

# Factor complexity and critical exponent of words in a Thue-Morse family

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# Sponsors



1 Setup

2 Klouda's method for bispecial factors

3 Main results

# A generalization of the Thue-Morse word

The binary numeration system is based on the linear recurrence sequence given by  $U_0 = 1$  and  $U_n = U_{n-1} + U_{i_{n-1}}$  for all  $n \geq 1$ . Any natural number can be written uniquely as a sum  $U_{i_{\ell-1}} + U_{i_{\ell-2}} \dots + U_0$  where  $\ell$  depends on  $n$  and  $i_j \geq i_{j-1} + 1$  for all  $j$ .

The Thue-Morse word  $\mathbf{t}$  is given by  $t_n = \ell(n) \bmod 2$ .

|   |   |   |   |   |   |   |   |   |    |   |   |   |   |   |   |   |     |
|---|---|---|---|---|---|---|---|---|----|---|---|---|---|---|---|---|-----|
| 1 |   | 2 | 4 |   | 8 |   |   |   | 16 |   |   |   |   |   |   |   |     |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0  | 0 | 1 | 0 | 1 | 1 | 0 | 1 | ... |

# A generalization of the Thue-Morse word

The **Zeckendorf** numeration system is based on the linear recurrence sequence given by  $U_0 = 1$ ,  $U_1 = 2$  and  $U_n = U_{n-1} + U_{n-2}$  for all  $n \geq 2$ . Any natural number can be written uniquely as a sum  $U_{i_{\ell-1}} + U_{i_{\ell-2}} \dots + U_0$  where  $\ell$  depends on  $n$  and  $i_j \geq i_{j-1} + 2$  for all  $j$ .

The **Fibonacci-Thue-Morse** word  $\mathbf{f}$  is given by  $f_n = \ell(n) \bmod 2$ .

|   |   |   |   |   |   |   |   |   |   |   |   |    |   |   |   |   |     |
|---|---|---|---|---|---|---|---|---|---|---|---|----|---|---|---|---|-----|
|   | 1 | 2 | 3 |   | 5 |   | 8 |   |   |   |   | 13 |   |   |   |   |     |
| 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1  | 1 | 0 | 0 | 0 | ... |

# A generalization of the Thue-Morse word

The **Narayana** numeration system is based on the linear recurrence sequence given by  $U_0 = 1$ ,  $U_1 = 2$ ,  $U_2 = 3$  and  $U_n = U_{n-1} + U_{n-3}$  for all  $n \geq 3$ . Any natural number can be written uniquely as a sum  $U_{i_{\ell-1}} + U_{i_{\ell-2}} \dots + U_0$  where  $\ell$  depends on  $n$  and  $i_j \geq i_{j-1} + 3$  for all  $j$ .

The **Allouche-Johnson** ('96) word **a** is given by  $a_n = \ell(n) \bmod 2$ .

|   |   |   |   |   |   |   |   |   |   |    |   |   |   |   |   |   |   |     |  |
|---|---|---|---|---|---|---|---|---|---|----|---|---|---|---|---|---|---|-----|--|
|   | 1 | 2 | 3 | 4 |   | 6 |   | 9 |   | 13 |   |   |   |   |   |   |   |     |  |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0  | 0 | 0 | 1 | 0 | 0 | 0 | 0 | ... |  |

## The word $\mathbf{x}_k$

Consider the numeration system based on the recurrence relation  $U_n = U_{n-1} + U_{n-k}$  and  $U_0, \dots, U_{k-1} = 1, \dots, k$ . Define  $\mathbf{x}_k$  by

