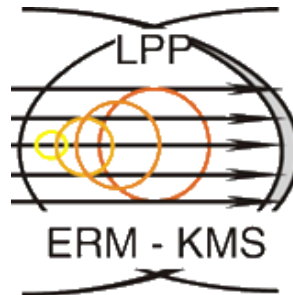


Numerical investigation of finite element formulations for ICRH plasma heating

Jules Zaleski^a, Bernard Reman^b, Philippe Lamalle^b and Christophe Geuzaine^a

^a Department of Electrical Engineering and Computer Science, University of Liège, Belgium

^b Laboratory for Plasma Physics, Royal Military Academy, Brussels, Belgium

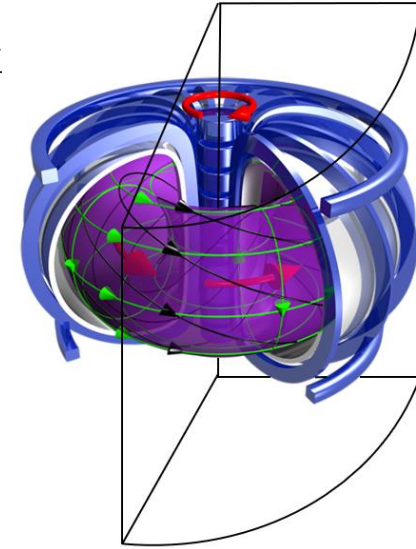


Advanced Computational Methods in ENgineering and Applied Mathematics

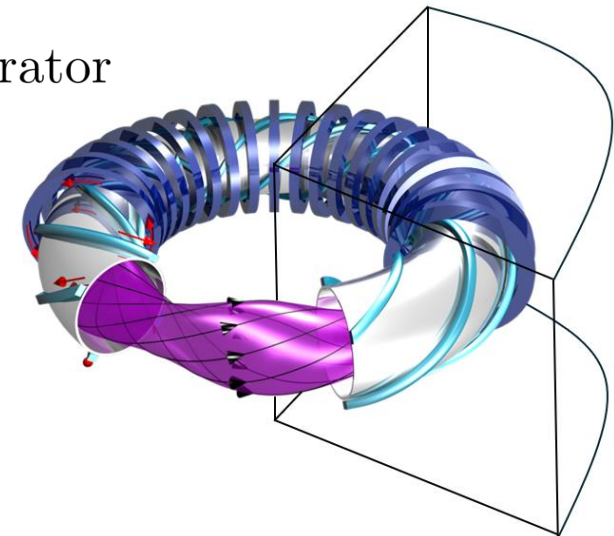
17th of September 2025

- Common **fusion reactors** are Tokamaks or Stellarators with high temperature confined plasma
- Both reactors use superconducting magnets to **confine the plasma** in the toroidal shape (Tokamak) or helical shape (Stellarator)
- One way to heat up the plasma is Ion Cyclotronic Resonance Heating (**ICRH**).
By exciting the particles at resonance frequency using electromagnetic waves
- The **electromagnetic field / plasma interaction** is complex and accurate numerical models are necessary

Tokamak

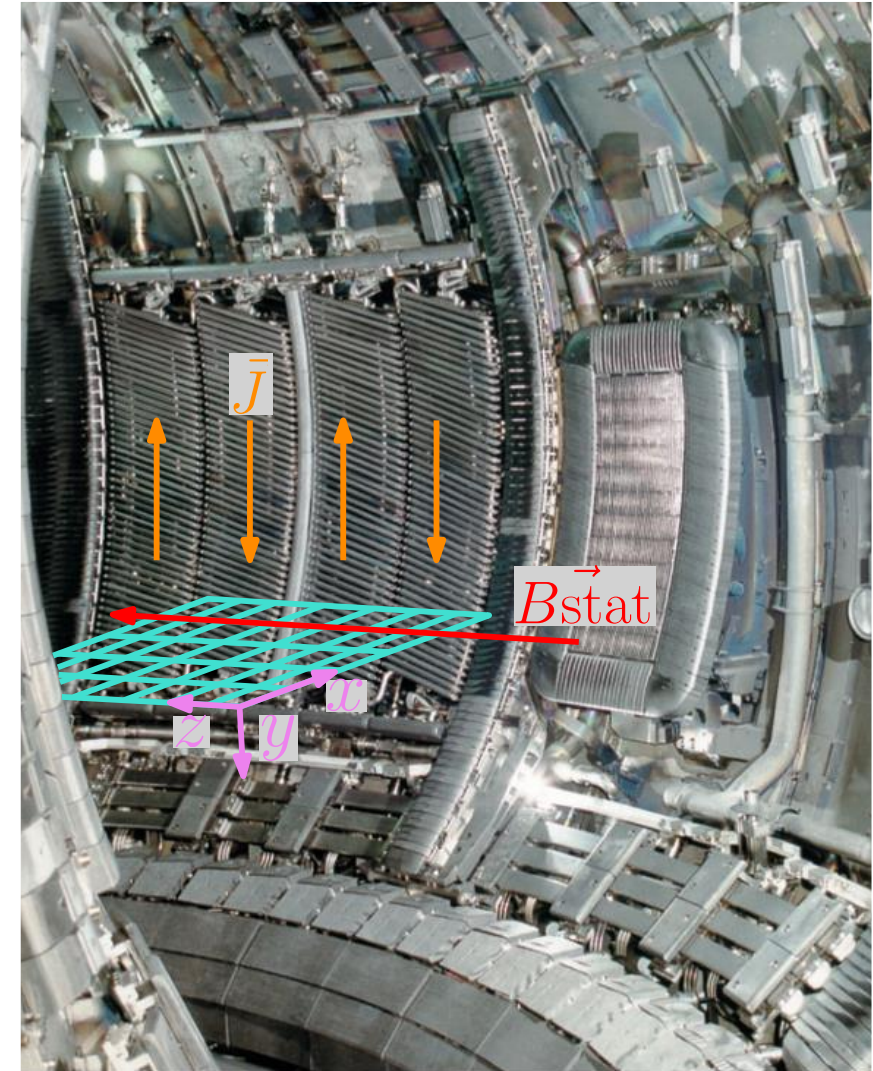


Stellarator



Aim of the project

- Investigate the use of finite element methods to simulate Ion Cyclotron resonance heating in complex geometries, as an alternative to simpler but limited spectral solvers written in Fourier space
- Allow for local mesh refinement at the resonance with high order elements
- Optimize the solver for large scale problems
- Allow for the coupling of this problem with available FEM antenna models



[1] ICRH antenna from JET

Nonlocal weak integral formulation

Mathematical formulation of the harmonic linearized Maxwell-Vlasov equation

Let us find $\mathbf{E}$, the electric field, such that:

$$\underbrace{\frac{i}{2} \int_V \left[\frac{1}{\omega \mu_0} (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{F})^* - \omega \varepsilon_0 \mathbf{E} \cdot \mathbf{F}^* \right] dV}_{M(\mathbf{E})} + \sum_{\beta} W_{\beta}(\mathbf{E}, \mathbf{F}) = -\frac{1}{2} \int_V \mathbf{j}_s \cdot \mathbf{F}^* dV \quad \forall \mathbf{F}$$

where $\mathbf{j}_s$ is an applied current density and ω the frequency.

Cold plasma

$$W_{\beta}(\mathbf{E}) = \frac{1}{2} \int_V \mathbf{E} \cdot \boldsymbol{\varepsilon}_{\beta}(B^{\text{stat}}) \cdot \mathbf{F}^* dr$$
$$\boldsymbol{\varepsilon}_{\beta} \propto \frac{1}{\omega - \omega_{c\beta}}$$

$\omega_{c\beta} = \frac{qB^{\text{stat}}}{m_{\beta}}$ is the cyclotronic frequency for the species β

Warm plasma^{[2][3]}

$$W_{\beta}[\Upsilon](\mathbf{z}, \mathbf{E}) = \int_V \int \Upsilon(\mathbf{E}, \mathbf{F}, z - z', B^{\text{stat}}) dS dz dz'$$

[2] P. U. Lamalle; Dielectric kernels for maxwellian tokamak plasmas.

[3] Machielsen, et al.(2023). Exact expression for the hot plasma conductivity kernel in configuration space.

Warm plasma: non local integral formulation

The warm plasma model is developed for any excitation mode of the antenna.

