Running maximum of a k-regular sequence

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April 10, 2026

Abstract

The k-regular sequences form a large class studied in number theory, combinatorics, and other parts of discrete mathematics. This class is known to be closed under many natural operations, such as term-by-term sum, product, running sum, and so forth, but it is not closed under running maximum. Proving the previously-known counterexample, involving the Stern sequence, required intricate arguments. In this note, we construct a significantly simpler example of a k-regular sequence whose running maximum is not k-regular.

1 Introduction

Understanding the closure properties of sequence classes is a central theme in discrete mathematics. For example, in algorithmic applications one often considers prefix-based operations such as running sums or running maximum, since these correspond to standard procedures in data analysis, signal processing, and string algorithms. Understanding whether these operations preserve membership in significant sequence classes is crucial for characterizing their combinatorial behavior.

One such class is the family of k-regular sequences, introduced by Allouche et al. [1] as a natural generalization of k-automatic sequences. The large class of k-regular sequences has been studied in number theory, combinatorics, and other parts of discrete mathematics [1, 2, 3, 6].

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[*] Research supported by NSERC grant 2024-03725.

Formally, let $k \geq 2$ be an integer. We say that a sequence $(a(n))_{n\geq 0}$ taking values in a ring R is k-regular if its k-*kernel*, the set of subsequences

$$\{(a(k^e n + f))_{n\geq 0} \ : \ e \geq 0, \ 0 \leq f < k^e\},$$

generates a finitely-generated module. In this paper we take $R = \mathbb{Q}$. Three classical examples of such sequences are as follows:

1. Define $s_k(n)$ to be the sum of the digits of n when represented in base k. Then we clearly have $s_k(kn + a) = s_k(n) + a$ for $0 \leq a < k$, which shows that the k-kernel of $(s_k(n))_{n\geq 0}$ is spanned by the sequence $(s_k(n))_{n\geq 0}$ and the constant sequence 1.

2. The *Stern sequence* $(d(n))_{n\geq 0}$ is defined by $d(0) = 0$, $d(1) = 1$, $d(2n) = d(n)$, and $d(2n+1) = d(n)+d(n+1)$ for $n \geq 0$; see, for example, [16, 12]. It is easy to verify that $d(4n + 1) = d(n) + d(2n + 1)$ and $d(4n + 3) = 2d(2n + 1) - d(n)$, and so the 2-kernel of $(d(n))_{n\geq 0}$ is spanned by the two sequences $(d(n))_{n\geq 0}$ and $(d(2n + 1))_{n\geq 0}$.

3. Let p be a polynomial. Then the sequence of values $(p(n))_{n\geq 0}$ is k-regular for every k, as it is easy to see its k-kernel is spanned by the sequences by

$$(1)_{n\geq 0}, \ (n)_{n\geq 0}, \ (n^2)_{n\geq 0}, \ \ldots, (n^d)_{n\geq 0},$$

   where d is the degree of p.

It is known that the class of k-regular sequences is closed under many different operations. Letting $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$ be two k-regular sequences, the term-by-term sum $(a(n) + b(n))_{n\geq 0}$, difference $(a(n)-b(n))_{n\geq 0}$, product $(a(n)b(n))_{n\geq 0}$, product by a scalar $(c\cdot a(n))_{n\geq 0}$, and linearly-indexed subsequences $(a(cn+d))_{n\geq 0}$ for integers $c, d \geq 0$ are all k-regular. This class is also closed under the operation of *running sum* (also called *prefix sum*), defined by $b(n) = \sum_{0\leq i\leq n} a(i)$. See [1, 4, 5, 9, 10] for these and other properties of k-regular sequences.

However, not all operations preserve k-regularity. The class of k-regular sequences is *not* closed under taking absolute values, or term-by-term maximum $(\max(a(n), b(n)))_{n\geq 0}$ and minimum [1, pp. 168–169]. This raises the obvious question of whether such sequences are closed under the operation of *running max*, defined by $b(n) = \max_{0\leq i\leq n} a(i)$. The running max operation is well-studied in the mathematical literature, and can also be found under the synonyms *prefix maximum* [8], *record statistics* [7, 14, 13], and *left-to-right maximum* [15, p. 23].

Recent work [11] answered this negatively, using the Stern sequence [12] as a counterexample. However, the proof is rather intricate and challenging. Hence it seems desirable to have a relatively simple example with a short and more accessible proof. Providing such a simple example is the goal of this note.

