The p-spectrum of Random Wavelet Series

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Joint work with

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Multifractal analysis of functions

Weierstrass' continuous nowhere differentiable functions :

$$W_{a,b}(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x) \quad \forall x \in \mathbb{R},$$

with $0 < a < 1$, b and odd integer and $ab > 1 + 3\pi/2$.

Given a function $f : \mathbb{R} \to \mathbb{R}$, we want to study

- the regularity in each of its points : $H_f(x) \ \ \forall x \in \mathbb{R}$,
- the significance of the different singularities by computing the Hausdorff dimension of the sets of points sharing a common regularity :

$$\dim_{\mathcal{H}} \left(\{x \in \mathbb{R} : H_f(x) = h\} \right) \quad \forall h \in \mathbb{R}\,.$$

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Wavelet basis

Orthonormal basis of $L^2(\mathbb{R})$ of the form

$$\left\{2^{\frac{j}{2}}\psi_{j,k} : j,k \in \mathbb{Z}\right\},$$

where

$$\psi_{j,k}(x) = \psi(2^j x - k).$$

To any function f, we can associate a sequence $\vec{c} = (c_{j,k})_{j\in\mathbb{N},\, k\in\{0,\ldots,2^j-1\}}$ such that

$$f = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}, \quad \text{with} \quad c_{j,k} = 2^j \int_0^1 f(x)\psi_{j,k}\, dx$$

(L^∞ normalization).

Notations :

- $\lambda_{j,k} = \left[k2^{-j}, (k+1)2^{-j}\right)$, j is the scale and k is the position
- $\psi_{\lambda_{j,k}} = \psi_{j,k}$ and $c_{\lambda_{j,k}} = c_{j,k}$
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Hölder regularity

Let $\alpha \geq 0, x_0 \in \mathbb{R}$ and f a function. Then $f \in C^\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$|f(x) - P(x)| \leq C\,|x - x_0|^\alpha \quad \forall x \in B(x_0, R),$$

i.e.

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha \quad \forall r \leq R.$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}.$$

Drawback : $f \in C^\alpha(x_0) \Rightarrow f$ bounded on a neighbourhood of x_0.

- Limited to locally bounded functions.
- Definition of the p-regularity by replacing L^∞ with L^p (Calderòn-Zygmud, 1961).

Hölder regularity

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p-Regularity and p-spectrum

Let $p \geq 1, \alpha \geq \frac{-1}{p}, x_0 \in \mathbb{R}$ and $f \in L^p_{\text{loc}}(\mathbb{R})$. Then $f \in T^p_\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$\left(\frac{1}{r}\int_{B(x_0,r)} |f(x) - P(x)|^p \ dx\right)^{\frac{1}{p}} \leq Cr^\alpha \quad \forall r \leq R,$$

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The p-exponent of f at x_0 is

$$h^{(p)}_f(x_0) = \sup\left\{\alpha \geq \frac{-1}{p} : f \in T^p_\alpha(x_0)\right\}.$$

We write

$$h^{(+\infty)}_f(x_0) = h_f(x_0).$$

p-Spectrum (called Hölder spectrum when $p = +\infty$) :

$$\mathscr{D}^{(p)}_f : h \in \left[\frac{-1}{p}, +\infty\right] \mapsto \dim_{\mathcal{H}}\left\{x \in \mathbb{R} : h^{(p)}_f(x) = h\right\}.$$

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Characterization of global regularity through wavelets

Uniform Hölder exponent :

$$h_f^{\min} = \liminf_{j \to +\infty} \frac{\log\left(\sup_{\lambda \in \Lambda_j} |c_\lambda|\right)}{\log(2^{-j})} = \sup\{s : f \in C^s_{\mathrm{loc}}(\mathbb{R})\}$$

- If $h_f^{\min} > 0$, then $f \in L^\infty_{\mathrm{loc}}$.
- If $h_f^{\min} < 0$, then $f \notin L^\infty_{\mathrm{loc}}$.

