

# Introduction to optical/IR/(sub-)mm/Radio interferometry

## History and basic principles

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- 2 Some reminders
  - Complex representation of an electromagnetic wave
  - Principle of Huygens-Fresnel
- 3 Brief history about the measurements of stellar diameters
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  - Newton
  - Fizeau-type interferometry
  - Home experiments: visualization of the Airy disk and the Young interference fringes
- 4 Light coherence
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  - Visibility of the interference fringes
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  - Zernicke-vanCittert theorem
  - Some remarkable properties of the Fourier transform and applications
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- 5 Some examples of optical interferometers
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  - The fundamental theorem: relation between the pupil and focal planes
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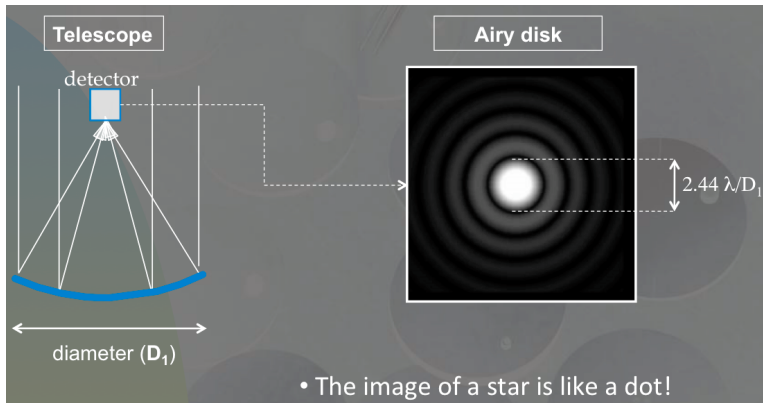
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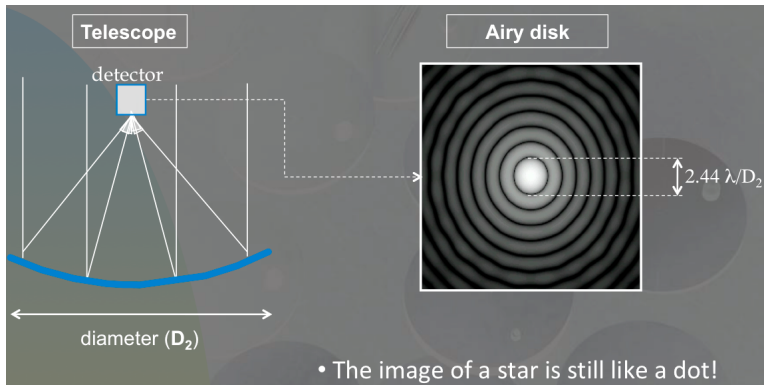
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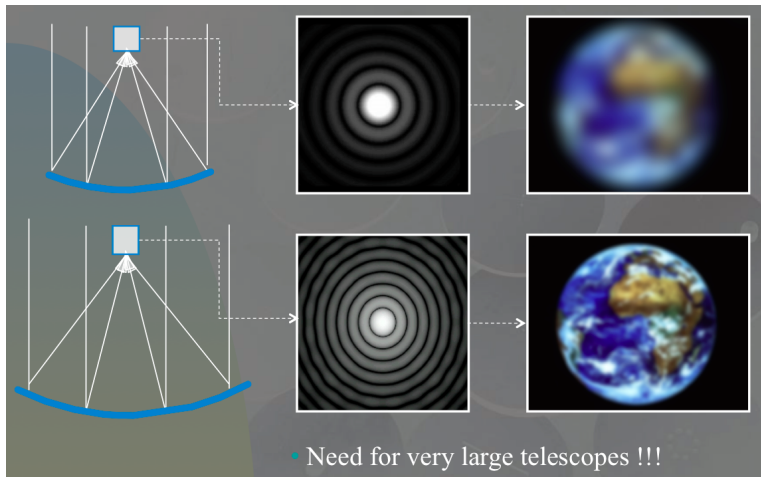
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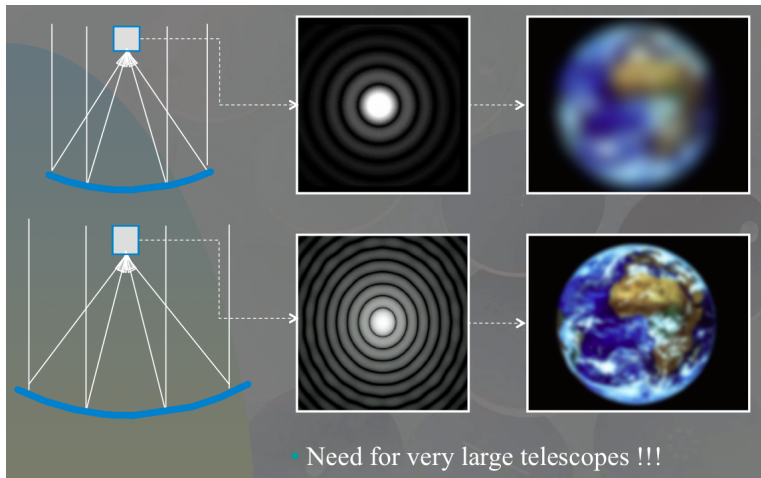
**Figure:** Airy disk of a point-like star recorded in the focal plane of a telescope with diameter  $D_1$ . The angular diameter of the Airy disk is  $2.44 \lambda/D_1$ .



