

Integral dielectric kernel implementation to model RF heating in toroidal plasmas

Bernard C. G. Reman^{1,*}, Philippe Lamalle¹, Jules Zaleski², Christoph Slaby^{3a}, Dirk Van Eester¹, Christophe Geuzaine², Fabrice Louche¹, Ernesto Lerche¹, Elena Moral Sánchez^{3b}, and Vincent Maquet¹

¹Laboratory for Plasma Physics, Royal Military Academy, Brussels, Belgium

²Dept. of Electrical Engineering and Computer Science, University of Liège, Belgium

³Max-Planck-Institut für Plasmaphysik, 17491 ^{3a}Greifswald and 85748 ^{3b}Garching Germany

Abstract. As discussed in Ref. [1], recent theoretical and numerical treatments [2, 3] have sought to express the plasma radiofrequency (RF) response as a nonlocal integral operator formulated in configuration space. Analytical expressions of the integral kernels are available for Maxwellian particle species. This approach enables (i) direct use of the finite element method (FEM) to model wave propagation and absorption in hot inhomogeneous fusion plasmas, (ii) local mesh-refinement, (iii) provides RF field representations suited to address tokamak geometry, and (iv) allows straightforward connection between plasma and antenna models. The present contribution focuses on the concrete application of this method, in an incremental way, developing codes and exploiting finite element codes/libraries.

1 Introduction

Waves in the ion cyclotron range of frequencies (ICRF) are an integral part of auxiliary heating systems, plasma startup as well as wall conditioning in magnetic confinement fusion plasmas, both tokamaks and stellarators [4]. Modelling ICRF heating poses challenges because its wavelength is on the typical order of the gradients length scales of the plasma, and this precludes the applicability of ray tracing methods. Therefore, accurate description of the anisotropic plasma response, including strong refractive index variation (at resonances, cut-offs, near the edge), requires a full-wave approach. Several codes have been devised which solve Maxwell's equations in the presence of a hot plasma as a boundary value problem, subject to the antennas excitations. They rest on a variety of formulations: spectral (AORSA [5]), mixed spectral-finite-element (TORIC [6], CYRANO [7], EVE [8], LEMan [9]), finite element (FEMIC [10], PETRAM [11], NGSolve [12]), and all aim at finding a balance between physics fidelity and numerical cost. Theoretical progress in the analytical expression of the nonlocal dielectric response of Maxwellian plasmas in configuration space has been ongoing with the work of Ref. [13], perpendicular to the local magnetic field, with stratified profiles ($\nabla \mathbf{B}_0 \cdot \mathbf{B}_0 = 0$). More recently, the homogeneous Maxwellian plasma response, both perpendicular and parallel to a uniform magnetic field has been derived [2], and was anticipated in the full guiding-center picture for general magnetic field equilibria [1, 3], see also [14, 15]. This approach allows to introduce the parallel magnetic field gradients. The parallel response to the absorbing part has

also been investigated nonlocally by means of Fast Fourier Transform evaluation of the $(\parallel, \parallel)$ component of the hot dielectric tensor [16, 17]. An overview of the different methods to incorporate the plasma response in full-wave modelling is given in Ref. [18]. The nonlocal description allows especially to use a FEM formulation, which captures the full antenna spectrum at once, provided it is suitably discretised, as well as to retain finite plasma temperature effects (warm $T_{\parallel} > 0$, hot $T_{\parallel} > 0$ and $T_{\perp} > 0$). It should enable local mesh refinement techniques that are ruled out with spectral approaches. While the current work seeks to exploit analytical expressions to ease the numerical workload, alternative approaches determine the dielectric plasma responses, for arbitrary distributions [19–21], by integrating the particles trajectories along the unperturbed orbits, as part of the full-wave equation solution. In addition to the theoretical results [2], the method was illustrated with the solution of the full finite Larmor radius (FLR) wave equation in configuration space, in one spatial dimension for the 3 components of the fields, retaining a Fourier mode expansion along the magnetic field. We explore, in a complementary way, the warm plasma effects along the magnetic field using the nonlocal formulation, in the limit of vanishing Larmor radius, in 2 dimensional slab geometry, as a first step, which allows to simulate minority heating scenarios. We describe the different avenues taken so far in the next sections, namely the development of a novel code in section 2, the exploitation of GMSH-FEM in section 3 and of PSYDAC in section 4, respectively.

*Corresponding author: bernard.reman@mil.be

2 PLIKES a Plasma Linear Integral Kernel Efield Solver

The model currently implemented in our PLIKES code, rests on a 2.5D straight-magnetic field lines (no $\nabla_{\parallel} \mathbf{B}_0$) solution of the weak-form of the linearised Maxwell-Vlasov system using the new nonlocal dielectric kernels [15], to lowest order in finite Larmor radius as a first step, i.e. with focus on wave dispersion effects along the equilibrium magnetic field and the associated fundamental cyclotron and Landau dampings. The polarisations $\mathcal{L} = +, -, \parallel$ are associated with the indices $L = 1, -1, 0$. The symbol β labels the plasma species, $\omega_{\beta p}$, $\omega_{\beta c}$ and $v_{\beta T}$, their plasma frequency, cyclotron frequency and thermal speed respectively, $\mathbf{j}_s$ is the antenna current, ω its angular frequency, $\lambda_{\beta p} = (\omega - p\omega_{\beta c})/2v_{\beta T}$. The functions $\mathbf{F}$ and $\mathbf{E}$ are the test and trial functions, respectively, $*$ denotes the complex conjugate, and the weak formulation eventually reads