These $k_{\parallel}$ come from:

$$J(k) = \varepsilon(k)E(k)$$

The nonlocality only appears in the direction parallel to the toroidal static magnetic field (without finite Larmor radius).

$$W = \frac{1}{2} \int_V \vec{F}^*(k_{\parallel}) \vec{j}^{\text{plasma}} dk^3 = \pi \int dr_{\perp}^2 \int \vec{F}^*(k_{\parallel}) \varepsilon(k_{\parallel}) \vec{E}(k_{\parallel}) dk_{\parallel}$$

With:

$$F^{\ddagger}(k_{\parallel}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(z) e^{-ik_{\parallel} z} dz \quad E^{\ddagger}(k_{\parallel}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E(z') e^{-ik_{\parallel} z'} dz'$$

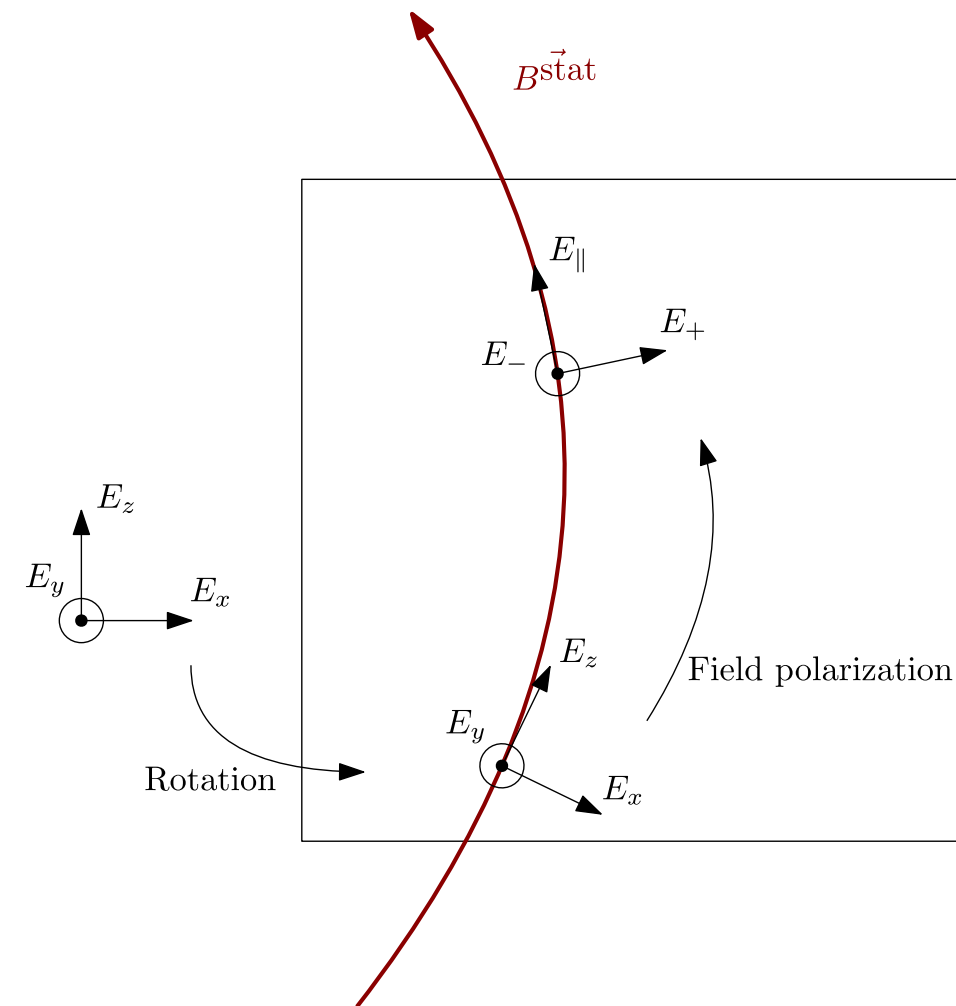
we find:

$$W_{\beta} = -i\pi\varepsilon_0 \int dr_{\perp}^2 \frac{\omega_{\text{plasma}}^2}{v_T} \int dz \int dz' \left\{ \begin{array}{l} F_{+}^{*}(z') \Upsilon_0(\kappa_{+1}|z' - z|) E_{+}(z) \\ + F_{-}^{*}(z') \Upsilon_0(\kappa_{-1}|z' - z|) E_{-}(z) \\ + F_{\parallel}^{*}(z') 2\Upsilon_2(\kappa_0|z' - z|) E_{\parallel}(z) \end{array} \right\}$$

Polarity along the curved field lines

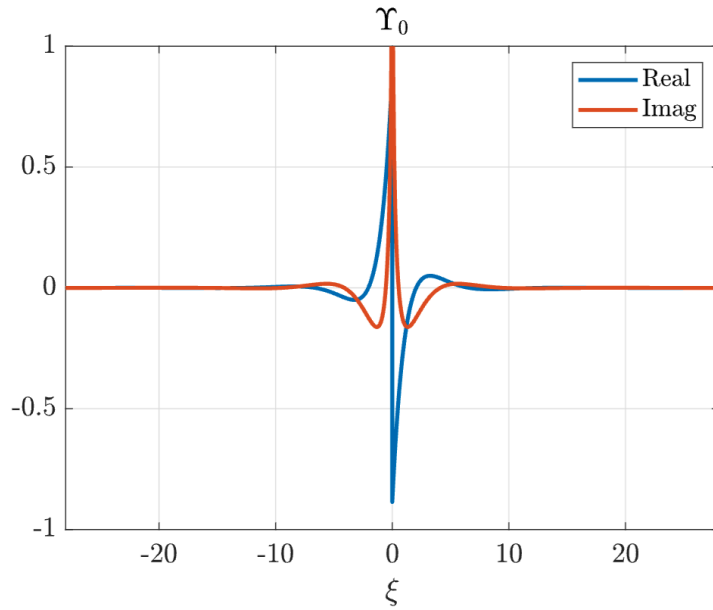
$$W_\beta = -i\pi\epsilon_0 \int dr_\perp^2 \frac{\omega_{\text{plasma}}^2}{v_T} \int dz \int dz' \left\{ \begin{array}{l} F_+^*(z') \Upsilon_0(\kappa_{+1}|z' - z|) E_+(z) \\ + F_-^*(z') \Upsilon_0(\kappa_{-1}|z' - z|) E_-(z) \\ + F_\parallel^*(z') 2\Upsilon_2(\kappa_0|z' - z|) E_\parallel(z) \end{array} \right\}$$

$$E_\pm = \frac{E_x \pm iE_y}{\sqrt{2}} \quad E_z = E_\parallel$$



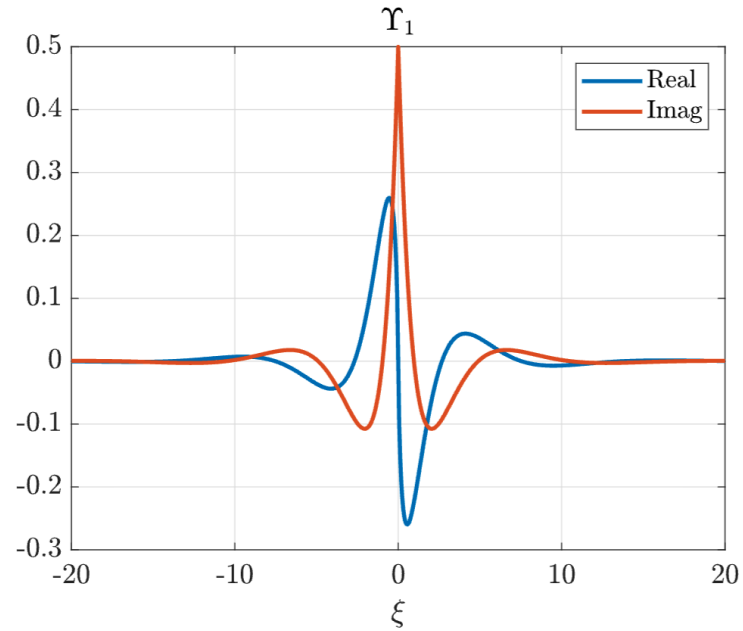
Warm plasma: kernel definition

Cyclotronic interaction

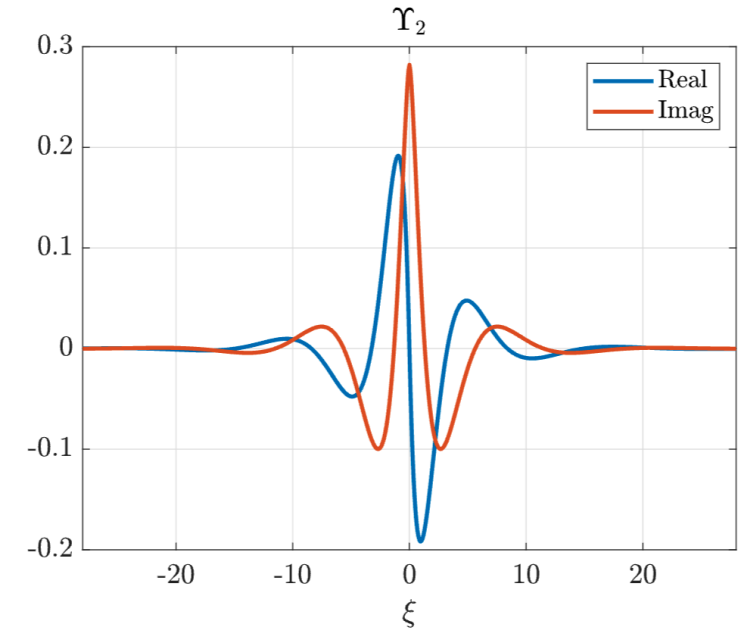


Log singularity for $\Im(\Upsilon_0)$

Finite Larmor radius



Landau damping



$$\Upsilon_\alpha(\xi) = \frac{i}{2\pi} G_{0,3}^{3,0} \left(\xi \left| 0, \frac{\alpha}{2}, \frac{1}{2} \right. \right)$$

Integration can be:

- Fully numerical
- Semi-analytical
- Fully analytical

Simplified warm model

- Simplified warm plasma for one excitation mode ($k_{\parallel}$)
- Natural treatment of resonances

