtt{1}^*\mathtt{2}^*$ over the ordered alphabet $\{\mathtt{1}, \mathtt{2}\}$. The first few words in the language are $\varepsilon, \mathtt{1}, \mathtt{2}, \mathtt{11}, \mathtt{12}, \mathtt{22}, \mathtt{111}$. We have for instance that $\mathrm{rep}_S(5) = \mathtt{22}$ and $\mathrm{val}_S(\mathtt{111}) = 6$.

Unfortunately, nothing is ever free, and to obtain something, a thing of equal value must be lost. One chapter ago we were working with positional numeration systems and fighting to establish their regularity, and now we have regular systems, but we are no longer sure that they are positional! For instance, observe that the numeration system built on $\mathtt{1}^*\mathtt{2}^*$ cannot be positional: indeed, we have $\mathrm{rep}_S(3) = \mathtt{11}$ and $\mathrm{rep}_S(5) = \mathtt{22}$, so there is no hope to find an integer sequence $(U_i)_{i \geq 0}$ such that $3 = 1 \cdot U_1 + 1 \cdot U_0$ and $5 = 2 \cdot U_1 + 2 \cdot U_0$ (see also [BR10, Example 3.1.12]). We thus raise the following question (See also [BR10, Exercise 3.13]):

Question 1. *What are the conditions for an abstract numeration system to be positional?*

This question is difficult to answer in its full generality, so we will consider a particular case with Question 2 below. Dumont–Thomas numeration systems, as defined in Sections 1.4 and 2.3, can be seen as a particular subset of abstract numeration systems, due to the correspondence between substitutions and automata implicit between Definitions 1.49 and 1.51 and presented in [RM02, Section 5.1]. These numeration systems were introduced by Dumont and Thomas in 1989 [DT89]. Since then, Dumont–Thomas numeration systems have been used to solve various problems as they provide a nice framework to work with. As the literature is quite vast, we focus on some recent papers from the 2020's only. Generalizations of Dumont–Thomas numeration systems as numeration systems *per se* can be found in [LL24a, Sur20]. In [GRS23, GRS24], classical Dumont–Thomas numeration systems are used to find string attractors for infinite words. In [MRST23], the authors use extensions of Dumont–Thomas numeration systems in the setting of random substitutions. In [MM24], the specific case of the Thue–Morse substitution is used to establish uniform bounds for the twisted correlations for all elements in the Thue–Morse subshift. In this chapter, we exhibit conditions on the underlying substitution for the corresponding Dumont–Thomas numeration for $\mathbb{Z}$ to be positional. We work on $\mathbb{Z}$ to stay in the spirit of the original article [KLS25b] even though most other results in this thesis stay in the context of nonnegative numbers.

The outline of the paper is as follows. In Section 7.2, we recall the necessary background and preliminary results, in particular the generalization of Dumont–Thomas numeration systems to biinfinite periodic points of substitutions. We then study which Dumont–Thomas numeration systems are positional to answer Question 2. We start with a sketch in Section 7.3, then we prove our main result in Section 7.4. We turn to particular cases in Section 7.5 and we finish by discussing the properties of our Dumont–Thomas numeration systems in relation to existing literature, e.g., the property of a numeration system to be Bertrand [BM89, CCS22]

7.2 Reminder on Dumont–Thomas numeration systems

In this section, we give a brief reminder on notation and on the definitions we will use through the rest of the chapter.

We let $\mathbb{D} \in \{\mathbb{N}, \mathbb{Z}_{<0}, \mathbb{Z}\}$ be the *domain* of our numeration system, that is, the set of numbers to be represented.

Definition 7.1. A numeration system over $\mathbb{D} = \mathbb{N}$ is *positional* if the underlying alphabet A is a set of consecutive integers $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{c}\}$ for some $\mathtt{c} \in \mathbb{N}$ and the evaluation map is of the form $\mathrm{val}\colon A^* \to \mathbb{N}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-1} w_i U_i$ for some sequence $U = (U_i)_{i\geq 0} \in \mathbb{N}^{\mathbb{N}}$. Over $\mathbb{D} = \mathbb{Z}$, a numeration system is *positional* if the underlying alphabet A is a set of consecutive integers $\{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{c}\}$ for some $\mathtt{c} \in \mathbb{N}$ and the evaluation map is of the form $\mathrm{val}\colon A^* \to \mathbb{Z}, w_{k-1} \cdots w_0 \mapsto \sum_{i=0}^{k-2} w_i U_i - w_{k-1} V_{k-1}$ for some sequences $U = (U_i)_{i\geq 0}, V = (V_i)_{i\geq 0} \in \mathbb{N}^{\mathbb{N}}$. The sequences U and V are the sequences of *weights* of the numeration system.

Every position has a given weight, while the presence of an additional sequence V helps deal with the representation of negative numbers. This definition is inspired by the usual two's complement numeration system described in the next example.

Example 7.2. The two's complement numeration system allows representations of all integers in a binary system using powers of 2. Let $A = \{\mathtt{0}, \mathtt{1}\}$. The evaluation map is defined as follows: for a word $w = w_{k-1}w_{k-2} \cdots w_0$ over A, we set $\mathrm{val}_{2c}(w) = -w_{k-1}2^{k-1} + \sum_{i=0}^{k-2} w_i 2^i$. Observe now that, for any word w over A, we have $\mathrm{val}_{2c}(\mathtt{00}w) = \mathrm{val}_{2c}(\mathtt{0}w)$ and $\mathrm{val}_{2c}(\mathtt{11}w) = \mathrm{val}_{2c}(\mathtt{1}w)$. So for any integer $n \in \mathbb{Z}$, there exists a unique word $w \in A^* \setminus (\mathtt{00}A^* \cup \mathtt{11}A^*)$ such that $n = \mathrm{val}_{2c}(w)$. We let $\mathrm{rep}_{2c}(n)$ denote this unique word and we call it the *two's complement representation* of n. The first few two's complement representations of integers are $(\mathrm{rep}_{2c}(n))_{-4\leq n\leq 4} = (\mathtt{100}, \mathtt{101}, \mathtt{10}, \mathtt{1}, \varepsilon, \mathtt{01}, \mathtt{010}, \mathtt{011}, \mathtt{0100})$. Observe that the two's complement numeration system is positional: we may use the sequence of powers of 2 as weights.

Dumont–Thomas numeration systems can be defined in two ways, using *admissible sequences* or using the *tree* $\mathcal{T}_{\mu,b|a}$ associated with a substitution μ.

Recall that a *periodic point* of a substitution μ is an infinite word $\mathbf{u}$ such that $\mu^p(\mathbf{u}) = \mathbf{u}$ for some p. The smallest such p is called *the period* of μ and $\mathbf{u}$ is a *fixed point* of μ if its period is 1. The *seed* of $\mathbf{u}$ is u_0 if $\mathbf{u}$ is right-infinite and $u_{-1}|u_0$ if $\mathbf{u}$ is biinfinite. Assuming that those letters are *growing* (that is, $\lim_{n\to\infty} |\mu^n(u_{-1})| = \lim_{n\to\infty} |\mu^n(u_{-1})| = \infty$), we have that $\mathbf{u} = \lim_{n\to\infty} \mu^{pn}(u_{-1}|u_0)$.

Definition 7.3. Let $\mu\colon A^* \to A^*$ be a substitution, $a \in A$ be a letter and $k \in \mathbb{N}$ be an integer. The sequence $((m_i, a_i))_{i=0,\dots,k} \in (A^* \times A)^{k+1}$ is *admissible with respect to* μ if for every $i \in \{1, \dots, k\}$, $m_{i-1}a_{i-1}$ is a prefix of $\mu(a_i)$. This sequence is *a-admissible with respect to* μ if it is admissible with respect to μ and $m_k a_k$ is a prefix of $\mu(a)$.

When the context is clear, we simply say *admissible* or *a-admissible* without specifying the substitution.

Definition 7.4. Let μ be a substitution and a, b be letters in A. The *tree* $\mathcal{T}_{\mu,a}$ is a rooted ordered tree with labeled nodes and edges, defined recursively as follows. The root of the tree is labeled a. Then, if a node has label x and $\mu(x) = y_0 \cdots y_{\ell-1}$, this node has ℓ children labeled $y_0, \dots, y_{\ell-1}$ in order, and the edge from x to y_k is labeled k. A node is said to be *in column n* if there are n nodes at the same depth to its left.

For the two-sided case, the tree $\mathcal{T}_{\mu,b|a}$ has a root node (labeled start in our figures) with a left child labeled b with edge labeled $\mathtt{1}$ and a right child labeled a with edge labeled $\mathtt{0}$, then the construction proceeds as above. A node is said to be *in column n* either if $n \geq 0$ and it is in the right subtree with n nodes of the same depth to its left in this subtree, or if $n < 0$ and it is in the right subtree with $-n-1$ nodes of the same depth to its right in this subtree.

When the context is clear, we drop the dependence on μ.

Example 7.5. Consider the substitution $\mu\colon a \mapsto abc, b \mapsto c, c \mapsto ac$ and the two-sided periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ with growing seed $c|a$. (Note that the period of $\mathbf{u}$ is 1, so it is actually a fixed point.) The first few levels of the corresponding tree $\mathcal{T}_{\mu,c|a}$ are depicted in Figure 7.1. The different columns are numbered below the tree. We have seen (Proposition 1.37) that admissible sequences correspond to paths in the tree. For instance, the c-admissible sequence given by $(m_0, a_0) = (ab, c)$ and $(m_1, a_1) = (\varepsilon, a)$ corresponds to the path going from the left child of the root to the beige node.

We can define one (or more, as we shall see) numeration system from these two definitions. With the point of view of trees, there is a clear correspondence between numbers and words: a path going to column n in the tree will be labeled with a word that can serve as the representation of n.

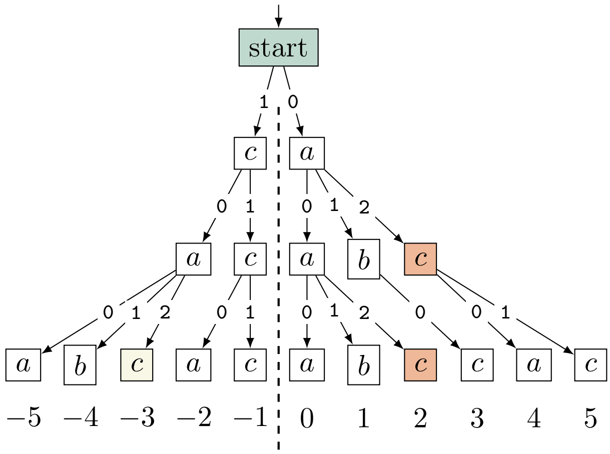


Figure 7.1: The tree $\mathcal{T}_{\mu,c|a}$ associated with the two-sided periodic point $\mathbf{u}$ of the substitution $\mu\colon a \mapsto abc, b \mapsto c, c \mapsto ac$ with growing seed $c|a$.

To guarantee uniqueness, we may choose the shortest such path (up to a question of congruence modulo p to deal with periodic points). On the side of admissible sequences, the property that $m_i a_i$ is a prefix of $\mu(a_{i+1})$ can be used inductively to conclude that $((m_i, a_i))_{i=0,\dots,k-1}$ being a-admissible implies that $\mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2})\cdots\mu^0(m_0)a_0$ is a prefix of $\mu^k(a)$. We can then, with a number n, associate an admissible sequence such that $\mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2})\cdots\mu^0(m_0)a_0$ is of length n, then extract the word $|m_{k-1}|\cdots|m_0|$ to serve as the representation of n (when n is nonnegative, in a way that can be adapted to negative n). The following two theorems then give us conditions to impose on this admissible sequence to obtain uniqueness.

Theorem 7.6 (Theorem 2.33)**.** *Let $\mu : A^* \to A^*$ be a substitution with growing letter $a \in A$. Consider a right-infinite periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{N}}(\mu)$ with $u_0 = a$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \dots, p-1\}$ modulo p and define $\mathbf{v}_r = \mu^r(\mathbf{u})$. For every integer $n \geq 0$, there exist a unique integer $k = k(n)$ with $k \equiv r \bmod p$ and a unique sequence $((m_i, a_i))_{i=0,\dots,k-1}$ such that the sequence is a-admissible,*

$$m_{k-1}m_{k-2}\cdots m_{k-p} \neq \varepsilon \text{ if } k \geq p, \tag{7.1}$$

and $(\mathbf{v}_r)_{[0,n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2})\cdots\mu^0(m_0)$.

Theorem 7.7 (Theorem 2.36)**.** *Let $\mu : A^* \to A^*$ be a substitution with growing letter $b \in A$. Consider a left-infinite periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}_{<0}}(\mu)$ with $u_{-1} = b$ and period $p \geq 1$. Fix a residue $r \in \{0, 1, \dots, p-1\}$ modulo p and define $\mathbf{v}_r = \mu^r(\mathbf{u})$. For every integer $n \leq -1$, there exist a unique integer $k = k(n)$ with $k \equiv r \bmod p$ and a unique sequence $((m_i, a_i))_{i=0,\dots,k-1}$ such that the sequence is b-admissible,*

$$\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b) \text{ if } k \geq p, \tag{7.2}$$

and $(\mathbf{v}_r)_{[-|\mu^k(b)|, n-1]} = \mu^{k-1}(m_{k-1})\mu^{k-2}(m_{k-2}) \cdots \mu^0(m_0)$.

These theorems were proved in Section 2.3. Together, they allow us to define a numeration system as follows.

Definition 7.8. Let $\mu \colon A^* \to A^*$ be a substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $u_{-1}|u_0$ and period $p \geq 1$. Let $r \in \{0, 1, \dots, p-1\}$ be a residue modulo p. Define $\mathtt{c} = \max_{a \in A} |\mu(a)| - 1$ and the set $D = \{\mathtt{0}, \mathtt{1}, \dots, \mathtt{c}\}$. We define the map $\mathrm{rep}_{\mathbf{u},r} \colon \mathbb{Z} \to \{\mathtt{0}, \mathtt{1}\}D^*, n \mapsto \mathrm{rep}_{\mathbf{u},r}(n)$ by

$$\mathrm{rep}_{\mathbf{u},r}(n) = \begin{cases} \mathtt{0} \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \geq 0; \\ \mathtt{1} \cdot |m_{k-1}| \cdot |m_{k-2}| \cdots |m_0|, & \text{if } n \leq -1; \end{cases}$$

where $k = k(n)$ is the unique integer congruent to r modulo p and where $((m_i, a_i))_{i=0,\dots,k-1}$ is the unique sequence obtained from Theorem 7.6 (resp. Theorem 7.7) applied on the right-infinite periodic point $\mathbf{u}|_{\mathbb{N}} = u_0u_1\cdots$ (resp. the left-infinite periodic point $\mathbf{u}|_{\mathbb{Z}_{<0}} = \cdots u_{-2}u_{-1}$) with period p.

This numeration system is called the *Dumont–Thomas complement numeration system associated with μ,* **u** *and r.* When the context is clear, we drop the dependence on μ, **u** and r.

As mentioned, trees give another interpretation of this numeration system.

Proposition 7.9 (Proposition 2.38)**.** *The Dumont–Thomas complement numeration system associated with μ,* **u** *and r can also be defined by letting $\mathrm{rep}_{\mu,\mathbf{u},r}(n)$ be the label of the shortest path of length $r+1 \bmod p$ from the root to column n in the tree $\mathcal{T}_{b|a}$.*

When working only on nonnegative numbers, we drop the leading sign bit from representations, which corresponds to considering the tree $\mathcal{T}_{\mu,a}$ instead of $\mathcal{T}_{\mu,b|a}$.

Example 7.10. Consider the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$. Take the periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{N}}(\mu)$ with growing seed $a|a$ and period $p = 2$. The tree $\mathcal{T}_{\mu,a|a}$ is depicted on Figure 7.2. Now depending on whether we want representations of even or odd lengths, we obtain different numeration systems as illustrated on the table in Figure 7.2.

Consider for instance the even-length (that is, with k odd in Definition 7.8) representation of -3. Although the sequence given by

$$(m_0, a_0) = (\varepsilon, c),\ (m_1, a_1) = (\varepsilon, a) \text{ and } (m_2, a_2) = (cc, d)$$

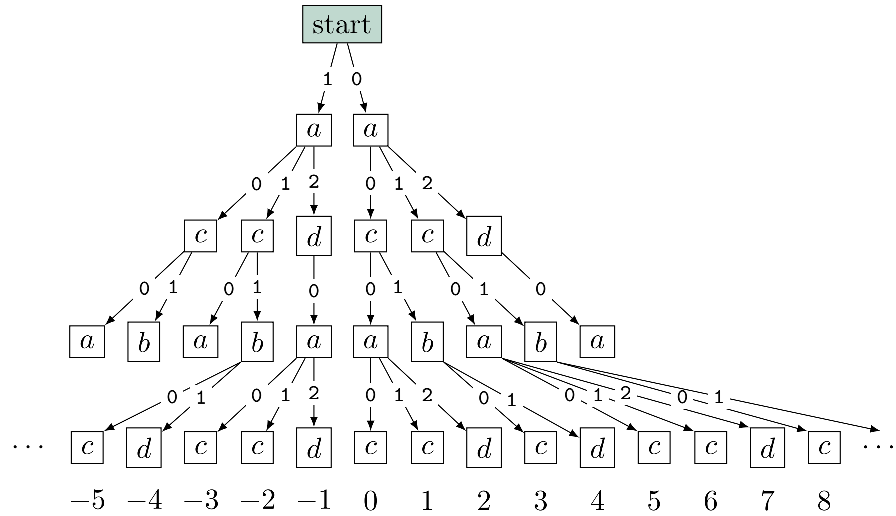
is a-admissible and verifies $\mu^2(m_2)\mu(m_1)m_0 = (\mathbf{v}_1)_{[-13,-4]}$, we have that $\mu(m_2)m_1a_1 = \mu^2(a)$ so it is not the appropriate admissible sequence to represent -3. The correct sequence is the one given by $(m_0, a_0) = (\varepsilon, c)$, where we have $m_0 = (\mathbf{v}_1)_{[-3,-4]} = \varepsilon$. So the even-length representation of -3 is $\mathtt{1}\cdot\mathtt{0}$ rather than $\mathtt{1}\cdot\mathtt{200}$.

As Dumont–Thomas numeration systems are a subset of abstract numeration systems with remarkable structure, it is natural to ask the question of positionality for only those systems. Lepšová did so in her thesis:

Question 2. [Lep24, Question 6.5.7] *What are the conditions for a complement Dumont–Thomas numeration system to be positional?*

This question is by no means easy, as the answer relies on the fine details of the substitution at play rather than, say, only its abelian properties or its eigenvalues.

Example 7.11. [Lep24, Example 6.5.6] Consider the *silver mean* $\beta = 1+\sqrt{2}$. We define the substitutions $\mu\colon a \mapsto aab, b \mapsto a$ and $\rho\colon a \mapsto abb, b \mapsto ab$. The characteristic polynomials of the corresponding adjacency matrices of μ and ρ are equal and have dominant root β. We now consider the two-sided periodic points $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ with growing seed $b|a$ and period 2 and $\mathbf{v} \in \mathrm{Per}_{\mathbb{Z}}(\rho)$ with growing seed $b|a$ and period 1. The representations of the few integers in the corresponding Dumont–Thomas numeration systems with $r = 0$ are given



n	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$\mathrm{rep}_{u,0}(n)$	101	110	111	1	0	001	010	011	020	00100	00101	00110
$\mathrm{rep}_{u,1}(n)$	1111	10	11	12	00	01	02	0010	0011	0100	0101	0102

Figure 7.2: On the top, the tree $\mathcal{T}_{\mu,a|a}$ for the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ and the periodic point $\mathbf{u}$ of period $p = 2$ and seed $a|a$. On the bottom, depending on the residue $r \in \{0,1\}$, we obtain a Dumont–Thomas numeration system and we give $(\mathrm{rep}_{\mathbf{u},r}(n))_{-4\leq n\leq 7}$ whose lengths are congruent to $r+1 \bmod p$.

in Table 7.3. We observe that the first numeration system is positional while the second is not. Indeed, if it were, there would be an evaluation map val such that $\mathrm{val}(\mathrm{rep}_{\mathbf{v},0}(n)) = n$ for all integers n. Since $\mathrm{rep}_{\mathbf{v},0}(3) = \texttt{010}$, we must have $U_1 = 3$, but then $\mathrm{val}(\mathrm{rep}_{\mathbf{v},0}(5)) = \mathrm{val}(\texttt{020}) = 6 \neq 5$, a contradiction.

We will now study when a substitution generates a Dumont–Thomas numeration system that is positional to answer Question 2. We state our theorem in the most general case, then present in Section 7.5 some particularizations as corollaries.

7.3 Sketch of the argument

The aim of this section is to informally sketch the argument that we will use to solve Question 2. We will then present examples where this argument

Table 7.3: The representations of the first few integers in the Dumont–Thomas numeration system associated with $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ and $\mathbf{v} \in \mathrm{Per}_{\mathbb{Z}}(\rho)$ and $r = 0$ where $\mu\colon a \mapsto aab, b \mapsto a$ and $\rho\colon a \mapsto abb, b \mapsto ab$, both with seed $b|a$.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathrm{rep}_{\mathbf{u},0}(n)$	10112	10120	100	101	1	0	001	002	010	011	012
$\mathrm{rep}_{\mathbf{v},0}(n)$	100	101	102	10	1	0	01	02	010	011	020

fails. This will allow us to motivate the technicalities that are introduced in Section 7.4 and to explain the reasoning without these technicalities getting in the way of the explanation.

Sketch 7.12. We let $\mu\colon A^* \to A^*$ be a substitution, $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point of μ with growing seed $b|a$ and period $p \geq 1$, and $r \in \{0, 1, \ldots, p-1\}$ be a residue. We also consider the corresponding Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$ and r.