$$\begin{aligned} & \frac{i}{2} \int_V \left[\frac{1}{\omega\mu_0} (\nabla \times \mathbf{F})^* \cdot (\nabla \times \mathbf{E}) - \omega\epsilon_0 \mathbf{F}^* \cdot \mathbf{E} \right] d^2\mathbf{r} \\ & + \sum_{\beta} \mathcal{W}_{\mathbf{F}\mathbf{E}\beta} = -\frac{1}{2} \int_V \mathbf{F}^* \cdot \mathbf{j}_s d^2\mathbf{r}, \\ & \mathcal{W}_{\mathbf{F}\mathbf{E}\beta} = \sum_{p=-\infty}^{+\infty} -\frac{i\epsilon_0}{2} \int dx \int dy \sum_{L=-1}^1 \delta_{L,p} 2^{\alpha/2} \\ & \times \int dy' F_{\mathcal{L}}^*(y') \frac{\omega_{\beta p}^2}{v_{\beta T}} \Upsilon_{\alpha}(\lambda_{\beta p} |y - y'|) E_{\mathcal{L}}(y), \alpha = 2\delta_{L,0}. \end{aligned} \quad (1)$$

The expressions are free from $k_{\parallel}$ dependence, and one notices instead the appearance of integrals connecting nonlocally points y, y' along the magnetic field. The symbol $\delta_{L,0}$ is the Kronecker-delta symbol, which yields $\alpha = 2$ when $L = 0$, corresponding to the Landau interaction and $\alpha = 0$ for $L = -1, 1$, the cyclotronic interactions. The mixed Landau-TTMP, corresponding to $\alpha = 1$, is not retained in the present study since it is a FLR effect. The functions Υ_0, Υ_1 and Υ_2 stem from the generalised definition

$$\Upsilon_{\alpha}(\xi) = \frac{1}{\pi} \int_0^{+\infty} Z^{(\alpha)}\left(\frac{2\xi}{t}\right) \begin{cases} \cos t \\ i \sin t \end{cases} \frac{dt}{t}, \quad \alpha \begin{cases} \text{even} \\ \text{odd} \end{cases}, \quad (2)$$

and $\text{Im}(\xi) > 0$ shown in Fig. 1, with $Z^{(\alpha)}$, the plasma dispersion functions

$$Z^{(\alpha)}(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^{\alpha} e^{-x^2}}{x - \zeta} dx. \quad (3)$$

In Eq. 2, when α is even Z^{α} is multiplied by $\cos(t)$ and when the index α is odd Z^{α} is multiplied by $i \sin(t)$. The Υ_{α} preserve the causality prescription and are connected to the functions S of [2], $S_1(|\lambda|y, \sigma_{\lambda}) = \Upsilon_0(\lambda|y|)$, $S_2(|\lambda|y, \sigma_{\lambda}) = -2\sigma_y \Upsilon_1(\lambda|y|)$ and $S_3(\lambda|y|, \sigma_{\lambda}) = -2\Upsilon_2(\lambda|y|)$, $\sigma_v = \text{sign}(v)$. They are further presented and studied in details in [22], in particular their mathematical properties and recurrence relations are established, the convergent series and asymptotic expansions are determined, together with a representation highlighting the logarithmic singularity of the imaginary part of Υ_0 . This requires due care at resonance and when $y = y'$.

2.1 Model

As introduced, the model is 2 dimensional in space and retains the 3 components of the electromagnetic fields, so-called 2.5D, in slab geometry. We now choose the confining magnetic field to be in the $\hat{y}$ direction, while the stratified profile varies in the $\hat{x}$ direction. In tokamak geometry, $\hat{x}$ would correspond to the radial direction and $\hat{y}$ to the toroidal direction. The curl operator $\nabla \times$ is set with $\nabla = (\partial_x, \partial_y, ik_z)$, and k_z , thought of as a poloidal mode number. The circularly polarised components

$$\mathbf{e}_{\pm} = \frac{\mathbf{e}_z \mp i\mathbf{e}_x}{\sqrt{2}}, \quad \mathbf{e}_{\parallel} = \mathbf{e}_y \quad (4)$$

provide

$$E_{\mathcal{L}} = \mathbf{e}_{\mathcal{L}}^* \cdot \mathbf{E}, \quad \mathcal{L} = +, -, \parallel. \quad (5)$$

The antenna is represented as an arbitrary number of infinitely narrow straps

$$\mathbf{j}_s = \mathbf{j}_s(x, y) \exp(ik_z z) = \mathbf{j}_s \delta(x - x_0) s(y) \exp(ik_z z) \quad (6)$$

and $s(y) = H(y - y_0 + w) - H(y - y_0 - w)$, $H(y)$ being the Heaviside function, w the strap half width. In some tests, the function $s(y)$ can be set to a desired profile, for example, a sine function which allows to simulate a single $k_{\parallel}$ spectrum.

2.2 General implementation

The PLIKES code is written in C and is so far hosted on a private Gitlab page. It uses complex numbers from the standard library, uses Gauss-Legendre quadrature of arbitrary orders (up to 16), as well as quadratures that adequately integrate log functions. This allows to treat the logarithmic singularity of $\text{Im}\Upsilon_0$. The matrix is constructed on a per element basis which introduces repetitions of global degrees of freedom (DoFs) entries in the system matrix. They can be gathered by means of a hash function [23]. The nonlocal coupling lengths are defined for each species and each polarisation, at each radial position x , based on the truncation constraint that $\lambda_{\beta,p}(x) |y - y'| < 20$, the argument beyond which the kernels are smaller than 10^{-6} , for a stratified, homogeneous magnetic field $\mathbf{B}_0 = \hat{y}B_0(x)$. Boundary conditions are introduced with Lagrange multipliers. The MUMPS library [24, 25] is used to solve the linear system and the results are written using the hdf5 library [26, 27] for post-processing. The curl-conforming first-kind Nédélec elements are used in the planar simulation domain [28] for the radial and "toroidal" components of the fields, and the "poloidal" components are described using the second order nodal Lagrange elements, resulting in an overall 21 DoFs per element. The power balance is obtained on a per species and/or per position basis by substituting $\mathbf{F} = \mathbf{E}$ in the weak form, with $\mathbf{E}$ the solution.