**Figure:** As the diameter of a telescope increases ( $D_2 > D_1$ ), the Airy disk of a point-like star gets smaller.



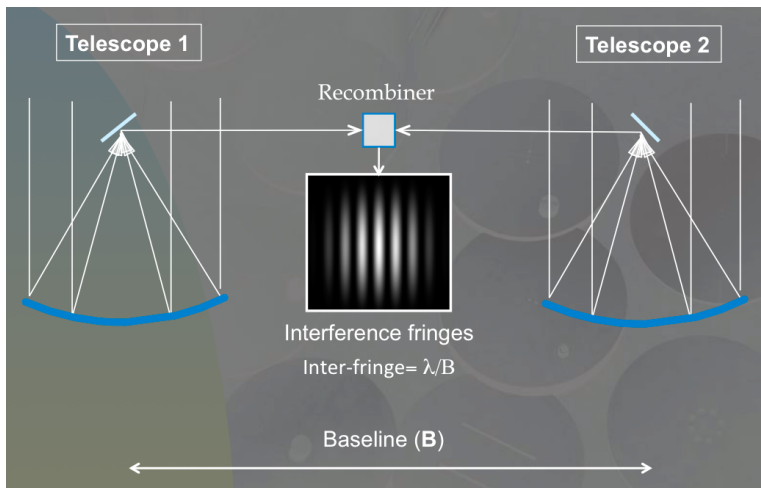
**Figure:** While observing an extended celestial object (cf. an Earth-like planet) above the atmosphere, we see more details as the diameter of the telescope increases.



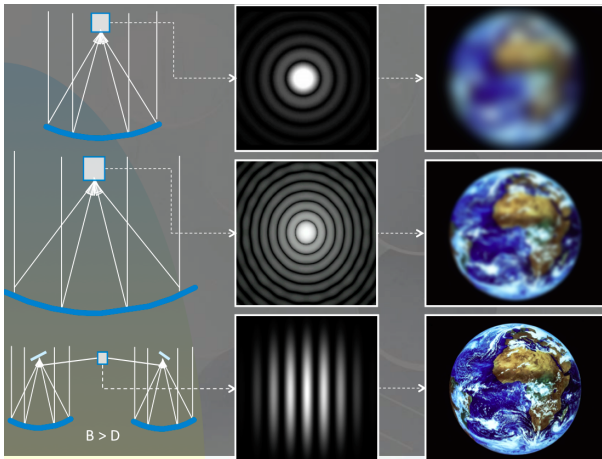
**Figure:** While observing an extended celestial object (cf. an Earth-like planet) above the atmosphere, we see more details as the diameter of the telescope increases.







**Figure:** When recombining the monochromatic light of two independent telescopes, there results the formation of a pattern of bright and dark fringes superimposed over the combined Airy disk. The angular inter-fringe separation is equal to  $\lambda/B$ .



**Figure:** Improvement expected in angular resolution while observing an extended celestial source (cf. an Earth-like planet) with telescopes of increasing size ( $D_2 > D_1$ ) and with an interferometer composed of two telescopes separated by a baseline  $B > D$ .