$$(\mathbf{x}_k)_n = |\text{rep}_U(n)|_1 \bmod 2.$$

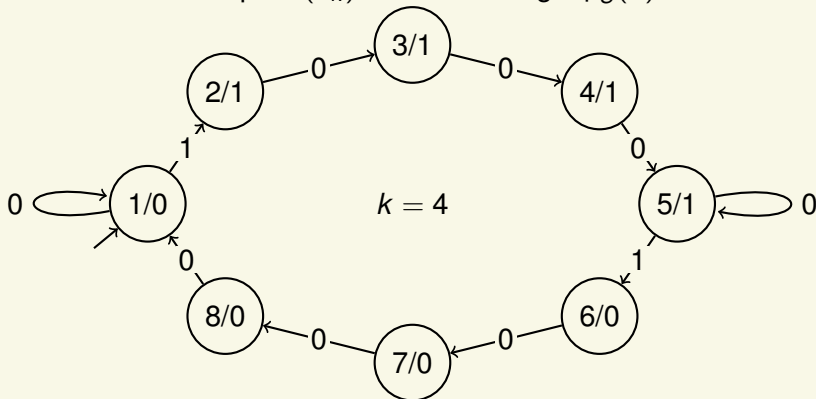
This word is  $U$ -automatic, in the sense that there exists an automaton that outputs  $(\mathbf{x}_k)_n$  when reading  $\text{rep}_U(n)$ .

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# The word $\mathbf{x}_k$

We know that  $U$ -automatic words in exotic numeration systems are morphic (Shallit '88 in this case, later Rigo '00).

We may define  $\mathbf{x}_k$  as the projection  $\pi(\mathbf{u}_k)$  where  $\mathbf{u}_k$  is the fixed point  $\xi_k^\omega(0)$ , with

$$\xi_k : \begin{array}{ll} 0 & \rightarrow 01 \\ 1 & \rightarrow 2 \\ 2 & \rightarrow 3 \\ & \vdots \\ (k-2) & \rightarrow (k-1) \\ (k-1) & \rightarrow 0' \end{array}, \quad \begin{array}{ll} 0' & \rightarrow 0'1' \\ 1' & \rightarrow 2' \\ 2' & \rightarrow 3' \\ & \vdots \\ (k-2)' & \rightarrow (k-1)' \\ (k-1)' & \rightarrow 0. \end{array} \quad (1)$$

and

$$\pi(a) = \begin{cases} 0, & \text{if } a = 0 \text{ or } a \in \{1', 2', \dots, (k-1)'\}; \\ 1, & \text{if } a = 0' \text{ or } a \in \{1, 2, \dots, k-1\}. \end{cases} \quad (2)$$

# The word $\mathbf{x}_4$

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|

 $\cdots \mathbf{u}_4$ 

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

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|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|

 $\cdots \mathbf{u}_4$   

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

 $\cdots \mathbf{x}_4$ 

In a March 2025 preprint, J. Shallit used `Walnut` to investigate properties of the sequence  $\mathbf{x}_3$ , as well as the sequence obtained from  $\mathbf{u}_3$  by deleting the 's (This sequence is linked to Hofstadter functions, see the March 11 One World talk by P. Letouzey). He formulated **two conjectures** on words in the family  $\mathbf{x}_k$ , but these cannot be verified for all  $k$  using just `Walnut` as the underlying automaton depends on  $k$ .

## Two conjectures

The *critical exponent* of a sequence  $\mathbf{x}$  is defined as

$$E(\mathbf{x}) = \sup\{r \in \mathbb{Q} : u^r \text{ is a non-empty factor of } \mathbf{x}\}.$$

For instance, since the Thue-Morse word  $\mathbf{t}$  contains squares but no overlaps,  $E(\mathbf{t}) = 2$ .

The asymptotic version is called the *asymptotic critical exponent*, denoted  $E^*(\mathbf{x})$ . If  $E(\mathbf{x}) = \infty$ , then  $E^*(\mathbf{x}) = \infty$ .

Otherwise, it is defined as

$$E^*(\mathbf{x}) = \lim_{n \rightarrow \infty} \sup\{r \in \mathbb{Q} : u^r \text{ is a non-empty factor of period } \geq n \text{ of } \mathbf{x}\}.$$

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## Conjecture (Shallit '25)

*Let  $k \geq 1$ . The sequence  $\mathbf{x}_k$  has critical exponent  $k + 1$ , which is attained by the words  $0^{k+1}$  and  $1^{k+1}$ . It contains no factor of length  $2n + k$  and period  $n$ , and therefore has asymptotic critical exponent equal to 2.*

# Two conjectures

The factor complexity of  $\mathbf{x}$  is given by  $p_{\mathbf{x}}(n) = |\text{Fac}(\mathbf{x}) \cap A^n|$ .  
Its first difference is  $\Delta p_{\mathbf{x}}(n) = p_{\mathbf{x}}(n+1) - p_{\mathbf{x}}(n)$ .