2.3 Kernel evaluation, numerical quadratures, matrix construction

Our simulation necessarily truncates the kernel to ξ_{max} , since it decays quickly which allows to tabulate it for fast

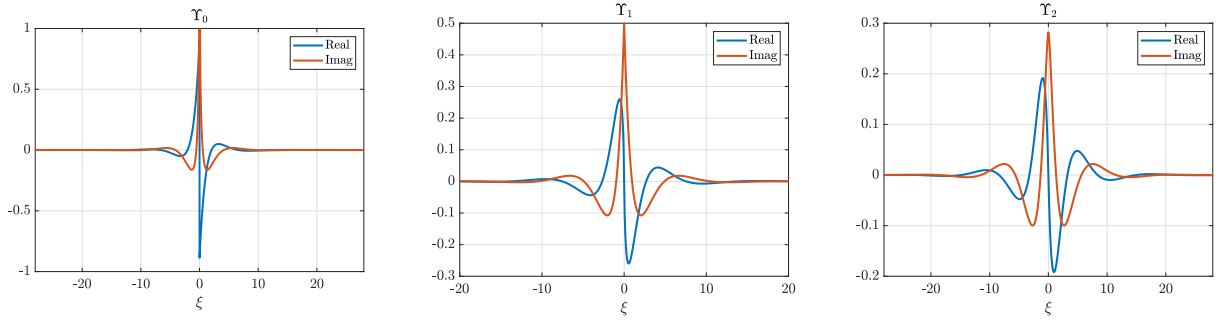


Figure 1. The real and imaginary parts of the Υ_α functions. The fundamental cyclotron damping is governed by Υ_0 while Υ_2 represents the Landau interaction. The Υ_1 function corresponds to mixed Landau-TTMP and intervenes for finite Larmor radius effects and will be exploited in the future.

evaluation as well as to limit the range of the nonlocal integrations. For each radial location x , species β and polarisation p , the quantity $\lambda_{\beta,p}(x)$ is evaluated. This implies that the kernel argument

$$\lambda_{\beta,p}(x)|y - y'| \leq \xi_{max}. \quad (7)$$

In turn, the nonlocal coupling lengths cannot be larger than $L_{max} = |y - y'|_{max} = \xi_{max}/\lambda_{\beta,p}$, as a result of this truncation criterion. At the cyclotron resonance, $\lambda_{\beta,+}(x_{res}) \rightarrow 0$ and therefore $L_{max} \rightarrow \infty$. We further require that the nonlocal coupling length be smaller than the domain length $L_{max} \leq n_y \Delta y$, with n_y and Δy , the number of elements and their (common) size along the magnetic field direction, respectively. In addition, away from resonance or when the temperature decreases, $\lambda_{\beta,p}(x)$ becomes large, such that $L_{max} = \xi_{max}/\lambda_{\beta,p}(x) \rightarrow 0$ and can then be smaller than Δy . This implies that the kernel support is contained over a single element which becomes challenging to integrate with a few number of Gauss points. However, the coupling being nonlocal, we should ideally also have $L_{max} \geq \Delta y$ to couple at least two elements. If $L_{max} \leq \Delta y$, the coupling becomes local. In this case, one can revert to the cold plasma dielectric tensor elements instead. Eventually $\Delta y \leq L_{max} \leq n_y \Delta y$. Therefore, the numerical quadrature should be accurate enough to integrate the support of the kernel over 1-2 elements. Since the number of quadrature points is fixed over an element, the greater L_{max} , the larger the number of integration points for the kernel support. The numerical integration rules have therefore been evaluated on Υ_0 , the most challenging due to its logarithmic divergence. Strategies to get around this are discussed in section 2.5. The result of the numerical evaluation of the indefinite integrals of Υ_0 using Gauss quadrature is compared against a dedicated quadrature that deals with the logarithmic singularity [29, 30]. The Gauss-quadrature struggles to correctly capture the integral over the whole support (truncated at $\xi_{max} = 20$) as seen on the left panel of Fig. 2, even with an order 20 integration rule, while starting from order 9, see rightmost panel, the log quadrature gives very good results, and higher orders are almost indistinguishable from the true values, central panel. This test can be thought of as integrating the kernel with the constant basis function for $\mathbf{F}$ and $\mathbf{E}$ while non constant basis functions will result in multiplication by polynomials

in y and y' that will smooth the logarithmic divergence.

The elements have been numbered with fastest index varying along the magnetic field direction such that the bandwidth of the sparse matrix will still be limited with the non-locally coupled elements. As a result, the matrix appears to be block sparse and the size of the block is related to the nonlocal coupling length, that is essentially related to the temperature for the sparsity patterns in the left and central panels in Fig. 3, and to both temperatures and magnetic field values for the right panel. In addition one sees the transition from the vacuum region to the plasma region.

2.4 Results

Ion cyclotron damping has been validated with PLIKES using the nonlocal kernel, setting uniform densities, temperatures and magnetic field. This allows direct comparison with the warm analytical dispersion relation, resulting in good agreement, see Fig. 4. Integral dielectric kernel to JET H-minority heating in D with the idealised sinewave antenna and with a JET-like 4-strap antenna in dipole phasing $(0, \pi, 0, \pi)$ are presented in Fig. 5, $\perp$ -stratified geometry, radial x , toroidal y , $f = 52.56\text{MHz}$, $B_0 = 3.45\text{T}$, $n_{e,0} = 3 \times 10^{19}\text{m}^{-3}$, $T_e = T_D = T_H = 2.0\text{keV}$, $\xi_D = n_D/n_e = 0.95$. We note that a spectral approach would need to sum the Fourier-modes of the 4-strap antenna, while the nonlocal kernel captures its spectrum at once.