Scaling function :

$$\eta_f : p > 0 \mapsto \liminf_{j \to +\infty} \frac{\log\left(2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p\right)}{\log(2^{-j})} = \sup\left\{s > 0 : f \in B^{\frac{s}{p},\infty}_{p,\mathrm{loc}}\right\}$$

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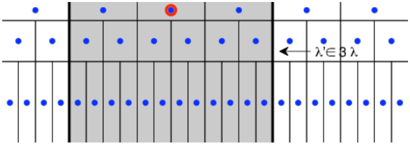
Leaders :

$$l_\lambda^{(+\infty)} = l_\lambda = \sup_{j' \geq j} \; \sup_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}| \quad \forall \lambda \in \Lambda_j \; \forall j \in \mathbb{N}$$



Characterization of pointwise regularity through wavelets

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- For every $p \in [1, +\infty)$ such that $\eta_f(p) > 0$,

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- Allows to define p-exponents when $p \in (0, 1)$ and $\eta_f(p) > 0$.

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$$l_\lambda^{(+\infty)} = l_\lambda = \sup_{j' \geq j} \sup_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}| \quad \forall \lambda \in \Lambda_j \ \forall j \in \mathbb{N}$$

- If $h_f^{\min} > 0$, then

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log\left(l_{\lambda_j(x_0)}\right)}{\log\left(2^{-j}\right)}, \quad \text{i.e. } l_{\lambda_j(x_0)} \sim 2^{-h_f(x_0)j}.$$

p-Leaders :

$$l_\lambda^{(p)} = \sup_{j' \geq j} \left(\sum_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \right)^{\frac{1}{p}} \quad \forall \lambda \in \Lambda_j \ \forall j \in \mathbb{N}$$

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Characterization of pointwise regularity through wavelets

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Wavelet density and profile

Wavelet density :

$$\rho_{\vec{c}}(\alpha) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log_2 \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq |c_\lambda| \leq 2^{-(\alpha-\varepsilon)j}\}}{j}$$

Wavelet profile :

$$\nu_{\vec{c}}(\alpha) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log_2 \#\{\lambda \in \Lambda_j : |c_\lambda| \geq 2^{-(\alpha+\varepsilon)j}\}}{j}$$

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$$\nu_{\vec{c}}(\alpha) = \sup_{\alpha' \leq \alpha} \rho_{\vec{c}}(\alpha')$$

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Hölder spectrum of Random Wavelet Series

Model : let

$$f=\sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k},$$

where the 2^j random variables $\frac{-\log_2|c_{j,k}|}{j}$ are drawn independently according to a given probability law $\boldsymbol{\rho_j}$, hence

$$\mathbb{P}\left(|c_{j,k}|\geq 2^{-\alpha j}\right)=\boldsymbol{\rho_j}((-\infty,\alpha]).$$

Theorem (Aubry-Jaffard, 2002)

If f is a Random Wavelet Series for which $h_f^{\min}>0$, then, almost surely, the support of $\mathscr{D}_f^{(+\infty)}$ is $[h_{\min},h_{\max}]$ and for every $h\in[h_{\min},h_{\max}]$,

$$\mathscr{D}_f^{(+\infty)}(h)=h\sup_{\alpha\in(0,h]}\frac{\rho_{\vec{c}}(\alpha)}{\alpha}=h\sup_{\alpha\in(0,h]}\frac{\nu_{\vec{c}}(\alpha)}{\alpha},$$

where $h_{\min}=\inf\{\alpha\in\mathbb{R}:\rho_{\vec{c}}(\alpha)\geq 0\}$ and $h_{\max}$ is the smallest h such that

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Model : for $\eta \in (0,1)$ and $\alpha \in \mathbb{R}$, let

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with $\xi_{j,k} \overset{\perp\!\!\!\perp}{\sim} \text{Bern}(2^{(\eta-1)j})$.