## Conjecture (Shallit '25)

*Let  $k \geq 1$ . The first difference of factor complexity of  $\mathbf{x}_k$ , for  $n$  large enough, takes the values  $4k - 2$  and  $4k$  only.*

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## Main result (Dvořáková, K., Pelantová '25)

Both conjectures are true.

# Elementary considerations

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|

We can make some elementary remarks about  $\mathbf{u}_k$  by observing its Rauzy graphs of small order.

If  $w$  is a factor of  $\mathbf{u}_k$ , we call *twin* of  $w$  the word obtained by swapping  $j$  and  $j'$  for all  $j$  in  $\{0, \dots, k-1\}$  (e.g.  $3'00' \leftrightarrow 30'0$ ). The twin of  $w$  is also a factor of  $\mathbf{u}_k$ .

# Elementary considerations

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|

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The letter  $j \in \{1, \dots, k-1\}$  only appears in  $\mathbf{u}_k$  as a suffix of the word  $01 \dots j$ , and  $0'$  when followed by  $0$  or  $0'$  can only occur as a suffix of  $01 \dots (k-1)0'$ . The projections of these  $k$  words and their twins form the set  $\{01^\ell : 1 \leq \ell \leq k\} \cup \{10^\ell : 1 \leq \ell \leq k\}$ , which is a suffix code.

# Bispecial factors

The study of *bispecial factors* will be our angle of attack for both conjectures.

## Definition

Let  $\mathbf{w} \in A^\omega$ . A factor  $y$  of  $\mathbf{w}$  is

- *left-special* if there exist  $a \neq b \in A$  such that  $ay$  and  $by$  are both factors of  $\mathbf{w}$ .
- *right-special* if there exist  $a \neq b \in A$  such that  $ya$  and  $yb$  are both factors of  $\mathbf{w}$ .
- *bispecial* if it is both left- and right-special.

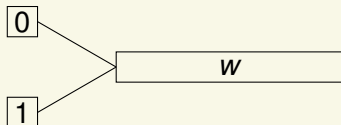
0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0' 1' 2' 3' 0 0 1]  $\cdots \mathbf{u}_4$

0 1 1 1 1 1 0 1 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1]  $\cdots \mathbf{x}_4$

# Bispecial factors for our purposes

## Proposition

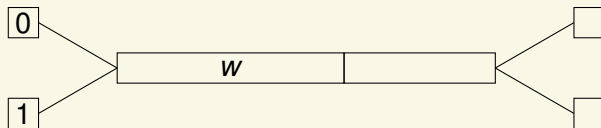
- *In a recurrent binary sequence,  $\Delta p_{\mathbf{x}}(n)$  is the number of left-special factors of length  $n$  of  $\mathbf{x}$ .*
- *In an aperiodic sequence, every left-special factor of  $\mathbf{x}$  is a prefix of exactly one shortest bispecial factor of  $\mathbf{x}$ .*



# Bispecial factors for our purposes

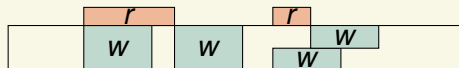
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# Bispecial factors for our purposes

Recall that a *return word* to  $w$  in  $\mathbf{x}$  is a word  $r$  such that  $rw$  starts and ends with  $w$  and contains no other occurrence of  $w$ .



## Proposition (Dolce, Dvořáková, Pelantová '23)

Let  $\mathbf{u}$  be a uniformly recurrent aperiodic sequence. Let  $(w_n)_{n \in \mathbb{N}}$  be the sequence of all bispecial factors in  $\mathbf{u}$  ordered by length. For every  $n \in \mathbb{N}$ , let  $r_n$  be the shortest return word to the bispecial factor  $w_n$  in  $\mathbf{u}$ . Then

$$E(\mathbf{u}) = 1 + \sup \left\{ \frac{|w_n|}{|r_n|} : n \in \mathbb{N} \right\} \quad \text{and} \quad E^*(\mathbf{u}) = 1 + \limsup_{n \rightarrow \infty} \frac{|w_n|}{|r_n|}.$$

# Klouta's method for bispecial factors

In a 2012 article, Klouta describes a method to find the bispecial factors within a circular non-pushy morphic word. The word  $\mathbf{u}_k$  fits this framework. We illustrate this method with  $\mathbf{u}_4$ .