2.5 Prospects

The present is a snapshot of ongoing work and subsequent development will ensue. The quadrature for the logarithmically-singular kernels are treated for the nearest-neighbour elements where $y = y'$ is allowed and proper treatment at the cyclotron resonance $\lambda_{\beta,p=1} = 0$, namely for the dx integration, is pending [29]. Indeed, the logarithmic singular parts of the Υ_α have been isolated. We have used the identity $\log(\lambda_{\beta,p}(x)|y - y'|) = \log(\lambda_{\beta,p}(x)) + \log(|y - y'|)$ to apply the dedicated log-quadrature on the latter term, and are working to apply it to the former. The recent discovery of closed expressions for the primitives of Υ_α [22] will be highly beneficial on the numerical aspects

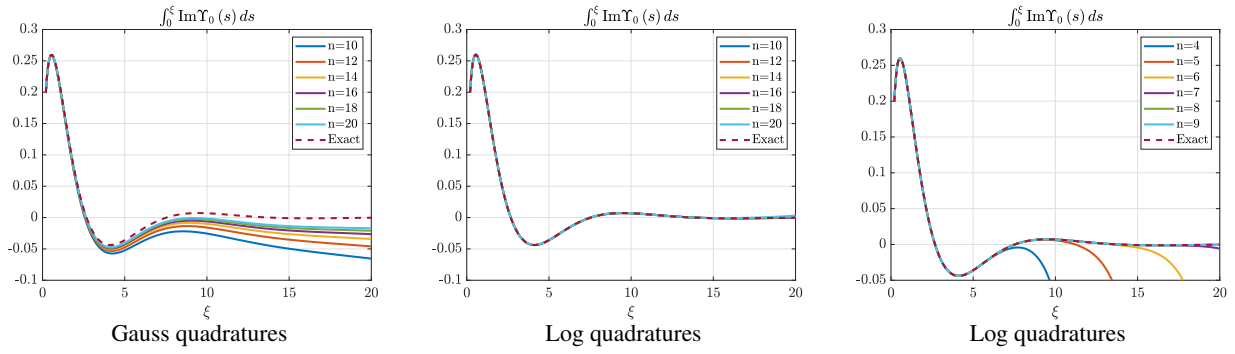


Figure 2. Comparison of Gauss quadratures with quadratures suited to treat logarithmic-singular integrand in the integration of the kernel function Υ_0 .

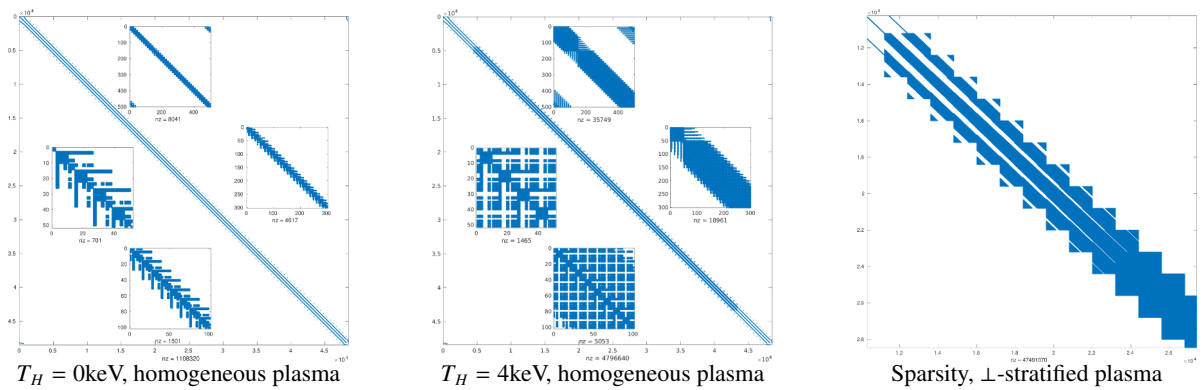


Figure 3. Indexing elements along the magnetic field retains sparsity by block, the size of which changes when moving from vacuum to plasma, left and middle panels, and with inclusion of profile temperature, as shown on the rightmost panel. The insets on the first 2 panels zoom in the main diagonal, clockwise, starting from the top. The top left structure on the rightmost panel corresponds to the vacuum region.

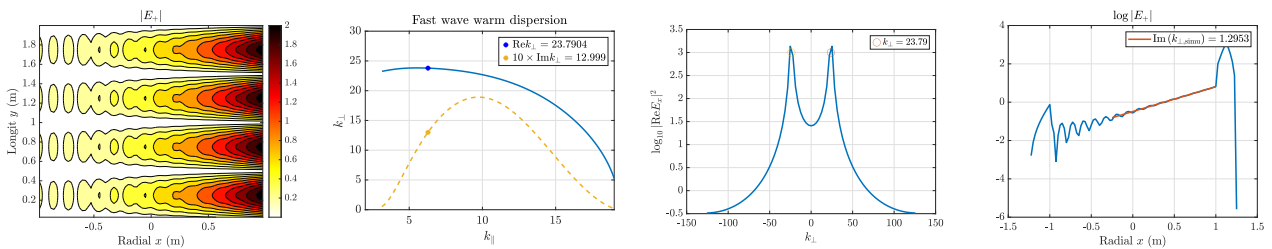


Figure 4. Cyclotron damping test case, homogeneous plasma, cold electrons and deuterons $n_D = 0.95n_e$, 2keV protons $n_H = 0.05n_e$, $\Omega_H/2\pi = 52.56\text{MHz}$, uniform magnetic field, sinewave antenna $k_{\parallel} = 2\pi$, $f_{\text{ant}} = 51.8\text{MHz}$. From left to right: modulus of the left-circularly polarised component of $\mathbf{E}$, analytical solution to warm dispersion relation, numerical E_x spectrum obtained in the radial direction, yielding $\text{Re } k_{\perp}$, and cyclotron damping from the simulation matching with analytical results $\text{Im } k_{\perp}$.

and will be exploited in next code development, because it allows to integrate directly $\Upsilon_{\alpha}(t)t^n$, in particular products of FEM basis functions with the kernels. The strategy currently adopted presets a non local coupling length in units of element along the magnetic field at each radial position, and this for each species and each polarisation, such that the outer loops are over species and polarisations. Therefore pairs of elements, and hence of DoFs, can be visited several times. This leads to repeated entries stored as Coordinate list. While this allows flexibility in terms of physics description, duplicate pairs quickly become a memory issue. Indeed revisiting the order of the

loops, further use the hash functions or identify nonlocally coupled DoFs prior to calculation could be beneficial.