Theorem (Abry et al., 2015)

Almost surely, for every $p < \begin{cases} \frac{\eta-1}{\alpha} & \text{if } \alpha < 0 \\ +\infty & \text{otherwise} \end{cases}$ *and every* $h \geq \frac{-1}{p}$,

$$\mathscr{D}^{(p)}_{f_{\alpha,\eta}}(h) = \begin{cases} \left(h + \frac{1}{p}\right) \frac{\eta}{\alpha + \frac{1}{p}} & \text{if } h \in \left[\alpha, \frac{\alpha}{\eta} + \left(\frac{1}{\eta} - 1\right)\frac{1}{p}\right] \\ -\infty & \text{otherwise} \end{cases} .$$

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Conjecture for the p-spectrum of Random Wavelet Series

- Hölder spectrum of Random Wavelet Series : for every $h \in [h_{\min}, h_{\max}]$,

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Main result

Theorem

If f is a Random Wavelet Series, then, almost surely, for every $p > 0$ such that $\eta_f(p) > 0$, the support of $\mathscr{D}_f^{(p)}$ is $\left[h_{\min}, h_{\max}^{(p)}\right]$ and for every $h \in \left[h_{\min}, h_{\max}^{(p)}\right]$,

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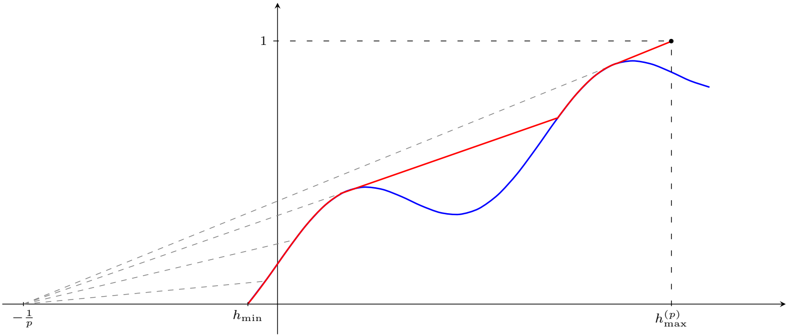

Upper bound

Theorem (Esser et al., ∼2014)
If $h_f^{\min} > 0$, then for every $h \geq 0$,

$$\mathscr{D}_f^{(+\infty)}(h) \leq \rho_{\vec{c}}^{(+\infty)}(h).$$

Similarly, if $\eta_f(p) > 0$, then for every $h \geq \frac{-1}{p}$,

$$\mathscr{D}_f^{(p)}(h) \leq \rho_{\vec{c}}^{(p)}(h).$$

Restricted p-leaders :

$$e_\lambda^{(p)} = \sup_{j' \geq j} \left(\sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \right)^{\frac{1}{p}}$$

Aim : prove that if f is a Random Wavelet Series and $\eta_f(p) > 0$, then for every $h \geq \frac{-1}{p}$,

$$\rho_{\vec{c}}^{(p),*}(h) \leq \left(h + \frac{1}{p} \right) \sup_{\alpha \in \left(\frac{-1}{p}, h\right]} \frac{\rho_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}}.$$

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Upper bound : Hölder case

If $e_\lambda \geq 2^{-hj}$, then $\exists j' \geq j,\ \lambda' \in \Lambda_{j'}$ such that $\lambda' \subseteq \lambda$ and $|c_{\lambda'}| \geq 2^{-hj}$.

But $\exists \alpha_0 > 0$ such that for all j' and all $\lambda' \in \Lambda_{j'}$, $|c_{\lambda'}| \leq 2^{-\alpha_0 j'}$; hence $j' \leq \frac{h}{\alpha_0} j$.

It follows that $\alpha_0 j' \leq hj \leq hj'$, hence $\exists \alpha(j') \in [\alpha_0, h]$ such that $hj = \alpha(j')j'$.