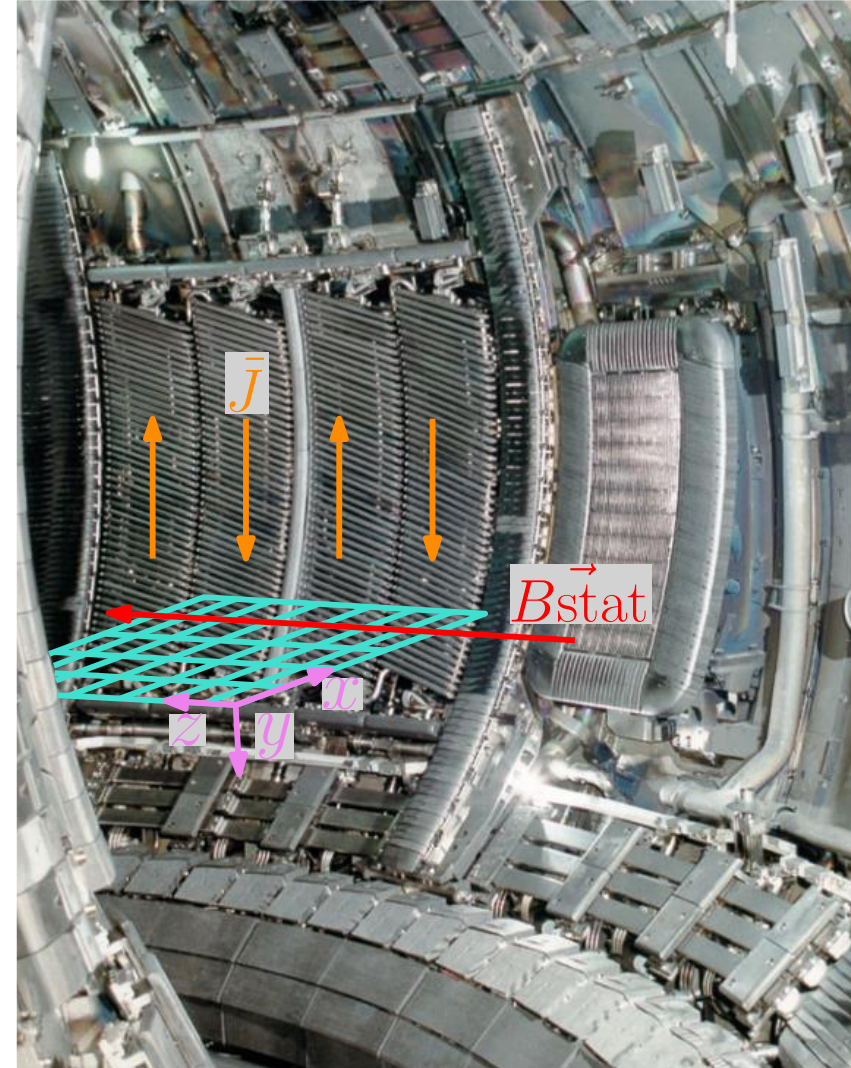
$$\left. \begin{aligned} W_{\beta}(\mathbf{E}) &= \sum_{\beta} \frac{1}{2} \int_V \mathbf{F}^* \cdot \boldsymbol{\epsilon}_{\beta}^{\text{plasma}} \cdot \mathbf{E} \, dr \\ E_{\pm} &= \frac{E_z \pm iE_x}{\sqrt{2}}, \quad E_y = E_{\parallel} \end{aligned} \right\} \quad \begin{aligned} \boldsymbol{\epsilon}_{\beta}^{\text{plasma}} &= \begin{pmatrix} R' + L' & 0 & -i(L' - R') \\ 0 & 2P' & 0 \\ i(L' - R') & 0 & R' + P' \end{pmatrix} \\ L' &= \sum_{\beta} \lambda_m Z_1 \quad R' = \sum_{\beta} \lambda_m Z_{-1} \quad P' = 2 \sum_{\beta} \lambda_m I_{20} \quad \lambda_m = \frac{\omega_p^2}{\omega} \frac{1}{k_{\parallel} v_T} \end{aligned}$$

The Z functions (Fried-Conte) are involved in the theoretical dispersion relations for plasma.

Simplified warm model: first results

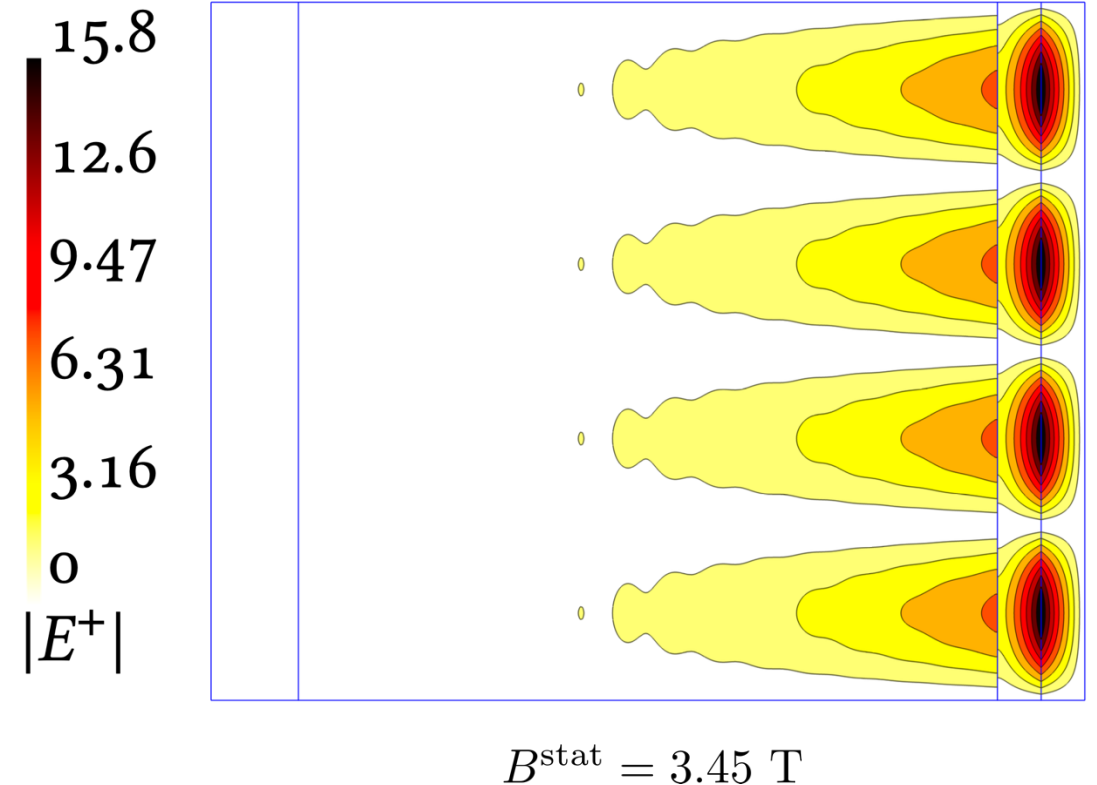
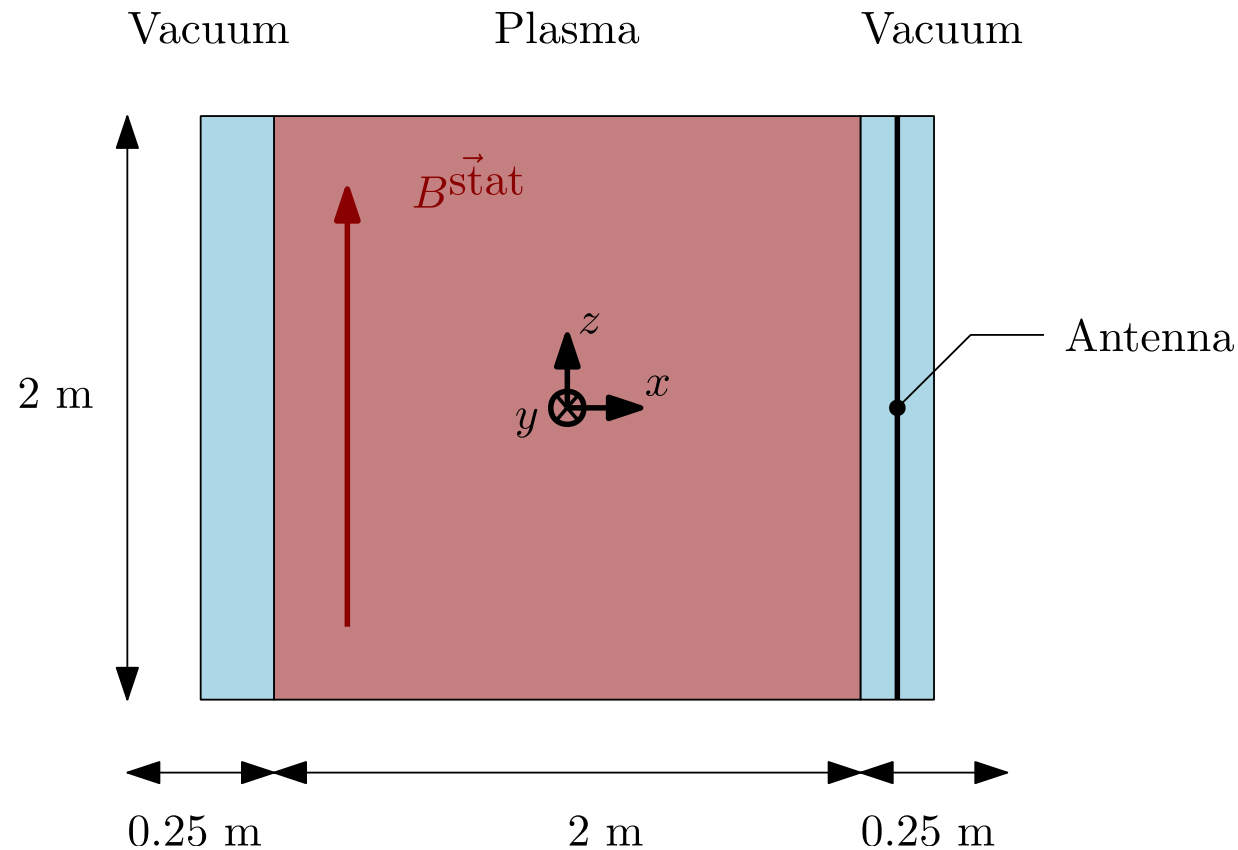
2D model from the 3D reactor