The near to mid-term goals are the investigation of physical scenario for which the background equilibrium magnetic field is inhomogeneous, namely has a parallel gradient, such as in a magnetic mirror [19]. A field-aligned coordinate system could be devised by transforming the curl-conforming Nédélec elements using curvilinear transformation map together with the transformation of the curl operator [31, 32]. The mid to long-term endeavour will be dedicated to mesh-refinement, and eventually extension to 3D tokamak and stellarator geometries. In GMSH-

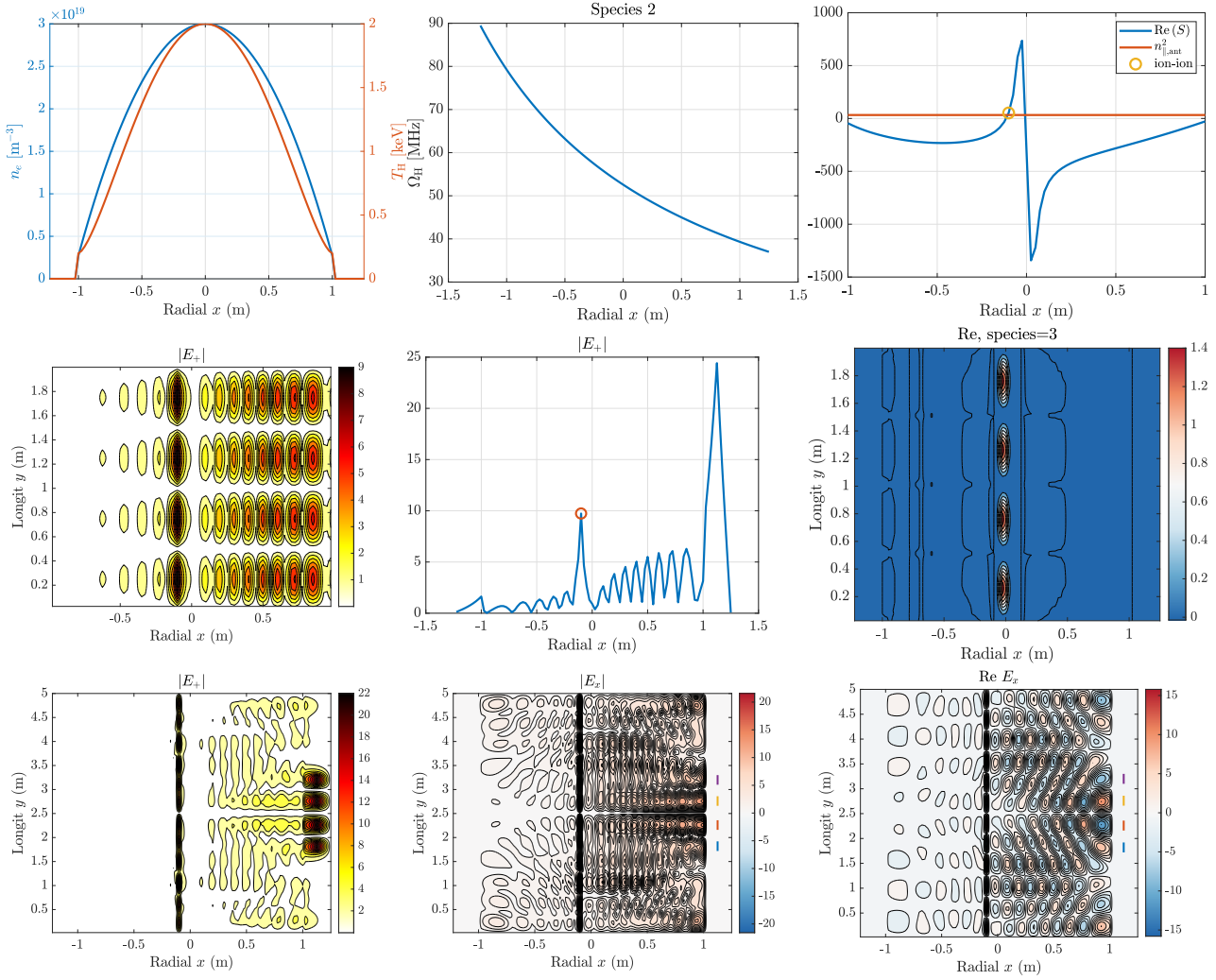


Figure 5. Minority heating simulations with JET-like profiles, and associated ion-ion hybrid resonance layer for a sinewave antenna characterised by $k_{\parallel} = 2\pi$ and $f_{ant} = 52.56\text{MHz}$, first row, and whose results appear in the second row showing enhancement of E_+ at the ion-ion hybrid layer and spatial profile of the wave-power absorption by the minority-H (species 3). Minority-heating using a 4 strap JET-like antenna in dipole phasing $(0, \pi, 0, \pi)$ is obtained in the last row, whose spectrum is captured at once thanks to the new nonlocal dielectric kernel formulation.

FEM, of which results appear in the next section, we plan especially to couple our solvers with distributed domain decomposition preconditioners such as Overlapping Restricting Additive Schwarz (ORAS) [33, 34].