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Idea: we aim to derive from  $\xi_4$  a transformation  $\Xi_4$  that maps bispecial factors to longer bispecial factors. However, we must work with *bispecial triplets*: a bispecial factor together with a pair of left- and right-extensions.

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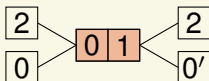
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Every bispecial factor will then be obtained within a triplet  $(\Xi_4)^n(t)$  where  $t$  is in a set of well-chosen initial triplets and  $n$  is a natural number.

# Method sketch

$\xi_4: 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0', \dots$

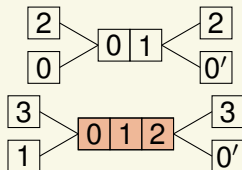
0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0' 1' 2' 3' 0 0 1 0' 1' 2' 3' 0 0 1 0 1 2 0' **u<sub>4</sub>**



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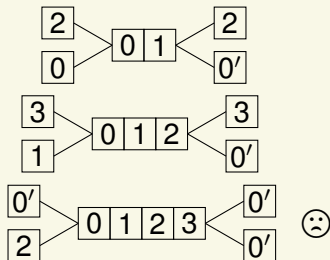
0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0' 1' 2' 3' 0 0 1 0' 1' 2' 3' 0 0 1 0 1 2 0' **u<sub>4</sub>**



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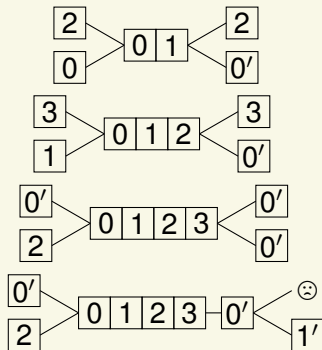
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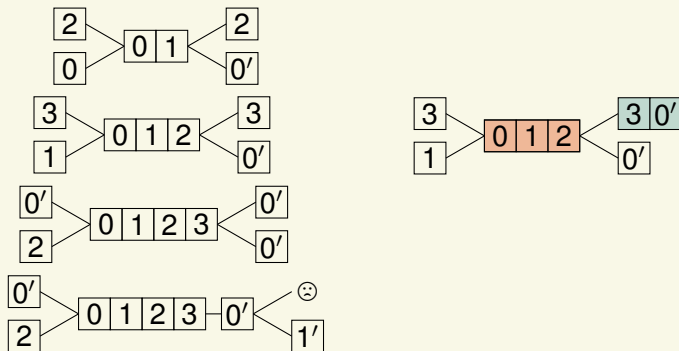
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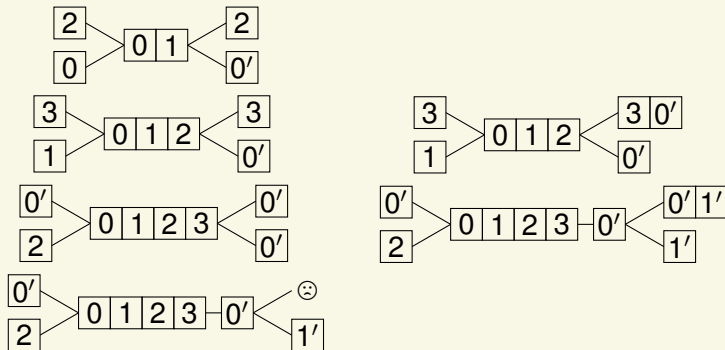
0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0' 1' 2' 3' 0 0 1 0' 1' 2' 3' 0 0 1 0 1 2 0' **u<sub>4</sub>**



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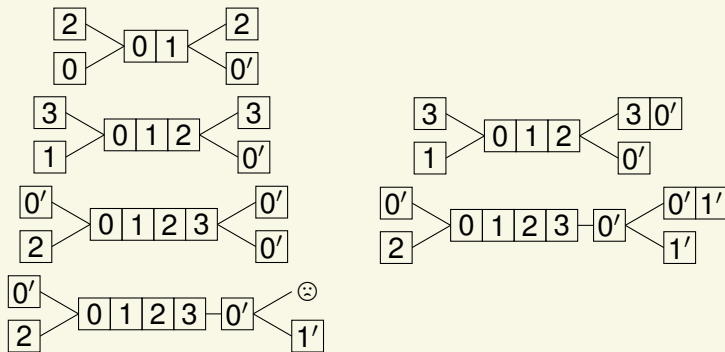
0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0 1' 2' 3' 0 0 1 0' 1' 2' 3' 0 0 1 0 1 2 0' **u<sub>4</sub>**



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0 1 2 3 0' 0' 1' 0' 1' 2' 0' 1' 2' 3' 0' 1' 2' 3' 0 0 1' 2' 3' 0 0 1 0' 1' 2' 3' 0 0 1 0 1 2 0' **u<sub>4</sub>**



We must **carefully** select pairs of left- and right-extensions. This will be formalized by the notion of (left-/right-)forky sets.