- 2D mesh spanning along the radius
- Approximated straight field lines from the curved one
- 4 straps ICRH antenna and their given polarity



Simplified warm model: first results

Plasma simulation of the electric field at $f = 51.8$ MHz with an in-plane constant static magnetic flux density, with three particle species: electrons (0 keV, $n_{\text{elec}} = 3 \cdot 10^{19} \text{ m}^{-3}$), hydrogen ($T_H = 2$ keV, $5 \% \cdot n_{\text{elec}}$) and deuterium (0 keV, $95 \% \cdot n_{\text{elec}}$).



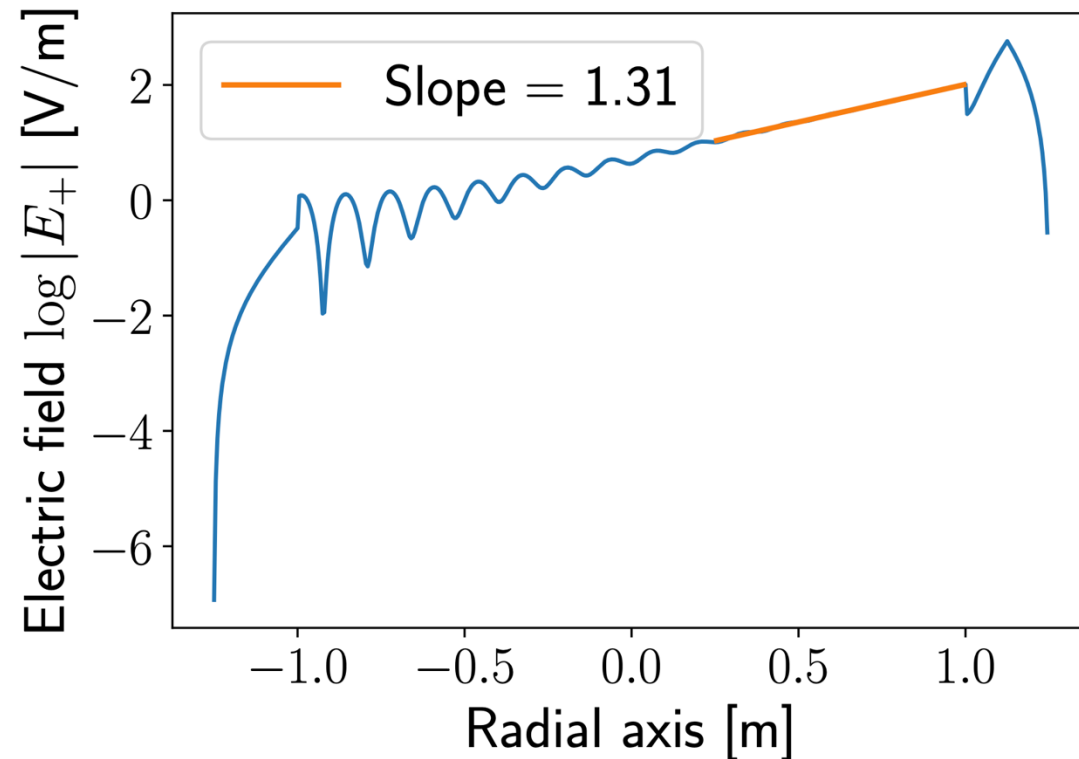
Simplified warm model: first results cont'

- Absorption rate validated against analytical dispersion relation

$$n_{\perp}^4 - \left(P(1 - \frac{n_{\parallel}^2}{S}) + \frac{RL}{S} - n_{\parallel}^2 \right) n_{\perp}^2 + P \left(\frac{n_{\parallel}^4 + RL}{S} - 2n_{\parallel}^2 \right) = 0 \quad \begin{pmatrix} S - n_{\parallel}^2 & -i(\frac{R-L}{2}) & n_{\parallel}n_{\perp} \\ i(\frac{R-L}{2}) & S - n_{\parallel}^2 - n_{\perp}^2 & 0 \\ n_{\perp}n_{\parallel} & 0 & P - n_{\perp}^2 \end{pmatrix}$$

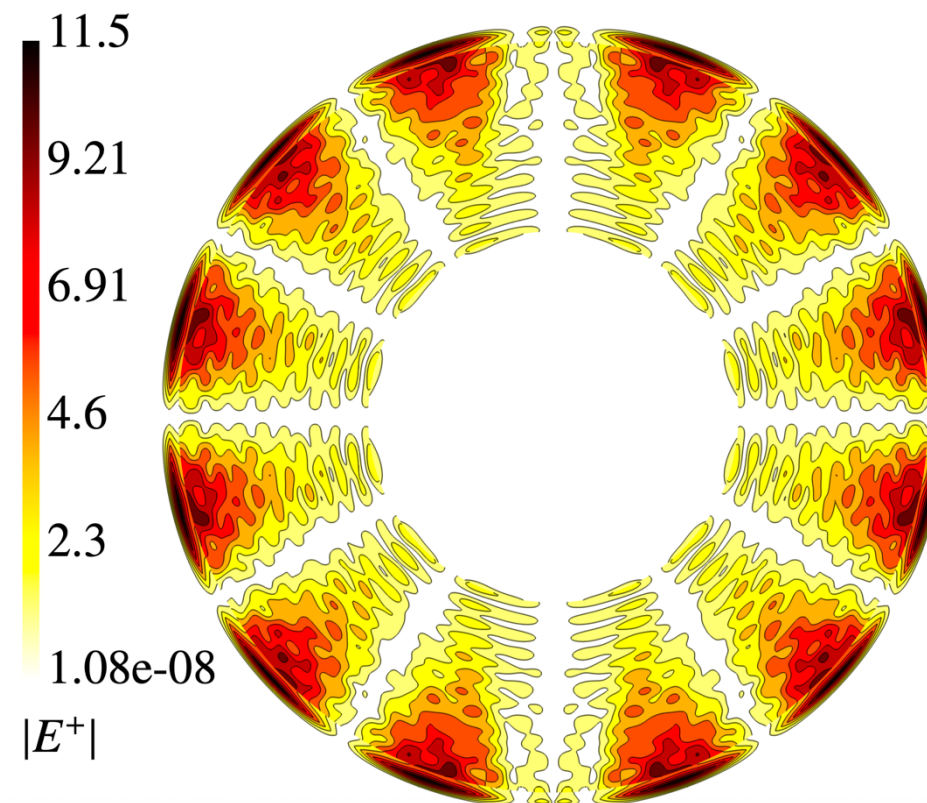
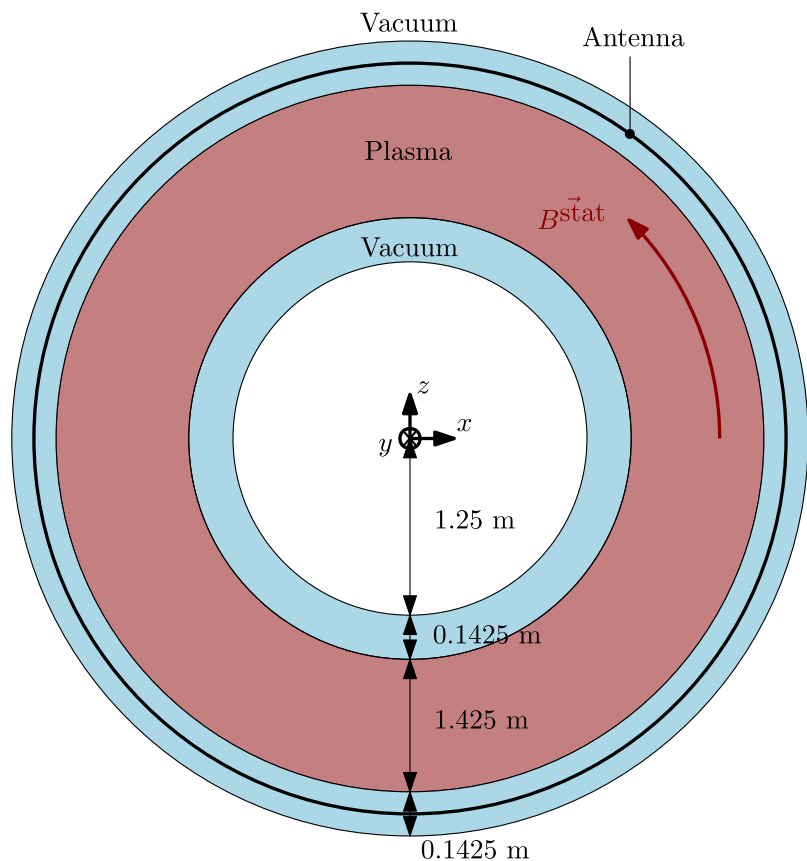
where $n = \frac{kc}{\omega}$