3 GMSH-FEM

Results obtained with PLIKES using the spectral plasma dielectric response and a sinewave antenna have been cross-validated with the open-source finite element solvers GetDP [35] and GMSH-FEM [36], which can further be used to simulate more complex geometrical configurations. As an example, Fig. 6 shows simulations with the same ionic species parameters as in subsection 2.4, in an annular configuration with the same dimensions as JET. Fourth order nodal and edge finite elements are used, and results are displayed for both cold and warm plasma cases, with constant and spatially varying applied static magnetic field, with single $k_{\parallel}$ as an intermediate step toward full tokamak geometry and inclusion of nonlocal kernels.

4 PSYDAC

PSYDAC [37] is a numerical Python library developed at IPP Garching which enables PDE formulation in FEM using tensor-product B-Spline elements [38]. The assembly of the symbolic expressions of the weak-form runs at Fortran/C speed [37, 39]. Complex domains are simulated through deformation of reference cube. The cold plasma dielectric tensor has been successfully implemented in 3D geometry, and validated against dispersion relation through FFT as shown in Fig. 7. The kernel implementation will follow suit.

5 Conclusions

The advantages of the method promise *i)* the use of the FE method to model wave propagation in warm inhomogeneous fusion plasmas, *ii)* the application of local mesh refinement, not possible in spectral methods, *iii)* to provide RF field representations suited to address tokamak

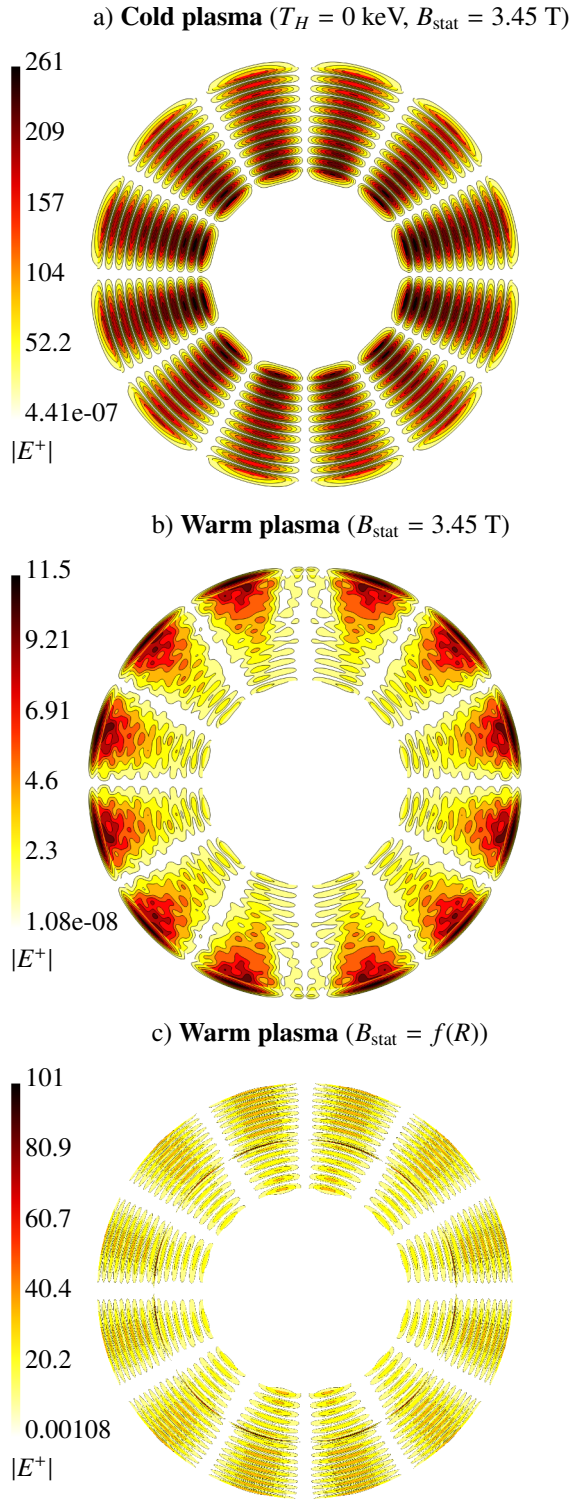


Figure 6. Plasma simulation at $f = 51.8$ MHz with: Electrons (0 keV, $n_{\text{elec}} = 3 \cdot 10^{19} \text{ m}^{-3}$); Hydrogen (H^+) ($5\% \cdot n_{\text{elec}}$); deuterium (0 keV, $95\% \cdot n_{\text{elec}}$). Two thin rings of vacuum surround the plasma at the inside and outside. At the inside and outside boundaries, the electric field is set to zero. The antenna makes a complete turn in the outer ring to guarantee a physical excitation mode of $k_{\parallel} = 2\pi$. **a)** $T_H = 0$ keV, azimuthal static field with constant amplitude $B_{\text{stat}}^{\theta} = 3.45$ T. **b)** $T_H = 2$ keV, azimuthal static field with constant amplitude $B_{\text{stat}}^{\theta} = 3.45$ T. **c)** $T_H = 2$ keV, azimuthal static field $B_{\text{stat}}^{\theta} = \frac{B_0 \cdot R_0}{R_0 + r} = \frac{3.45}{1+r/2.97}$ T.

geometry, *iv*) to allow straightforward connection between plasma and antenna models, directly applicable to the study of minority heating in realistic geometry. While the present efforts are dedicated to enhance the description of the parallel dynamics, they should at a later stage be complemented by ongoing efforts focusing on the perpendicular one, as with NGSolve [2, 12, 18].

