Regularity of Gaussian Fields from Kernel Conditions via Wavelets

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Context and motivations

- The regularity of Gaussian processes and fields has been studied for a long time (see e.g. [1, 2, 5]). In the case of the fractional Brownian motion, precise results have already been obtained [5].
- Our main goal is to extend these results to a large class of Gaussian fields. Mainly, we study fields of the form

$$X_t = \int_{\mathbb{R}^n} (e^{it\cdot\xi} - 1)K(\xi)\, d\hat{W}(\xi)$$

where W is a Gaussian white noise.

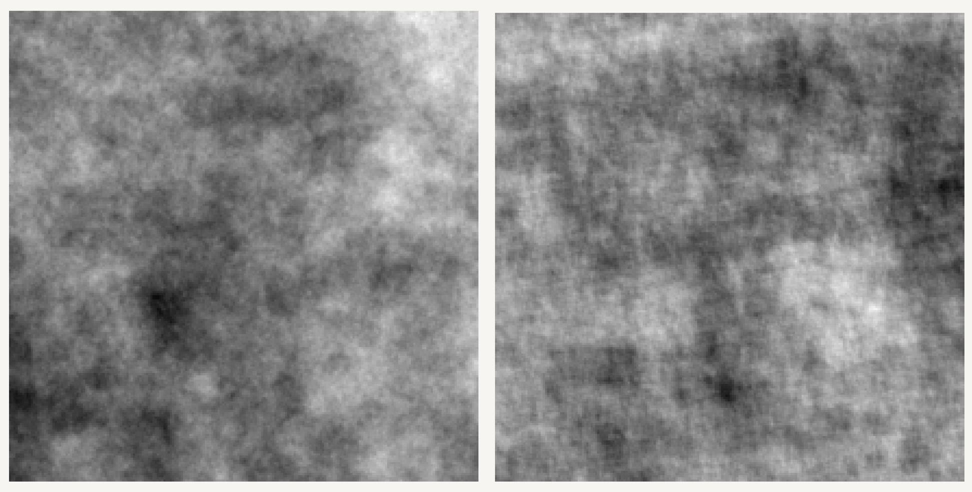

Figure 1: Simulation of the FBF (left) and of the field X_t (right) [3, 6]

Slow, rapid and ordinary points

Our main result determines the exact pointwise regularity and reveals three types of behavior.

Theorem 1. *Almost surely, the process X_t satisfies the following property for every non-empty open set I of $\mathbb{R}^n$:*

- *almost every $t \in I$ is ordinary:*

$$0 < \limsup_{s\to t} \frac{|X_t - X_s|}{|t-s|^H \sqrt{\log\log|t-s|^{-1}}} < +\infty,$$

- *there exists $t \in I$ which is fast:*

$$0 < \limsup_{s\to t} \frac{|X_t - X_s|}{|t-s|^H \sqrt{\log|t-s|^{-1}}} < +\infty,$$

- *there exists $t \in I$ which is slow:*

$$0 < \limsup_{s\to t} \frac{|X_t - X_s|}{|t-s|^H} < +\infty.$$

Main tool : The Lemarie-Meyer Wavelet

We will use the Lemarie-Meyer wavelet basis :

$$\left\{\psi_{p,j,k} = 2^{jn/2}\psi_p(2^j\cdot - k) : p \in \{1,\dots,2^n-1\}, j\in\mathbb{Z}, k\in\mathbb{Z}^n\right\}.$$

Where the functions ψ_p are in the Schwartz class. Their Fourier transforms are compactly supported :

$$\text{Supp}\left(\hat{\psi}_{p,j,k}\right) \subset \left[-\frac{2^{j+3}\pi}{3}, \frac{2^{j+3}\pi}{3}\right]^n \setminus \left(-\frac{2^{j+1}\pi}{3}, \frac{2^{j+1}\pi}{3}\right)^n.$$