# What is a forky set?

**Definition 18.** Let  $(w_1, w_2)$  and  $(v_1, v_2)$  be unordered pairs of words. We say that

- (i)  $(w_1, w_2)$  is a prefix (suffix) of  $(v_1, v_2)$  if either  $w_1$  is a prefix (suffix) of  $v_1$  and  $w_2$  of  $v_2$ , or  $w_1$  is a prefix (suffix) of  $v_2$  and  $w_2$  of  $v_1$ ;
- (ii)  $(w_1, w_2)$  and  $(v_1, v_2)$  are L-aligned if

$$(v_1 = uw_1 \text{ or } w_1 = uv_1) \quad \text{and} \quad (v_2 = u'w_2 \text{ or } w_2 = u'v_2)$$

or

$$(v_1 = uw_2 \text{ or } w_2 = uv_1) \quad \text{and} \quad (v_2 = u'w_1 \text{ or } w_1 = u'v_2)$$

for some words  $u, u'$ .

**Definition 20.** Let  $\varphi$  be a morphism with a fixed point  $\mathbf{u}$ . A finite set  $\mathcal{B}_L$  of unordered pairs  $(w_1, w_2)$  of nonempty factors of  $\mathbf{u}$  is called L-forky if all the following conditions are satisfied:

- (i) the last letters of  $w_1$  and  $w_2$  are different for all  $(w_1, w_2) \in \mathcal{B}_L$ ,
- (ii) no distinct pairs  $(w_1, w_2)$  and  $(w'_1, w'_2)$  from  $\mathcal{B}_L$  are L-aligned,
- (iii) for any  $v_1, v_2 \in \mathcal{L}(\mathbf{u}) \setminus \{\epsilon\}$  with distinct last letters there exists  $(w_1, w_2) \in \mathcal{B}_L$  such that  $(w_1, w_2)$  and  $(v_1, v_2)$  are L-aligned,
- (iv) for any  $(w_1, w_2) \in \mathcal{B}_L$  there exists  $(w'_1, w'_2) \in \mathcal{B}_L$  such that

$$(w'_1 f_L(w_1, w_2), w'_2 f_L(w_1, w_2))$$

is a suffix of  $(\varphi(w_1), \varphi(w_2))$ .

Analogously we define an R-forky set.

A *right forky set* is a set of pairs of words that represent right-extensions of bispecial factors. We want:

- All possible pairs of extensions to be represented
- With no duplicates. (For instance,  $\{10, 01\}$  and  $\{1, 012\}$  cannot both be in a right forky set. In which category would we put  $\{10, 012\}$ ?)
- When we apply  $\xi$  then factor out the longest common prefix, the pair we obtain is also seen in the forky set.

Left forky sets are defined similarly. For all circular non-pushy morphic words, left and right forky sets exist.

# A right forky set for $\mathbf{u}_4$

$$\xi_4: 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0', \dots$$

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |    |    |    |    |   |   |   |   |   |   |    |                |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0 | 1 | 2 | 0' | $\mathbf{u}_4$ |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|

Pictured is the subset of a right forky set for  $\mathbf{u}_4$  with all pairs that have a 0 as prefix of one element.

$$\begin{Bmatrix} 00 \\ 3'0' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 1'0' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 2'0' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 2'3' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 1'2' \end{Bmatrix}$$

$$\begin{Bmatrix} 01 \\ 3'0' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 2'0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 1'0 \end{Bmatrix}$$

$$\begin{Bmatrix} 00' \\ 3'0' \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 3'0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 3 \end{Bmatrix}$$