Acknowledgements

This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

References

- [1] P. Lamalle, Dielectric kernels for maxwellian tokamak plasmas, AIP Conference Proceedings **2254**, 100001 (2020). <https://doi.org/10.1063/5.0014257>
- [2] M. Machielsen, J. Rubin, J. Graves, Exact expression for the hot plasma conductivity kernel in configuration space, Fundamental Plasma Physics **3**, 100008 (2023). <https://doi.org/10.1016/j.fpp.2023.100008>
- [3] P. Lamalle, Integral dielectric kernels for Maxwellian tokamak plasmas, 20th European Fusion Theory Conference, 2-3 October 2023, Padova, Italy (2023). <https://indico.global/event/7904/contributions/69727/contribution.pdf>
- [4] J. Ongena, Y.O. Kazakov, B. Schweer, F. Louche, A. Messiaen, M. Vervier, V. Borsuk, R. Bilato, D. Hartmann, K. Hollfeld et al., Physics and applications of ICRH on W7-X, pp. 17–22 (2017). https://scipub.euro-fusion.org/wp-content/uploads/WPS1CP16_15030_submitted.pdf
- [5] E. Jaeger, L. Berry, E. D’Azevedo, D. Batchelor, M. Carter, K. White, H. Weitzner, Advances in full-wave modeling of radio frequency heated, multi-dimensional plasmas, Physics of Plasmas **9**, 1873 (2002). <https://doi.org/10.1063/1.1455001>
- [6] M. Brambilla, Numerical simulation of ion cyclotron waves in tokamak plasmas, Plasma Physics and Controlled Fusion **41**, 1 (1999). <https://doi.org/10.1088/0741-3335/41/1/002>
- [7] P. Lamalle, On the radiofrequency response of tokamak plasmas, Plasma physics and controlled fusion **39**, 1409 (1997). <https://doi.org/10.1088/0741-3335/39/9/011>
- [8] R. Dumont, Variational approach to radiofrequency waves in magnetic fusion devices, Nuclear Fusion **49**, 075033 (2009). <https://doi.org/10.1088/0029-5515/49/7/075033>
- [9] P. Popovich, W.A. Cooper, L. Villard, A full-wave solver of the Maxwell’s equations in 3d cold plasmas, Computer Physics Communications **175**, 250 (2006). <https://doi.org/10.1016/j.cpc.2006.04.001>

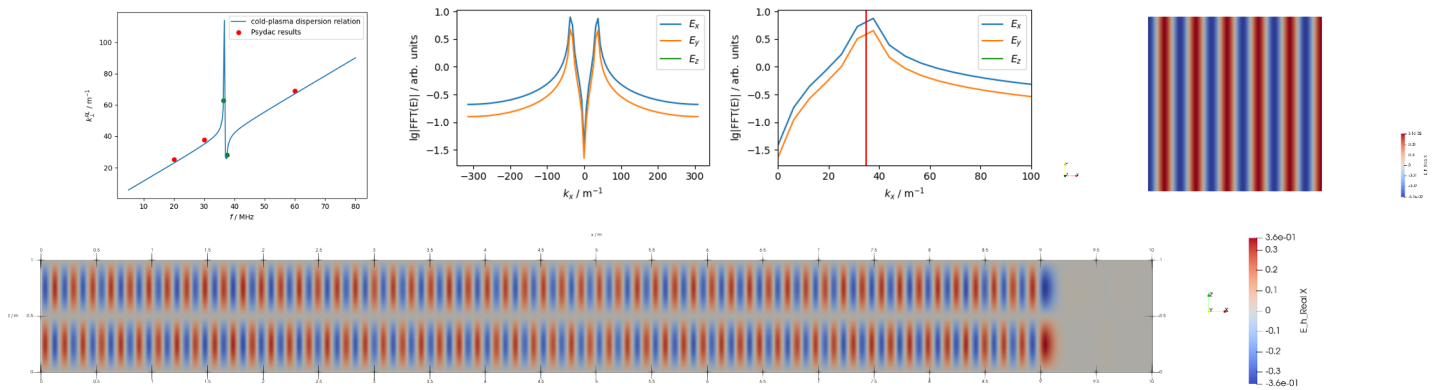


Figure 7. Outputs of a full-wave simulation applying the Silver-Müller boundary condition (top). Field resulting from a sine-wave antenna (single $k_{\parallel}$), metallic wall, and a vacuum region between plasma and antenna (n_x, n_y, n_z) = (200, 20, 20), 64 cores (bottom).