Assumptions on K

- $\forall t \in \mathbb{R}^n$, $\xi \mapsto (e^{it\cdot\xi} - 1)K(\xi) \in L^2(\mathbb{R}^n)$;
- K is self-similar : $\forall a > 0$, $K(a\xi) = a^{-H-n/2}K(\xi)$ where $H \in (0,1)$;
- It follows that there exists a function ρ defined on the unit sphere S^n such that

$$K(\xi) = |\xi|^{-H-\frac{n}{2}}\rho\left(\frac{\xi}{|\xi|}\right);$$

- For every $p \in \{1,\dots,2^n-1\}$, let

$$\Psi_p(t) = \int_{\mathbb{R}^n} e^{it\cdot\xi}K(\xi)\hat{\psi}_p(\xi)\, d\xi.$$

We will require, at a first step, $\Psi_p \in S(\mathbb{R}^n)$. $(*)$

Wavelet decomposition of X_t

- Using $(*)$, we obtain [1] :

$$X_t = \sum_{p=1}^{2^n-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} 2^{-jH}\varepsilon_{p,j,k}\left(\Psi_p(2^jt-k) - \Psi_p(-k)\right),$$

where $\varepsilon_{p,j,k}$ are independent $\mathcal{N}(0,1)$.
This allows us to find an upper bound for $|X_t - X_s|$.

- If

$$\Psi_p^\perp(t) = \int_{\mathbb{R}^n} e^{it\cdot\xi}|\xi|^{-H-\frac{n}{2}}\hat{\psi}_p(\xi)\, d\xi,$$

then

$$\langle X_t, \Psi^\perp_{p,j,k}(t)\rangle = X^1_{p,j,k} + X^2_{p,j,k},$$

where $X^2_{p,j,k}$ is negligible thanks to $(*)$.
It allows us to derive a lower bound for $|X_t - X_s|$.

About condition $(*)$

- Let $(c_{p,j,k})_{p,j,k}$ be a sequence such that for all $N \in \mathbb{N}, \exists C > 0$ such that

$$\forall p,j,k : |c_{p,0,k}| \le C(1+|k|)^{-N} \text{ and } c_{p,j,k} = 2^{-jH}c_{p,0,k}.$$

- We define the tempered distribution

$$\mathcal{K} : \varphi \mapsto \sum_{p,j,k} c_{p,j,k}\int_{\mathbb{R}^n}\hat{\psi}_{p,j,k}(x)\varphi(x)\, dx.$$

- There exists a function K associated to $\mathcal{K}$ such that X_t is well-defined and the functions $\Psi_p \in S(\mathbb{R}^n)$ for all $p \in \{1,\dots,2^n-1\}$.
- Conversely, from a function K satisfying the assumptions, we can retrieve the distribution $\mathcal{K}$.

Perspectives

- What can be said about which functional space X_t belongs to (e.g. Besov spaces) [4] ?
- Can we weaken the hypothesis on K ?
- What about more general fields like the OSSRF [2] ?
- Can we generalize these results to a general class of non-Gaussian fields (in particular to fields in a Wiener chaos [4]) ?
- Can we use these results to construct estimators of the Hurst index H with "good properties" ?

References

[1] A. Ayache, *Multifractional stochastic fields*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019, Wavelet strategies in multifractional frameworks.

[2] H. Biermé and C. Lacaux, *Hölder regularity for operator scaling stable random fields*, Stochastic Process. Appl. 119(7):2222-2248, 2009.

[3] H. Biermé, M. Moisan, and F.J.P. Richard. A turning-band method for the simulation of anisotropic fractional Brownian field. J. Comput. Graph. Statist., 24(3):885–904, 2015.

[4] Z. Ciesielski, G. Kerkyacharian and B. Roynette, Quelques espaces fonctionnels associés à des processus gaussiens, Studia Math. 107 (2), 171–204 (1993).

[4] L. Daw and L. Loosveldt, *Wavelet methods to study the pointwise regularity of the generalized Rosenblatt process*, Electronic Journal of Probability, 27:1-45,2022.

[5] C. Esser and L. Loosveldt, *Slow, ordinary and rapid points for Gaussian wavelets series and application to fractional Brownian motions*, Latin American Journal of Probability and Mathematical Statistics, 19(2):1471-1495, 2022.

[6] F.J.P. Richard. PyAFBF: a Python library for sampling image textures from the anisotropic fractional Brownian field. Journal of Open Source Software, 7(75):3821, 2022.