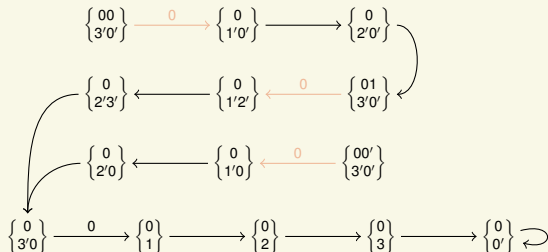
$$\begin{Bmatrix} 0 \\ 0' \end{Bmatrix}$$

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$$\xi_4: 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0', \dots$$

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |    |    |    |    |   |   |   |   |   |   |    |                |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0 | 1 | 2 | 0' | $\mathbf{u}_4$ |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|

Pictured is the subset of a right forky set for  $\mathbf{u}_4$  with all pairs that have a 0 as prefix of one element. Red arrows indicate moving to the corresponding twin pair. Arrows are labeled with the longest common prefix that must be factored out.

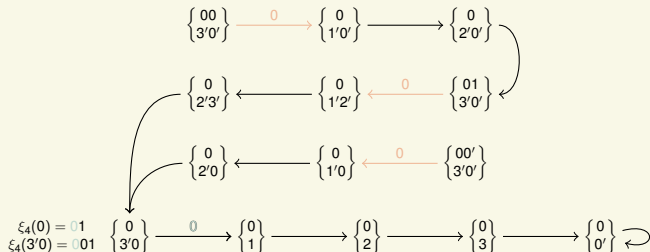


# A right forky set for $\mathbf{u}_4$

$$\xi_4: 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0', \dots$$

|   |   |   |   |    |    |    |    |    |    |    |    |    |    |    |    |    |    |   |    |    |    |    |   |   |   |    |    |    |    |   |   |   |   |   |   |    |                |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0 | 1 | 2 | 0' | $\mathbf{u}_4$ |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|

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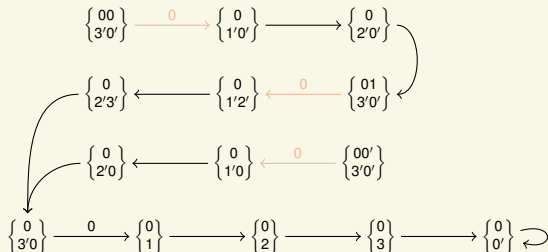


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|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|
| 0 | 1 | 2 | 3 | 0' | 0' | 1' | 0' | 1' | 2' | 0' | 1' | 2' | 3' | 0' | 1' | 2' | 3' | 0 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0' | 1' | 2' | 3' | 0 | 0 | 1 | 0 | 1 | 2 | 0' | $\mathbf{u}_4$ |
|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---|----|----|----|----|---|---|---|----|----|----|----|---|---|---|---|---|---|----|----------------|

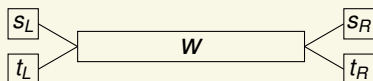
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The left forky set is simply given by all the pairs of different letters.

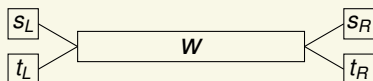
# Bispecial triplets

A *bispecial triplet* of  $\mathbf{u}$  is given by a bispecial factor  $w$ , and pairs of extensions  $\{s_L, t_L\}$  and  $\{s_R, t_R\}$  that are in the chosen left (resp. right) forky set and such that  $s_L w s_R$  and  $t_L w t_R$  are factors of  $w$  (or maybe  $s_L w t_R$  and  $t_L w s_R$ ).



# Bispecial triplets

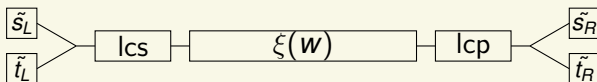
A *bispecial triplet* of  $\mathbf{u}$  is given by a bispecial factor  $w$ , and pairs of extensions  $\{s_L, t_L\}$  and  $\{s_R, t_R\}$  that are in the chosen left (resp. right) forky set and such that  $s_L w s_R$  and  $t_L w t_R$  are factors of  $w$  (or maybe  $s_L w t_R$  and  $t_L w s_R$ ).



The bispecial triplet  $(\{s_L, t_L\}, w, \{s_R, t_R\})$  can be transformed into the triplet

$$(\{\tilde{s}_L, \tilde{t}_L\}, \text{lcs}(\xi(s_L), \xi(t_L))\xi(w) \text{lcp}(\xi(s_R), \xi(t_R)), \{\tilde{s}_R, \tilde{t}_R\})$$

where  $\{\tilde{s}_L, \tilde{t}_L\}$  is the pair in the left forky set that is reached after factoring out the longest common suffix of the images.