Absorption rate: $\alpha \approx 1.299$



Simplified warm model: complex geometry

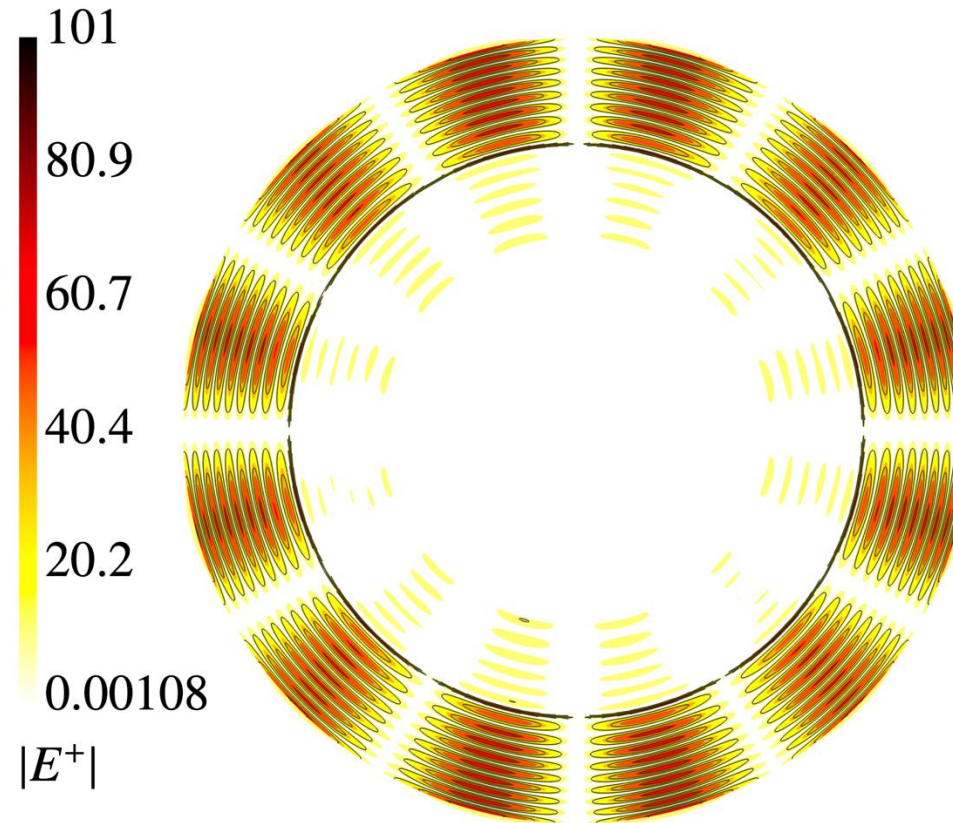
Plasma simulation of the electric field at $f = 51.8$ MHz with an in-plane static magnetic flux density B^y , with three particle species: electrons (0 keV, $n_{\text{elec}} = 3 \cdot 10^{19} \text{ m}^{-3}$), hydrogen ($T_H = 2$ keV, $5 \% \cdot n_{\text{elec}}$) and deuterium (0 keV, $95 \% \cdot n_{\text{elec}}$).



$T_H = 2$ keV, azimuthal static field with constant amplitude $B_{\theta}^{\text{stat}} = 3.45$ T.

Simplified warm model: resonance handling

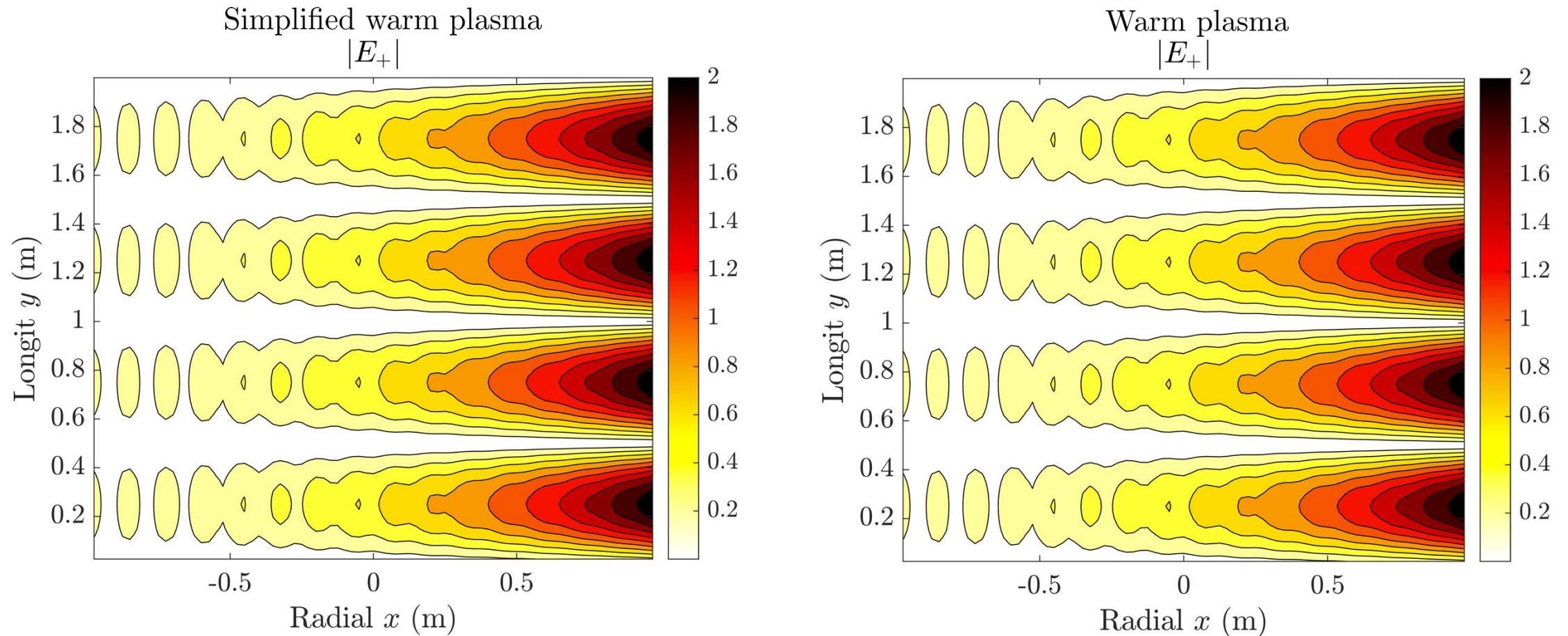
- Same case with natural treatment of resonance with: $Z(\xi)$



$$T_H = 2 \text{ keV, azimuthal static field}$$
$$B_{\text{stat}}^\theta = \frac{B_0 \cdot R_0}{R_0 + r} = \frac{3.45}{1 + r/2.97} \text{ T.}$$