- [10] P. Vallejos, T. Johnson, R. Ragona, T. Hellsten, L. Frassinetti, Effect of poloidal phasing on ion cyclotron resonance heating power absorption, *Nuclear Fusion* **59**, 076022 (2019). <https://doi.org/10.1088/1741-4326/ab1ab7>
- [11] S. Shiraiwa, J. Wright, P. Bonoli, T. Kolev, M. Stowell, RF wave simulation for cold edge plasmas using the MFEM library, *EPJ Web of Conferences* **157**, 03048 (2017). <https://doi.org/10.1051/epjconf/201715703048>
- [12] V. Maquet, B. Reman, D. Van Eester, R. Ragona, Implementation of a fast 2D wave-solver using the BUDE method with NGSolve, 25th Topical Conference on Radio-Frequency Power in Plasmas, Schloss Hohenkammer, Germany, May 19-22 (2025).
- [13] O. Sauter, Nonlocal analyses of electrostatic and electromagnetic waves in hot, magnetized, nonuniform, bounded plasmas, *Ecole Polytechnique Federale de Lausanne* (1992). <https://inis.iaea.org/records/tge38-1ay77>
- [14] A. Fukuyama, Kinetic full wave analysis in inhomogeneous plasmas using integral form of dielectric tensor, *Radiation Effects and Defects in Solids* **179**, 1520 (2024). <https://doi.org/10.1080/10420150.2024.2434501>
- [15] P.U. Lamalle, B.C.G. Reman, C. Slaby, D. Van Eester, F. Louche, e.a. Geuzaine, C., Integral dielectric kernel approach to modelling RF heating in toroidal plasmas, 25th Topical Conference on Radio-Frequency Power in Plasmas, Schloss Hohenkammer, Germany, May 19-22 (2025).
- [16] O. Meneghini, S. Shiraiwa, R. Parker, Full wave simulation of Lower Hybrid waves in Maxwellian plasma based on the finite element method, *Physics of Plasmas* **16** (2009). <https://doi.org/10.1063/1.3216548>
- [17] S. Shiraiwa, O. Meneghini, R. Parker, P. Bonoli, M. Garrett, M. Kaufman, J. Wright, S. Wukitch, Plasma wave simulation based on a versatile finite element method solver, *Physics of Plasmas* **17** (2010). <https://doi.org/10.1063/1.3396371>
- [18] D. Van Eester, V. Maquet, B. Reman, H. Carpioux, B. Valentin, E. Lerche, P. Lamalle, Almost-off-the-shelf tools for ICRH modelling, 25th Topical Conference on Radio-Frequency Power in Plasmas, Schloss Hohenkammer, Germany, May 19-22 (2025).
- [19] V. Svidzinski, J. Kim, J. Spencer, L. Zhao, S. Galkin, E. Evstatiev, Hot plasma dielectric response to radio-frequency fields in inhomogeneous magnetic field, *Physics of Plasmas* **23** (2016). <https://doi.org/10.1063/1.4966638>
- [20] V. Svidzinski, L. Zhao, J. Kim, N. Barov, Full wave modeling of radio-frequency beams in tokamaks in the electron cyclotron frequency range, *Physics of Plasmas* **31** (2024). <https://doi.org/10.1063/5.0194838>
- [21] D.L. Green, L.A. Berry, A.B. Simpson, T.R. Younkin, Kinetic-j: A computational kernel for solving the linearized Vlasov equation applied to calculations of the kinetic, configuration space plasma current for time harmonic wave electric fields, *Computer Physics Communications* **225**, 122 (2018). <https://doi.org/10.1016/j.cpc.2018.01.002>
- [22] P.U. Lamalle, B. Reman (2025), to be submitted for publication
- [23] M. Asp  n, A. Signell, J. Westerholm, Efficient assembly of sparse matrices using hashing, *International Workshop on Applied Parallel Computing* pp. 900–907 (2006). https://doi.org/10.1007/978-3-540-75755-9_107
- [24] P.R. Amestoy, I.S. Duff, J.Y. L'Excellent, J. Koster, A fully asynchronous multifrontal solver using distributed dynamic scheduling, *SIAM Journal on Matrix Analysis and Applications* **23**, 15 (2001). <https://doi.org/10.1137/S0895479899358194>
- [25] P.R. Amestoy, A. Guermouche, J.Y. L'Excellent, S. Pralet, Hybrid scheduling for the parallel solution of linear systems, *Parallel computing* **32**, 136 (2006). <https://doi.org/10.1016/j.parco.2005.07.004>
- [26] M. Folk, G. Heber, Q. Koziol, E. Pourmal, D. Robinson, An overview of the HDF5 technology suite and its applications, *EDBT/ICDT 2011*

- workshop on array databases pp. 36–47 (2011). <https://doi.org/10.1145/1966895.1966900>
- [27] HDF Group Hierarchical Data Format version 5, <https://github.com/HDFGroup/hdf5>
- [28] M.W. Scroggs, online; accessed: 16-May-2025, <https://defelement.org>
- [29] H. Bateman, A. Erdélyi, Higher transcendental functions, volume II, Bateman Manuscript Project) Mc Graw-Hill Book Company **410** (1953). <https://authors.library.caltech.edu/records/cnd32-h9x80>
- [30] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions: with formulas, graphs, and mathematical tables, Vol. 55 (Courier Corporation, 1965)
- [31] J.A. Stratton, Electromagnetic theory (John Wiley & Sons, 2007)
- [32] P. Solin, K. Segeth, I. Dolezel, Higher-order finite element methods (Chapman and Hall/CRC, 2003), <https://doi.org/10.1201/9780203488041>
- [33] O.G. Ernst, M.J. Gander, Why it is difficult to solve Helmholtz problems with classical iterative methods, Numerical analysis of multiscale problems pp. 325–363 (2011). https://doi.org/10.1007/978-3-642-22061-6_10
- [34] A. St-Cyr, M.J. Gander, S.J. Thomas, Optimized multiplicative, additive, and restricted additive Schwarz preconditioning, SIAM Journal on Scientific Computing **29**, 2402 (2007). <https://doi.org/10.1137/060652610>
- [35] P. Dular, C. Geuzaine, F. Henrotte, W. Legros, A general environment for the treatment of discrete problems and its application to the finite element method, IEEE transactions on magnetics **34**, 3395 (2002). <https://doi.org/10.1109/20.717799>
- [36] A. Royer, E. Béchet, C. Geuzaine, Gmsh-Fem: An efficient finite element library based on Gmsh, 14th World Congress on Computational Mechanics (WCCM), ECCOMAS Congress 2020 (2021). <https://hdl.handle.net/2268/262394>
- [37] Y. Güçlü, S. Hadjout, A. Ratnani, PSYDAC: a high-performance IGA library in python, 8th European Congress on Computational Methods in Applied Sciences and Engineering (2022).
- [38] E.M. Sánchez, M.C. Pinto, O. Maj et al., Time-splitting methods for the cold-plasma model using Finite Element Exterior Calculus, Journal of Computational Physics p. 114305 (2025). <https://doi.org/10.1016/j.jcp.2025.114305>
- [39] E. Bourne, Y. Güçlü, S. Hadjout, A. Ratnani, Pyccl: a Python-to-X transpiler for scientific high-performance computing, Journal of Open Source Software **8**, 4991 (2023). <https://doi.org/10.21105/joss.04991>