Naming this transformation  $\Xi$ , all bispecial factors can be found within a bispecial triplet obtained by iterating  $\Xi$  from a well-chosen initial set.

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A *synchronizing point* is a point in a factor of  $\mathbf{u}$  that always separates images of letters when desubstituting  $\xi$ . For instance, in the Thue-Morse word,  $0 \cdot 0$  is a synchronizing point but  $0 \cdot 1$  is not.

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In the case of  $\mathbf{u}_4$ , there is a synchronizing point to the left of every letter except 1 and  $1'$ , and to the right of every letter except 0 and  $0'$ . Therefore, the only initial bispecial triplets are those for which  $w = \varepsilon$ .

# Application to $\mathbf{u}_4$

Things only get interesting when we depart from triplets that have  $\varepsilon$  as their factors. Our four starting points are (from bottom to top on the figure, a star means any letter can be used):

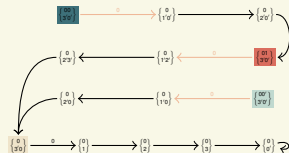
$$T_1 = (\{*, 2'\}, \varepsilon, \{0, 3'0'\})$$

$$T_3 = (\{3', 2'\}, \varepsilon, \{00', 3'0'\})$$

$$T_2 = (\{*, 2'\}, \varepsilon, \{01, 3'0'\})$$

$$T_4 = (\{3', 2'\}, \varepsilon, \{00, 3'0'\}).$$

From there, we can obtain all bispecial factors of  $\mathbf{u}_k$ .



$$\xi_4: 0 \mapsto 01, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 0', \dots$$

As an example, here are all the bispecial triplets arising from  $T_1$ :

$$\begin{aligned} T_1 &= (\{*, 2'\}, \varepsilon, \{0, 3'0\}) \\ \Xi_4(T_1) &= (\{*, 3'\}, 0, \{1, 0\}) \\ \Xi_4^2(T_1) &= (\{*, 0\}, \xi_4(0), \{2, 0\}) \\ \Xi_4^3(T_1) &= (\{*, 1\}, \xi_4^2(0), \{3, 0\}) \\ \Xi_4^4(T_1) &= (\{*, 2\}, \xi_4^3(0), \{0', 0\}) \\ &\vdots \end{aligned}$$

Therefore, every  $\xi_4^n(0)$  is bispecial in  $\mathbf{u}_4$ , with every letter as a left extension and two known right extensions.

# Bispecial factors of $\mathbf{u}_k$

## Proposition

*The bispecial factors within  $\mathbf{u}_k$  are the  $\xi_k^n(0)$  for any  $n$  and the  $\xi_k^{n+2k-1}(0)\xi_k^n(0)$  for any  $n$ , plus their twins. Moreover,  $\xi_k^n(0)$  can be extended to the left by any of the  $2k$  letters, while  $\xi_k^{n+2k-1}(0)\xi_k^n(0)$  only has two left-extensions.*

| $n$                                           | 0        | 1        | 2       | ... |
|-----------------------------------------------|----------|----------|---------|-----|
| Left-extensions of $\xi_4^{n+7}(0)\xi_4^n(0)$ | $1', 2'$ | $2', 3'$ | $3', 0$ | ... |

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| $n$                                           | 0        | 1        | 2       | ... |
|-----------------------------------------------|----------|----------|---------|-----|
| Left-extensions of $\xi_4^{n+7}(0)\xi_4^n(0)$ | $1', 2'$ | $2', 3'$ | $3', 0$ | ... |

Through a study of the return words to these bispecial factors, we obtain the following corollary.

## Corollary

*The word  $\mathbf{u}_k$  is overlap-free.*

# Projecting to $\mathbf{x}_k$

Bispecial factors in  $\mathbf{u}_k$  and  $\mathbf{x}_k$  do not necessarily match. The following lemmas bridge the gap between  $\mathbf{u}_k$  and  $\mathbf{x}_k$ .