Therefore,

$$\begin{aligned}
\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-hj}\} &\leq \sum_{j'=j}^{hj/\alpha_0} \#\{\lambda' \in \Lambda_{j'} : |c_{\lambda'}| \geq 2^{-hj}\} \\
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&= \sum_{j'=j}^{hj/\alpha_0} \#\{\lambda' \in \Lambda_{j'} : |c_{\lambda'}| \geq 2^{-\alpha(j')j'}\} \\
&= \sum_{j'=j}^{hj/\alpha_0} 2^{\nu_{\vec{c}}(\alpha(j'))j'} \\
&= \sum_{j'=j}^{hj/\alpha_0} 2^{h\frac{\nu_{\vec{c}}(\alpha(j'))}{\alpha(j')} j} \\
&\leq Cj 2^{h \sup_{\alpha \in [\alpha_0, h]} \frac{\nu_{\vec{c}}(\alpha)}{\alpha} j}.
\end{aligned}$$

Upper bound : Hölder case

If $e_\lambda \geq 2^{-hj}$, then $\exists j' \geq j,\ \lambda' \in \Lambda_{j'}$ such that $\lambda' \subseteq \lambda$ and $|c_{\lambda'}| \geq 2^{-hj}$.

But $\exists \alpha_0 > 0$ such that for all j' and all $\lambda' \in \Lambda_{j'}$, $|c_{\lambda'}| \leq 2^{-\alpha_0 j'}$; hence $j' \leq \frac{h}{\alpha_0} j$.

It follows that $\alpha_0 j' \leq hj \leq hj'$, hence $\exists \alpha(j') \in [\alpha_0, h]$ such that $hj = \alpha(j')j'$.

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Upper bound : difficulty in the case $p < +\infty$

If $e_\lambda^{(p)} \geq 2^{-hj}$, then $\exists j' \geq j$ such that

$$\sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \geq 2^{-hpj}.$$

Moreover, $\exists \alpha_0 > 0$ such that $2^{-j'} \sum_{\lambda' \in \Lambda_{j'}} |c_{\lambda'}|^p \leq 2^{-\alpha_0 p j'}$; hence $j' \leq \frac{h+1/p}{\alpha_0} j$.

- Since

$$\sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \leq \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p,$$

  $\exists \lambda' \in \Lambda_{j'}$ such that $\lambda' \subseteq \lambda$ and $|c_{\lambda'}| \geq 2^{-hj}$.
  - $h \geq -1/p$ can be negative, in which case $hj \geq hj'$.
  - Even for $h \geq 0$, we recover the large-deviation formalism.

- Assume that $\exists \lambda' \in \Lambda_{j'}$ such that $\lambda' \subseteq \lambda$ and

$$|c_{\lambda'}| \geq 2^{-hj} 2^{\frac{j'-j}{p}} = 2^{-\left(h+\frac{1}{p}\right)j} 2^{\frac{j'}{p}}.$$

Since $\alpha_0 j' \leq (h+1/p)\, j \leq (h+1/p)\, j'$, $\exists \alpha(j') \in [\alpha_0 - 1/p, h]$ such that

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Upper bound : solution for the case $p < +\infty$

For $\lambda \in \Lambda_j$ such that $e_\lambda^{(p)} \sim 2^{-hj}$, $\exists j'(\lambda) \in [j, Cj]$ such that

$$\sum_{\lambda' \subseteq \lambda} |c_{\lambda'}|^p \, 2^{-(j'(\lambda)-j)} \sim 2^{-hpj}$$

and $\exists \alpha(\lambda) \in \left[\alpha_0 - \frac{1}{p}, h\right]$ such that

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It follows that

$$\# \left\{ \lambda' \subseteq \lambda \,:\, |c_{\lambda'}| \sim 2^{-\alpha(\lambda) j'(\lambda)} \right\} \sim 2^{-(hp+1)j} 2^{(\alpha(\lambda)p+1)j'(\lambda)}.$$

For infinitely many scales j,

$$\#\{\lambda \in \Lambda_j : e_\lambda^{(p)} \sim 2^{-hj}\} \sim 2^{\rho_{\vec{c}}^{(p),*}(h) j}.$$