Now consider two words $wt0^\ell$ and $w(t+1)0^\ell$ that label two paths in the tree $\mathcal{T}_{\mu,b|a}$ (see Figure 7.4, where we have arbitrarily chosen to represent the situation with $\mathcal{T}_{\mu,a}$ instead) and assume these words are the representations of the integers n_1 and n_2 respectively, i.e., the paths end on columns n_1 and n_2, in addition to which we have some conditions for admissibility. We also let c and d be the letters attained after reading wt and $w(t+1)$ respectively.

If the numeration system is positional, then $n_2 - n_1$ must equal U_ℓ, the weight in position ℓ, as the representations of n_1 and n_2 differ only by one unit in position ℓ. However, $n_2 - n_1$ is the number of columns between those two nodes in the tree, which is $\left|\mu^\ell(c)\right|$, i.e., the number of level-ℓ descendants of c in the tree (see again Figure 7.4). Since this reasoning could be applied to any letter c that has a sibling to its right in $\mathcal{T}_{\mu,b|a}$, this points towards the following implication: *If the Dumont–Thomas numeration system associated with μ, $\mathbf{u}$, and r is positional, then $\left|\mu^\ell(c)\right| = U_\ell$ for every letter c that has a younger sibling in $\mathcal{T}_{\mu,b|a}$.*

The converse implication is also justified, almost by the definition of a Dumont–Thomas numeration system. If the positive integer $n \geq 1$ is represented by the word $\mathtt{0} \cdot w_{k-1} \cdots w_0$, this means that there is an a-admissible

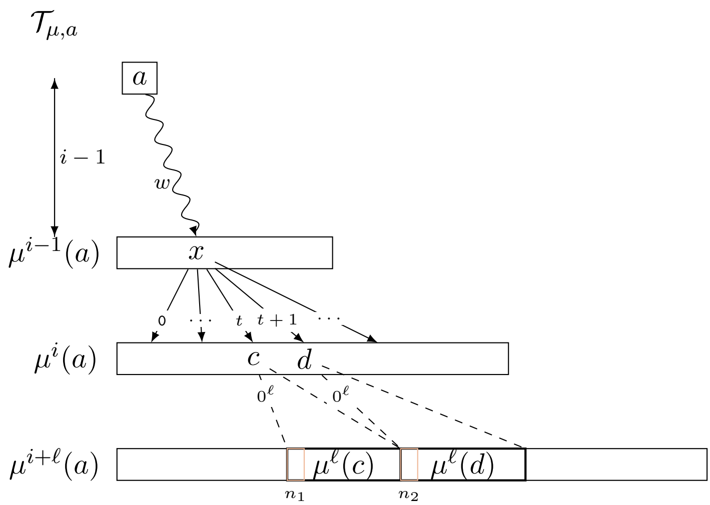


Figure 7.4: Comparing the values of $wt0^\ell$ and $w(t+1)0^\ell$ in the right part of $\mathcal{T}_{\mu,b|a}$.

sequence $((m_i, a_i))_{i=0,\dots,k-1}$ such that

$$\mathbf{u}_{[0,n-1]} = \mu^{k-1}(m_{k-1})\dots\mu^0(m_0) \tag{7.3}$$

and $|m_i| = w_i$ for every $i \in \{0,\dots,k-1\}$. Because all letters in m_i have a_i as a younger sibling, their image by μ^ℓ has length U_ℓ by assumption. Taking the length in (7.3) yields $n = \sum_{i=0}^{k-1} U_i w_i$, which corresponds to the numeration system being positional with weights $(U_i)_{i\geq 0}$. The case of negative numbers is similar, with one correcting term corresponding to the value of V_{k-1}.

The above sketch leads us to formulate the next conjecture: *The Dumont–Thomas numeration system associated with μ, $\mathbf{u}$, and r is positional if and only if $c \mapsto \left|\mu^\ell(c)\right|$ is constant on all letters c that have a younger sibling in $\mathcal{T}_{\mu,b|a}$, in which case this constant is the weight U_ℓ.*

However, trying to prove this conjecture shows two issues with Sketch 7.12, which the following examples highlight.

Example 7.13. Recall the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ from Example 7.10. Observe that μ is not primitive and is built by intertwining on distinct alphabets the Fibonacci substitution $x \mapsto xy, y \mapsto x$ and the substitution $x \mapsto xxy, y \mapsto xy$ associated with the squared golden ratio.

Furthermore, μ has periodic points with period $p = 2$.

The letters that have a younger sibling in $\mathcal{T}_{\mu,a}$ are a and c, but the sequences of the lengths of their consecutive images under μ are respectively

$$(|\mu^j(a)|)_{j\geq 0} = 1, 3, 5, 13, 21, 55, 89, 233, 377, \ldots$$
$$\text{and } (|\mu^j(c)|)_{j\geq 0} = 1, 2, 5, 8, 21, 34, 89, 144, 377, \ldots.$$

Our tentative condition for positionality is not satisfied. Despite this, the corresponding Dumont–Thomas numeration system is positional for both values of r, with the weight sequence $1, 2, 5, 8, 21, 34, \ldots$ for $r = 0$ and $1, 3, 5, 13, 21, 55, \ldots$ for $r = 1$, both of which are obtained by taking two out of every three terms in the Fibonacci sequence.

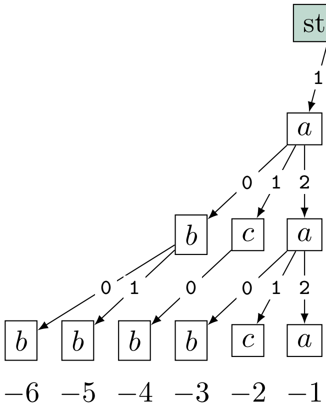
The way to see this is to examine our proof for the converse implication in the above sketch. With the same notation, for $r = 0$ we get that m_i is a power of a if i is even and a power of c if i is odd. Therefore, we obtain the sequence of weights by choosing $U_i = |\mu^i(a)|$ if i is even, and $U_i = |\mu^i(c)|$ if i is odd. Since the letters a and c are never present at the same level of the tree, the difference in image length is not a problem in our case.

Example 7.13 illustrates that, in the case where μ is not primitive, not all letters may appear at every level of the tree, and as such we may only control some of their image lengths. This will be the purpose of the sets E_j in the following section with Definition 7.16.

Example 7.14. Consider the substitution $\mu\colon a \mapsto bca$, $b \mapsto bb$, $c \mapsto b$ and the seed $a|b$. The fragment of the associated numeration system over $\mathbb{Z}^{<0}$ represented in Figure 7.5. The letters b and c both appear in the tree with a younger sibling, and they have images of different lengths. However, the numeration system is still positional. One can show that the weights are $U_i = 2^i$ and $V_0 = 1$, $V_i = 3 \cdot 2^{i-1}$ for every $i \geq 1$ by showing that the language of this numeration system is $\mathtt{1}\{\mathtt{0},\mathtt{1}\}^* \setminus \mathtt{111}\{\mathtt{0},\mathtt{1}\}^*$.

Our sketched argument fails, because if wt is a path to a node labeled by c in the tree, then $w(t+1)0^\ell$ is never the representation of any number, due to Condition (7.2) from Theorem 7.7. Thus we cannot constrain the lengths of the images of c.

The sort of argument of Example 7.14 can happen for letters that only appear in column -2 in $\mathcal{T}_{\mu,b|a}$ as $w(t+1)$ will lead to a node in column -1 and



(a)

n	$\mathrm{rep}_{u,0}(n)$
-1	$1{\cdot}\varepsilon$
-2	1·1
-3	1·0
-4	1·10
-5	1·01
-6	1·00

(b)

Figure 7.5: The tree $\mathcal{T}_{\mu,a|\cdot}$ and the first few representations of negative integers in the numeration system associated with the substitution $\mu\colon a \mapsto bca, b \mapsto bb, c \mapsto b$, the left-infinite periodic point $\mathbf{u} = \cdots bbca$ and residue $r = 0$.

paths extending this one may not correspond to representations of numbers because of Condition (7.2). In fact, such letters may lead to even stranger behavior.

Example 7.15. Consider the eight-letter substitution μ defined by $a_1 \mapsto bca_2$, $f \mapsto b^2$, $a_2 \mapsto a_3$, $b \mapsto d^2$, $c \mapsto d^2e$, $a_3 \mapsto a_1$, $d \mapsto f^2$, $e \mapsto f^4$, and its periodic point $\mathbf{u}$ with seed $a_1|\cdot$ with period $p = 3$ (specifying the right part is not of importance for the sequel). When considering the residue class $r = 2$, this numeration system is not positional as we have $\mathrm{rep}_{\mathbf{u},r}(-6) = \mathtt{100}$, $\mathrm{rep}_{\mathbf{u},r}(-4) = \mathtt{110}$ and $\mathrm{rep}_{\mathbf{u},r}(-1) = \mathtt{120}$, leading to both $U_1 = 2$ and $U_1 = 3$, a contradiction. However, if we change $\mu(c)$ to de instead of d^2e, the numeration system becomes positional for $r = 2$ despite the fact that the lengths of most images under μ of b and c are still different.

These examples motivate the detours and technical details present in the next section, notably Definition 7.17. Keeping these hurdles in mind, we now move on to proving the main result (Theorem 7.19), using the sketch above but proceeding in a more careful manner.

7.4 Main result

To state the main result, we need to define some particular sets of letters. The setting of this section is as follows: we consider a substitution $\mu\colon A^* \to A^*$ and a two-sided periodic point $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ of μ with growing seed $b|a$ and period $p \geq 1$. We draw the tree $\mathcal{T}_{\mu,b|a}$. For a fixed residue $r \in \{0, \dots, p-1\}$, we also consider the corresponding Dumont–Thomas numeration system $\mathrm{rep}_{\mathbf{u},r}\colon \mathbb{Z} \to \{\mathtt{0}, \mathtt{1}\}D^*$ from Definition 7.8.

Definition 7.16. Let $j \in \{0, \dots, p-1\}$. We let E_j be the set of letters $c \in A$ such that there exist some integer $k \geq 1$ and some sequence $((m_i, a_i))_{i=0,\dots,k-1} \in (A^* \times A)^k$ that verify the following:

- the sequence $((m_i, a_i))_{i=0,\dots,k-1}$ is a- or b-admissible;
- we have $k \equiv j \bmod p$;
- the letter c appears at the end of the word m_0;
- if $((m_i, a_i))_{i=0,\dots,k-1}$ is b-admissible, then $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 \neq \mu^k(b)$.

Thinking in terms of the tree $\mathcal{T}_{\mu,b|a}$, the first three conditions simply describe letters that appear in the tree at some level congruent to $j \bmod p$ and have a younger sibling on that level. Going back to Figure 1.2 for visual support if needed, it can be seen that the last condition excludes letters c that only appear in column -2 in the tree $\mathcal{T}_{\mu,b|a}$. To deal with these letters, a dedicated condition is required, as follows. Recall that the jth level of the tree $\mathcal{T}_{\mu,b|a}$ corresponds to the jth iteration of μ on $b|a$ (by convention, the root of $\mathcal{T}_{\mu,b|a}$ is on level -1).

Definition 7.17. Let $j \in \{0, \dots, p-1\}$. On level j in the tree $\mathcal{T}_{\mu,b|a}$, we let c be the letter in column -2 and d be the letter in column -1 (i.e., d is immediately to the right of c). If c and d share the same parent, we consider two cases to modify E_j. If $\left|\mu^{p-j}(d)\right| > 1$, then we add c to E_j if it was not already present. If $\left|\mu^{p-j}(d)\right| = 1$ and $j \leq r < p$, then we add the following condition:

$$\left|\mu^{r-j}(c)\right| \text{ must be equal to } \left|\mu^{r-j}(e)\right| \text{ for every letter } e \in E_j. \tag{7.4}$$

As we will see below, all these values are equal if the numeration system is positional.

Example 7.18. For the substitution μ from Example 7.10, consider its two-sided periodic point with seed $a|a$, so $p = 2$. In this case, we obtain $E_0 = \{a\}$ and $E_1 = \{c\}$. Note that the letters b, d do not have younger siblings in $\mathcal{T}_{\mu,a|a}$. For $j = 1$, we observe that c is on level j and column -2. It also has d as younger sibling and we have $\left|\mu^{p-j}(d)\right| = |\mu(d)| = 1$. Condition (7.4) is trivially satisfied for $r = 1$.

Consider the substitution $\mu\colon a \mapsto bcd$, $d \mapsto ba$, $b \mapsto b^2$, $c \mapsto b$, its two-sided periodic point of period 2 with seed $a|b$ and the residue $r = 0$. If we go only by Definition 7.16, we will find $E_0 = E_1 = \{b\}$, but if we add Definition 7.17, c is added to E_1, and we now correctly find that the system is not positional (which we can also see from the representations of -5, -3 and -2).

We can now state the main result.

Theorem 7.19. *Let $\mu\colon A^* \to A^*$ be a substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $b|a$ and period $p \geq 1$. The Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$, and r is positional if and only if for every $j \in \{0, \ldots, p-1\}$, the map $c \mapsto \left|\mu^{\ell}(c)\right|$ is constant over E_j for every ℓ such that $\ell + j \equiv r \bmod p$, and Condition* (7.4) *is satisfied for the letters where it was added.*

In this case, the sequences U, V of weights of the numeration system are given as follows: for every $\ell \geq 0$, we define $U_\ell = \left|\mu^{\ell}(c)\right|$ for a letter $c \in E_j$ where $j \in \{0, \ldots, p-1\}$ and $\ell + j \equiv r \bmod p$; and $V_\ell = \left|\mu^{\ell}(b)\right|$ for every $\ell \in \mathbb{N}$.

Remark 7.20. It may be, although rarely, that E_j is empty. (For instance, consider the case of $\mu\colon a \mapsto b$, $b \mapsto aa$ and the set E_1.) In this case, if Condition (7.4) applies to some letter c, we may give the weight $\left|\mu^{r-j}(c)\right|$ to position $r - j$. Otherwise, this means that only the digit 0 appears at positions congruent to $j \bmod p$ in this numeration system, and as such the weight given to these positions is arbitrary.

The proof of our main result relies on the following technical lemma. Note that the conditions in the statement are those of Theorems 7.6 and 7.7.

Lemma 7.21. *Let $j \in \{0, \ldots, p-1\}$. Fix a letter c that is in E_j according to Definition 7.16.*

If there exists a b-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ that verifies the four conditions in Definition 7.16, then there exists a b-admissible sequence that verifies those conditions, plus either $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$ or $k < p$.

If there exists an a-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ that verifies the three conditions in Definition 7.16, then there exists an a-admissible sequence that verifies those conditions, plus the extra condition that $m_{k-1} \cdots m_{k-p} \neq \varepsilon$ or $k < p$.

Proof. For the first part, if we assume on the contrary that $k \geq p$ and $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} = \mu^p(b)$, then the fact that $\mathbf{u}$ has period p implies that $a_{k-p} = b$. The cropped sequence $((m_i, a_i))_{i=0,\ldots,k-p-1}$ is thus again b-admissible. We also have $k - p \equiv j \bmod p$ and c is the last letter of m_0. We also have $\mu^{k-p-1}(m_{k-p-1}) \cdots \mu^0(m_0)a_0 \neq \mu^{k-p}(b)$ for otherwise

$$\begin{aligned}\mu^k(b) &= \mu^{k-p}(\mu^p(b)) = \mu^{k-p}(\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})b)\\ &= \mu^{k-1}(m_{k-1}) \cdots \mu^{k-p}(m_{k-p})\mu^{k-p}(b)\\ &= \mu^{k-1}(m_{k-1}) \cdots \mu^{k-p}(m_{k-p})\mu^{k-p-1}(m_{k-p-1}) \cdots \mu^0(m_0)a_0,\end{aligned}$$

which is forbidden by the condition of Definition 7.16. We have showed that the shorter sequence $((m_i, a_i))_{i=0,\ldots,k-p-1}$ also satisfies the condition of the previous definition. Iterating this process, we get the existence of the required sequence.

Similarly, if the sequence is a-admissible with $m_{k-1}m_{k-2} \cdots m_{k-p} = \varepsilon$ and $k \geq p$, then the fact that $\mathbf{u}$ has period p implies that $a_{k-p} = a$. We conclude in the same way: the cropped sequence $((m_i, a_i))_{i=0,\ldots,k-p-1}$ is thus again a-admissible, we also have $k - p \equiv j \bmod p$, and c is the last letter of m_0. □

We now prove our main result.

Proof of Theorem 7.19. We start by assuming that the Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$ and r is positional and we let U, V be its sequences of weights as in Definition 7.1. Fix some

$j \in \{0, \ldots, p-1\}$. We prove that $c \mapsto \left|\mu^\ell(c)\right|$ is constant over E_j for all suitable values of ℓ. Indeed, consider some letter $c \in E_j$. From Definition 7.16 and Definition 7.17, we know that there is some a- or b-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ with $k \equiv j \bmod p$ and where c is the last letter of m_0. Additionally, if the sequence is b-admissible, then we may assume that one of the following holds:

- we have $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 \neq \mu^k(b)$ and one of the two conditions $k < p$ or $\mu^{p-1}(m_{k-1}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$ (by Definition 7.16 and Lemma 7.21);
- we have $\mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0)a_0 = \mu^k(b)$ (by Definition 7.17).

On the other hand, if the sequence is a-admissible, then from Lemma 7.21 we may assume that $m_{k-1} \cdots m_{k-p} \neq \varepsilon$ or $k < p$.

We define two sequences $((m'_n, a'_n))_{n=0,\ldots,k-1+\ell}$ and $((m''_n, a''_n))_{n=0,\ldots,k-1+\ell}$ of elements of $A^* \times A$ as follows. The first sequence is defined by

$$(m'_n, a'_n) = \begin{cases} (m_{n-\ell}, a_{n-\ell}), & \text{if } n \in \{\ell, \ldots, k-1+\ell\}; \\ (\varepsilon, \mu(a'_{n+1})_0), & \text{if } n \in \{0, \ldots, \ell-1\}. \end{cases}$$

For the second sequence, the definitions of m'' and a'' are the same, except at index $n = \ell$ for which we set $m''_\ell c = m'_\ell$ and $a''_\ell = c$. (Note that $m'_\ell = m_0$ ends with c, so m''_ℓ is the prefix of m_0 without its last letter, which we may write $m_0 c^{-1}$.) Thinking in terms of the tree, these sequences correspond to the paths labeled by $w(t+1)0^\ell$ and $wt0^\ell$ from our Sketch 7.12.

Now, notice that the sequence $((m'_n, a'_n))_{n=0,\ldots,k-1+\ell}$ satisfies the conditions of Theorem 7.7 in the b-admissible case and the conditions of Theorem 7.6 in the a-admissible case. Indeed, the only case where this is not trivial is the b-admissible case with $k < p$, in which case the result is obtained by considering

$$\begin{aligned} \left|\mu^{p-1}(m_{k-1}) \cdots \mu^{p-k}(m_0)\right| &\leq \left|\mu^{p-1}(m_{k-1}) \cdots \mu^{p-k}(m_0)\mu^{p-k}(a_0)\right| - 1 \\ &\leq \left|\mu^{p-k}(\mu^k(b))\right| - 1, \end{aligned}$$

where one of the two inequalities must be strict. The same result is true for the sequence $((m''_n, a''_n))_{n=0,\ldots,k-1+\ell}$, with the sole exception of sequences of this form with $k < p$, $m_{k-1} = \cdots = m_1 = \varepsilon$, and $m_0 = c$.

From Definition 7.8, we obtain the following. On the one hand, if the sequences are b-admissible, we get that $\mathtt{1} \cdot |m_{k-1}| \cdots |m_0| \cdot \mathtt{0}^\ell$ is the representation of the integer

$$\left|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0)\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)\right| - \left|\mu^{k+\ell}(b)\right|, \tag{7.5}$$

while $\mathtt{1} \cdot |m_{k-1}| \cdots \left|m_0 c^{-1}\right| \cdot \mathtt{0}^\ell$ is the representation of the integer

$$\left|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0 c^{-1})\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)\right| - \left|\mu^{k+\ell}(b)\right|. \tag{7.6}$$

Subtracting (7.6) from (7.5), we get that $U_\ell = \left|\mu^\ell(c)\right|$ (recall that U_ℓ is the weight in position ℓ in the numeration system). On the other hand, if the sequences are a-admissible, we get that $\mathtt{0} \cdot |m_{k-1}| \cdots |m_0| \cdot \mathtt{0}^\ell$ is the representation of

$$\left|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0)\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)\right|, \tag{7.7}$$

while $\mathtt{0} \cdot |m_{k-1}| \cdots \left|m_0 c^{-1}\right| \cdot \mathtt{0}^\ell$ is the representation of

$$\left|\mu^{k-1+\ell}(m_{k-1}) \cdots \mu^\ell(m_0 c^{-1})\mu^{\ell-1}(\varepsilon) \cdots \mu^0(\varepsilon)\right|. \tag{7.8}$$

Subtracting again (7.8) from (7.7) leads to the same conclusion. In the exceptional case that was singled out above, (7.8) does not hold anymore but (7.7) becomes that $\mathtt{0}^{k-1}\mathtt{1}\mathtt{0}^\ell$ is a representation of $\left|\mu^\ell(c)\right|$, which leads to the same conclusion once more. All in all, for every $c \in E_j$ and every ℓ such that $j + \ell \equiv r \bmod p$, we have shown that $\left|\mu^\ell(c)\right| = U_\ell$, thus the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over E_j as desired.