**Convolution theorem**

$$I(\zeta, \eta) = \iint PSF(\zeta - \zeta', \eta - \eta') O(\zeta', \eta') d\zeta' d\eta' = PSF(\zeta, \eta) \otimes O(\zeta, \eta)$$

$$FT(I(\zeta, \eta))(u, v) = FT(PSF(\zeta, \eta))(u, v) FT(O(\zeta, \eta))(u, v)$$

$u = B_u / \lambda, v = B_v / \lambda$

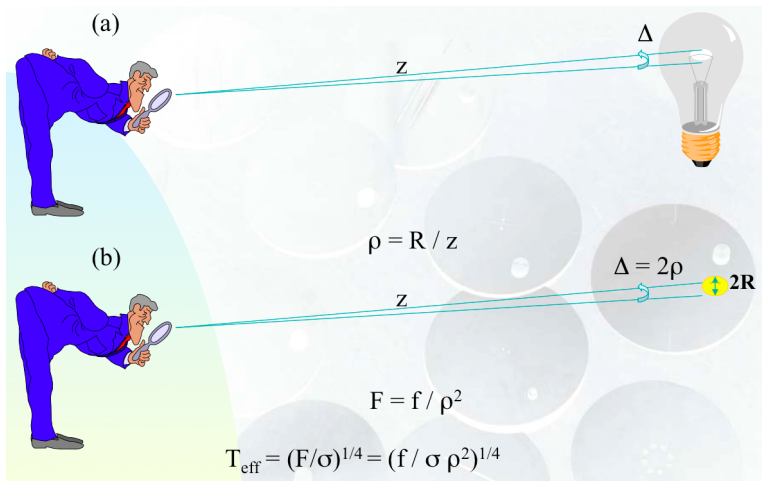
$$O(\zeta, \eta) = FT(-1)FT(O(\zeta, \eta)) = FT(-1)(FT(I(\zeta, \eta)) / FT(PSF(\zeta, \eta)))$$

**Figure:** The image  $I(\zeta, \eta)$  we observe in the focal plane of an instrument (cf. single dish telescope) from a distant extended source as a function of its angular coordinates  $\zeta, \eta$  is the convolution product of the real source image (cf. the extended Earth-like planet,  $O(\zeta, \eta)$ ) by the point spread function  $PSF(\zeta, \eta)$  of the telescope.

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# Some reminders



**Figure:** Resolving the angular diameter of a star (b) is alike trying to estimate the angular size of the filament of a light bulb (a).

$$E = a \cos(2\pi(\nu t - z/\lambda)) \quad (1)$$

where

$$\lambda = c T = c/\nu \quad (2)$$

$c$ ,  $\lambda$ ,  $\nu$ ,  $T$  and  $a$  representing the speed of light, the wavelength, the frequency, the period and the amplitude of the electromagnetic vibrations, respectively (see Figure 11).

# Some reminders: Complex representation of an electromagnetic wave

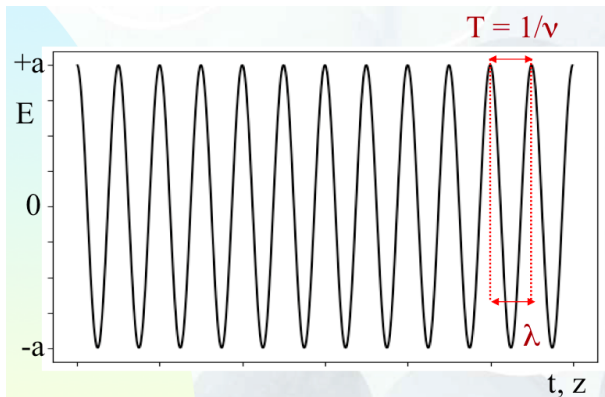


Figure: Representation of an electromagnetic wave.

We know how convenient it is to rewrite the previous relation in complex notation:

$$E = \text{Re}\{a \exp[i2\pi(\nu t - z/\lambda)]\} \quad (3)$$

where  $\text{Re}$  represents the real part of the expression between the two curly braces.

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$$E = a \exp(-i\phi) \exp(i2\pi\nu t) \quad (4)$$

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$$\phi = 2\pi z/\lambda. \quad (5)$$

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The intensity  $I$  is proportional to the temporal average of the square of the electric field:

$$I = \langle E^2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E^2 dt = \frac{1}{2T} \int_{-T}^{+T} A \exp(i2\pi\nu t) A^* \exp(-i2\pi\nu t) dt = A A^* . \quad (8)$$