There exist A and α such that for each of those scales j, there exists $j' \sim Aj$ for which

$$\#\{\lambda \in \Lambda_j : e_\lambda^{(p)} \sim 2^{-hj} \text{ with } j'(\lambda) = j' \text{ and } \alpha(\lambda) = \alpha\} \sim 2^{\rho_{\vec{c}}^{(p),*}(h) j}.$$

At those scales j,

$$\# \left\{ \lambda' \in \Lambda_{j'} : |c_{\lambda'}| \sim 2^{-\alpha j'} \right\} \geq 2^{\rho_{\vec{c}}^{(p),*}(h) j} \times 2^{-(hp+1)j} 2^{(\alpha p+1)j'}.$$

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$$\#\left\{\lambda' \subseteq \lambda \,:\, |c_{\lambda'}| \sim 2^{-\alpha(\lambda) j'(\lambda)}\right\} \sim 2^{-(hp+1)j} 2^{(\alpha(\lambda)p+1)j'(\lambda)}.$$

For infinitely many scales j,

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There exist A and α such that for each of those scales j, there exists $j' \sim Aj$ for which

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Upper bound : solution for the case $p < +\infty$

For $\lambda \in \Lambda_j$ such that $e_\lambda^{(p)} \sim 2^{-hj}$, $\exists j'(\lambda) \in [j, Cj]$ such that

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Aim : prove that if f is a Random Wavelet Series and $\eta_f(p) > 0$, then for every $h \in \left[h_{\min}, h_{\max}^{(p)}\right]$,

$$\mathscr{D}_f^{(p)}(h) \geq \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left(\frac{-1}{p}, h\right]} \frac{\nu_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}}.$$

Theorem (Daviaud, 2025)

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of balls of $[0,1]$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of $(0,1]$. If

$$s = \sup\left\{\gamma : \mathcal{L}\left(\limsup_{n\,:\,\gamma_n \leq \gamma} B_n^{\gamma_n}\right) = 1\right\},$$

then there exists a gauge function ξ such that

$$\mathcal{H}^{\xi}\left(\limsup_{n\to+\infty} B_n\right) > 0 \quad \textit{and} \quad \lim_{r\to 0^+} \frac{\log \xi(r)}{\log r} = s.$$

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$$\limsup_{n\to+\infty} B_n \subseteq \left\{x \in [0,1] : h_f^{(p)}(x) h\right\}.$$

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0 1

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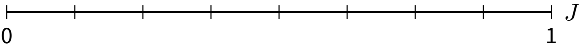

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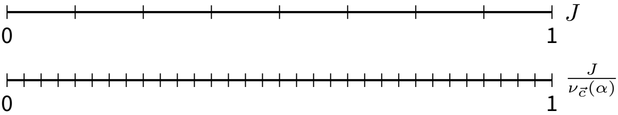

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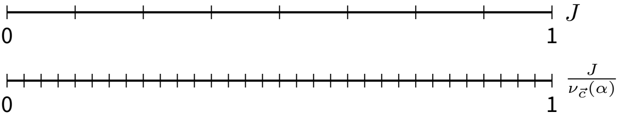

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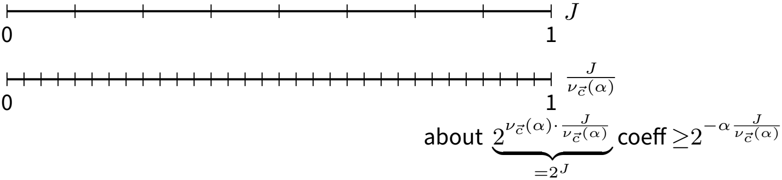

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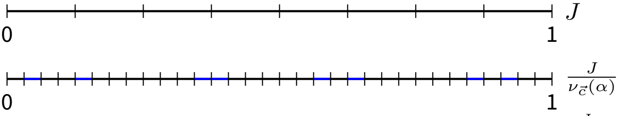