To end the proof of this implication, we now show that Condition (7.4) is satisfied for the required letters, still in the case of a positional numeration system. We let c be a letter for which the hypotheses of Condition (7.4) are fulfilled. As c is on the jth level of the tree $\mathcal{T}_{\mu,b|a}$, there exists some b-admissible sequence $((m_n, a_n))_{n=0,\ldots,j-1}$ with c being the last letter of m_0 and $\mu^{j-1}(m_{j-1}) \cdots \mu^0(m_0)a_0 = \mu^j(b)$. Since $\left|\mu^{p-j}(a_0)\right| = 1$ and $j \le r < p$, it follows that the sequence

$$(m_{j-1}, a_{j-1}), \ldots, (m_0, a_0), (\varepsilon, \mu(a_0)), \ldots, (\varepsilon, \mu^{r-j}(a_0))$$

is b-admissible and

$$\mu^{r-1}(m_{j-1}) \cdots \mu^0(m_0)\mu^{r-j}(a_0) = \mu^r(b). \tag{7.9}$$

Therefore, the representation of -1 must be $\mathtt{1} \cdot |m_{j-1}| \cdots |m_0| \cdot \mathtt{0}^{r-j}$. On the other hand, we have as above that $\mathtt{1} \cdot |m_{j-1}| \cdots |m_0 c^{-1}| \cdot \mathtt{0}^{r-j}$ is the representation of

$$\left|\mu^{r-1}(m_{j-1}) \cdots \mu^{r-j}(m_0 c^{-1}) \mu^{r-j-1}(\varepsilon) \cdots \mu^0(\varepsilon)\right| - |\mu^r(b)|. \tag{7.10}$$

Utilizing (7.9) to substitute $\mu^r(b)$, the integer in (7.10) is $-1 - \left|\mu^{r-j}(c)\right|$. Comparing the two representations we discussed, we obtain that $\left|\mu^{r-j}(c)\right| = U_{r-j}$. This value is also the value of $\left|\mu^{r-j}(e)\right|$ for every letter $e \in E_j$ as proven above, so Condition (7.4) is satisfied as expected.

We turn to the converse implication. We assume that the map $c \mapsto \left|\mu^{\ell}(c)\right|$ is constant over E_j for all suitable values of j and ℓ and that Condition (7.4) is satisfied, and we prove the positionality of the numeration system, i.e., we find suitable sequences of weights. We let U_ℓ be the constant value imposed on μ^ℓ by the discussed conditions and we also define $V_\ell = \left|\mu^{\ell}(b)\right|$ for every $\ell \in \mathbb{N}$.

We now consider an integer $n \in \mathbb{Z}$ and we divide the argument into three different cases.

Case 1. Assume that n is nonnegative. Then (7.8) implies that

$$\operatorname{rep}_{\mathbf{u},r}(n) = \mathtt{0} \cdot |m_{k-1}| \cdots |m_0| \tag{7.11}$$

for some a-admissible sequence $((m_i, a_i))_{i=0,\ldots,k-1}$ satisfying

$$\mu^k(a)_{[0,n-1]} = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0).$$

In particular, we have that

$$n = \sum_{i=0}^{k-1} \left|\mu^i(m_i)\right|. \tag{7.12}$$

If c is a letter in m_i, we may write $m_i a_i = p_i c s_i$ for some words p_i, s_i and we note that the existence of the sequence $(m_{k-1}, a_{k-1}), \ldots, (p_i c, (s_i)_0)$ means that c is in $E_{k-i \bmod p}$. Since $i + k - i \equiv k \equiv r \bmod p$, we obtain that $\left|\mu^i(c)\right| = U_i$ for any letter c of m_i. As a result, (7.12) yields that $n = \sum_{i=0}^{k-1} |m_i| \, U_i$. Due to (7.11), we obtain that the numeration system is positional with weights $(U_i)_{i \in \mathbb{N}}$ as expected.

Case 2. Assume that n is negative and different from -1. Again Definition 7.8 implies that

$$\operatorname{rep}_{\mathbf{u},r}(n) = \mathtt{1} \cdot |m_{k-1}| \cdots |m_0| \tag{7.13}$$

for some b-admissible sequence $((m_i, a_i))_{i=0,\dots,k-1}$ satisfying

$$\mu^k(b)_{[-|\mu^k(b)|,n-1]} = \mu^{k-1}(m_{k-1}) \cdots \mu^0(m_0).$$

A bit of care is required to prove that all letters in m_i are in $E_{k-i \bmod p}$. Indeed, if c is such a letter with $m_i a_i = p_i c s_i$ for some words p_i, s_i, then we must verify that either we have $\mu^{k-i-1}(m_{k-1}) \cdots \mu^0(p_i c)(s_i)_0 \neq \mu^{k-i}(b)$ or we have an equality but $k-i \leq p$ and $\left|\mu^{p-k+i}(s_i)\right| > 1$. The only case where the equality can hold is if $p_i c = m_i$ and $s_i = a_i$. Next, remember that the sequence $((m_i, a_i))_{i=0,\dots,k-1}$ must satisfy $\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} \neq \mu^p(b)$. Thus, the equality can only occur if $k-i < p$. But in this case, it must be that $\left|\mu^{p-k+i}(a_i)\right| > 1$, otherwise we would have

$$\begin{aligned}
&\mu^{k-i-1}(m_{k-1}) \cdots \mu^0(m_i)a_i = \mu^{k-i}(b) \\
\Rightarrow &\mu^{p-k+i}\left(\mu^{k-i-1}(m_{k-1}) \cdots \mu^0(m_i)a_i\right) = \mu^p(b) \\
\Rightarrow &\mu^{p-1}(m_{k-1}) \cdots \mu^{p-k+i}(m_i)\mu^{p-k+i}(a_i) = \mu^p(b) \\
\Rightarrow &\mu^{p-1}(m_{k-1})\mu^{p-2}(m_{k-2}) \cdots \mu^0(m_{k-p})a_{k-p} = \mu^p(b)
\end{aligned}$$

as $\mu^{p-k+i-1}(m_{i-1}) \cdots a_{k-p}$ is a prefix of length at least 1 of $\mu^{p-k+i}(a_i)$, which has length 1. The last equality we obtain is a contradiction, so $k-i < p$ and $\left|\mu^{p-k+i}(a_i)\right| > 1$. Thus, we have proven that c belongs to $E_{k-i \bmod p}$ in all cases. From there, we get that

$$n = -\left|\mu^k(b)\right| + \sum_{i=0}^{k-1} \left|\mu^i(m_i)\right| = -\left|\mu^k(b)\right| + \sum_{i=0}^{k-1} |m_i|\, U_i,$$

where the second equality follows because all the letters of m_i are in $E_{k-i \bmod p}$. Due to (7.13), we get that the numeration system is positional with sequences U, V of weights as defined above.

Case 3. Only the case where $n = -1$ remains. Here, we have a sequence $((m_i, a_i))_{i=0,\dots,r-1}$ with $\mu^{r-1}(m_{r-1}) \cdots \mu^0(m_0)a_0 = \mu^r(b)$. All the letters of m_i except the last one are guaranteed to be in $E_{k-i \bmod p}$ as in the previous case. For the last letter of m_i, either it is in $E_{k-i \bmod p}$ or Condition (7.4) applies. In any case, we have $\left|\mu^i(c)\right| = U_i$, and the rest of the argument is as for Case 2.

In all three cases, we have shown the numeration system to be positional: for every $i \geq 0$, the weights U_i are defined by the constant value of $c \mapsto \left|\mu^i(c)\right|$ over $E_{r-i \bmod p}$, while the weights V_i are equal to $\left|\mu^i(b)\right|$. This concludes the proof of the main result. □

Remark 7.22. Note that replacing the substitution by one of its powers may lead to the loss of positionality of the corresponding numeration system. For instance, consider the substitution $\mu\colon a \mapsto ccd, b \mapsto cd, c \mapsto ab, d \mapsto a$ from Example 7.10 and its square renamed $\nu\colon a \mapsto ababa, b \mapsto aba, c \mapsto ccdcd, d \mapsto ccd$. From before, we already know that the Dumont–Thomas numeration system associated with μ and the periodic point of μ starting with a of period 2 is positional. However, that associated with ν and its fixed point starting with a is not. Indeed, drawing the first two levels of the tree $\mathcal{T}_{\nu,a}$ we obtain that the representation of 5 is 10 and that of 8 is 20, making it impossible to find a suitable evaluation map for the numeration system to be positional. This is as expected from our Theorem 7.19 as a and b are both in E_0 but have images of different lengths.

7.5 Particular cases

In this section, we highlight some general cases where the technicalities of Section 7.4 do not occur, leading to results that are more concise and legible. We also discuss possible simplifications of the substitution at play, as well as a parallel to Bertrand numeration systems defined in Section 1.5.

From now on, we assume that the alphabet A of the substitution μ is minimal, i.e, all the letters in A are present in $\mu^n(b|a)$ for some n, where $b|a$ designates the seed of the periodic point $\mathbf{u}$ of μ. Given a substitution μ, we say that a letter c is *nonfinal* if there exist $d \in A$, $x \in A^*$, and $y \in A^+$ such that $\mu(d) = xcy$. We let E_μ denote the set of nonfinal letters of μ. When there is no ambiguity, we drop the subscript.

We start with the case where the domain $\mathbb{D}$ is equal to $\mathbb{N}$ and the substitution μ has a fixed point. In this case, the period p is equal to 1 and there is only one sequence of weights, with no particular care given to the most significant digit in a representation.

Corollary 7.23. *Let $\mu\colon A^* \to A^*$ be a substitution and $\mathbf{u} = u_0u_1\cdots$ be a right-infinite fixed point of μ with growing seed a. Then the Dumont–Thomas numeration system (for $\mathbb{N}$) associated with μ, $\mathbf{u}$ and $r = 0$ is positional if and only if the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over E_μ for every ℓ, in which case the sequence of weights is equal to $U_\ell = \left|\mu^\ell(a)\right|$ for every $\ell \geq 0$.*

Proof. Note that since a is the growing seed of a fixed point of μ, we must have $\mu(a) = ay$ for some nonempty word y, so $a \in E_\mu$. Note also that since the domain is $\mathbb{N}$, Definition 7.17 does not apply. It then suffices to apply the proof of Theorem 7.19 restricted to the domain $\mathbb{N}$, taking into account that, since $p = 1$ and $r = 0$ in our case, the set E_μ is equal to the set E_0 of Theorem 7.19, which is the only set for which we have conditions to check. □

Another special case is that of primitive substitutions, even on the complete domain $\mathbb{D} = \mathbb{Z}$. Recall that the definition of primitive substitutions was given in Section 1.1.2, where we mentioned that if μ is primitive then there exists k such that $a \in \mu^\ell(b)$ for all $a, b \in A$ and $\ell \geq k$.

Corollary 7.24. *Let $\mu\colon A^* \to A^*$ be a primitive substitution and let $\mathbf{u} \in \mathrm{Per}_{\mathbb{Z}}(\mu)$ be a two-sided periodic point with growing seed $b|a$ and period $p \geq 1$. The Dumont–Thomas complement numeration system associated with μ, $\mathbf{u}$, and r is positional if and only if the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over E_μ for every $\ell \geq 0$. In this case, for every $\ell \geq 0$, U_ℓ is the constant value of $\left|\mu^\ell(\cdot)\right|$ over E_μ and $V_\ell = \left|\mu^\ell(b)\right|$.*

Proof. Let k be an integer such that, for all letters $c, d \in A$, for all $\ell > k$, c appears in $\mu^\ell(d)$. We then know that every letter appears at every sufficiently large level in the tree $\mathcal{T}_{\mu,a}$. In particular, b appears in $\mu^k(a)$, so $\mathcal{T}_{\mu,b}$ appears as a subtree of $\mathcal{T}_{\mu,a}$. Therefore, any letter that appears with a younger sibling on column -2 also appears with a younger sibling on some column with nonnegative index. As a result, Definition 7.17 can be skipped when determining the sets E_j and Condition (7.4) is never relevantly added to any letter.

Next, we show that, for every $j \in \{0, \ldots, p-1\}$, $E_j = E_\mu$. Fix some $j \in \{0, \ldots, p-1\}$. As the inclusion $E_j \subseteq E_\mu$ is clear by Definition 7.16 we show the other one. If $c \in E_\mu$, there exist $d \in A$, $x \in A^*$, and $y \in A^+$ such that $\mu(d) = xcy$. Since μ is primitive, d appears in $\mu^{j-1+\ell p}(a)$ for some sufficiently large ℓ, so c appears with a younger sibling at some level $j + \ell p$ in the tree $\mathcal{T}_{\mu,b|a}$. This implies that $c \in E_j$, as desired.

Finally, notice that since a is a growing letter of μ, at least one letter must have an image by μ that is of length at least 2, thus E_μ is nonempty.

Now that we have shown that $E_j = E_\mu \neq \emptyset$ for every $j \in \{0, \ldots, p-1\}$ and that Condition (7.4) is not relevant to any letter, applying Theorem 7.19

leads to the statement. □

The substitutions that generate the positional Dumont–Thomas numeration systems discussed in the two corollaries above turn out to have a special form as we will see. This form lends itself well to simplifications and will allow us to link these positional numeration systems to some known families of numeration systems. The following lemma shows that these substitutions are equivalent (in the sense that they generate the same Dumont–Thomas numeration system) to a substitution that contains only one nonfinal letter.

Lemma 7.25. *Let $\mu\colon A^* \to A^*$ be a substitution such that the map $c \mapsto \left|\mu^\ell(c)\right|$ is constant over E_μ for every integer $\ell \geq 0$. Then there exist an alphabet $B \subseteq A$, a substitution $\nu\colon B^* \to B^*$ such that $|E_\nu| = 1$ and some letters $a', b' \in B$ such that the trees $\mathcal{T}_{\mu,b|a}$ and $\mathcal{T}_{\nu,b'|a'}$ differ only by their labeling.*

Proof. Before showing that the statement holds, we start with some notation. For an integer $k \geq 0$ and a letter $a \in A$, we let $\mathcal{T}_{\mu,a}^{\leq k}$ denote the first k levels of the tree $\mathcal{T}_{\mu,a}$ (agreeing that the root is on level 0). Additionally, we extend our definition of trees associated with substitutions to forests associated with substitutions: if $w = w_1 \ldots w_n$ is a word over A, then $\mathcal{T}_{\mu,w}$ is the ordered forest where the first tree is $\mathcal{T}_{\mu,w_1}$, the second tree is $\mathcal{T}_{\mu,w_2}$, and so on. We define similarly $\mathcal{T}_{\mu,w}^{\leq k}$ for any integer $k \geq 0$.

Now we prove the following claim: for every integer $k \geq 0$ and for every pair of letters $c, d \in E_\mu$, the trees $\mathcal{T}_{\mu,c}^{\leq k}$ and $\mathcal{T}_{\mu,d}^{\leq k}$ differ only by their labeling. Proving this for every k will also show it for the entire trees. We proceed by induction on $k \geq 0$. The case $k = 0$ is trivial as we start with letters. The case $k = 1$ is obtained directly from the hypothesis for $\ell = 1$. Indeed, c and d must have the same number of children as they are both in E_μ by assumption. Now, assume that the claim holds for k and let us prove it for $k + 1$. We inductively define $\mu(c) = x_1 c_1$ and $\mu(c_i) = x_{i+1} c_{i+1}$ for every $i \geq 1$, where $x_i \in A^*$ and $c_i \in A$ for every $i \geq 1$ (see Figure 7.6). Similarly, we define $\mu(d) = y_1 d_1$ and $\mu(d_i) = y_{i+1} d_{i+1}$ for every $i \geq 1$ with the same constraints on y_i and d_i. The induction hypothesis ensures that $\mathcal{T}_{\mu,c}^{\leq k}$ and $\mathcal{T}_{\mu,d}^{\leq k}$ only differ by their labeling. Still from the induction hypothesis, the same is true for $\mathcal{T}_{\mu,x_1}^{\leq k}$ and $\mathcal{T}_{\mu,y_1}^{\leq k}$ and, more generally, for $\mathcal{T}_{\mu,x_i}^{\leq k+1-i}$ and $\mathcal{T}_{\mu,y_i}^{\leq k+1-i}$ with $i \in \{1, \ldots, k\}$ (letters of the words x_i, y_i are nonfinal by definition).

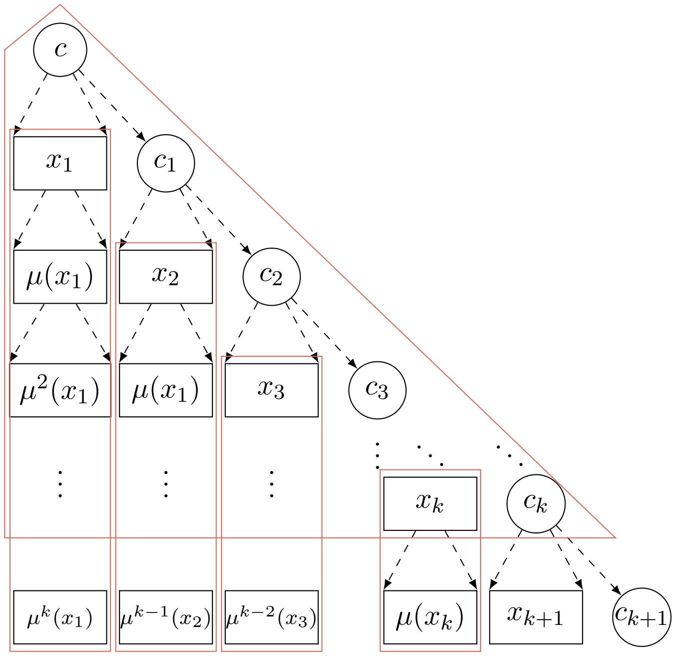


Figure 7.6: In the proof of Lemma 7.25, given a letter c, we inductively define $\mu(c) = x_1c_1$ and $\mu(c_i) = x_{i+1}c_{i+1}$ for every $i \geq 1$, where x_i is a word and c_i is a letter for every $i \geq 1$. We draw the tree $\mathcal{T}_{\mu,c}^{\leq k+1}$ as defined in the proof of Lemma 7.25. Black rectangles correspond to words (note that we do not give information on the length here) and black circles to letters. The induction hypothesis tells us that, up to labeling, every color-framed subtree is equal to its corresponding subtree in $\mathcal{T}_{\mu,d}^{\leq k+1}$.

See Figure 7.6 for a more visual explanation.

As a result, to conclude the proof of the claim, we only need to check that c_k and d_k have the same number of children, i.e., $|\mu(c_k)| = |\mu(d_k)|$. Since

$$\left|\mu^{k+1}(c)\right| = \sum_{i=1}^{k} \left|\mu^{k+1-i}(x_i)\right| + |\mu(c_k)|$$

$$\text{and} \quad \left|\mu^{k+1}(d)\right| = \sum_{i=1}^{k} \left|\mu^{k+1-i}(y_i)\right| + |\mu(d_k)|,$$

we have

$$|\mu(c_k)|=\left|\mu^{k+1}(c)\right|-\sum_{i=1}^{k}\left|\mu^{k+1-i}(x_i)\right| = \left|\mu^{k+1}(d)\right|-\sum_{i=1}^{k}\left|\mu^{k+1-i}(y_i)\right| = |\mu(d_k)|\,,$$

where the second equality results from our hypothesis (all the letters in x_i and y_i belong to E_μ, as do c and d). Thus the claim is proved.

Now that we have proven that $\mathcal{T}_{\mu,c}$ and $\mathcal{T}_{\mu,d}$ only differ by their labeling, we may replace every instance of the letter d in an image of μ by the letter c and this will only change the labeling of the tree $\mathcal{T}_{\mu,b|a}$, not its shape. If we proceed like this for every letter d in $E_\mu \setminus \{c\}$ and then remove any such superfluous letter from the alphabet A, we will have constructed the desired substitution ν. □

Example 7.26. Consider the primitive substitution $\mu\colon a \mapsto ab, b \mapsto ba$ (often referred to as the Thue–Morse substitution). We note that both a, b are nonfinal letters and $\left|\mu^\ell(c)\right| = 2^\ell$ for $c \in \{a, b\}$ and for every $\ell \geq 0$. Applying our result gives the substitution $\nu\colon a \mapsto a^2$. The corresponding Dumont–Thomas numeration system (over $\mathbb{N}$) is the usual binary system.

As a consequence, when considering positional Dumont–Thomas numeration systems associated with either primitive substitutions or fixed points on the domain $\mathbb{N}$ (namely, the assumptions of Corollaries 7.23 and 7.24) we may look only at substitutions having one single nonfinal letter e by Lemma 7.25. In such a substitution, the image of a letter is determined by the number of repetitions of the nonfinal letter e and the choice of final letter.

If we let a be the seed of the periodic point $\mathbf{u}$, it must be that $\mu^\ell(a)$ starts with e for some ℓ. In order for $\mathbf{u}$ to be a periodic point, we must have either $e = a$ (Case 1) or a chain of letters that all have images of length 1, say $e \mapsto b_1 \mapsto b_2 \mapsto \cdots \mapsto b_k \mapsto a$ (Case 2). In all other cases, no iterated image of e starts with a, and $\mathbf{u}$ cannot be a periodic point as a result.

In what follows, we will try to link these substitutions to positional (and in particular Bertrand) numeration systems. These results were already mentioned in Section 1.5 but the proofs were delayed as they appeared in this article. Note that in positional numeration systems, representations can have any length. Thus we should consider $p = 1$ for a link to appear, which corresponds to Case 1 above, with the nonfinal letter being equal to the seed, $e = a$. In this case, with the extra condition to operate on the minimal alphabet, the substitution is equivalent to one of the form

$$\mu : a_1 \mapsto a_1^{d_1} a_2,\ a_2 \mapsto a_1^{d_2} a_3,\ \ldots,\ a_n \mapsto a_1^{d_n} a_{k+1}, \tag{7.14}$$

where n is a nonnegative integer, $\{a_1, \ldots, a_n\}$ is the alphabet of the substitution, $k \in \{0, \ldots, n-1\}$, $d_1 > 0$ and d_i is a nonnegative integer for every $i \in \{1, \ldots, n\}$. An example of such a substitution is the Fibonacci (resp. Tribonacci) substitution given by $\varphi\colon a \mapsto ab, b \mapsto a$ (resp. $\tau\colon a \mapsto ab, b \mapsto ac, c \mapsto a$), for which $n = 2$, $a_1 = a$, $a_2 = b$, $d_1 = 1$, $d_2 = 0$, and $k = 0$ (resp. $n = 3$, $a_1 = a$, $a_2 = b$, $a_3 = c$, $d_1 = d_2 = 1$, $d_3 = 0$, and $k = 0$). See also their generalization to *generic n-bonacci* substitutions in [Rig14b, Ex 2.11].