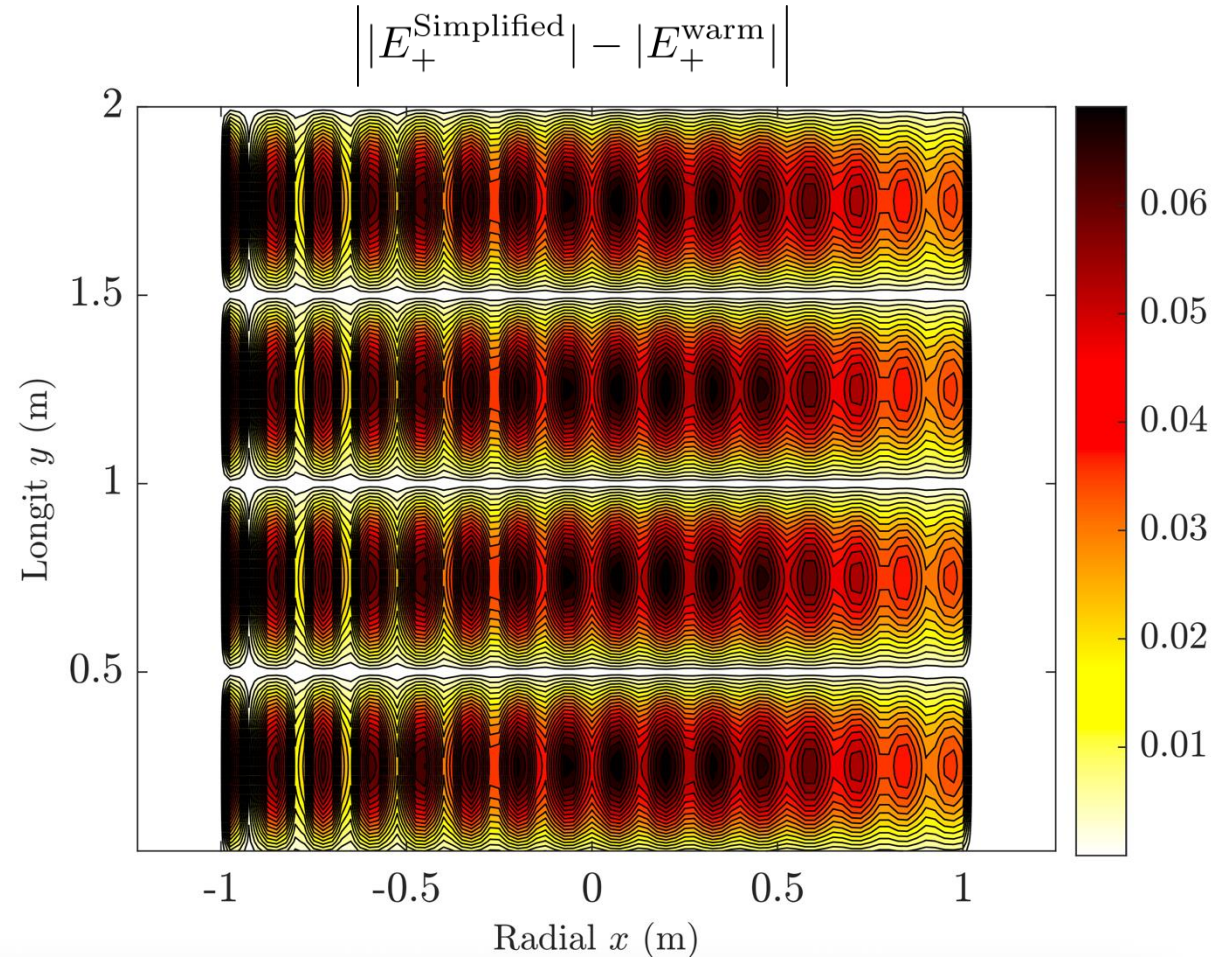
Warm model: first results

- Simulate a first warm model with the kernel semi-analytical integration in the same configuration as previously ^{PLIKES code [4]}



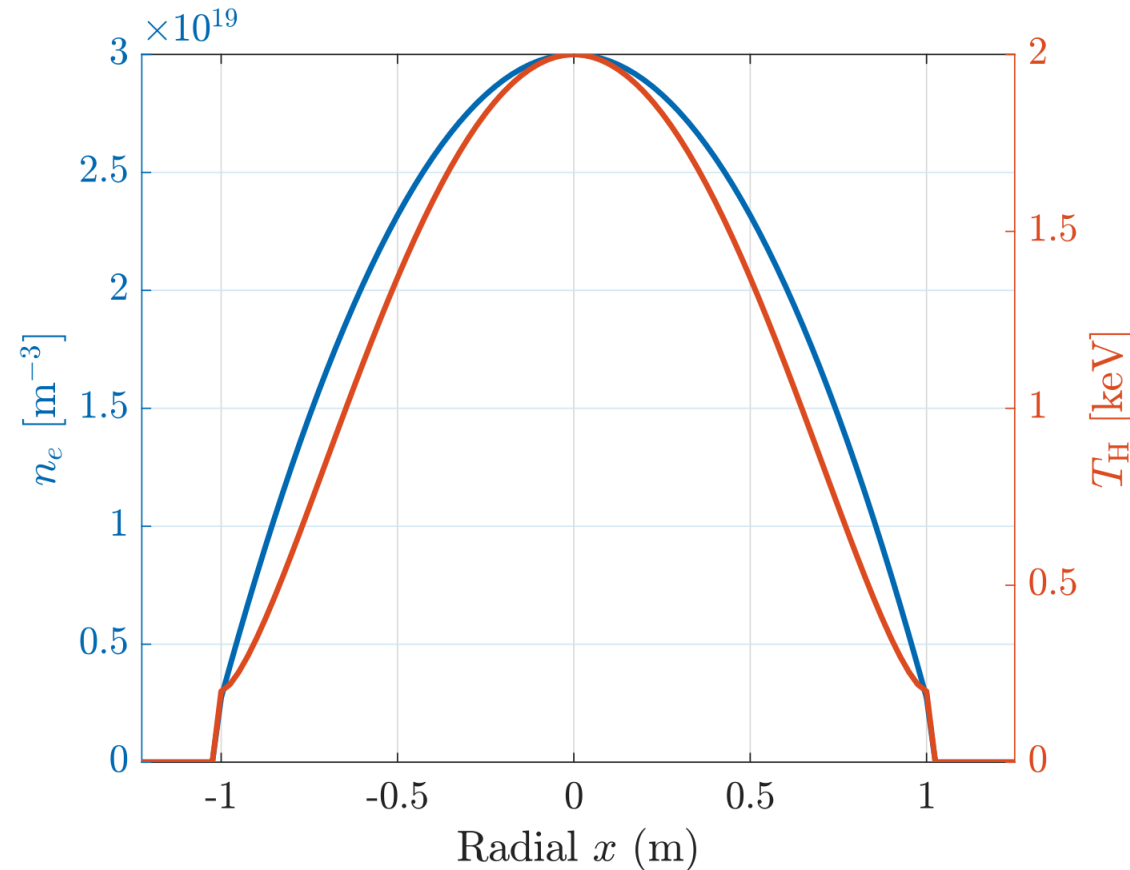
Warm model: first results

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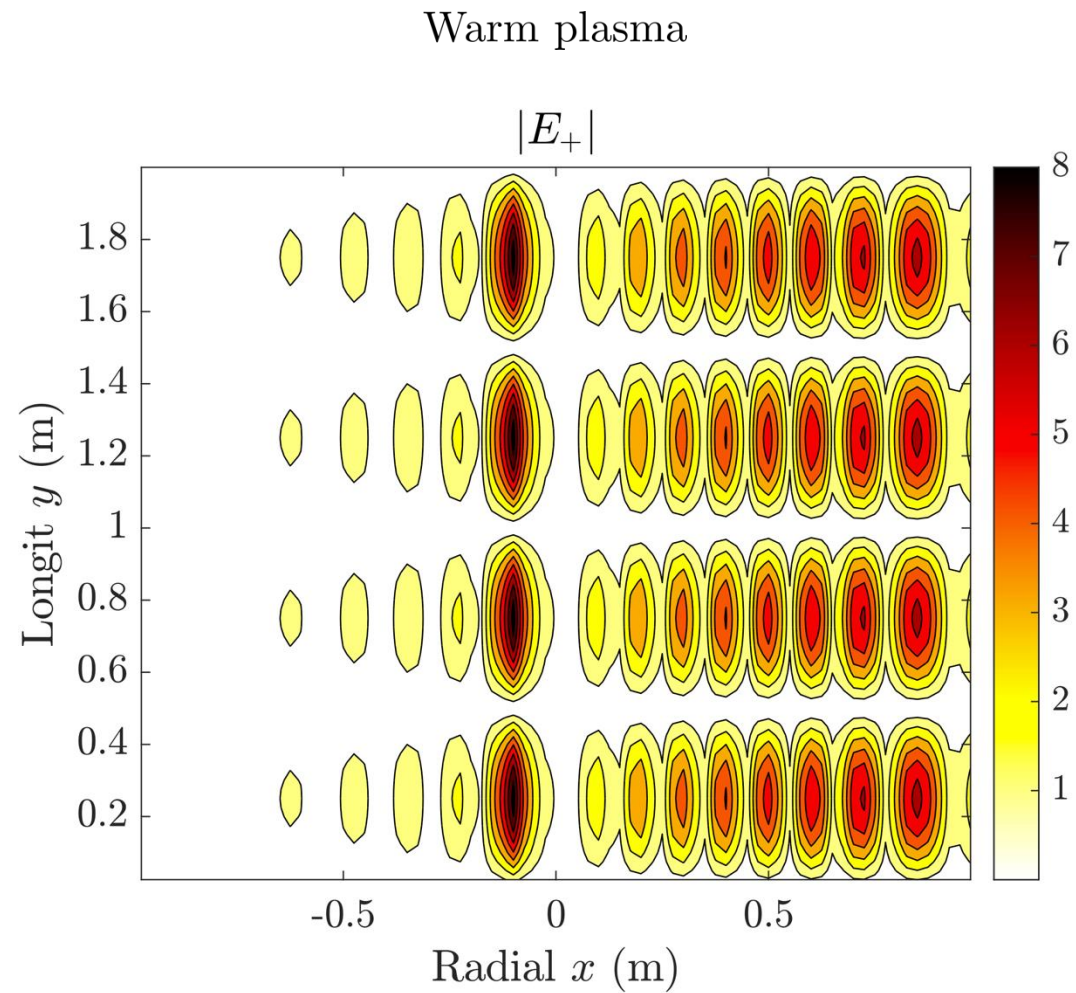
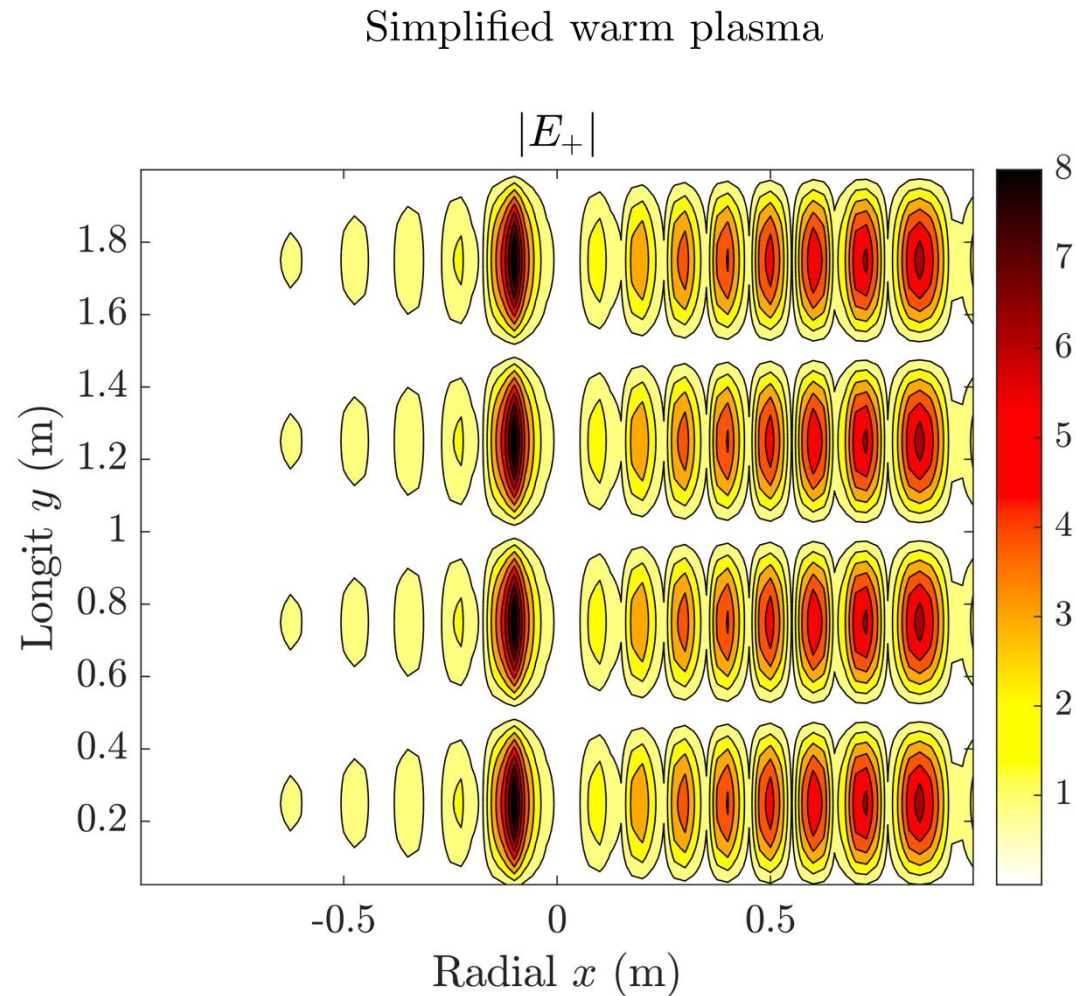