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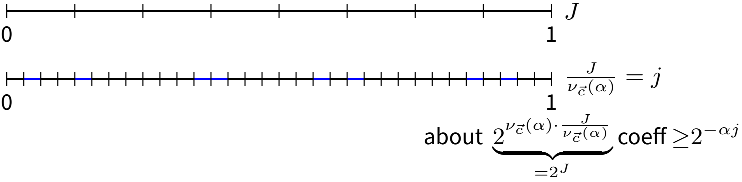

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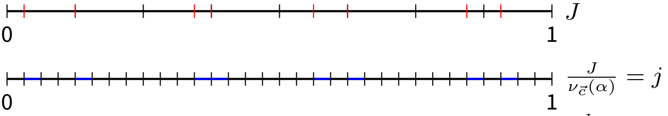

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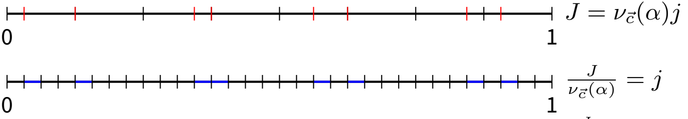

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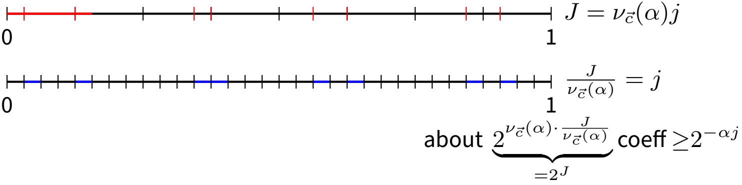

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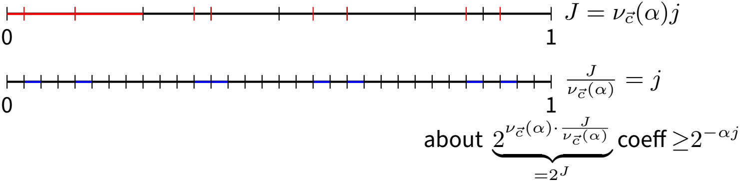

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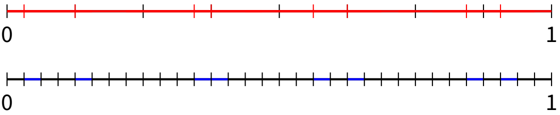

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Table of contents

Idea

Theorem (Aubry-Jaffard, 2002)

If $h_f^{\min} > 0$, then for every $h \geq 0$,

$$\mathscr{D}_f^{(+\infty)}(h) \leq h \sup_{\alpha \in (0,h]} \frac{\rho_{\vec{c}}(\alpha)}{\alpha} = h \sup_{\alpha \in (0,h]} \frac{\nu_{\vec{c}}(\alpha)}{\alpha}.$$

Question : if $\eta_f(p) > 0$, is it true that

$$\mathscr{D}_f^{(p)}(h) \leq \rho_{\vec{c}}^{(p),*}(h) \leq \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left(\frac{-1}{p}, h\right]} \frac{\rho_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}}$$

for every $h \geq \frac{-1}{p}$?

Answer : it is true for every $h \geq 0$ but not necessarily for $h < 0$.

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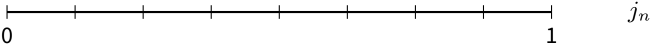

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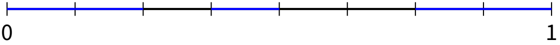

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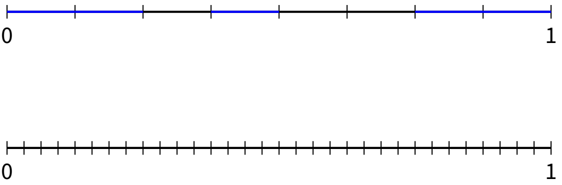