The similarity with the substitutions studied by Fabre in [Fab95] is striking, so we will call *Fabre-like* the substitutions of the form (7.14). Also note that the substitutions of Fabre correspond to the μ_β of Definition 1.51. We will now study this particular class of substitutions in additional detail. First, we note the following property.

Proposition 7.27. *Let μ and μ' be Fabre-like substitutions and consider their respective fixed points $\mathbf{u}$ and $\mathbf{u}'$ with respective seeds a and a'. If the Dumont–Thomas numeration systems over $\mathbb{N}$ corresponding to $\mathbf{u}$ and $\mathbf{u}'$ respectively and $r = 0$ have the same sequence of weights, then every natural number has the same representation in both numeration systems.*

In a sense, the weight sequence completely determines the numeration system provided that it is based on a Fabre-like substitution. In what follows, we will call two numeration systems over $\mathbb{N}$ *equal* if every natural number is represented by the same word in both systems. Note that this property does not hold in general as shown in the next example.

Example 7.28. Consider the substitutions $\mu\colon a \mapsto bcd$, $b \mapsto ef$, $c \mapsto e^2g$, $d \mapsto d$, $e \mapsto ad$, $f \mapsto a^2d$, $g \mapsto a$ and μ' defined in the same way but with $\mu'(a) = cbd$ instead of bcd. Then, if $\mathbf{u}$ and $\mathbf{u}'$ are the periodic points of period $p = 3$ starting with a of μ and μ' respectively and $r = 0$, the corresponding Dumont–Thomas numeration systems associated with $\mathbf{u}$ and r and with $\mathbf{u}'$ and r have the same sequence of weights, but the representations of some numbers differ: for example, we have $\mathrm{rep}_{\mathbf{u},0}(4) = \texttt{012}$, $\mathrm{rep}_{\mathbf{u}',0}(4) = \texttt{020}$, $\mathrm{rep}_{\mathbf{u},0}(9) = \texttt{120}$, and $\mathrm{rep}_{\mathbf{u}',0}(9) = \texttt{112}$.

Sketch of the proof of Proposition 7.27. From the equality of the weight sequences, we deduce that $|\mu^\ell(a)| = |\mu'^\ell(a')|$ for every integer ℓ as $a \in E_\mu$ and $a' \in E_{\mu'}$. Then, we prove that, for every $k \geq 0$, the pair of trees $\mathcal{T}_{\mu,a}^{\leq k}$ and $\mathcal{T}_{\mu',a'}^{\leq k}$ differ only by their labeling, using the same method as for the claim in Lemma 7.25. Since the two trees differ only by their labeling, $\text{rep}_{\mathbf{u},0}(n) = \text{rep}_{\mathbf{u}',0}(n)$ for every natural number n. □

We now recall the central notions around Bertrand numeration systems. See also Section 1.5.

With a Rényi numeration system with $d_\beta^*(1) = d_1 d_2 \cdots$, we can associate a sequence of weights by setting $U_0 = 1$ and

$$U_i = d_1 U_{i-1} + d_2 U_{i-2} + \cdots + d_i U_0 + 1$$

for all $i \geq 1$. This numeration system is called the *canonical Bertrand numeration system* associated with β, after Bertrand-Mathis who defined it in [BM89]. Later, Charlier, Cisternino and Stipulanti in [CCS22] added more systems to the family of Bertrand numeration systems. If β is a simple Parry number with $d_\beta(1) = t_1 \cdots t_\ell 0^\omega$, and setting $t_i = 0$ for $i > \ell$, we can define the *noncanonical* Bertrand numeration system associated with β by setting $U_0 = 1$ and

$$U_i = t_1 U_{i-1} + t_2 U_{i-2} + \cdots + t_i U_0 + 1$$

for all $i \geq 1$. Finally, the *trivial* Bertrand numeration system is the one defined by $U_i = i + 1$ for all i.

Bertrand numeration systems enjoy a variety of properties, such as having the same set of factors as the language of the Rényi numeration system with base β. Two other notable properties of such numeration systems are the following (see also Definition 1.44, where we used them as the definition of those systems instead):

Property 7.29. *A word w is the representation of some natural number if and only if $w0$ is.*

Property 7.30. *The lexicographically greatest representations of every length are all prefixes of one another.*

Between Bertrand-Mathis' paper and Charlier and coauthors' additions, Fabre [Fab95] introduced another way to view canonical Bertrand numeration systems. If β is a Parry number with $d^*_\beta(1) = d_1 \cdots d_n(d_{n+1} \cdots d_{n+m})^\omega$ for some m, n, we introduce the substitution

$$\mu_\beta \colon 1 \mapsto 1^{d_1}2, \ldots, (n+m-1) \mapsto 1^{d_{n+m-1}}(n+m),\ (n+m) \mapsto 1^{d_{n+m}}(n+1). \tag{7.15}$$

Note that Fabre defines his substitutions not by using the word $d^*_\beta(1)$ like we do but using the word $d_\beta(1)$. Still, the substitution in (7.15) is the same as the one defined by Fabre. The case where $d_\beta(1)$ is finite corresponds to the case where $n = 0$ with our notation.

Fabre then shows that $|\mu^\ell_\beta(1)|$ is the integer U_ℓ defined for the canonical Bertrand numeration system, and his [Fab95, Theorem 2] establishes the equality between the Dumont–Thomas numeration system associated with μ_β and the canonical Bertrand numeration system associated with β.

As an aside, we note that the term "conjugate substitutions" used by Fabre in [Fab95, Section 3.1] can be thought of in terms of the trees associated with those substitutions. Following Fabre's notation, for two substitutions $\mu \colon A^* \to A^*$ and $\nu \colon B^* \to B^*$ with seeds $a \in A$ and $b \in B$ respectively, we write $\mu \to \nu$ if there exists a morphism $h \colon A^* \to B^*$ such that $h(a) = b$ and $h(\mu(c)) = \nu(h(c))$ for every $c \in A$. We then say that μ and ν are *conjugates* if there exists a third substitution τ such that $\tau \to \mu$ and $\tau \to \nu$.

Proposition 7.31. *Two substitutions $\mu \colon A^* \to A^*$ and $\nu \colon B^* \to B^*$ with respective seeds a and b are conjugate exactly when the trees $\mathcal{T}_{\mu,a}$ and $\mathcal{T}_{\nu,b}$ are equal up to relabeling.*

Proof. On the one hand, if the trees $\mathcal{T}_{\mu,a}$ and $\mathcal{T}_{\nu,b}$ are equal up to relabeling, we may construct a tree with the same graph structure but with labels in $A \times B$ obtained by simply joining the labels of the corresponding nodes in the two trees. Then, this new tree gives a substitution $\mu \times \nu$ on $A \times B$ that is such that $\mu \times \nu \to \mu$ and $\mu \times \nu \to \nu$, which implies that μ and ν are conjugate.

On the other hand, if $\tau \to \mu$ for some substitution τ, applying the morphism h on the label of every node in the tree associated with τ gives the tree associated with μ (which must have the same shape since we only have changed the labels). By applying this result twice, we conclude that the trees associated with two conjugate substitutions have the same shape and differ only by their labeling. $\square$

With this recalled, let us go back to the study of Dumont–Thomas numeration systems based on Fabre-like substitutions. We first note that all Bertrand numeration systems associated with Parry numbers form a particular case of Dumont–Thomas numeration systems (and not just the canonical ones as proven by Fabre).

Proposition 7.32. *Every Bertrand numeration system associated with a Parry number is equal to some Dumont–Thomas numeration system associated with a Fabre-like substitution.*

Proof. It is clear that the substitution μ_β in (7.15) introduced by Fabre is Fabre-like when β is a Parry number (note that the first digit of $d^*_\beta(1)$ is nonzero, so $d_1 > 1$ as expected). Therefore, the canonical Bertrand numeration systems can all be seen as special cases of our Dumont–Thomas numeration systems.

The noncanonical Bertrand numeration systems introduced in [CCS22] also fall in this framework. If $d^*_\beta(1) = (d_1 \cdots d_m)^\omega$, we create μ'_β from μ_β by adding the letter $m+1$ to the alphabet, setting $\mu'_\beta(m) = 1^{d_m+1}(m+1)$ instead of $\mu_\beta(m) = 1^{d_m+1}$, and setting $\mu'_\beta(m+1) = (m+1)$. In a sense, this corresponds to improperly setting $d^*_\beta(1) = d_1 \cdots d_{m-1}(d_m+1)0^\omega$, then defining a Fabre substitution from this ultimately periodic expansion.

Since the trivial Bertrand numeration system based on the weight sequence $(i+1)_{i\geq 0}$ is the Dumont–Thomas numeration system associated with the substitution $\mu\colon a \mapsto ab$, $b \mapsto b$, which is Fabre-like, we conclude the proof of the statement. □

Although our Dumont–Thomas numeration systems clearly verify Properties 7.29 and 7.30 that characterize Bertrand numeration systems, the converse of Proposition 7.32 is not true, as we see in the following example.

Example 7.33. Consider the Fabre-like substitution $\mu\colon a \mapsto aab$, $b \mapsto aaaa$ with fixed point $\mathbf{u} = aabaabaaaa\cdots$. The corresponding Dumont–Thomas numeration system is positional, with the sequence of weights starting by $1, 3, 10, 32, \ldots$. We note that $\mathrm{rep}_{\mathbf{u},0}(9) = \mathtt{23}$, but this cannot happen in a Bertrand numeration system as the representation of 9 would be $\mathtt{30}$ with the given sequence of weights.

The catch, of course, is that Dumont–Thomas numeration systems, while positional, do not represent numbers by applying a greedy algorithm on the sequence of weights, but rather by applying a greedy algorithm on the factorization of their fixed point, which can lead to different results. This can be understood with an adaptation of the *Parry condition*, which governs what words can be expansions of 1 in Rényi numeration systems. To recall, this condition (first seen in [Par60, Corollary 1] and presented as Theorem 1.25 in this thesis) states that a word $d_1 d_2 \cdots$ is equal to $d_\beta(1)$ for some $\beta > 1$ if, and only if, $d_1 d_2 \cdots >_{\mathrm{lex}} 10^\omega$ and $d_1 d_2 \cdots >_{\mathrm{lex}} d_i d_{i+1} \cdots$ for every $i \geq 1$. In the case of Example 7.33, if the substitution μ were the Fabre substitution associated with some Parry number β, we would have $d^*_\beta(1) = (23)^\omega$ and $d_\beta(1) = 240^\omega$, which contradicts the Parry condition. Thus the system cannot be equal to a Bertrand numeration system.

In fact, the Parry condition (adapted for use with $d^*_\beta(1)$ instead of $d_\beta(1)$) is all that is necessary to guarantee that the system is greedy and therefore equal to a Bertrand numeration system, as we will see in the following result.

Proposition 7.34. *Let μ be a Fabre-like substitution as in* (7.14)*. Construct the word $d_1 d_2 \cdots = d_1 \cdots d_k (d_{k+1} \cdots d_n)^\omega$. The Dumont–Thomas numeration system associated with μ and the seed a_1 is equal to a Bertrand numeration system if and only if we have $d_i d_{i+1} \cdots \leq_{\mathrm{lex}} d_1 d_2 \cdots$ for each $i \geq 1$.*

Proof. Assume first that $d_i d_{i+1} \cdots >_{\mathrm{lex}} d_1 d_2 \cdots$ for some $i \geq 1$. In this case, we must have $d_i \cdots d_{j-1} >_{\mathrm{lex}} d_1 \cdots d_{j-i}$ for some i, j. However, it is not hard to see that the lexicographically greatest representations of length j and $j-i$ in our numeration system must be $d_1 \cdots d_j$ and $d_1 \cdots d_{j-i}$ respectively. Since the suffix of length $j-i$ of the former word is lexicographically greater than the latter, our system cannot be equal to a greedy positional numeration system (it is not suffix-closed), and therefore cannot be equal to a Bertrand numeration system.

For the other direction, assume that

$$d_i d_{i+1} \cdots \leq_{\mathrm{lex}} d_1 d_2 \cdots, \quad \forall i \geq 1. \tag{7.16}$$

holds. We consider two cases depending on the periodicity of $d_1 d_2 \cdots$.

As a first case, if $d_1 d_2 \cdots$ is not purely periodic, the inequality sign can be replaced by a strict inequality in (7.16) (which corresponds to the Parry condition). If $d_1 d_2 \cdots = 10^\omega$, then $\mu\colon a \mapsto ab$, $b \mapsto b$ and our Dumont–

Thomas system is equal to the trivial Bertrand numeration system associated with weights $(i+1)_{i\geq 0}$. If $d_1d_2\cdots$ ends with 0^ω and is not 10^ω, then $d_1d_2\cdots$ is equal to $d_\beta(1)$ for some simple Parry number β. Then, μ is equal to the Fabre substitution μ'_β modified to fit the noncanonical Bertrand system (as in the proof of Proposition 7.32) and our Dumont–Thomas numeration system is equal to the noncanonical Bertrand numeration system associated with β. Finally, if $d_1d_2\cdots$ does not end in 0^ω, then it is equal to $d_\beta(1) = d^*_\beta(1)$ for some nonsimple Parry number β. In this case, $\mu = \mu_\beta$ and our Dumont–Thomas numeration system is equal to the canonical Bertrand numeration system associated with β.

As a second case, if $d_1d_2\cdots$ is purely periodic with minimal period ℓ and is equal to $(d_1\cdots d_\ell)^\omega$, then $d_1\cdots d_{\ell-1}(d_\ell+1)0^\omega$ also verifies (7.16), this time with a strict inequality (see [CCS22, Lemma 4]). This new word is equal to $d_\beta(1)$ for some simple Parry number β, thus $d_1d_2\cdots$ is equal to $d^*_\beta(1)$ for the same β, our μ is the associated Fabre substitution, and our numeration system is the canonical Bertrand numeration system associated with β.

In all cases, we have shown that the Dumont–Thomas numeration system associated with μ is equal to some Bertrand numeration system, as desired.

□

Chapter 8

Additional results

This chapter contains two additional results than were produced when studying the contents of Chapter 6. They were separated since they do not contribute directly to this already long chapter, but are also too small to deserve their own chapter. The first is a result on linear recurrence sequences, examining under what conditions the limits of quotients of consecutive terms in these sequences exist "p steps by p steps". The results of this first section were obtained with Émilie Charlier and were sent for submission to Uniform Distribution Theory without prepublication. The second is a study of the initial conditions that produce regularity, in the setting of Hollander and in the case where he doesn't produce a criterion. These results were obtained with Émilie Charlier and remain unsubmitted.

Contents

8.1 On the Kepler limit for sequences without a dominant root **258**

8.1.1 Introduction and statement of main result 258

8.1.2 Proof of Theorem 8.1 260

8.2 Initial conditions where L_U is regular when U has a dominant root **266**

8.2.1 Introduction 266

8.2.2 A reminder on Hollander's and our results 267

8.2.3 Main results 270

8.2.4 Examples 273

8.1 On the Kepler limit for recurrence sequences without a dominant root

8.1.1 Introduction and statement of main result

Linear recurrence sequences, chief among them the Fibonacci sequence, have applications in diverse fields of science [Les45, FT87, Jag21]. In those applications, the ratio $\frac{U_{n+1}}{U_n}$ and its limit as n tends to infinity, where (U_n) is the linear recurrence sequence in question, is of frequent relevance. This limit is sometimes called the *Kepler limit* [FV11, BK22, BK25], after Kepler who computed it for the Fibonacci sequence. The article [FV11] characterizes sequences for which this limit exists.

However, the Kepler limit is not sufficient for all applications. In this thesis, especially in Chapter 6, we have worked with linear recurrence sequences that do not admit a Kepler limit. See also [CV73] where the dynamics of a biological population are such that the Kepler limit does not exist. In these cases, we must instead consider limits of the form

$$\lim_{n\to\infty} \frac{U_{n+p}}{U_n}, \tag{8.1}$$

or of the form $\lim_{n\to\infty} \frac{U_{pn+i+1}}{U_{pn+i}}$ which we will see convey similar information. In this chapter, we provide a full characterization for the existence of the limit (8.1) when $(U_n)_{n\geq 0}$ is a linear recurrence sequence of complex numbers, provided that the sequence of moduli $(|U_n|)_{n\geq 0}$ is eventually increasing. In our setting, this will be the case as base sequences for positional numeration systems are assumed to be increasing. This characterization has a very simple form, and is based on the roots of the minimal polynomial of the sequence $(U_n)_{n\geq 0}$. Whenever $p = 1$, we obtain again the case of the Kepler limit. In this case, the hypothesis on the increasing moduli is not needed.

We will work with the notions related to linear recurrence sequences (minimal polynomial, eigenvalues, multiplicity,...) that were defined near the beginning of Section 1.2. We also refer the reader to [BR11] for more details on linear recurrence sequences, and rational series in general. A linear recurrence sequence over K is said to be *strict* if it has only nonzero eigenvalues. (We note that this is the setting considered in [FV11].) As we are only concerned with an asymptotic property here, we do not restrict ourselves to strict linear recurrence sequences. Any linear recurrence sequence that is not ultimately zero has at least one nonzero eigenvalue, and hence, our result will be stated in terms of the nonzero eigenvalues only.

Our proof technique mostly relies on ideas developed in [FV11]. However, our result is based on the *minimal polynomial* of the sequence $(U_n)_{n\geq 0}$ and not on some *characteristic polynomial* satisfied by this sequence. Our point of view is therefore quite different from the one presented in [FV11]. Let us explain why in some more detail. In their paper, Fiorenza and Vincenzi consider a given polynomial of some prescribed degree k with a nonzero constant term, together with a list of k initial conditions. These data completely determine a linear recurrence sequence, and the considered polynomial is called the characteristic sequence of the recurrence relation satisfied by the obtained sequence. In general, it may be that this sequence also satisfies other linear recurrence relations with characteristic polynomials of smaller degrees. If this is the case, then some roots of the initially considered polynomial will be useless in the closed-formed expression of the sequence. In this paper, we take a reversed perspective. We start with a given linear recurrence sequence $(U_n)_{n\geq 0}$ of complex numbers, and we work with the unique monic polynomial P_U with the smallest possible degree which is the characteristic polynomial of a recurrence relation satisfied by $(U_n)_{n\geq 0}$. This polynomial P_U indeed depends on the full sequence $(U_n)_{n\geq 0}$, in the sense that for another set of initial conditions, the sequence obtained by following the same recurrence relation might satisfy another recurrence relation of smaller order. So we cannot speak of *agreement* between the initial conditions and the minimal polynomial as it is done in [FV11].

Our main result is the following one.

Theorem 8.1. *Let $(U_n)_{n\geq 0}$ be a linear recurrence sequence of complex numbers that is eventually nonzero and let p be a positive integer. Let $\alpha_1, \ldots, \alpha_d$ be the eigenvalues of $(U_n)_{n\geq 0}$ with maximal multiplicity among the eigenvalues with maximal modulus. Consider the following assertions.*

(a) The limit $\lim_{n\to\infty} \frac{U_{n+p}}{U_n}$ *exists.*

(b) The p limits $\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}}$ *exist, for* $i \in \{0, \ldots, p-1\}$.

(c) We have $\alpha_1^p = \alpha_2^p = \cdots = \alpha_d^p$.

We have (b)⇒(a) and (a)⇒ (c). If the sequence $(|U_n|)_{n\geq 0}$ *is eventually increasing or if* $p=1$*, then we also have (c)⇒(b). Furthermore, if (a) holds, then the limit is* α_1^p*.*

Let us emphasize that for $p \geq 2$, the additional condition that $(|U_n|)_{n\geq 0}$ is eventually increasing is necessary in order to obtain the implications (c)⇒(a) and (c) ⇒(b). Consider the sequence given by the recurrence relation $U_{n+3} = U_{n+2}+4U_{n+1}-4U_n$ for $n \geq 0$ and the initial conditions $U_0 = 3, U_1 = 1, U_2 = 9$. The minimal polynomial of this sequence is $(X^2-4)(X-1)$ and we have the closed formula $U_n = 2^n + (-2)^n + 1$ for all $n \geq 0$. This sequence satisfies the condition (c) of Theorem 8.1 with $p = 2$, but $\frac{U_{2n+3}}{U_{2n+1}} = 1$ for all n and $\frac{U_{2n+2}}{U_{2n}}$ converges to 4 as n goes to infinity. Thus, the condition (a) is not satisfied, and hence neither is the condition (b). In fact, as n tends to infinity, the quotients $\frac{U_{2n+1}}{U_{2n}}$ and $\frac{U_{2n+2}}{U_{2n+1}}$ respectively converge to 0 and to infinity.

Whenever $p = 1$, that is, whenever (8.1) is the Kepler limit, the condition (c) of Theorem 8.1 reduces to say that $d = 1$. In particular, it can be restated as follows, which, in essence, is the main result of Fiorenza and Vincenzi from [FV11].

Corollary 8.2. *Let* $(U_n)_{n\geq 0}$ *be a linear recurrence sequence of complex numbers that is eventually nonzero. The following assertions are equivalent.*

(a) The limit $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n}$ *exists.*

(b) There is exactly one eigenvalue of U *with maximal multiplicity among the eigenvalues of* U *with maximal modulus.*

Moreover, whenever it exists, the limit is equal to the unique eigenvalue with maximal multiplicity among the eigenvalues with maximal modulus.

8.1.2 Proof of Theorem 8.1

We mostly follow the lines of the proof of Fiorenza and Vincenzi, which we have to adapt to our more general framework. In particular, we will make use of the following lemma proved in [FV11].