Warm model: variable parameters

- The density and the temperature are varying as in JET with a profile in the plasma and vacuum on the side



Warm model: varying parameters



Conclusion

- Simulations on complex geometries with the simplified warm model
- First results with the warm model

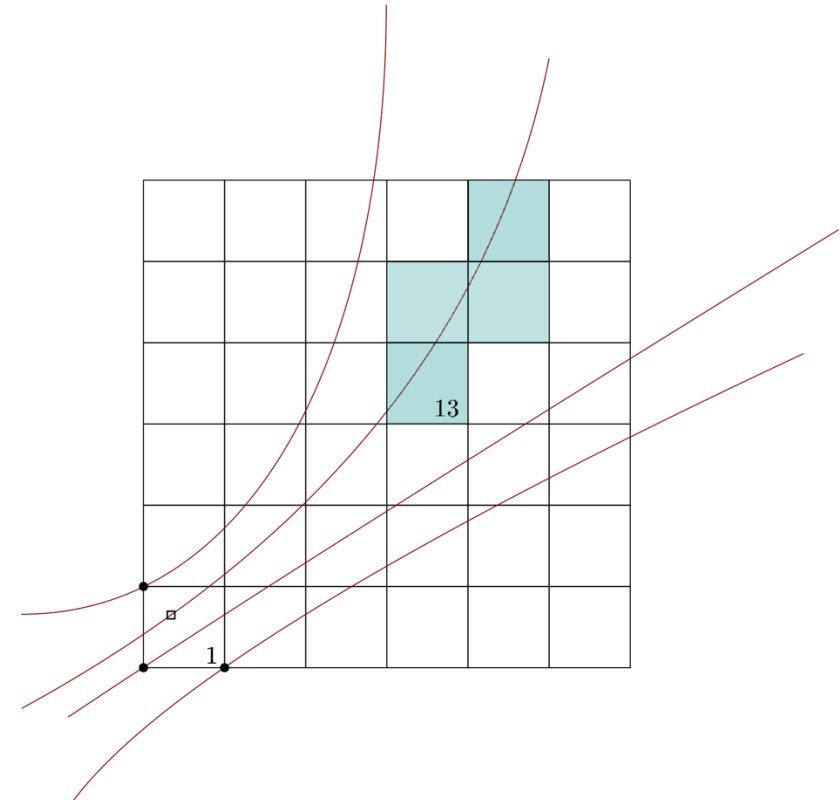
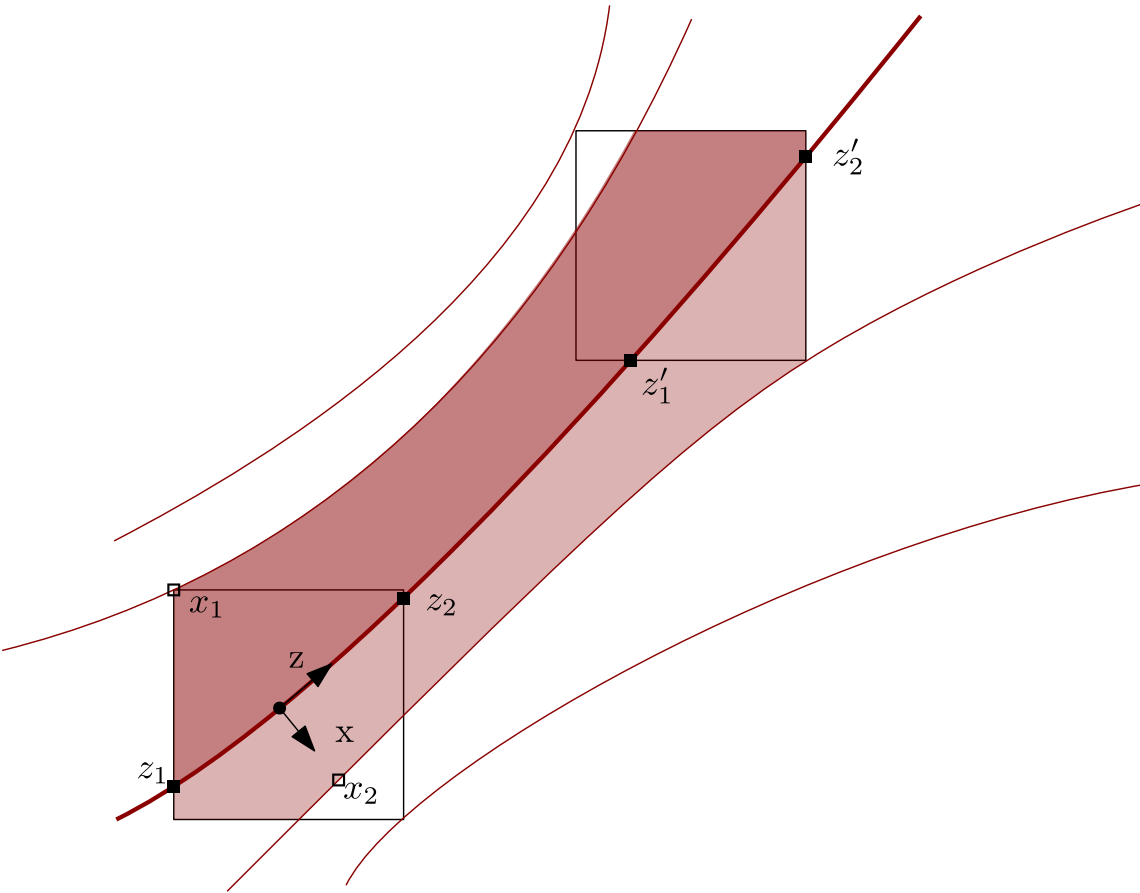
This nonlocal approach is promising as it is validated in several cases with a well-known simplified model

Perspectives:

- Determine whether a field aligned mesh is advantageous
- Deal with the computationally expensive dense blocks with the kernel