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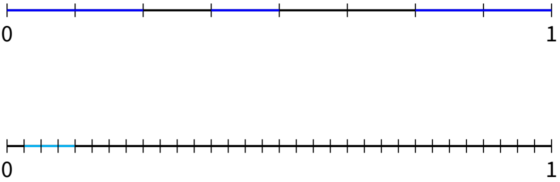

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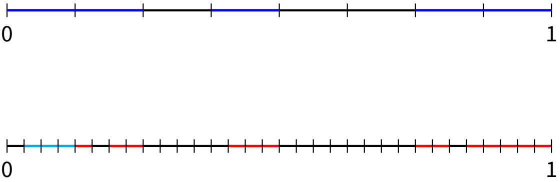

j_n

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$j'_n = \frac{h}{\alpha}\, j_n$

$2^{-(hp+1)j_n}\, 2^{(\alpha p+1)j'_n}$ coeff $2^{-\alpha j'_n}$

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Counter-example in the case $h < 0$

Fix $\alpha < 0, 0 < \eta < \alpha p + 1$ and $\frac{\alpha}{1-\eta} \leq h < \alpha$. We construct a function f such that

- $\eta_f(p) > 0$,
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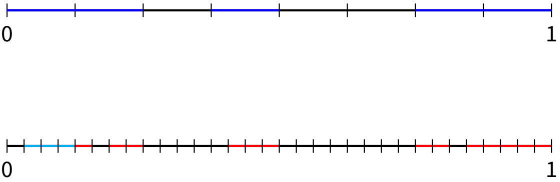

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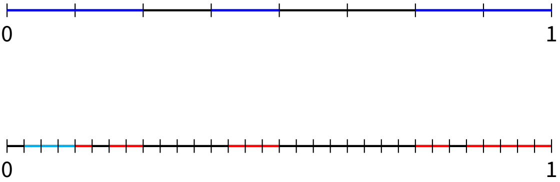

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Theorem (Leonarduzzi et al., 2016)

Let $p \geq 1$. For every $f \in L^p$ and every $h \leq 0$,

$$\mathscr{D}_f^{(p)}(h) \leq hp + 1.$$

Theorem

Let $p \geq 1$. For a prevalent and residual subset of $f \in L^p$, for every $h \leq 0$,

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A property $\mathcal{P}$ over a Polish space E is said to be prevalent if there exists a stochastic process X such that almost surely, $X \in E$ and for every $f \in E$, $X + f$ satisfies property $\mathcal{P}$ (Hunt-Sauer-Yorke, 1992).

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Dyadic approximation

A non-dyadic point $x_0 \in [0,1]$ is said to be α-approximable by dyadics ($\alpha \geq 1$) if there exists a sequence $(j_n, k_n)_{n\in\mathbb{N}}$ such that

$$k_n \in \{0,\ldots,2^{j_n}-1\} \setminus 2\,\mathbb{N}_0 \quad \text{and} \quad \left| x_0 - \frac{k_n}{2^{j_n}} \right| \leq \frac{1}{2^{\alpha j_n}} \quad \forall n \in \mathbb{N}\,.$$

We write F_α the set of points that are α-approximable by dyadics, i.e.

$$F_\alpha = \limsup_{j\to+\infty} \bigcup_{k\in\{0,\ldots,2^j-1\}\setminus 2\,\mathbb{N}_0} \left[\frac{k}{2^j} - \frac{1}{2^{\alpha j}}, \frac{k}{2^j} + \frac{1}{2^{\alpha j}} \right].$$

Theorem (Jaffard, 2000)

For every $\alpha \geq 1$

$$\mathcal{H}^{h_{1/\alpha}}(F_\alpha) > 0, \quad \textit{where } h_a(x) = \log^2(x)\,|x|^a\,.$$

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Saturating function

We define the saturating function

$$F = \sum_{j\in\mathbb{N}_0}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}, \quad \text{where } c_{j,k} = \frac{1}{j^a}2^{\frac{j}{p}}2^{\frac{-J}{p}} \quad \text{and} \quad a = \frac{1}{p}+\frac{2}{q}$$

if $J \leq j$ is such that there exists $K \in \{0,\dots,2^J-1\}$ for which $\frac{K}{2^J}$ is the irreducible form of $\frac{k}{2^j}$ (Jaffard, 2000).