Lemma 8.3. *For all $n \geq 0$, let*

$$G_n = \sum_{j=1}^{r} c_j \mu_j^n$$

where $r \geq 2$, $c_1, \dots, c_r$ are nonzero complex numbers and $\mu_1, \dots, \mu_r$ are pairwise distinct nonzero complex numbers. Then for all $n \geq 0$, there exists $s \in \{n, n+1, \dots, n+r-1\}$ such that $G_{n+1}G_{n-1} \neq G_n^2$.

Proof of Theorem 8.1. The implication (b)⇒(a) is easy. Indeed, suppose that

$$\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}} = \beta_i$$

for all $i \in \{0, \dots, p-1\}$. Then for all such i, we have

$$\lim_{n\to\infty} \frac{U_{np+i+p}}{U_{np+i}} = \lim_{n\to\infty} \frac{U_{np+i+p}}{U_{np+i+p-1}} \frac{U_{np+i+p-1}}{U_{np+i+p-2}} \cdots \frac{U_{np+i+1}}{U_{np+i}} = \prod_{i=0}^{p-1} \beta_i.$$

Since the latter limit does not depend on i, we get that (a) holds.

Let us show the two other implications. Let $\alpha_{d+1}, \dots, \alpha_e$ be the other nonzero eigenvalues of U and for each $j \in \{1, \dots, e\}$, let m_j be the multiplicity of α_j. From Theorem 1.8, we know that

$$U_n = \sum_{j=1}^{e} P_j(n)\alpha_j^n$$

for all sufficiently large n, where P_j is a complex polynomial of degree $m_j - 1$ for each j.

First, we assume that the sequence $(|U_n|)_{n\geq 0}$ is eventually increasing and that (c) holds, and we prove that the p limits of (b) exist. For $i \in \{0, \dots, p-1\}$, we have

$$U_{np+i} = Q_i(n)\alpha_1^{np} + \sum_{j=d+1}^{e} \alpha_j^i P_j(np+i)\alpha_j^{np} \tag{8.2}$$

where

$$Q_i(n) = \sum_{j=1}^{d} \alpha_j^i P_j(np+i).$$

One of the polynomials $Q_0, \dots, Q_{p-1}$ has to be nonzero. Since the sequence U has eventually increasing moduli, they must share the same degree, and hence are all nonzero. We let $q_0, \ldots, q_{p-1}$ be the leading coefficients of $Q_0, \ldots, Q_{p-1}$ respectively. Then

$$\lim_{n\to\infty} \frac{U_{np+i+1}}{U_{np+i}} = \lim_{n\to\infty} \frac{Q_{i+1}(n)\alpha_1^{np}}{Q_i(n)\alpha_1^{np}} = \frac{q_{i+1}}{q_i}$$

for each $i \in \{0, \ldots, p-2\}$ and

$$\lim_{n\to\infty} \frac{U_{np+p}}{U_{np+p-1}} = \lim_{n\to\infty} \frac{U_{(n+1)p}}{U_{np+p-1}} = \lim_{n\to\infty} \frac{Q_0(n+1)\alpha_1^{(n+1)p}}{Q_{p-1}(n)\alpha_1^{np}} = \frac{q_0\alpha_1^p}{q_{p-1}},$$

which proves (b). Whenever $p = 1$, we do not need the additional hypothesis that $(|U_n|)_{n\geq 0}$ is eventually increasing since in this case, there is only one polynomial Q_i to consider (which simply is $Q_0 = P_1$ since $d = 1$ by the hypothesis (c)).

Now, we turn to the implication (a)$\Rightarrow$(c). We assume that $\lim_{n\to\infty} \frac{U_{n+p}}{U_n}$ exists and we show that $\alpha_1^p = \alpha_2^p = \cdots = \alpha_d^p$. Without loss of generality, we may suppose that $\alpha_1^p, \alpha_2^p, \cdots, \alpha_c^p$ are pairwise distinct and such that for all $j \in \{c+1, \ldots, d\}$, there exists $k \in \{1, \ldots, c\}$ such that $\alpha_j^p = \alpha_k^p$. We have to show that $c = 1$.

Let $R = |\alpha_1| = \cdots = |\alpha_d|$ and let M be the common multiplicity of $\alpha_1, \ldots, \alpha_d$. First, we observe that

$$\begin{aligned}\frac{U_{n+p}U_{n-p} - U_n^2}{n^{2(M-1)}R^{2n}} &= \frac{U_{n-p}^2}{n^{2(M-1)}R^{2n}} \left(\frac{U_{n+p}}{U_n}\frac{U_n}{U_{n-p}} - \frac{U_n^2}{U_{n-p}^2} \right) \\ &= \left(\frac{U_{n-p}}{(n-p)^{M-1}R^{n-p}} \right)^2 \left(\frac{(n-p)^{M-1}}{n^{M-1}R^p} \right)^2 \left(\frac{U_{n+p}}{U_n}\frac{U_n}{U_{n-p}} - \frac{U_n^2}{U_{n-p}^2} \right). \quad (8.3)\end{aligned}$$

Since the first two factors are bounded and the third converges to 0 as n tends to infinity (since we have assumed (a)), we obtain that

$$\lim_{n\to\infty} \frac{U_{n+p}U_{n-p} - U_n^2}{n^{2(M-1)}R^{2n}} = 0. \quad (8.4)$$

Now, for $j \in \{1, \ldots, d\}$, we let

$$\lambda_j = \frac{\alpha_j}{R}$$

and we let c_j be the leading coefficient of the polynomial P_j. Then we define

$$V_n = \sum_{j=1}^{d} c_j\lambda_j^n$$

for all $n \geq 0$. Thus, we have

$$V_n = \frac{U_n}{n^{M-1}R^n} + o(1).$$

We now consider the sequence $(W_n)_{n\geq 0}$ defined by $W_n = V_{n+p}V_{n-p} - V_n^2$. Then

$$\begin{aligned} W_n = &\left(\frac{U_{n+p}}{(n+p)^{M-1}R^{n+p}} + o(1)\right)\left(\frac{U_{n-p}}{(n-p)^{M-1}R^{n-p}} + o(1)\right) \\ &- \left(\frac{U_n}{n^{M-1}R^n} + o(1)\right)^2 \\ =&\frac{U_{n+p}}{(n+p)^{M-1}R^{n+p}}\frac{U_{n-p}}{(n-p)^{M-1}R^{n-p}} - \left(\frac{U_n}{n^{M-1}R^n}\right)^2 + o(1) \\ =&\frac{U_{n+p}U_{n-p}}{n^{2(M-1)}R^{2n}}\left(\frac{n^2}{n^2-p^2}\right)^{M-1} - \left(\frac{U_n}{n^{M-1}R^n}\right)^2 + o(1). \end{aligned}$$

Using (8.4) and the fact that $\frac{U_{n+p}U_{n-p}}{n^{2(M-1)}R^{2n}}$ is bounded, we obtain that

$$\lim_{n\to\infty} W_n = \lim_{n\to\infty} \frac{U_{n+p}U_{n-p} - U_n^2}{n^{2(M-1)}R^{2n}} = 0,$$

which will be a useful observation later on in the proof.

For now, we can rewrite V_n as

$$V_n = \sum_{j=1}^{c}\sum_{q=0}^{p-1} c_{j,q}\left(\exp\left(i\frac{2q\pi}{p}\right)\lambda_j\right)^n.$$

Observe that for each $j \in \{1,\dots,c\}$, we have $c_{j,0} = c_j \neq 0$. For each $r \in \{0,\dots,p-1\}$, we have

$$V_{np+r} = \sum_{j=1}^{c} d_{j,r}\lambda_j^{np}.$$

where we have set

$$d_{j,r} = \sum_{q=0}^{p-1} c_{j,q}\left(\exp\left(i\frac{2q\pi}{p}\right)\lambda_j\right)^r.$$

For all $j \in \{1, \dots, c\}$, by letting $\mu_{j,q} = \exp\left(i\frac{2q\pi}{p}\right)\lambda_j$, we obtain the matrix equality

$$\begin{pmatrix} d_{j,0} \\ d_{j,1} \\ \vdots \\ d_{j,p-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mu_{j,0} & \mu_{j,1} & \dots & \mu_{j,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{j,0}^{p-1} & \mu_{j,1}^{p-1} & \dots & \mu_{j,p-1}^{p-1} \end{pmatrix} \begin{pmatrix} c_{j,0} \\ c_{j,1} \\ \vdots \\ c_{j,p-1} \end{pmatrix}.$$

The involved matrix is a Vandermonde matrix. Since $\mu_{j,0}, \dots, \mu_{j,p-1}$ are pairwise distinct, we get that this matrix is invertible. Since $c_{j,0} \neq 0$, we obtain that for each $j \in \{1, \dots, c\}$, there must exist some $r \in \{0, \dots, p-1\}$ such that $d_{j,r} \neq 0$. For $r \in \{0, \dots, p-1\}$, consider the following properties $C_1(r)$ and $C_2(r)$.

$$\begin{aligned} C_1(r) &: \text{ There exists } j \in \{1, \dots, c\} \text{ such that } d_{j,r} \neq 0,\, d_{k,r} = 0 \text{ for all } k \neq j. \\ C_2(r) &: \text{ For all } k \in \{1, \dots, c\}, \text{ we have } d_{k,r} = 0. \end{aligned}$$

If $C_1(r)$ holds then

$$U_{np+r} = (np+r)^{M-1} R^{np+r} \big(V_{np+r} + o(1)\big) = (np+r)^{M-1} R^{np+r} \big(d_{j,r}\lambda_j^{np} + o(1)\big)$$

and it follows that

$$\lim_{n\to\infty} \frac{U_{np+r+p}}{U_{np+r}} = (R\lambda_j)^p = \alpha_j^p.$$

Since the limit $\lim_{n\to\infty} \frac{U_{n+p}}{U_n}$ exists and since $\alpha_1^p, \dots, \alpha_c^p$ are pairwise distinct, this can happen for at most one j across all r such that $C_1(r)$ holds. We consider the following two cases.

First, suppose that for all r, either $C_1(r)$ or $C_2(r)$ holds. In view of the previous paragraph, for all r such that $C_1(r)$ holds, there exists one common $j \in \{1, \dots, c\}$ such that $d_{j,r} \neq 0$ and $d_{k,r} = 0$ for all $k \neq j$. If $c \geq 2$, then we get that there exists $k \in \{1, \dots, c\}$ such that $d_{k,r} = 0$ for all r, which we know is not the case. Therefore, we get $c = 1$ in this case, as desired.

Now we suppose that there is some r such that neither $C_1(r)$ nor $C_2(r)$ holds. Consider such an r and set $G_n = V_{np+r}$ for all $n \geq 0$. We have

$$G_{n+1}G_{n-1} - G_n^2 = V_{np+r+p}V_{np+r-p} - V_{np+r}^2 = W_{np+r}.$$

By contradiction, we suppose that $c \geq 2$. Then there exist two distinct j, k such that $d_{j,r}$ and $d_{k,r}$ are not zero. By Lemma 8.3, we obtain that the sequence $(W_{np+r})_{n\geq 0}$ is not eventually zero, and neither is the sequence $(W_n)_{n\geq 0}$. Moreover, since

$$\begin{aligned} W_n &= \left(\sum_{j=1}^{d} c_j\lambda_j^{n+p}\right)\left(\sum_{j=1}^{d} c_j\lambda_j^{n-p}\right) - \left(\sum_{j=1}^{d} c_j\lambda_j^{n}\right)^2 \\ &= \sum_{1\leq j,k\leq d} c_jc_k\lambda_j^{n+p}\lambda_k^{n-p} - \sum_{j=1}^{d} c_j^2\lambda_j^{2n} - \sum_{1\leq j<k\leq d} 2c_jc_k\lambda_j^n\lambda_k^n \\ &= \sum_{1\leq j<k\leq d} c_jc_k\left(\lambda_j^p\lambda_k^{-p} + \lambda_j^{-p}\lambda_k^p - 2\right)(\lambda_j\lambda_k)^n. \end{aligned}$$

for all $n \geq 0$, we obtain that $(W_n)_{n\geq 0}$ is a linear recurrence sequence whose eigenvalues are simple and all of modulus 1. Therefore, we can write

$$W_n = \sum_{j=1}^{f} d_j\mu_j^n$$

for all $n \geq 0$, where $f \geq 1$, $d_1, \ldots, d_f$ are nonzero complex numbers and $\mu_1, \ldots, \mu_f$ are pairwise distinct complex numbers of modulus 1.

Now, for all $n \geq 0$, we consider the following linear system in the f unknowns $X_1^{(n)}, \ldots, X_f^{(n)}$:

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \mu_1 & \mu_2 & \ldots & \mu_f \\ \vdots & \ddots & \vdots & \vdots \\ \mu_1^{f-1} & \mu_2^{f-1} & \ldots & \mu_f^{f-1} \end{pmatrix}\begin{pmatrix} X_1^{(n)} \\ X_2^{(n)} \\ \vdots \\ X_f^{(n)} \end{pmatrix} = \begin{pmatrix} W_n \\ W_{n+1} \\ \vdots \\ W_{n+f-1} \end{pmatrix}.$$

This is a Cramer system and thus it has the unique solution

$$\begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_f^{(n)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \mu_1 & \mu_2 & \ldots & \mu_f \\ \vdots & \ddots & \vdots & \vdots \\ \mu_1^{f-1} & \mu_2^{f-1} & \ldots & \mu_f^{f-1} \end{pmatrix}^{-1}\begin{pmatrix} W_n \\ W_{n+1} \\ \vdots \\ W_{n+f-1} \end{pmatrix}.$$

Since the sequence $(W_n)_{n\geq 0}$ tends to zero, we get that $\lim_{n\to\infty} x_j^{(n)} = 0$ for each $j \in \{1, \ldots, f\}$. But the vector

$$\begin{pmatrix} d_1\mu_1^n \\ d_2\mu_2^n \\ \vdots \\ d_f\mu_f^n \end{pmatrix}$$

is clearly a solution of this system. So we also get that $x_j^{(n)} = d_j\mu_j^n$ for all $j \in \{1, \ldots, f\}$ and $n \geq 0$. As $|\mu_j| = 1$ for each j, it cannot be that $x_j^{(n)}$ tends to zero as n tends to infinity. We have thus reached a contradiction, and hence we must have $c = 1$ as desired. The implication (a)⇒(c) is proven.

Finally, assuming that the limit $\lim_{n\to\infty} \frac{U_{n+p}}{U_n}$ exists, let us argue that it must be equal to α_1^p. Since we have shown that (a)⇒(c), we may use the equality (8.2) in order to get

$$\lim_{n\to\infty} \frac{U_{n+p}}{U_n} = \lim_{n\to\infty} \frac{U_{(n+1)p}}{U_{np}} = \lim_{n\to\infty} \frac{Q_0((n+1)p)\alpha_1^{(n+1)p}}{Q_0(np)\alpha_1^{np}} = \alpha_1^p.$$

□

8.2 Initial conditions where L_U is regular when U has a dominant root

8.2.1 Introduction

In his article [Hol98], Hollander gives some necessary and some sufficient conditions on the linear recurrence sequence U that has a dominant root for the numeration language L_U to be regular. However, Hollander's criteria do not offer a complete answer to the problem at hand. Indeed, Hollander focuses his efforts on the procurement of a criterion that relies on the *recurrence relations* satisfied by the sequence U. However, as he notes, there exist sequences that have the same minimal polynomial but different initial conditions, and that give rise to languages whose regularity depends on those initial conditions. Hollander mentions the example of the relation $U_{n+2} = 4U_{n+1} - 3U_n$, where the initial conditions $(1, 2)$ give a nonregular language whereas the conditions $(1, 4)$ give a regular one.

In Chapter 6, we have developed new criteria that completely decide the regularity of the language L_U, even if U has no dominant root. We also recover Hollander's results as a consequence of ours. However, our criteria are less tractable than those of Hollander. They of course depend on the initial conditions of the sequence U, and they necessitate the computation of auxiliary sequences.

In this section, we wish to provide criteria in the spirit of those of Hollander, that rely more on the characteristic polynomial of the sequence itself. We also study more precisely the set of initial conditions that lead to a regular language.

8.2.2 A reminder on Hollander's and our results

In this subsection, we recall some of Hollander's results from [Hol98] and our results from Chapter 6. We consider a linear numeration system with base sequence $(U_n)_{n\in\mathbb{N}}$ and assume that this base sequence has a dominant root β. As proved by Hollander, dependency to the initial conditions can only occur when the dominant root of the sequence U is a simple Parry number.

Definition 8.4. Let $\beta > 1$ be a simple Parry number with $d_\beta(1) = t_1 t_2 \cdots t_\ell$. The *canonical β-polynomial* is the polynomial

$$P_{0,\ell} = X^\ell - \sum_{k=1}^{\ell} t_k X^{\ell-k}.$$

The *extended β-polynomials* are the family given by

$$P_{N,M\ell} = X^N(1 + X^\ell + \ldots + X^{(M-1)\ell})P_{0,\ell} \text{ for } N \geq 0,\ M \geq 1.$$

Remark 8.5. The reason to define the family of extended β-polynomials as such, besides their use in proofs, is to mirror the definition for non-simple Parry numbers. Recall from the end of Section 6.5 that if $d_\beta(1) = t_1 \cdots t_q (t_{q+1} \cdots t_{q+m})^\omega$, we define the β-polynomial by

$$P_{q,m} = \left(X^{q+m} - \sum_{k=1}^{q+m} t_k X^{q+m-k}\right) - \left(X^q - \sum_{k=1}^{q} t_k X^{q-k}\right). \qquad (8.5)$$

In the case of a simple Parry number, we have $d^*_\beta(1) = (t_1 \cdots t_{\ell-1}(t_\ell - 1))^\omega$. Using this word as if it were the greedy expansion of 1 in a nonsimple Parry base, we find back the polynomial $P_{0,\ell}$ as defined above. However, writing $d^*_\beta(1)$ as $d_1 d_2 \cdots$, it is also the case that $d^*_\beta(1) = (d_1 \cdots d_N)(d_{N+1} \cdots d_{N+M\ell})^\omega$ for any $N \geq 0$ and $M \geq 1$. Using these larger preperiod and period in (8.5), we find the extended β-polynomials.

Hollander proved the following, which we also reobtained as Corollary 6.44.

Proposition 8.6 ([Hol98, Lemmas 7.2 and 7.5])**.** *Let U be a numeration system with a dominant root $\beta > 1$ such that $d_\beta(1)$ is finite of length ℓ, i.e., β is a simple Parry number.*

- *If the numeration language L_U is regular, then the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial multiplied by $(X^\ell - 1)$.*
- *If the base sequence U satisfies a linear recurrence relation whose characteristic polynomial is given by an extended β-polynomial, then the numeration language L_U is regular.*

As can be seen, this result does not decide regularity when the minimal polynomial of U is of the form $P_{N,M\ell}Q$, where Q is a nonconstant divisor of $X^\ell - 1$. In this case, we can still resort to the results of Chapter 6, notably Corollary 6.41 which we reproduce below. We first recall that the sequence $(\Delta_n)_{n\geq\ell}$ is defined by

$$\Delta_n = U_n - \sum_{k=1}^{\ell} t_k U_{n-k}.$$

In fact, it is a specific instance of a more general family of operations on sequences that we will use below.

Definition 8.7. Let U be a sequence and $P = X^\ell - \sum_{k=1}^{\ell} c_k X^{\ell-k}$. We define the sequence $\Delta_P(U)$ by

$$(\Delta_P(U))_n = U_{n+\ell} - \sum_{k=1}^{\ell} c_k U_{n+\ell-k}.$$

Note that we have changed the indexing between the Δ used in Chapter 6 and the definition of $\Delta_P(U)$ here. Whereas the former started with ℓ undefined terms, this one is defined for all natural n, and is offset by ℓ positions to compensate. The reason to proceed like so is that in Chapter 6, we are only working with eventual properties of Δ and do not care about initial conditions, and it is more important to illustrate the correspondence to the greedy algorithm, so it is more natural to ask that $(\Delta_i)_n$ corresponds to U_{pn+i}. Here however, we will shortly be very interested in the first few values of various $\Delta_P(U)$, and the fact that we are working with $p = 1$ means that the link to the sequence U can be slightly obscured without losing too much clarity in the argument. As such, the sequence $\Delta_{P_{0,\ell}}(U)$ that we will use is $\sigma^\ell(\Delta)$, where Δ is the one used in Chapter 6 and the σ^ℓ removes the undefined values. We have the following immediate properties on the sequence $\Delta_P(U)$, the third being a consequence of the other two. They are already mentioned in [Hol98, Lemma 3.1]

Proposition 8.8.

- *A sequence U satisfies the recurrence relation associated with a polynomial P if and only if $\Delta_P(U) = 0$.*
- *If P and Q are polynomials, then $\Delta_{PQ}(U) = \Delta_P(\Delta_Q(U))$.*
- *The sequence U satisfies the recurrence relation associated with PQ if and only if $\Delta_P(U)$ satisfies the recurrence relation associated with Q.*

Let us now recall our result from Chapter 6, adapted to the case where $p = 1$.

Proposition 8.9 (Corollary 6.41)**.** *Let U be a positional numeration system with a dominant root $\beta \geq 1$ such that $d_\beta(1)$ is finite of length ℓ.*

- *If the numeration language L_U is regular, then the sequence $\Delta_{P_{0,\ell}}(U)$ is ultimately periodic.*
- *Assume that the sequence $\Delta_{P_{0,\ell}}(U)$ is ultimately periodic with a preperiod N and a period $M\ell$ with $M \geq 1$. Then the numeration language L_U is regular if and only if*

$$\sum_{j=0}^{M-1} (\Delta_{P_{0,\ell}}(U))_{n-j\ell} \geq 0$$

for all $n \in \{N + (M-1)\ell, \ldots, N + M\ell - 1\}$.

8.2.3 Main results

In this section, we deduce corollaries from Proposition 8.9 that give criteria in a spirit closer to Hollander's. We begin with a case where no matter the initial conditions, the language cannot be regular.