2 A simple counterexample to closure of k-regular sequences under running maximum

Theorem 1. *Let $s_2(n)$ be the sum of digits in the base-2 representation of n, and define $f(n) = n + s_2(n)$. Then $(f(n))_{n\geq 0}$ is 2-regular, but its running max sequence $(g(n))_{n\geq 0}$ defined by*

$$g(n) = \max_{0\leq i\leq n} f(i)$$

is not 2-regular.

Proof. We saw above that the sequences $(s_2(n))_{n\geq 0}$ and $(n)_{n\geq 0}$ are both 2-regular, and hence by a known result, their term-by-term sum $(f(n))_{n\geq 0}$ is also.

Define $\Delta(n) = g(n) - f(n)$. Consider the subsequence $\Delta_j(n) = \Delta(2^{2^j-1}n + 2^j - 1)$, which belongs to the 2-kernel of $(\Delta(n))_{n\geq 0}$. We will show that the sequences $(\Delta_j(n))_{n\geq 0}$ are linearly independent over $\mathbb{Q}$. It follows that the module generated by the k-kernel of $(\Delta(n))_{n\geq 0}$ is not finitely generated, and hence $(\Delta(n))_{n\geq 0}$ is not 2-regular. Thus $(g(n))_{n\geq 0}$ is not 2-regular.

First we show by induction on j that $g(2^j-1) = f(2^j-1)$ for all $j \geq 0$. For the base case, we see that $g(0) = f(0)$ by definition. Now, for the induction step, let $j > 0$ and assume that $g(2^{j-1} - 1) = f(2^{j-1} - 1)$. We have

$$g(2^{j-1} - 1) = f(2^{j-1} - 1) = 2^{j-1} - 1 + j - 1 < 2^j - 1 + j = f(2^j - 1),$$

while for $\ell \in \{0, \ldots, 2^{j-1} - 1\}$ we have

$$\begin{aligned} f(2^{j-1} + \ell) &= 2^{j-1} + \ell + (1 + s_2(\ell)) \\ &\leq 2^{j-1} + 2^{j-1} - 1 + (1 + j - 1) \\ &\leq 2^j + j - 1 = f(2^j - 1). \end{aligned}$$

Hence $g(2^j - 1) = f(2^j - 1)$. This completes the induction.

Next, we show that

$$\Delta_j(2^i) = 0 \text{ and } \Delta_j(2^{j+3}) = 1$$

for $j \geq 0$ and $i \in \{2, \ldots, j+2\}$, as illustrated in Figure 1 for $j = 1$. To see this, first note that for all $i \geq 2$ and $\ell \in \{0, \ldots, 2^j - 1\}$, we have

$$\begin{aligned} f(2^{2^j+i-1} + \ell) &= 2^{2^j+i-1} + \ell + (1 + s_2(\ell)) \\ &\leq 2^{2^j+i-1} + 2^j + j = f(2^{2^j+i-1} + 2^j - 1). \end{aligned}$$

Now consider $i \in \{2, \ldots, j+2\}$. From above we get

$$\begin{aligned} g(2^{2^j+i-1} - 1) &= f(2^{2^j+i-1} - 1) \\ &= 2^{2^j+i-1} + 2^j + i - 2 \\ &\leq 2^{2^j+i-1} + 2^j + j = f(2^{2^j+i-1} + 2^j - 1). \end{aligned}$$