Then $F \in B^0_{p,q} \subseteq L^p$. Indeed,

$$\sum_{j\in\mathbb{N}}\left(\sum_{\lambda\in\Lambda_j}|c_\lambda|^p\,2^{-j}\right)^{\frac{q}{p}} \leq \sum_{j\in\mathbb{N}}\left(j^{1-ap}\right)^{\frac{q}{p}} < +\infty.$$

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p-Spectrum of the saturating function

Aim : prove that

$$F_\alpha \subseteq \left\{ x \in [0,1] : h_F^{(p)}(x) \le \frac{1}{p\alpha} - \frac{1}{p} \right\}.$$

Let $x \in F_\alpha$.

There exists $(J_n, K_n)_{n\in\mathbb{N}}$ such that

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Prevalence

Randomization of F :

$$X = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k} \xi_{j,k} \psi_{j,k}, \quad \text{where } \xi_{j,k} \overset{\text{i.i.d.}}{\sim} \text{Unif}\,[-1,1]\,.$$

Clearly, $X \in L^p$.

Aim : prove that, almost surely,

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Almost surely, $\exists J \in \mathbb{N}$ such that for all $j \geq J$ and all $k \in \{0, \ldots, 2^j - 1\}$, $\exists m \in \{0, \ldots, \log_2 j\}$ such that $|\xi_{\lambda_{j,k}^{(m)}}| \geq 2^{\frac{-j}{\log_2 j}}$, where $\lambda_{j,k}^{(m)} = \lambda_{j+m, 2^m k}$.

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Prevalence

Randomization of F :

$$X = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\xi_{j,k}\psi_{j,k}, \quad \text{where } \xi_{j,k} \overset{\text{i.i.d.}}{\sim} \text{Unif}\,[-1,1]\,.$$

Clearly, $X \in L^p$.

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Prevalence

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Prevalence

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It follows that

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Prevalence

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Baire categories

Fix $(f_n)_{n\in\mathbb{N}}$ a dense sequence in L^p of coefficients $(c_\lambda^{(n)})_{\lambda\in\Lambda}$.

Let

$$g_n = \sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1} d_{j,k}^{(n)}, \quad \text{where } d_{j,k}^{(n)} = \begin{cases} c_{j,k}^{(n)} & \text{if } j < n \\ c_{j,k} & \text{if } j \geq n \end{cases}$$

and

$$A = \bigcap_{N\in\mathbb{N}}\bigcup_{n\geq N} B(g_n, r_n), \quad \text{where } r_n = \frac{1}{2} n^{-a} 2^{\frac{-n}{p}}.$$

Since $(g_n)_{n\in\mathbb{N}}$ is dense, A is residual.

For every

$$f = \sum_{j\in\mathbb{N}}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k} \in B(g_n, r_n)$$

and every $\lambda \in \Lambda_n$, $|d_\lambda - c_\lambda| \leq \frac{|c_\lambda|}{2}$.

Indeed,

$$|d_\lambda - c_\lambda| \leq 2^{\frac{n}{p}} \left\| \left(\left(d_\lambda - d_\lambda^{(n)} \right)^2 \chi_\lambda \right)^{\frac{1}{2}} \right\|_{L^p} \leq 2^{\frac{n}{p}} \left\| \left(\sum_{j\in\mathbb{N}}\sum_{\lambda\in\Lambda_j} \left(d_\lambda - d_\lambda^{(n)} \right)^2 \chi_\lambda \right)^{\frac{1}{2}} \right\|_{L^p}$$

$$= 2^{\frac{n}{p}} \|f - g_n\|_{L^p} \leq \frac{1}{2} n^{-a} \leq \frac{|c_\lambda|}{2}.$$

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