Corollary 8.10. *Let U be a linear recurrence sequence with dominant root $\beta > 1$ with $d_\beta(1) = t_1 \dots t_\ell$ and let U satisfy the recurrence relation associated with the polynomial $X^N(1+X+X^2+\dots+X^{M\ell-1})P_{0,\ell}$ for some $N \geq 0$ and $M \geq 1$. Then, L_U is regular if and only if it also satisfies the recurrence relation associated with $P_{N,M\ell} = X^N(1+X^\ell+\dots+X^{(M-1)\ell})P_{0,\ell}$.*

Equivalently, if the minimal polynomial of U is of the form $P_{N,M\ell}Q$ with Q a nonconstant divisor of $X^\ell - 1$, and if L_U is regular, then Q must be a multiple of $X - 1$. This can be seen by contraposition: if Q is not a multiple of $X - 1$, then U satisfies the recurrence relation of polynomial $X^N(1 + X + \cdots + X^{M\ell-1})P_{0,\ell}$, so it must satisfy the recurrence relation of polynomial $P_{N,M\ell}$ and Q is therefore constant.

Proof. The condition is sufficient by the work of Hollander. We show that it is necessary. Assume that U satisfies the recurrence relation associated with the polynomial $X^N(1 + X + X^2 + \dots + X^{M\ell-1})P_{0,\ell}$ and L_U is regular. We find that $\Delta_{P_{0,\ell}}(U)$ satisfies the recurrence relation of polynomial $X^N(1+X+X^2+\dots+X^{M\ell-1})$ and is therefore ultimately periodic with preperiod N and period $M\ell$. Proposition 8.9 applies, and we have

$$0 = \sum_{k=N}^{N+M\ell-1} (\Delta_{P_{0,\ell}}(U))_k = \sum_{n=N+(M-1)\ell}^{N+M\ell-1} \sum_{j=0}^{M-1} (\Delta_{P_{0,\ell}}(U))_{n-j\ell}.$$

In order for L_U to be regular, all M inner sums must be greater than or equal to zero by Proposition 8.9. So they must all equal zero. Thus $\Delta_{P_{0,\ell}}(U)$ satisfies the recurrence relation associated with $X^N(1+X^\ell+\dots+X^{(M-1)\ell})$, and U must satisfy the recurrence relation associated with $X^N(1+X^\ell+\dots+X^{(M-1)\ell})P_{0,\ell}$, as expected. □

In general, when the recurrence polynomial is of the form $P_{N,M\ell}Q$ with Q multiple of $X - 1$, the set of initial conditions giving a regular language becomes harder to predict. We can still give a geometric description of this set.

Definition 8.11. A *polyhedral cone* $\mathcal{C}$ in the Euclidean space $\mathbb{R}^q$ is an intersection of a finite number of half-spaces that have the origin on their boundary. Alternatively, it is the set of points x such that Ax has nonnegative components, for some matrix $A \in \mathbb{R}^{m\times q}$. Such a cone is called *salient* if $\mathcal{C} \cap -\mathcal{C} = \{0\}$.

This is precisely the shape taken by the set of initial conditions that give a regular language.

Proposition 8.12. *Let U be a linear recurrence sequence that has a dominant root $\beta > 1$ with $d_\beta(1)$ finite of length ℓ. Let U satisfy the recurrence relation of polynomial QR, where Q is a multiple of $P_{0,\ell}$ and a divisor of $P_{N,M\ell}$, and R is a multiple of $X-1$ and a divisor of $X^\ell - 1$. Let q and r be the respective degrees of Q and R. Consider the linear recurrence sequence V satisfying the recurrence relation of polynomial Q, with initial conditions $V_i = U_i$ for $i \in \{0,\dots,q-1\}$.*

Then there exists a polyhedral cone $\mathcal{C} \subset \mathbb{R}^r$ such that L_U is regular if and only if $(U_q,\dots,U_{q+r-1})$ is in $\mathcal{C} + (V_q,\dots,V_{q+r-1})$. This cone can be effectively computed, is salient and has nonempty interior.

Proof. Since the sequence U satisfies the recurrence relation associated with QR, the sequence $\Delta_{P_{0,\ell}}(U)$ satisfies the recurrence relation associated with $\frac{Q}{P_{0,\ell}}R$. Since $\frac{Q}{P_{0,\ell}}$ divides $X^N(1+X^\ell+\ldots+X^{(M-1)\ell})$ and R divides $X^\ell-1$, we get that $\frac{Q}{P_{0,\ell}}R$ divides $X^N(X^{M\ell}-1)$ and the sequence $\Delta_{P_{0,\ell}}(U)$ must be ultimately periodic with preperiod N and period $M\ell$. Therefore, by Proposition 8.9, we must study the sign of the ℓ sums

$$\sum_{j=0}^{M-1} (\Delta_{P_{0,\ell}}(U))_{n-j\ell} \tag{8.6}$$

for n in $\{N+(M-1)\ell,\dots,N+M\ell-1\}$.

Note that since the sequence V satisfies the recurrence relation associated with $P_{N,M\ell}$, the sequence $\Delta_{P_{0,\ell}}(V)$ satisfies the recurrence relation associated with $X^N(1+X^\ell+\cdots+X^{(M-1)\ell})$. In particular, the ℓ sums $\sum_{j=0}^{M-1}(\Delta_{P_{0,\ell}}(V))_{n-j\ell}$ for $n \in \{N+(M-1)\ell,\dots,N+M\ell-1\}$ must all be zero.

The sequence $U-V$ satisfies the same recurrence relation as U does, so the periodicity of the sequence $\Delta_{P_{0,\ell}}(U-V)$ is the same as that of $\Delta_{P_{0,\ell}}(U)$.

Since $\Delta_{P_{0,\ell}}(U) = \Delta_{P_{0,\ell}}(U-V) + \Delta_{P_{0,\ell}}(V)$, we deduce from the previous paragraph that we can consider the sums

$$\sum_{j=0}^{M-1} (\Delta_{P_{0,\ell}}(U-V))_{n-j\ell} \tag{8.7}$$

for n in $\{N+(M-1)\ell, \dots, N+M\ell-1\}$ instead of those of (8.6), as they have the same value. We stand to gain that the sequence $U-V$ starts with q zero terms, and has only r initial conditions that can be nonzero. The initial conditions $(U_q, \dots, U_{q+r-1})$ that make L_U regular are the translates by $(V_q, \dots, V_{q+r-1})$ of the initial conditions $((U-V)_q, \dots, (U-V)_{q+r-1})$ that make the ℓ sums in (8.7) nonnegative. Let us study these initial conditions.

Since $U-V$ is a linear recurrence sequence, and since $(U-V)_0 = \cdots = (U-V)_{q-1} = 0$, the terms $(U-V)_{q+r}, \dots, (U-V)_{N+M\ell+\ell-1}$ are each an integer linear combination of the terms in the r-tuple

$$a = ((U-V)_q, \dots, (U-V)_{q+r-1}).$$

As a result, so are each of the quantities $(\Delta_{P_{0,\ell}}(U-V))_n$ for n in $\{N, \dots, N+M\ell-1\}$ and so are each of the ℓ sums of (8.7). Therefore, to have the regularity of L_U, we must have the nonnegativity of ℓ integer linear combinations of the components of a. This corresponds exactly to requiring that a belong to a polyhedral cone $\mathcal{C}$, as expected. Since the linear combinations involved at each step can be effectively computed, so can $\mathcal{C}$.

To show that $\mathcal{C}$ is salient, we must show that if a belongs to $\mathcal{C}$, then $-a$ doesn't. Since moving to $-a$ changes the sign of all the involved linear combinations, this corresponds to requiring that the zero q-tuple is the only one for which the ℓ sums of (8.7) are all zero. If those sums are all zero, then $U-V$ satisfies the linear recurrence relation associated with the polynomial $P_{N,M\ell}$, and it also satisfies the relation associated with $\gcd(P_{N,M\ell}, QR)$. This greatest common divisor is Q, as Q is a multiple of $P_{0,\ell}$ and the roots of R are ℓ-th roots of unity, which are not roots of $\frac{P_{N,M\ell}}{P_{0,\ell}}$. Since we know that the first q terms of $U-V$ are zero, $U-V$ must be the zero sequence, which indeed gives us that a is zero.

Finally, to show that $\mathcal{C}$ has nonempty interior, it suffices to show that there exists a q-tuple a such that the ℓ sums of (8.7) are all strictly positive. We construct such a q-tuple. Assume that $U-V$ satisfies the recurrence relation associated with $(X-1)Q$ and that $(U-V)_q$ is positive. Let a be the r-tuple formed from $(U-V)_q, \ldots, (U-V)_{q+r-1}$ (which can be computed with just the extra initial condition $(U-V)_q$). In such a case, the sequence $\Delta_{P_{0,\ell}}(U-V)$ satisfies the recurrence relation of the polynomial $X^N(1+X^\ell+\cdots+X^{(M-1)\ell})(X-1)$. Therefore, the sequence

$$\Delta_{X^N(1+X^\ell+\cdots+X^{(M-1)\ell})}\left(\Delta_{P_{0,\ell}}(U-V)\right)$$

satisfies the recurrence relation associated with $X-1$: it is constant. The ℓ sums of (8.7), which are the first ℓ terms of this sequence, must have the same value. This value constant cannot be zero since the cone is salient, so either a (if this value is positive) or $-a$ (if this value is negative) is in the interior of $\mathcal{C}$, as expected. □

Remark 8.13. Note that the cone in question can contain both q-tuples of integers with negative entries and nonincreasing q-tuples. It is therefore not the case that any point in this cone actually corresponds to a numeration system.

8.2.4 Examples

In this subsection, we present some applications of the results of Section 8.2.3.

Example 8.14. Let us consider the golden ratio φ. Its canonical β-polynomial is X^2-X-1, with $\ell=2$. We consider the polynomials $Q(X)=(X^2-X-1)(X^4-X^3+X^2-X+1)$, which is a divisor of $P_{0,5\ell}$, and $R(X)=X^2-1$. Consider the sequence U that satisfies the recurrence relation with polynomial QR and initial conditions $(1,2,4,7,10,20,u_6,u_7)$. For which values of u_6 and u_7 does this sequence have an associated regular language L_U?

First note that the sequence V from Proposition 8.12 satisfies the recurrence relation $(X^2-X-1)(X^4-X^3+X^2-X+1)$ with the initial conditions $(1,2,4,7,10,20)$. Its next two terms are 34 and 53. Thus, we will focus on the sequence $U-V$, which satisfies the same recurrence relation as U but with the initial conditions $(0,0,0,0,0,0,a_1,a_2)$, where a_1 is u_6-34 and a_2 is u_7-53.

We know that $\Delta_{P_{0,\ell}}(U-V)$ satisfies the recurrence relation of polynomial $(X^4-X^3+X^2-X+1)(X^2-1)$, which divides $X^{10}-1$. It is thus purely periodic with period 10. Using the recurrence relation, we find that the first

12 terms of $U - V$ are

$$0,\, 0,\, 0,\, 0,\, 0,\, 0,\, a_1,\, a_2,\, 2a_2,\, -a_1 + 4a_2,\, -2a_1 + 7a_2,\, -5a_1 + 12a_2.$$

As a result, the first 10 terms of the sequence $(\Delta_{P_{0,\ell}}(U))_n = (U_{n+2} - U_{n+1} - U_n)_n$ are

$$0,\, 0,\, 0,\, 0,\, a_1,\, a_2 - a_1,\, a_2 - a_1,\, a_2 - a_1,\, a_2 - a_1,\, a_2 - 2a_1.$$

The two sums that interest us are thus equal to $2a_2 - a_1$ and $3a_2 - 4a_1$. The initial conditions for U that are such that L_U is regular are exactly those where

$$\begin{cases} 2(u_7 - 53) - (u_6 - 34) & \geq 0 \\ 3(u_7 - 53) - 4(u_6 - 34) & \geq 0. \end{cases}$$

See an illustration on Figure 8.1. The beige and black crosses correspond to initial conditions such that QR is not the minimal polynomial of the sequence U. For blue crosses, this polynomial is $(X-1)Q$, while it is $(X+1)Q$ for purple crosses. Note that this corresponds to respectively

$$2(u_7 - 53) - (u_6 - 34) = 3(u_7 - 53) - 4(u_6 - 34)$$

and

$$2(u_7 - 53) - (u_6 - 34) = -3(u_7 - 53) + 4(u_6 - 34).$$

Note also that none of the black crosses apart from the origin of the axes give a regular language, illustrating Corollary 8.10. Finally, remark that this cone includes pairs (u_6, u_7) whose entries are both negative, as mentioned in Remark 8.13.

Example 8.15. In this example, we study a family of systems rather than just one. Our dominant root will still be the golden ratio. Let q be an odd prime number. We consider the cyclotomic polynomial $\phi_q = X^{q-1} + X^{q-2} + \cdots + X + 1$ and $Q(X) = (X^2 - X - 1)\phi_q(X)$, which divides $P_{0,2q}$, and we let $R(X) = X^2 - 1$. Note that q is not the degree of Q in this example.

A sequence whose minimal polynomial is QR is determined by $q+3$ initial conditions. Let us fix the first $q+1$ and study for which values of U_{q+1} and U_{q+2} the language L_U is regular. As above, we rather consider the sequence $U - V$, whose initial conditions are $q+1$ zeros followed by a_1 and a_2. We

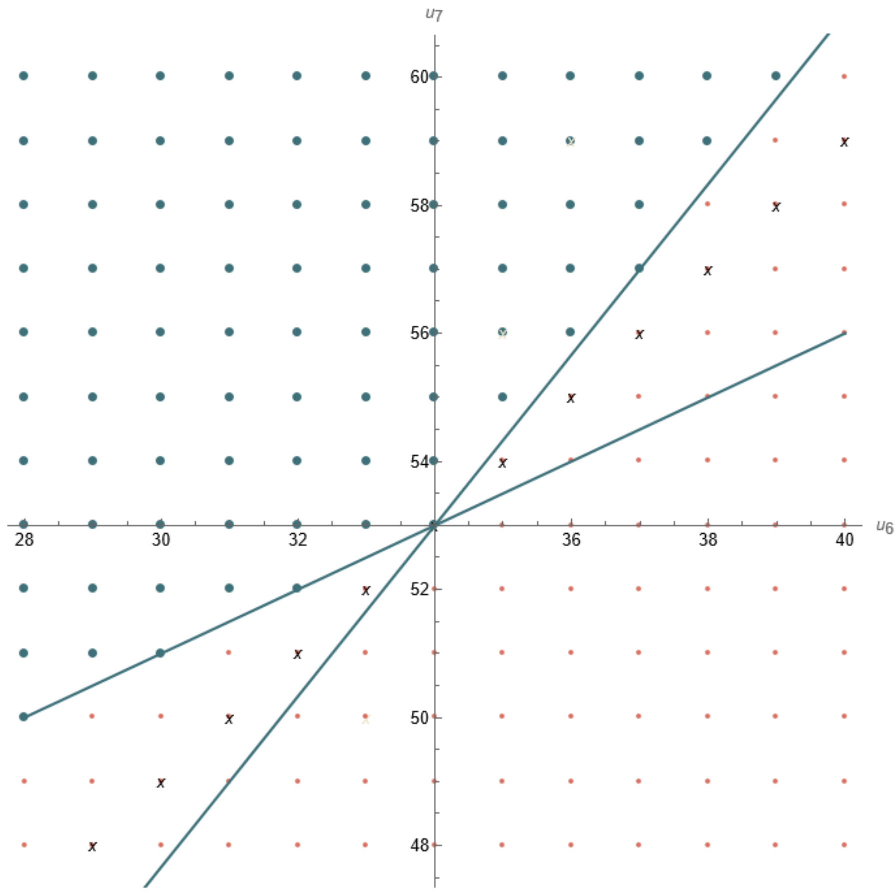


Figure 8.1: Impact of initial conditions u_6 and u_7 on the regularity of the language generated by the linear recurrence of polynomial $(X^2-X-1)(X^4-X^3+X^2-X+1)(X^2-1)$ with initial conditions $(1, 2, 4, 7, 10, 20, u_6, u_7)$.
Teal dots: initial conditions where the language L_U is regular.
Red dots: initial conditions where the language L_U is not regular.
Beige crosses: initial conditions where U satisfies the polynomial $(X^2-X-1)(X^4-X^3+X^2-X+1)(X-1)$.
Black crosses: initial conditions where U satisfies the polynomial $(X^2-X-1)(X^4-X^3+X^2-X+1)(X+1)$.
The two teal lines have equations $2(u_7-53)-(u_6-34)=0$ and $3(u_7-53)-4(u_6-34)=0$.

note that $\Delta_{P_{0,\ell}}(U-V)$ starts with $q-1$ zeros, then a_1 and a_2-a_1. We are now interested in the quantities

$$\sum_{k=0}^{q-1}(\Delta_{P_{0,\ell}}(U-V))_{2k} \quad \text{and} \quad \sum_{k=0}^{q-1}(\Delta_{P_{0,\ell}}(U-V))_{2k+1}. \tag{8.8}$$

Seeing that the sequence $\Delta_{P_{0,\ell}}(U-V)$ satisfies the recurrence relation associated with $(X^2-1)\phi_q(X)$, we know that there must exist constants c_j with $j=1,\dots,q-1$, c_+ and c_- such that

$$(\Delta_{P_{0,\ell}}(U-V))_k = c_+ + c_-(-1)^k + \sum_{j=1}^{q-1} c_j \left(e^{\frac{2i\pi j}{q}}\right)^k. \tag{8.9}$$

When summing terms of this form to reach (8.8), the terms associated with roots of unity other than 1 or -1 cancel out, and we find

$$\sum_{k=0}^{q-1}(\Delta_{P_{0,\ell}}(U-V))_{2k} = q(c_+ + c_-)$$

and

$$\sum_{k=0}^{q-1}(\Delta_{P_{0,\ell}}(U-V))_{2k+1} = q(c_+ - c_-)$$

We will now deduce the values of c_+ and c_-. The constants $c_1,\dots,c_{q-1}$ and c_+, c_- must satisfy the system of equations

$$\begin{pmatrix} 1 & \cdots & 1 & 1 & 1 \\ e^{\frac{2i\pi 1}{q}} & \cdots & e^{\frac{2i\pi(q-1)}{q}} & -1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \\ e^{\frac{2i\pi 1(q-1)}{q}} & \cdots & e^{\frac{2i\pi(q-1)(q-1)}{q}} & 1 & 1 \\ e^{\frac{2i\pi 1 q}{q}} & \cdots & e^{\frac{2i\pi(q-1)q}{q}} & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{q-1} \\ c_- \\ c_+ \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_1 \\ a_2-a_1 \end{pmatrix}$$

We can use Cramer's rule, together with the formulas for Vandermonde determinants and generalized Vandermonde determinants seen in [Hei29], to find

$$c_+ = \frac{(a_2-a_1) - a_1((\sum_{j=1}^{q-1} e^{\frac{2i\pi j}{q}}) - 1)}{2\prod_{j=1}^{q-1}(1-e^{\frac{2i\pi j}{q}})}.$$

Similarly, we find

$$c_- = \frac{a_1(\sum_{j=1}^{q-1} e^{\frac{2i\pi j}{q}} + 1) - (a_2 - a_1)}{2\prod_{j=1}^{q-1}(-1 - e^{\frac{2i\pi j}{q}})}.$$

Noticing that $\prod_{j=1}^{q-1}(X - e^{\frac{2i\pi j}{q}}) = \phi_q(X)$, we find that the two denominators are respectively $2q$ and 2. We also have $\sum_{j=1}^{q-1} e^{\frac{2i\pi j}{q}} = -1$. Writing that $c_+ + c_- \geq 0$ and $c_+ - c_- \geq 0$ and simplifying, we obtain the two equations

$$(1+q)a_1 + (1-q)a_2 \geq 0 \quad \text{and} \quad (1-q)a_1 + (1+q)a_2 \geq 0.$$

We notice that these cones get increasingly narrow as q tends to infinity, with their limit being just one line. Similarly, we could have done the same with the polynomial $\phi_{2q} = (1 - X + X^2 - \ldots + X^q)$, a case that covers the previous example. We would have found that in this case the cone gets increasingly large as q goes to infinity, with its limit being an open half-plane plus one point.

To close out this section, let us remark that the methods of the previous example can be adapted to the original case that Hollander used to illustrate its point.

Proposition 8.16. *Let U be a linear recurrence sequence that has a dominant root $\beta > 1$ with $d_\beta(1)$ finite of length ℓ. Let U satisfy the recurrence relation of polynomial $(X-1)Q$, where Q is of degree q, a multiple of $P_{0,\ell}$ and a divisor of $P_{N,M\ell}$. Consider the linear recurrence sequence V satisfying the recurrence relation of polynomial Q, with initial conditions $V_i = U_i$ for $i \in \{0, \ldots, q-1\}$.*

Then L_U is regular if and only if $U_q \geq V_q$.

Sketch of proof. Similarly to the previous results of this section, we must evaluate terms of the form $\sum_{j=0}^{M-1}(\Delta_{P_{0,\ell}}(U-V))_{n-j\ell}$ and test whether they are positive. Similarly to (8.9), we can write $(\Delta_{P_{0,\ell}}(U-V))_n$ as a weighted sum of powers of roots of unity, plus some coefficient c_+ corresponding to the root 1. When summing the terms appropriately, all the terms cancel out except c_+, and we find that the ℓ sums in consideration are all equal to Mc_+ (which is the constant mentioned in the last paragraph of the proof of Proposition 8.12, whose value was not studied precisely at the time).

As in Example 8.15, we can use Cramer's rule and Vandermonde determinants to compute c_+. We find

$$c_+ = \frac{U_q - V_q}{\prod(1-\omega_k)}$$

where the ω_k are the roots of unity that are roots of $\frac{Q}{P_{0,\ell}}$. Since these roots (except possibly -1) come in conjugate pairs and all have real part less than 1, c_+ is positive exactly when $U_q - V_q$ is. Thus, L_U is regular exactly when $U_q - V_q$ is nonnegative. □

Thus, in the discussion in the last paragraph of the proof of Proposition 8.12, it was actually always a that was in the interior of $\mathcal{C}$. It can also be seen on Figure 8.1 that the beige crosses that correspond to regular languages are those to the right of the y-axis.