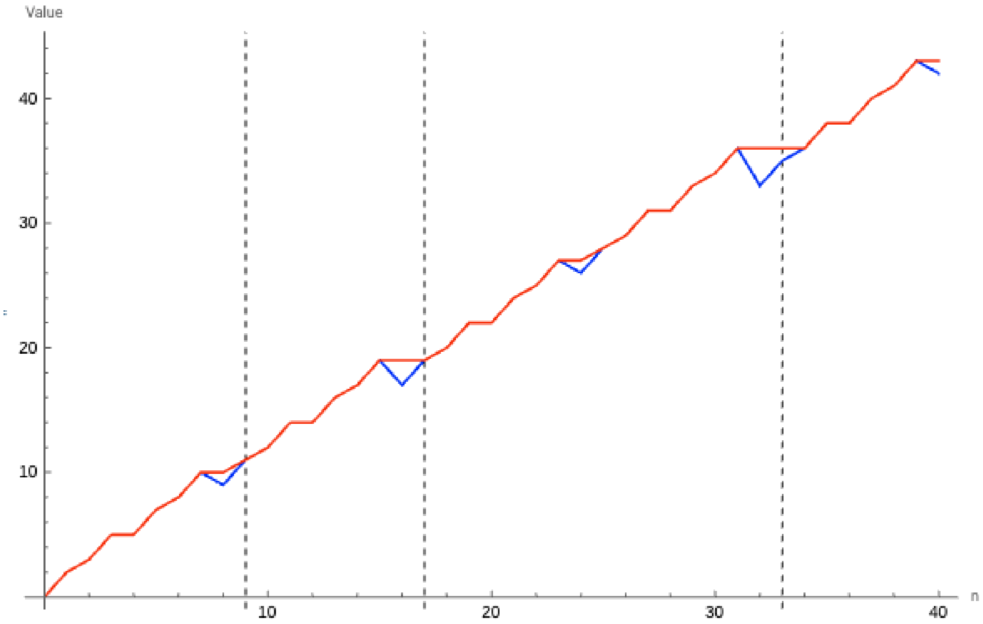

Figure 1: The sequence f in blue, along with its running max g in red. The vertical lines indicate the points corresponding to $\Delta_j(2^i)$ with $j = 1$ and $i \in \{2, \ldots, j+3\}$.

Hence $g(2^{2^j+i-1} + 2^j - 1) = f(2^{2^j+i-1} + 2^j - 1)$ and $\Delta_j(2^i) = 0$. Since we have

$$\begin{aligned} g(2^{2^j+j+2} - 1) &= 2^{2^j+j+2} - 1 + 2^j + j + 2 \\ &= 2^{2^j+j+2} + 2^j + j + 1 = f(2^{2^j+j+2} + 2^j - 1) + 1, \end{aligned}$$

it follows that $g(2^{2^j+j+2} + 2^j - 1) = f(2^{2^j+j+2} + 2^j - 1) + 1$ and $\Delta_j(2^{j+3}) = 1$. Thus all the sequences $(\Delta_j(n))_{n\geq 0}$ are linearly independent, and this completes the proof. $\square$

Naturally, by replacing g with $-g$, the same result holds for running min.

Acknowledgments

We are grateful to the editor and the referees for their helpful comments.

References

[1] J.-P. Allouche and J. Shallit. The ring of k-regular sequences. *Theoret. Comput. Sci.* **98** (1992), 163–197.

[2] J.-P. Allouche and J. Shallit. *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[3] J.-P. Allouche and J. Shallit. The ring of k-regular sequences, II. *Theoret. Comput. Sci.* **307** (2003), 3–29.

[4] J. P. Bell. On the values attained by a k-regular sequence. *Adv. Appl. Math.* **34** (2005), 634–643.

[5] J. P. Bell, M. Coons, and K. G. Hare. The minimal growth of a k-regular sequence. *Bull. Austral. Math. Soc.* **90** (2014), 195–203.

[6] J. Berstel and C. Reutenauer. *Noncommutative Rational Series with Applications*, Vol. 137 of *Encyclopedia of Mathematics and Its Applications.* Cambridge University Press, 2011.

[7] K. N. Chandler. The distribution and frequency of record values. *J. Royal Stat. Soc. Ser. B* **14** (1952), 220-228.

[8] S. Chaudhuri and J. Radhakrishnan. The complexity of parallel prefix problems on small domains. In *FOCS 1992: 33rd Annual Symposium.* IEEE Press, 1992, pp. 638–647.

[9] M. Coons and J. Lind. Radial asymptotics of generating functions of k-regular sequences. *Bull. Aust. Math. Soc.* **110** (2024) 528–534.

[10] J. Honkala. Quasi-universal k-regular sequences. *Theor. Comput. Sci.* **891** (2021), 84–89.

[11] A. Keramatipour and J. Shallit. Record-setters in the Stern sequence. *Discrete Math.* **347** (2024), 113852.

[12] D. H. Lehmer. On Stern's diatomic series. *Amer. Math. Monthly* **36** (1929), 59–67.

[13] V. B. Nevzorov. Records. *Theor. Probab. Appl.* **32** (1988), 201–228.

[14] A. Rényi. Théorie des éléments saillants d'une suite d'observations. *Ann. Sci. Univ. Clermont Math.* **8** (1962), 7–13.

[15] R. P. Stanley. *Enumerative Combinatorics*, Vol. I. 2nd edition. Cambridge University Press, 2012.

[16] M. A. Stern. Über eine zahlentheoretische Funktion. *J. Reine Angew. Math.* **55** (1858), 193–220.