## Lemma (1)

*Let  $u, v \in \text{Fac}(\mathbf{u}_k)$ . If  $\pi_k(u) = \pi_k(v)$  and  $u, v$  disagree on every letter, then  $|u| \leq k + 1$ . If furthermore  $u0, v0 \in \text{Fac}(\mathbf{u}_k)$ , or  $u0', v0' \in \text{Fac}(\mathbf{u}_k)$ , then  $|u| \leq k - 1$  and  $\pi_k(u) = 0^{|u|}$  or  $1^{|u|}$ .*

## Lemma (2)

*Let  $w \neq \varepsilon$  be a bispecial factor of  $\mathbf{u}_k$ . Given two right-extensions of  $w$ , their projections can agree on at most*

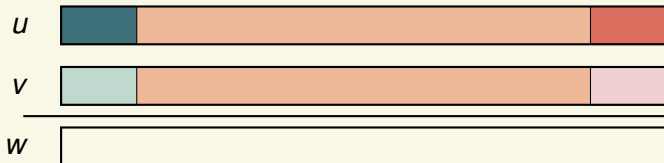
- *the first two letters, if  $w = \xi_k^\ell(0)$  with  $0 \leq \ell \leq k - 2$ .*
- *the first letter, if  $w = \xi_k^\ell(0)$  with  $k \leq \ell \leq 2k - 2$ .*
- *no letters, otherwise.*

# Factor complexity

To recall, the first difference of factor complexity  $\Delta p_{\mathbf{x}_k}(n)$  is given by the number of left-special factors of length  $n$  in  $\mathbf{x}_k$ .

## Lemma (3)

*Let  $w$  be a left special factor of  $\mathbf{x}_k$  such that  $|w|$  is at least  $|\xi^{2k-2}(0)| + k + 2$ . Then there exist  $i \in \mathbb{N}$ ,  $1 \leq i \leq k$ , and a left special factor  $f$  of  $\mathbf{u}_k$  such that  $w$  or its twin equals  $w = 0^{i-1}\pi(f)$ .*



- If  $f$  is a prefix of  $\xi_k^n(0)$ , it can be extended to the left by any letter. Examining further extensions to the left, we obtain that all words  $0^\ell \pi(f)$  and  $1^\ell \pi(f)$  are left-special in  $\mathbf{x}_k$  for  $\ell = 0, \dots, k - 1$ . Counting words related to the twin of  $f$ , we find  $4k - 2$  left-special words for any sufficiently large length.

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- If  $f$  is instead a prefix of  $\xi_k^{n+2k-1}(0)\xi_k^n(0)$  but not of  $\xi_k^n(0)$ , it has only two left extensions. This and its twin sporadically give an extra  $2$  left-special factors, with no overlap between different values of  $n$ . Thus we have proven our first conjecture.

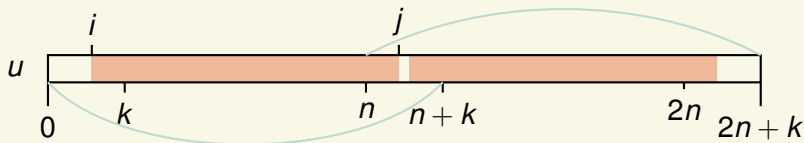
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## Theorem (Dvořáková, K., Pelantová '25)

*Let  $k \geq 1$ . The first difference of factor complexity of  $\mathbf{x}_k$ , for  $n$  large enough, takes the values  $4k-2$  and  $4k$  only.*

# Overlaps and asymptotic critical exponent

Assume on the contrary that an overlap exists, i.e. a word of length  $2n + k$  and period  $n$ , and consider one of its preimages.



The **shaded** areas represent equal words. The leftmost and rightmost nonshaded areas must be small due to our lemmas, but the shaded areas cannot overlap since there is no overlap in  $u_k$ . This gives strong enough conditions on  $i$  and  $j$  to exclude all possibilities by casework.

## Theorem (Dvořáková, K., Pelantová '25)

*Let  $k \geq 1$ . The sequence  $\mathbf{x}_k$  contains no factor of length  $2n + k$  and period  $n$ , and therefore has asymptotic critical exponent equal to 2. Since  $0^{k+1}$  and  $1^{k+1}$  are factors of  $\mathbf{x}_k$ , the critical exponent is  $k + 1$ .*

- Klouda's method can identify bispecial factors, and is especially tractable in words where synchronization points are abundant.
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Thank you for your attention!