Intuitively, what happens is the following: the total of each of the ℓ sums considered in Proposition 8.9 is equal to the sum of all the elements in the period of $\Delta_{P_{0,\ell}}(U)$. If $X-1$ is not a divisor of the recurrence polynomial, this latter sum is 0. Each of the ℓ smaller sums must be balanced perfectly for the language to be regular. Multiplying the recurrence polynomial by a divisor of $X^\ell - 1$ that is not multiple of $X-1$ will not change the sum of all elements in the period of $\Delta_{P_{0,\ell}}(U)$, but it will imbalance the different sums of arithmetic progressions, making the language not regular (Corollary 8.10). When the recurrence polynomial is multiplied by $X-1$, the sum of all elements in the period can change, and it increases as the last initial condition increases, thus the language stays regular (Proposition 8.16). When both factors are present, their influences work in opposite ways: multiplying by $X-1$ may increase the total sum of the period, giving more room for imbalances between sums along subsequences, making it easier for L_U to be regular, while the other factors create imbalances between the subsequences, bringing some of them closer to negative values and making it harder for L_U to be regular. Since all these effects are linear, we obtain a polyhedral cone. As the differing behavior between the two examples illustrates, the dynamic between these opposite forces varies depending on the rest of the recurrence polynomial.

Chapter 9

Perspectives

In this final chapter, we open the discussion by considering results that were left unproven and potential future directions for research. We present four directions for more coherent projects that would prolong our work, as well as some more isolated questions. The level of vagueness of these questions varies greatly depending on the topic.

Contents

9.1 Bertrand numeration systems in the case without a dominant root . . . **279**

9.2 More work on Dumont–Thomas numeration systems . . . **283**

9.3 The language $\mathrm{Max}(L_U)$ when U has a dominant root . . **285**

9.4 Study of $\mathcal{B}$-integers . . . **291**

9.5 Other perspectives . . . **292**

9.1 Bertrand numeration systems in the case without a dominant root

As illustrated in Example 2.41, the results regarding the link between positional Dumont–Thomas and Bertrand numeration systems, that were presented in Section 1.5 and proved in Section 7.5, seem tantalizingly within reach in the alternate case. Although we have not yet conducted the related research, we present a program in the form of a list of conjectures. It should be noted that we do not have strong evidence for the validity of these conjectures, and they merely form an potential path to a generalization of the above link.

As explained in Section 1.5, notably Lemma 1.43 and Theorem 1.45, Bertrand numeration systems have multiple definitions. They are numeration systems whose language is stable by adding or deleting a zero to the right, as well as systems where the language of maximal words is the set of prefixes of a given infinite word with some lexicographic conditions. Additionally, they are the numeration systems whose set of factors is the set of factors of a β-shift for some β. Each of these criteria has a likely generalization. The next conjecture generalizes Lemma 1.43, where the case $p = 1$ corresponds to that lemma.

Conjecture 9.1. *Consider a numeration system U and an integer $p \geq 1$. The two following properties are equivalent.*

(a) For all words $w \in A_U^*$, $w \in L_U \Leftrightarrow w0^p \in L_U$.

(b) There exist p infinite words $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0 \in A_U^\omega$ such that

$$\mathrm{rep}_U(U_{np+i} - 1) = \mathrm{Pref}_{np+i}(\mathbf{a}_i)$$

for all $n \in \mathbb{N}$ and $i \in \{0, \ldots, p-1\}$.

In this case, the words $\mathbf{a}_{p-1}, \ldots, \mathbf{a}_0$ satisfy $\sigma^j(\mathbf{a}_i) \leq_{\mathrm{lex}} \mathbf{a}_{i-j}$ for all $i \in \{0, \ldots, p-1\}$ and $j \in \mathbb{N}$, where the indices are taken modulo p.

Definition 9.2. A positional numeration system is *p-Bertrand* if it satisfies one of the two above conditions.

This is the case of the numeration system considered in Example 2.41.

Unfortunately, the connection to alternate bases is harder than in the case where $p = 1$, as intermediate representations of 1 can now appear as the words $\mathbf{a}_i$ used above, which was not the case previously: see [CCS22, Theorem 2], where only the greedy and quasi-greedy expansions of 1 are mentioned. For instance, the language $M = \{\varepsilon, 1\} \cup 11(00)^\omega \cup 111(00)^\omega$ is the language of maximal words of a positional numeration system by Lemma 1.12, and this numeration system would be 2-Bertrand by item (b) of Conjecture 9.1. However, no alternate base of period 2 has 110^ω and 1110^ω as its greedy expansions of 1. Rather, we have that the base $(3/2, 2)$, that was already mentioned in passing in item (e) of Section 5.5.3, has $\mathbf{w}_{0,0} = 110^\omega$ and $\mathbf{w}_{1,1} = 1110^\omega$. As a consequence, we cannot directly use the results of Chapter 3, such as Theorem 3.19. We would need an additional lemma of the following form.

Conjecture 9.3. *Let $\mathbf{a}_{p-1}, \dots, \mathbf{a}_0$ be p words that satisfy the lexicographic inequalities $\sigma^j(\mathbf{a}_i) \leq_{\text{lex}} \mathbf{a}_{i-j}$ for all $j \in \mathbb{N}$ and $i \in \{0, \dots, p-1\}$ where the indices are taken modulo p. Then there exists an alternate base $(\beta_{p-1}, \dots, \beta_0)$ and some indices $j_i \in \mathbb{N}$ for $i \in \{0, \dots, p-1\}$ such that*

$$\mathbf{a}_i = \mathbf{w}_{i,j_i}$$

where the intermediate representations $\mathbf{w}_{i,j}$ were defined in Definition 2.12.

Compared to Theorem 3.19, we no longer require strict inequalities for finite representations, and we no longer require representations to be greater than 10^ω, to obtain the analogues of the trivial Bertrand numeration system, the first item of Theorem 1.45.

Assuming that Conjecture 9.3 or a similar enough conjecture holds, every Bertrand numeration system is associated with an alternate base. Since we can define the $\mathcal{B}$-shift for an alternate base $\mathcal{B}$ (see e.g. [Cha23]), and since Conjecture 9.1 deals with the maximal words in the language, it should be possible to generalize the characterization of Bertrand numeration systems as those whose language is the language of a β-shift. As in the case $p = 1$, we will need to pay close attention to noncanonical shifts (see [CCS22]). We suggest the following definition.

Definition 9.4. Let $\mathcal{B}$ an alternate base and select intermediate representations $\mathbf{w}_{i,j_i}$ for i in $\{0, \dots, p-1\}$, allowing the quasi-greedy representation as the intermediate representation of index ∞ if it is not already present. Assume that the chosen representations $\mathbf{w}_{i,j_i}$ satisfy $\sigma^k(\mathbf{w}_{i,j_i}) \leq_{\text{lex}} \mathbf{w}_{i-k,j_{i-k}}$ for all k in $\mathbb{N}$ and i in $\{0, \dots, p-1\}$, counting the indices modulo p. Let

$$S_{\mathcal{B}} = \{\mathbf{x} : \sigma^i(\mathbf{x}) \leq_{\text{lex}} \mathbf{w}_{i,j_i} \ \forall i\} \quad \text{and} \quad \Sigma_{\mathcal{B}} = \bigcup_{i=0}^{p-1} S_{\sigma^i(\mathcal{B})}$$

where the terms in the union are all based on the same choice of intermediate representations. Then $\Sigma_{\mathcal{B}}$ is a *noncanonical $\mathcal{B}$-shift.*

Conjecture 9.5. *A numeration system is Bertrand if and only if there exists an alternate base $\mathcal{B}$ and a noncanonical $\mathcal{B}$-shift $\Sigma_\mathcal{B}$ such that $L_U = \mathrm{Fac}(\Sigma_\mathcal{B})$.*

Let us now consider the side of Dumont–Thomas numeration systems. From a candidate regular language of maximal words, such as

$$\{\mathrm{Pref}_{np+i}(\mathbf{w}_{i,j_i}) : 0 \leq i \leq p-1,\, n \geq 0\},$$

we know that we can build an automaton accepting words whose every suffix is lexicographically less than or equal to the corresponding maximal word (see the proof of Proposition 6.15, illustrated in Example 6.19). Notice that in the case of a p-Bertrand numeration system, this construction can be adapted to not need the auxiliary automata P_j and S_j from the proof, leaving us with a simple automata, as was seen in Figure 2.3 coming from the other direction. Note that this construction only stands for Parry alternate bases. This automaton can then be converted into a substitution using usual methods. From the shape of the automaton, it can be seen that at a given level in any associated tree $\mathcal{T}_{\mu,a}$ (defined in Section 1.4), only one nonfinal letter can be found (nonfinal letters were defined in Section 7.5). As a result, the Dumont–Thomas numeration system defined from such a substitution associated with such a Bertrand numeration system would be positional, and would have a periodic point of period p.

Let us call *p-Fabre-like* substitutions similar to the ones defined by the above process: the alphabet is divided into p groups numbered 0 through $p-1$, each group contains one nonfinal letter a_i, and the image of a letter in group i is a power of a_{i-1} followed by some nonfinal letter belonging to group $i-1$, where group -1 is identified with group $p-1$. Those substitutions have a periodic point of period p and define Dumont–Thomas numeration systems that are positional. It is not unreasonable to make the following conjectures, generalizing those of Section 7.5.

Conjecture 9.6.

(a) All positional Dumont–Thomas numeration systems over $\mathbb{N}$ that correspond to a periodic point of period p are obtained from p-Fabre-like substitutions.

(b) There is a lexicographic condition on the coefficients of a p-Fabre-like substitution, such that a positional Dumont–Thomas numeration sys-

tem associated with such a substitution corresponds to a p-Bertrand numeration system (up to the number of leading zeros) exactly when the substitution satisfies this condition.

(c) *Every p-Bertrand numeration system associated with a Parry alternate base is, up to the number of leading zeros, equal to such a Dumont–Thomas numeration system.*

Based on observations, we conjecture one final link, where Dumont–Thomas systems, by corresponding to p systems depending on the chosen length of representations, make apparent a connection between p different Bertrand numeration systems.

Conjecture 9.7. *Consider an alternate base $\mathcal{B}$ and a selection of intermediate representations $\mathbf{w}_{i,j_i}$ giving rise to a substitution and a positional Dumont–Thomas numeration system. The p systems obtained by changing the lengths of representations in the Dumont–Thomas system correspond to the p shifts of the underlying alternate base.*

The conjectures in this section seem approachable and will be our next priority.

9.2 More work on Dumont–Thomas numeration systems

The generalization of Dumont–Thomas systems started in [LL24a] is not completely understood, and some choices made deserve further attention and point to possible further generalizations. We present a number of linked problems.

Problem 1. We have proved in Proposition 7.32 that Bertrand numeration systems *associated with Parry numbers* correspond to Dumont–Thomas numeration systems. When β is not a Parry number and has $d_\beta(1) = d_1 d_2 \cdots$, one can construct a substitution on an infinite alphabet by setting $\mu(n) = 0^{d_{n+1}}(n+1)$ for all n. While we have only defined Dumont–Thomas numeration systems for substitutions on finite alphabet, there is no clear objection to their definition on substitutions with infinite alphabets. What can be said about these systems? The study of Bertrand numeration systems associated with non-Parry numbers could be a motivation for the study of such objects.

The method of defining substitutions in Chapter 7 and Section 9.1 above is reminiscent of the substitutions that S-adically define $\mathcal{B}$-integers in [CCMP25]. This inspires two remarks.

Problem 2. First, it is sometimes possible to factorize a substitution with a periodic point of period p into p substitutions that alternate between p alphabets. When starting from a positional Dumont–Thomas numeration system, can these p factored substitutions be interpreted in any way?

Problem 3. Second, we can associate a sequence of trees with an S-adic sequence, as well as we can with a substitution. Consider the expression $\lim_{n\to\infty} \mu_0\mu_1\cdots\mu_{n-1}(a_n)$. We associate with $\mu_0\mu_1\cdots\mu_{n-1}(a_n)$ a tree of height n, where the root is a_n and the children of a node labeled b at height h are given by the image $\mu_{n-1-h}(b)$. If $\mu_n(a_{n+1})$ starts with a_n, the trees obtained in this fashion are nested into a graph that extends infinitely upwards, and we can define the representation of n as the label of the shortest path from a node in column 0 to the node in column n in the last row of the tree. In a sense, rather than extending the tree downwards by iterating a substitution, we extend the tree upwards using a S-adic process. What does the numeration system we obtain look like, and what properties does it have?

Through personal communications by Émilie Charlier and Olivier Carton, we are aware that a team composed of Olivier Carton, Jake Sudbery and Reem Yassawi is currently working on numeration systems based on Bratteli diagrams. This goal could be related to Dumont–Thomas systems, with Bratteli diagrams acting as a generalization of trees. This will have to be revisited once the results of this team are published.

Problem 4. Dumont–Thomas numeration systems were originally defined for real numbers as well [DT89]. It would be interesting to explore this aspect of our generalized Dumont–Thomas systems. We know from personal communications with Sébastien Labbé that he is personally interested in this type of question, as he is quite familiar with connections between the original Dumont–Thomas systems and various other combinatorial objects.

Problem 5. Finally, we must question a choice that has not been discussed yet as it was less relevant to the core of this thesis. We have created representations of negative numbers in Dumont–Thomas systems by prefixing 1 as a sign bit to the representations of negative numbers. This was inspired by the two's complement numeration system. However, it is not clear that this way of proceeding is appropriate for more general systems. Consider for instance a 10-uniform substitution and the associated Dumont–Thomas system. With our definition, the paths that go to column -1 in the tree are those with a label in $1 \cdot 9^*$. Would it not make more sense to consider 9 as a sign bit here, similarly to how $\cdots 999$ is the 10-adic representation of -1? This question seems to tie back to the question of "upwards-expanding trees" and could also be linked to another aspect of Carton, Sudbery and Yassawi's research, being the extension of p-adic numbers to U-systems.

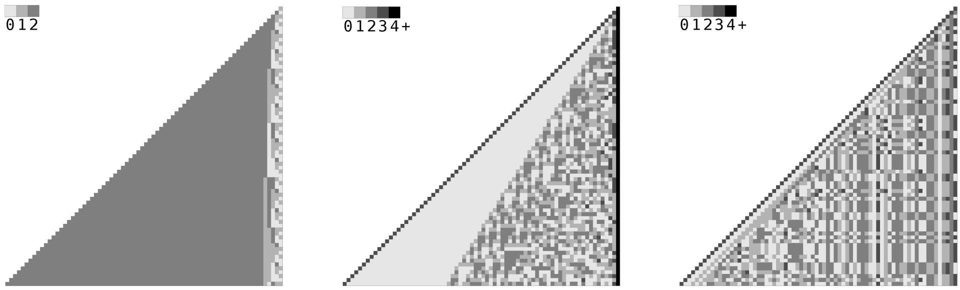
Similarly, we have used the term Dumont–Thomas *complement* numeration systems by analogy with the two's complement numeration system, but we have not verified whether the main properties that make two's complement useful are preserved by our more general family of systems. In particular, can the addition of integers be performed similarly to the addition of natural numbers? For the analogue of the two's complement numeration system in the Fibonacci context, this question was answered positively in [LL23].

9.3 The language $\mathrm{Max}(L_U)$ when U has a dominant root

In Chapter 6, we have used the language of maximal words $\mathrm{Max}(L_U)$ as a tool to study the regularity of the language L_U. However, the language $\mathrm{Max}(L_U)$ itself exhibits striking structure, and deserves supplemental study, even in the case where $p = 1$.

In this section, the figures depict an initial segment of the language $\mathrm{Max}(L_U)$ as a colored discrete triangle. Row n of the triangle has n squares and represents the word $\mathrm{rep}_U(U_n - 1)$. Each square is colored in a different color according to the corresponding letter of the word $\mathrm{rep}_U(U_n - 1)$. See Figure 9.1 for an example.

Problem 6. Let us put ourselves in the case of a U-system with a dominant



(a) $U_0 = 1$, $U_1 = 2$
$U_n = 4U_{n-1} - 3U_{n-2}$

(b) $U_0 = 1$, $U_1 = 6$
$U_n = 5U_{n-1} - 6U_{n-2}$

(c) $U_0 = 1$, $U_1 = 4$
$U_n = 6U_{n-1} - 9U_{n-2}$

Figure 9.1: Three example numeration systems and their language of maximal words. The eigenvalues of the three systems are respectively 3 and 1, 3 and 2, and 3 with multiplicity 2.

root β. Theorem 1.41 ensures that $\text{rep}_U(U_n - 1)$ shares a common prefix with some intermediate representation $\mathbf{w}_j$ of 1. The length of this common prefix can be made arbitrarily large, if one increases n enough.

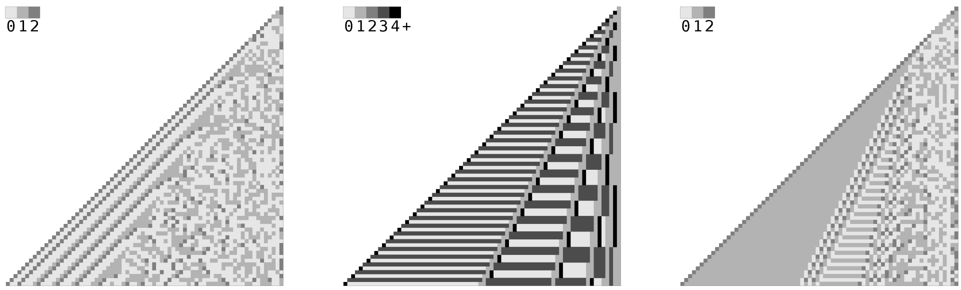
This result does not, however, provide more information on the length of this common prefix. For a finite word x, let $\alpha(x)$ be the longest common prefix between x and some $\mathbf{w}_j$ ($j \in \mathbb{N}$), and let $\omega(x)$ be such that $x = \alpha(x)\omega(x)$. What can we then say on $|\alpha(\text{rep}_U(U_n - 1))|$ as a function of n?

As seen on Figure 9.1, it appears that this quantity is asymptotically linear in n. Other examples, displayed on Figure 9.2, appear to support the following conjecture.

Conjecture 9.8. *Let U be a linear positional numeration system, with a dominant root β that is a Parry number. Let γ be the dominant root of the minimal polynomial of the recurrence sequence $\Delta_P(U)$, where P is the canonical β-polynomial, or $\gamma = 1$ if $\Delta_P(U)$ is the zero sequence. Then*

$$\lim_{n\to\infty} \frac{|\omega(\text{rep}_U(U_n - 1))|}{n} = \frac{\log(\gamma)}{\log(\beta)}.$$

Unfortunately, this conjecture cannot be true as is, because the "second dominant root" γ can itself exhibit alternation, and therefore appear only at some classes modulo p', where p' occupies a role similar to p but for the second dominant root. See Figure 9.3 for a case where this behavior occurs.



(a) Based on a Pisot number studied by Boyd

(b) Based on the polynomial $(x-4)(x-2)$

(c) Based on the polynomial $(x^2-x-1)(x^2-3x+1)$

Figure 9.2: Representations of $\mathrm{Max}(L_u)$ for three numeration systems.
Left: System based on a number studied by Boyd [Boy96a]. The Pisot number β that is the positive root of the polynomial $P(x) = x^5 - 2x^4 - x^3 + x^2 + x - 1$ is such that $d_\beta(1) = 200(20001)^\omega$. Its canonical β-polynomial is $P(x)(x^3 + x + 1)$. The associated Bertrand system justifies the choice of the canonical rather than the minimal polynomial of β in Conjecture 9.8. The displayed system is based on the polynomial $P(x)(x^3 + x + 1)(x^2 - x - 1)$ and the initial conditions $(1, 3, 6, 11, 24, 50, 110, 224, 550, 1000)$. The prefix matches $200(20001)^\omega$ on a $(1 - \frac{\log\varphi}{\log\beta})$-fraction of the maximal word.
Center: System obtained from $U_n = \frac{2}{3}4^n + \frac{1}{3}(-2)^n$. The changes in behavior ("cuts") occur at fractions $1/2, 3/4, 7/8, \ldots$ of the maximal word.
Right: System obtained from the product of the minimal polynomials of φ and φ^2, with the initial conditions $(1, 3, 8, 13)$. The cuts occur at fractions $1/2$ and $3/4$ of the maximal word.

Depending on the parity of n, $\dfrac{|\omega(\mathrm{rep}_U(U_n - 1))|}{n}$ converges to 0 or 1/2.

We offer three other questions related to the structure of the language $\mathrm{Max}(L_U)$.

Problem 7. First, let us consider the language

$$\{\omega(\mathrm{rep}_U(U_n - 1)) : n \in \mathbb{N}\}.$$

We will left-align those words and consider the right-infinite words $\omega(\mathrm{rep}_U(U_n - 1))0^\omega$. What can be said about them? Going back to Figure 9.1, on the

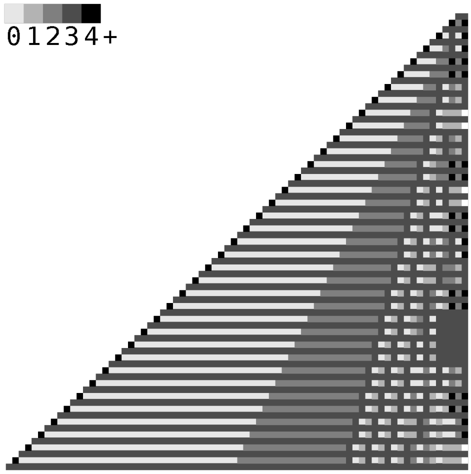
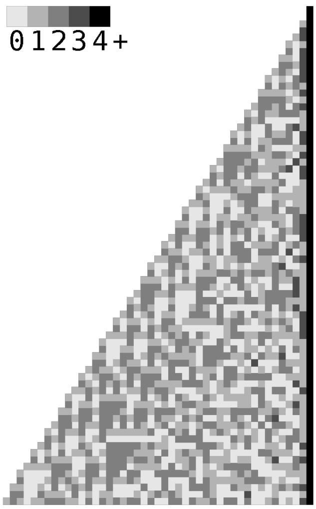


Figure 9.3: Representation of $\mathrm{Max}(L_U)$ for $U_n = 3 \cdot 4^n - 2^n - (-2)^n$. The sequence $\Delta_{x-4}(U)$ doesn't have a dominant root, since the roots of its minimal polynomial are 2 and -2. The behavior of $\mathrm{rep}_U(U_n - 1)$ depends on the parity of n.