Perspectives: Coupled domain of integration

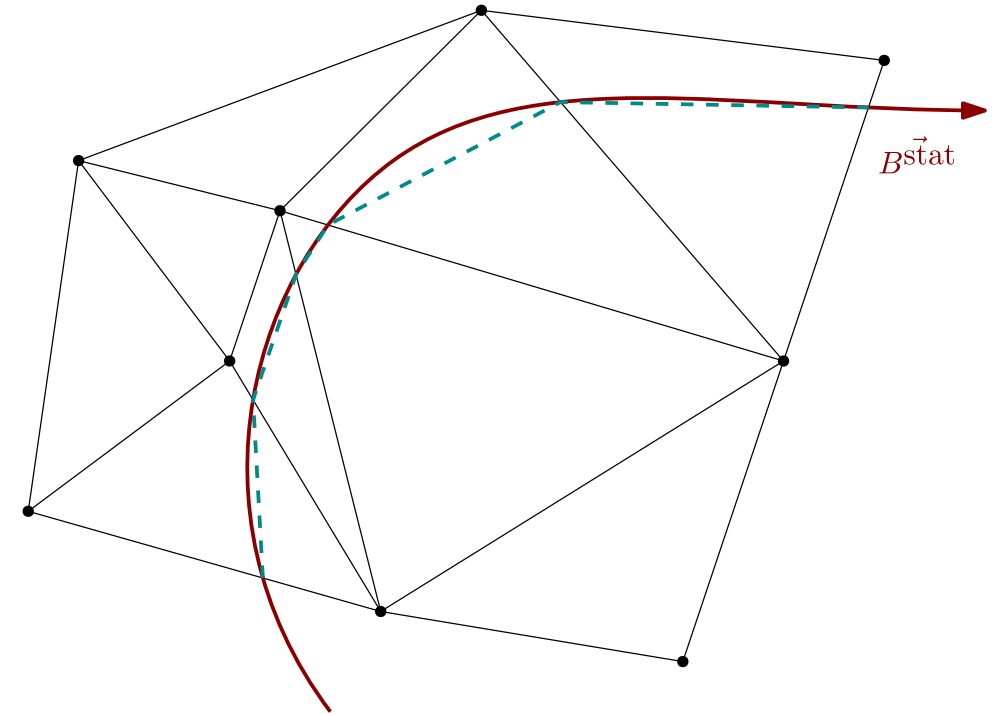
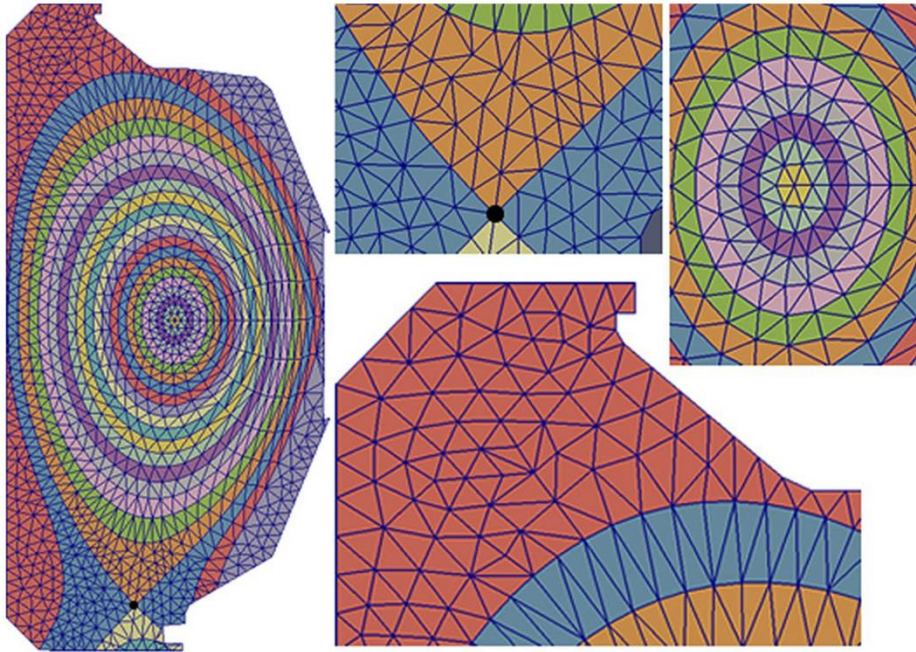
$$W_{\beta} = -i\pi\epsilon_0 \int dr_{\perp}^2 \frac{\omega_{\text{plasma}}^2}{v_T} \int dz \int dz' \left\{ \begin{aligned} &F_{+}^{*}(z') \Upsilon_0(\kappa_{+1}|z' - z|) E_{+}(z) \\ &+ F_{-}^{*}(z') \Upsilon_0(\kappa_{-1}|z' - z|) E_{-}(z) \\ &+ F_{\parallel}^{*}(z') 2\Upsilon_2(\kappa_0|z' - z|) E_{\parallel}(z) \end{aligned} \right\}$$



Perspectives: Magnetic field lines integration

To integrate along the magnetic field lines:

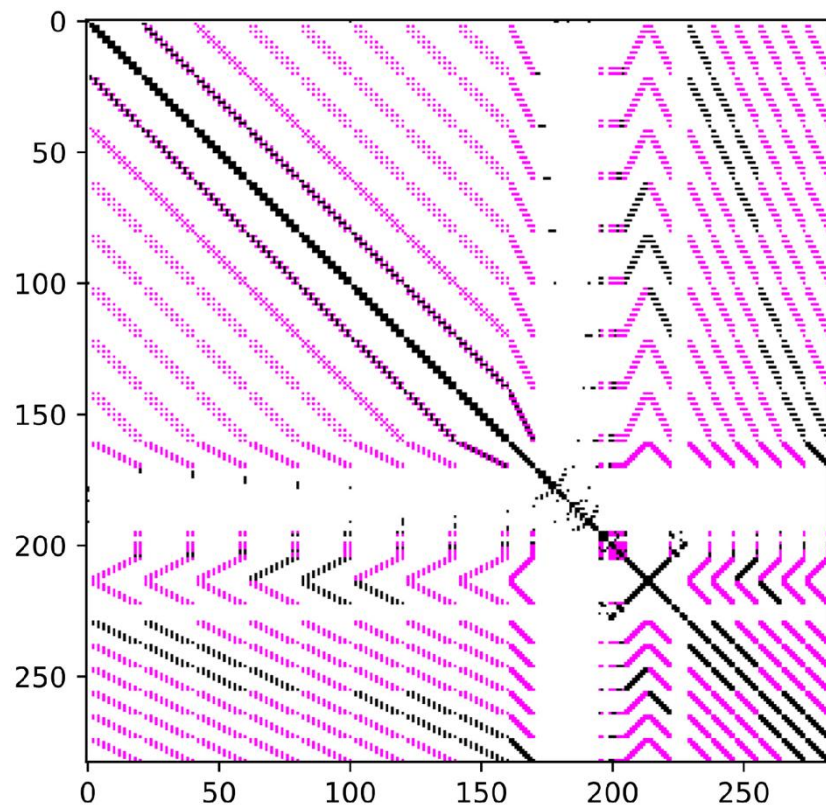
- Create a field aligned mesh as in [5] but must be deal with never-closing field lines
- Recreate the lines for each pair of elements



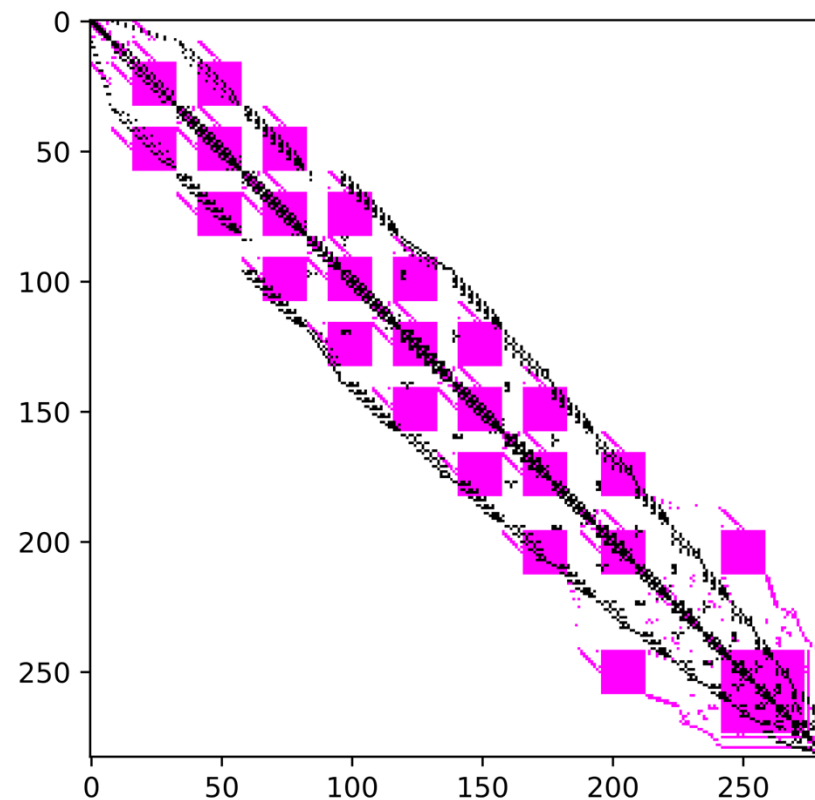
Perspectives: Nonlocal sparsity pattern

Black: Cold plasma pattern

Pink: Addition of warm plasma interaction



Reordering of the dofs using
Cuthill-McKee algorithm



Hierarchical matrix algorithm could approximate this dense matrix blocks $\rightarrow$ Htool-DDM^[6]

Conclusion

- Simulations on complex geometries with the simplified warm model
- First results with the warm model

This nonlocal approach is promising as it is validated in several cases with a well-known simplified model

Perspectives:

- Determine whether a field aligned mesh is advantageous
- Deal with the computationally expensive dense blocks with the kernel