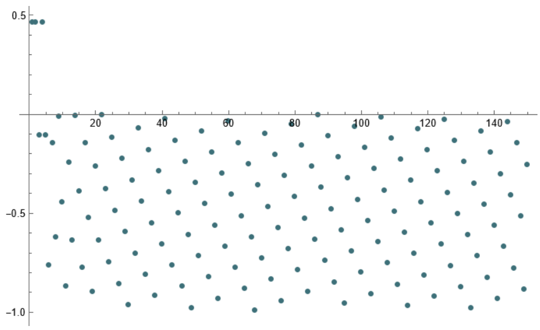
leftmost example, all words $\omega(\mathrm{rep}_U(U_n-1))$ are ordered in decreasing lexicographic order when grouping them by length. On the rightmost example, the same is true when grouping the words by common length of $\alpha(\mathrm{rep}_U(U_n-1))$ instead. Finally, on the middle example, the situation is more complicated. We can study the value of the words $\omega(\mathrm{rep}_U(U_n-1))0^\omega$ in the Rényi base β which is 3 in this case. Since $\alpha(\mathrm{rep}_U(U_n-1))$ is of the form 30^* in this case, $\omega(\mathrm{rep}_U(U_n-1))$ must start in 1 or 2, and thus has a value between 1/3 and 1. Taking the logarithm in base 3 to normalize the values between -1 and 0, we observe the appearance of a rotation by an angle of $\frac{\log 2}{\log 3}$, as seen in Figure 9.4.

Problem 8. Second, we have considered the regularity of the language $\mathrm{Max}(L_U)$. After a seminar on this topic, Michel Rigo asked us the seemingly innocent question, "When is this language context-free"? Just as slender regular languages are those of the form $\bigcup_{i=1}^k x_i y_i^* z_i$, slender context-free languages are the ones of the form

$$\bigcup_{i=1}^{k} \{u_i v_i^n w_i w_i^n y_i : n \in \mathbb{N}\},$$

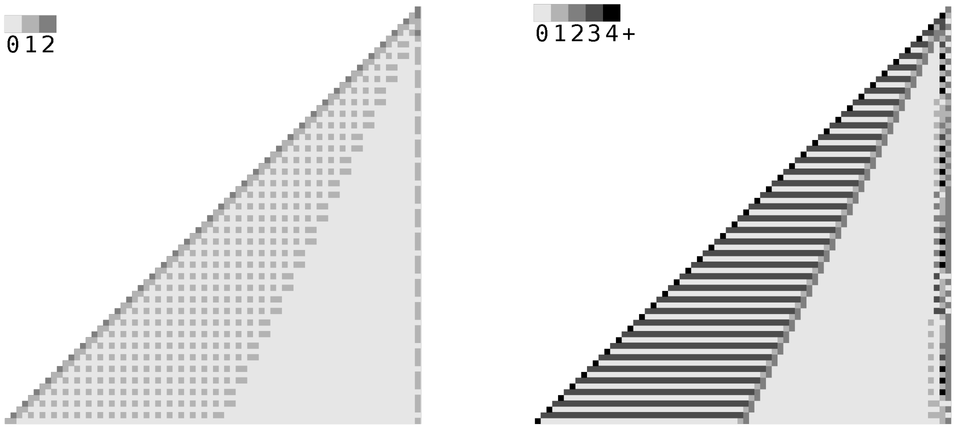


(a) Suffixes $\omega(\text{rep}_U(U_n - 1))$ in item 9.1b in Figure 9.1.



(b) Plot of $\log_3(\text{val}_3(\omega(\text{rep}_U(U_n - 1))0^\omega))$ as a function of n.

Figure 9.4: Rotation phenomenon appearing in the values of the suffixes $\omega(\text{rep}_U(U_n - 1))$. The right figure corresponds also to the plot of $T^n(x)$, where T is a circle rotation of angle $\frac{\log 2}{\log 3}$.



(a) $U_0 = 1$,
$U_{2n+1} = 3U_{2n} - n$,
$U_{2n} = U_{2n-1} + U_{2n-2} + 2$

(b) $U_n = \frac{4}{3}4^n + \frac{2}{3}(-2)^n - 1$

Figure 9.5: Left: a numeration system without a dominant root where the language $\mathrm{Max}(L_U)$ is regular.
Right: a near-miss in the case with a dominant root.

as shown by Ilie [Ili94]. Seeing how, according to Conjecture 9.8, the words $\mathrm{rep}_U(U_n - 1)$ decompose in two "phases", a prefix matching an intermediate β-representation of 1 and a suffix whose behavior is not completely understood, perhaps some of the languages $\mathrm{Max}(L_U)$ could be context-free. In the case without a dominant root, we have found such an example, presented in Figure 9.5a. However, it is not obvious if such languages exist in the case with a dominant root. The context-freeness condition imposes both strong conditions on the eigenvalues of the sequence U and on the coefficients of its minimal polynomial. It is unclear if both these strong, seemingly independent conditions can be simultaneously satisfied. We have been unable to provide either an example or a proof of impossibility.

Problem 9. Finally, we pose one much more particular question. Let us consider the system displayed on Figure 9.1c. The associated language is clearly not regular as the sequence $\Delta_{x-3}(U)$ is not bounded, hence not ultimately periodic. Yet, the set $\mathrm{Max}(L_U)$ appears to possess some kind of order. In particular, it appears that the *suffixes* of the $\mathrm{rep}_U(U_n - 1)$ also converge to some left-infinite word, not just the prefixes. For instance, the words $\operatorname{rep}_U(U_n - 1)$ agree on a suffix of length 500 for all n in $\{1500, 2000\}$. We have not been able to replicate or understand this behavior. Can we explain it and characterize the systems that show it?

9.4 Study of $\mathcal{B}$-integers

The set of $\mathcal{B}$-integers has been a regular presence in this document, appearing in Section 3.7 and Chapter 5 and through its defining substitutions in Chapter 7. As proved in [CCMP25] and mentioned in Section 3.7, the set of $\mathcal{B}$-integers can be coded by an infinite word. In case that $p = 1$, these words are known under the name *Parry words*, or *k-bonacci-like words*, and many combinatorial properties of these words have been described, e.g., factor complexity in [FMP04, KP09, BMP07], palindromic complexity in [AMPF06, BM09], abelian complexity in [BBT11, Tur13, Tur15], critical exponent in [BKP09, BKP11], etc.

Problem 10. These above properties are particularly nice in the case where the base is chosen to be a confluent Parry number. For example, the infinite word is closed under reversal as seen [FMP04], and it has constant number of return words as seen in [BPS08]. As the maximal digit property of Chapter 5 generalizes the concept of confluent Parry numbers, it is intriguing to study the combinatorial features of infinite words coding the $\mathcal{B}$-integers in alternate bases with MDP.

In a similar vein, can we extend the MDP further and obtain similar properties for nonalternate Cantor bases? What does the word coding $\mathcal{B}$-integers look like in this case?

Problem 11. In Section 3.7, we have explained how some Arnoux-Rauzy words can be obtained as faithful codings of $\mathcal{B}$-integers in some Cantor bases. The specific case of standard Sturmian sequences, singled out in Corollary 3.26, was already proved in [CCMP25]. In the same article, even some nonstandard Sturmian sequences were presented as coding of $\mathcal{B}$-integers. It would be interesting to describe how diverse the set of Sturmian words is hidden among the words encoding $\mathcal{B}$-integers.

9.5 Other perspectives

In this section, we present a more disconnected assortment of questions that arose in the context of our research, but that do not belong to a more elaborate research program like the four above.

Problem 12. *Abstract numeration systems* are defined by fixing an infinite regular language $L \subset A^*$ and ordering it using the radix order. Then, if $L = \{w_0, w_1, w_2, \ldots\}$ with $w_0 <_{\text{rad}} w_1 <_{\text{rad}} w_2 <_{\text{rad}} \cdots$, the representation map simply sends j to w_j, and the evaluation map sends w_j to j. These systems were introduced in [LR01] as a way to generalize recognizability to nonstandard numeration systems. It is a long-standing problem to find which abstract numeration systems are positional in the sense of the evaluation map being expressible as a dot product with a sequence of weights, see for instance [LR10, Exercise 3.13].

The arguments developed in Chapter 7 could provide additional insight on this problem. As a first step, languages that are *prefix-closed* and *right-extendable* can be represented by a tree, in such a way that the representation of n is the label of a shortest path to column n in the tree. In this setting, we say that a language is *prefix-closed* if the prefixes of any word in the language are still in the language, and it is *right-extendable* if for any word w in the language there exists a letter a such that wa is still in the language.

Do the arguments used in the proof of Theorem 7.19 transpose to this case? If yes, can they be generalized further to provide criteria for the positionality of more general abstract numeration systems?

Problem 13. In Chapter 4, the condition that the elements $\beta_{p-1}, \ldots, \beta_0$ of an alternate base be inside the extension of the field $\mathbb{Q}$ by the product δ of the bases was quite relevant. This condition appeared once again in Example 6.57, where it was crucial for our methods to express all relevant quantities within $\mathbb{Q}(\delta)$.

This raises the following question: when considering an alternate base *associated with some U-system*, is it necessarily the case that the base elements lie in $\mathbb{Q}(\delta)$, where δ is the product $\lim_{n\to\infty} \frac{U_{n+p}}{U_n}$ of the base elements? Computer experiments point to the answer being yes, but we have not been able to prove it. Of course, if the base is not associated with a U-system, the base elements can be freely chosen, and there is no reason for them to lie in

$\mathbb{Q}(\delta)$. We know that independently of a link to U-systems, this condition is satisfied for Parry alternate bases [CCMP23, Theorem 14].

Problem 14. In the article that was built upon in Chapter 4, Schmidt conjectured that for real bases β that are *Salem* numbers rather than Pisot, we still have that $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$ [Sch80]. Even though some work has been done, this problem is still open today, as well as the partial problem to know whether all Salem numbers are Parry numbers; see for instance [Boy89, Boy96b, Hic14, Váv21].

Analogous questions can be asked in the framework of alternate bases. Namely, is it true that if the product of the bases $\delta = \prod_{i=0}^{p-1} \beta_i$ is a Salem number and if $\beta_{p-1}, \ldots, \beta_0 \in \mathbb{Q}(\delta)$, then $\mathrm{Per}(\mathcal{B}) = \mathbb{Q}(\delta) \cap [0, 1)$? More specifically, is it true that under the same conditions, the alternate base $\mathcal{B}$ is Parry?

Problem 15. To close out Chapter 6, we explained that in general it was impossible to obtain criteria in the style of Hollander for systems without a dominant root. Nevertheless, is it possible to obtain partial results, or results in the style of Section 8.2 in this case? For example, can we point to some polynomials for which the associated U-systems definitely have a nonregular language, no matter what initial conditions are chosen?

Problem 16. In Section 8.1, our proof mimics that of the article [FV11] and is therefore quite extensive. Does there exist a shorter, more elegant proof that leverages the results of [FV11] directly, rather than walking a lengthy parallel path?

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Index

N-continued fractions, 99
S-adic sequence, 94, 284
U-system, *see* positional numeration system
$\mathcal{B}$-integers, *see* alternate base numeration system, Cantor numeration system
β-expansion, 19
  of 1, 22
  quasi-greedy, 22
β-polynomial, 38, 185, 206, 267
  canonical, 38
  extended, 38
β-shift, 24
  noncanonical, 25
β-transformation, 24
a-admissible sequence, 27

adjacency matrix, 9
admissible sequence, 27
algebraic number, 103
alternate base numeration system, 42–51, 73–160
  $\mathcal{B}$-integers, 127, 291
  $\mathcal{B}$-shift, 281
  associated rewriting system, 135, 142, 143
  associated to a positional numeration system, 54
  evaluation map, 42
  existence and uniqueness, 73–94
  graph G, 46
  greedy algorithm, 43
  maximal digit property, 119–160
  optimality, 128
  Parry, 49, 177
  quasi-greedy algorithm, 46
  spectrum, 112, 127
  two-way, 50
  ultimately periodic expansions, 101–115
automaton
  $\mathcal{A}_\beta$, 35
  constructed from maximal words, 173–177
  deterministic finite (DFA), 6
  nondeterministic finite (NFA), 6

Bertrand numeration system, 32–38, 171, 251, 279–283
  noncanonical, 34
Binet's formula, 15

Cantor numeration system, 49–51, 76
  $\mathcal{B}$-expansion, 77
  $\mathcal{B}$-integers, 94–100
  evaluation map, 77
  two-way, 50

combinatorics on words, 3–10
cone, *see also* Furstenberg theorem, 271
confluence, *see also* alternate base numeration system, Parry number, rewriting system, 134–158
core rule, *see* rewriting system
cylinder, 8

dominant root, 31, 185, 203–206, 266–278, 285–291
Dumont–Thomas numeration system, 26–30, 59–71, 225–255, 282–285
  associated with μ, $\mathbf{u}$ and r, 65, 67
  complement, 61, 284
  positionality, 233–255
  representation map, 28, 29, 61, 65

evaluation map, 10

Fabre substitution, *see* substitution
faithful coding, 94–100
finiteness property, *see* Property (F), Rényi numeration system
Furstenberg theorem, 84

Galois conjugate, 103
genealogical order, 5

Hollander's theorem, *see* dominant root

intermediate representations, 22

Kakeya sequence, 159
Kepler limit, 258
Kleene star, 7

language, 5
  factorial, 5
  of maximal words, *see* positional numeration system
  of numeration system, *see* numeration system
  prefix-closed, 5
  regular, *see also* positional numeration system, 7
  right-extendable, 5
  slender, 172
  suffix-closed, 5
  thin, 177
lexicographic order, 5
lexicographically maximal words, *see* positional numeration system
linear recurrence sequence, 14, 258–266
  characteristic polynomial, 14
  eigenvalue, 15
  minimal polynomial, 14
  strict, 15

maximal digit property, *see* alternate base numeration system
Meyer set, 121
morphism, *see also* substitution, 9
  fixed point of, 10
  periodic point of, 10
  primitive, 9
  seed, 10

Newman's lemma, 150
normalization, 21, 121, 138
numeration system, 10
  $(\varphi, 3, \varphi)$, 44, 47, 55, 57
  $d_0 = 201$, $d_1 = 11$, 44, 47, 54, 57, 67

$d_0 = 3121$, $d_1 = 231$, $d_2 = 123$, 126
$d_0 = 533$, $d_1 = 353$, $d_2 = 435$, 126, 128, 129, 137
p-Bertrand, 71, 279–283
abstract, 163, 226, 292
alternate base, *see* alternate base numeration system
Bertrand, *see* Bertrand numeration system
Cantor, *see* Cantor numeration system
Dumont–Thomas, *see* Dumont–Thomas numeration system
Fina, 13, 18, 32, 34, 39
golden base, 19, 24–26, 32, 34, 107, 112
language of, 13
linear, 170
positional, *see also* positionality, *see* positional numeration system
Rényi, *see* Rényi numeration system
two's complement, 228
Zeckendorf, 13, 17, 32, 34, 38, 39

optimality
alternate base numeration system, 128
Kakeya sequence, 159
Rényi numeration system, 122

Parikh vector, 4
Parry alternate base, *see* alternate base numeration system
Parry conditions, 23, 48, 79–81
Parry number, 34
confluent, 121
Pisot number, 103
positional numeration system, 12–18, 51–59, 161–223
associated alternate base, 54
evaluation map, 12
greedy algorithm, 12
lexicographically maximal words, 15, 31, 56, 171
regularity of language, 161–223, 293
with dominant root, *see* dominant root
positionality, *see also* Dumont–Thomas numeration system, 12, 228
Property (F), 120, 158

Rényi numeration system, 18–26, 115–117, 154–158
β-integers, 121
associated rewriting system, 154
evaluation map, 18
finiteness property, 120
greedy algorithm, 19
optimality, 122
spectrum, 116
two-way, 26
radix order, 5
regular operations, 7
representation map, 10
rewriting system, 135
allows normalization, 138
associated with a Parry number, 154
associated with an alternate base, 135, 142, 143
confluence, 135

core rule, 135, 142, 143, 154
rule width, 144

Salem number, 103, 293
seed, 10
shift, 9, 45
spectrum, *see* alternate base numeration system, Rényi numeration system
subshift, 9
of finite type, 9
sofic, 9
substitution, *see also* morphism, 9, 27
p-Fabre-like, 282
associated tree, 28, 60
conjugate, 252
Fabre, 37, 98
Fabre-like, 249
Fibonacci, 29, 38, 249
nonfinal letter, 245
Tribonacci, 98, 249

words
Arnoux-Rauzy, 97
distance on, 8
dynamics on, 8
generalities on, 3
indexing of, 4
infinite, 4
language of, 5
morphism on, 9
order on, 5
topology on, 8
two-sided, 4

Index of notation

$(\gamma_n)_{n\in\mathbb{Z}}$ (Furstenberg), 84
$(f_n)_{n\in\mathbb{Z}}$ (Furstenberg), 84
A^*, A^+, 3
$A^{\mathbb{N}}$, $A^{\mathbb{Z}}$, 4
L^*, 7
$L_{\mathcal{A}}$, 6
M_μ, 9
Pref, Pref_i, 3
Suff, Suff_i, 3
• (separator), 4
$\cdot$ (concatenation), 3
$\cdots u_{-3}u_{-2}u_{-1} \mid u_0u_1u_2\cdots$, 4
$|w|$, 3
$|w|_a$, 4
$<_{\mathrm{lex}}$, $\leq_{\mathrm{lex}}$, 5
$<_{\mathrm{rad}}$, 5
$\mathcal{A}$ (automaton), 6
μ (morphism), 9
$\oplus$, 135
ψ (Parikh vector), 4
rep, 10
σ (shift), 9
val, 10
ε, 3
${}^\omega w$, 5
w^ω, 5
w^{-1}, 3
w_n, $w_{[i,j]}$, 4
alternate base numeration system
$(\eta_m)_{m\geq 1}$, 106
$(r_n)_{n\geq 0}$, 103
$A_{\mathcal{B}}$, 44
B (alphabet), 135
G (graph), 46
$X^{\Delta}(\delta)$, 112
$X_{\mathcal{B}}$, 127
$\mathcal{B}$, 42
$\mathrm{Dig}(\mathcal{B})$, 106
$\mathbb{N}_{\mathcal{B}}$, 127
$\mathrm{Per}(\mathcal{B})$, 104
$\mathbf{d}_i$, d_i, d'_i, $\mathbf{d}^*_i$, 46
δ (product), 42
ℓ_i (length), 46
$\langle\cdot\rangle_{\mathcal{B}}$, 50
μ (routing), 46
$\rho_{\mathcal{B}}$, 135
$\rho'_{\mathcal{B}}$, 142
$\rho''_{\mathcal{B}}$, 143
$\sigma^n(\mathcal{B})$, 45
$\mathbf{w}_{i,j}$, 46
$d_{i,j}$ (digits), 46
$f_{\mathcal{B}}$, 105
$k_{i,j}$ (length), 46
$m_{i,j}$ (routing), 46, 169
$t_{i,j}$ (digits), 46
$w_{i,j}, w'_{i,j}$, 143

Cantor numeration system
- $(\varphi_m)_{m\in\mathbb{N}}$ (S-adic sequence), 94
- $\mathcal{B}$, 76
- $\Delta_{m,n}$ ($\mathcal{B}$-integers), 94
- $\mathbf{d}_n$, $\mathbf{d}_n^*$, 78
- $\langle x\rangle_{\mathcal{B}}$, 77
- π_m ($\mathcal{B}$-integers), 94
- $\sigma(\mathcal{B})$, 77
- $d_{n,i}$, 78
- $t_{n,i}$, 78

Dumont–Thomas numeration system
- $((m_i, a_i))_{i\in\{0,\dots,k\}}$, 27
- E_j (set), 238
- E_μ, 245
- $\mathcal{T}_{\mu,a}$, 28
- $\mathcal{T}_{\mu,b|a}$, 60
- μ_β, μ'_β, 37
- $\mathrm{rep}_{\mu,r}$, 231

positional numeration system
- A_U, 13
- L_U, 13
- L_i, 178
- $L_{i,c}$, 180
- M, M' (routing), 193, 208
- $P_{q,m}$, 185
- Δ_i ($\mathbf{d}_i$ finite), 186
- $\Delta_i^{(j)}$ ($\mathbf{d}_i$ finite), 198
- Δ_P, 268
- $\Delta_{i,e,j}$ ($\mathbf{d}_i$ finite), 200
- $\Delta_{i,q,m}$ ($\mathbf{d}_i$ infinite), 181
- $\Gamma_{i,e,j}$, 210
- $\mathrm{Max}(L)$, 15
- $\mathrm{rep}_{i,c}(n)$, 55
- $x_e y_e^* z_e$, 187
- $x_{h,e} y_{h,e}^* z_{h,e}$, 193, 208

Rényi numeration system
- A_β, 19
- S_β, 24
- S'_β, 25
- T_β, 24
- $\mathrm{Fin}(\beta)$, 120
- $\mathrm{Per}(\beta)$, 102
- $\langle x\rangle_\beta$, 26
- $\mathcal{A}_\beta$, 35
- μ_β, μ'_β, 37
- $\mathbf{w}_j$, 22
- $d_\beta(x)$, 19
- $d_\beta^*(1)$, 22