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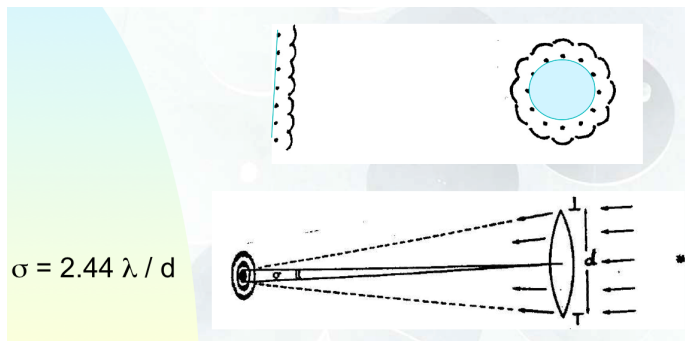
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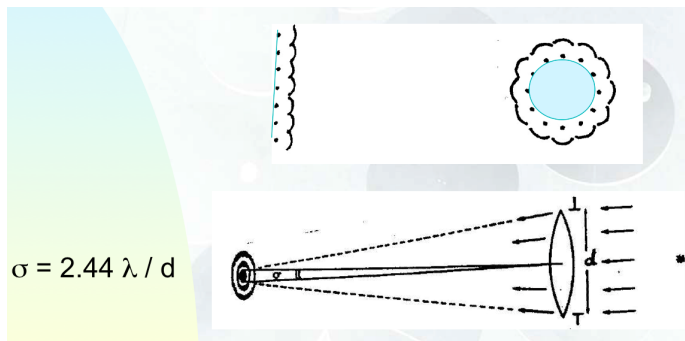
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# Some reminders: Principle of Huygens-Fresnel



**Figure:** Illustration of the Huygens-Fresnel principle during the propagation of a plane or circular wavefront and diffraction of light which encounters a converging lens.

# Some reminders: Principle of Huygens-Fresnel

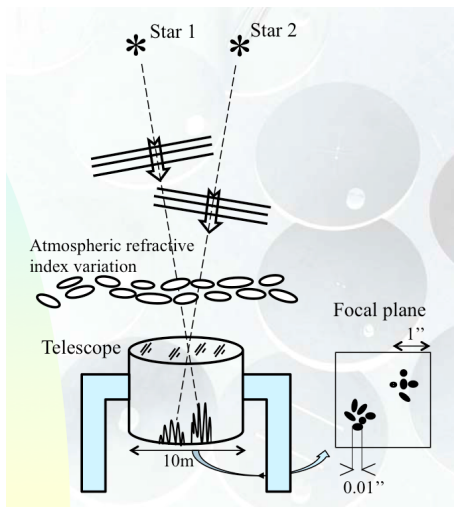


**Figure:** Illustration of the Huygens-Fresnel principle during the propagation of a plane or circular wavefront and diffraction of light which encounters a converging lens.

$$\sigma = 2.44 \lambda / d \quad (10)$$

where  $\lambda$  is the wavelength of light and  $d$  is the linear diameter of the aperture.

# Some reminders: Principle of Huygens-Fresnel



**Figure:** Atmospheric agitation above the objective of a large telescope causing the seeing effects seen in its focal plane.

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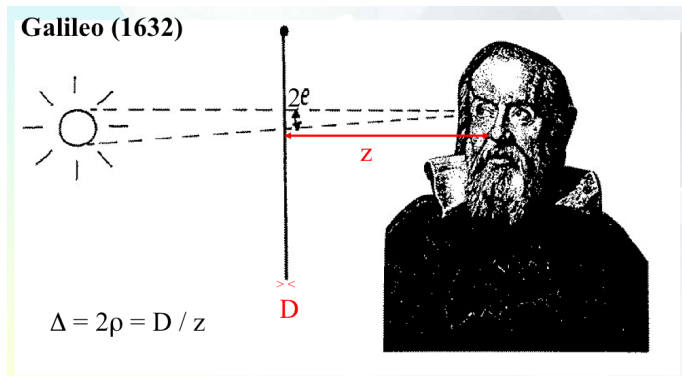


Figure: Experimental measurement by Galileo of the angular diameter of a star (see text).

A theoretical estimate of the angular dimension of a star of magnitude zero (cf. that of the star Vega) was performed by Newton. His approach was as follows: the angular diameter of the Sun  $\Delta$  placed at the distance of Vega ( $V = 0$ ) is given by the product of the apparent angular diameter of the Sun  $\Delta_{\odot}$  times the factor  $10^{V_{\odot}/5}$  where  $\Delta_{\odot} = 30'$  and the visual apparent magnitude of the Sun,  $V_{\odot} = -26.7$ .

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry

Let us first remind the results obtained in the Young double hole experiment (1803, see Fig. 15).

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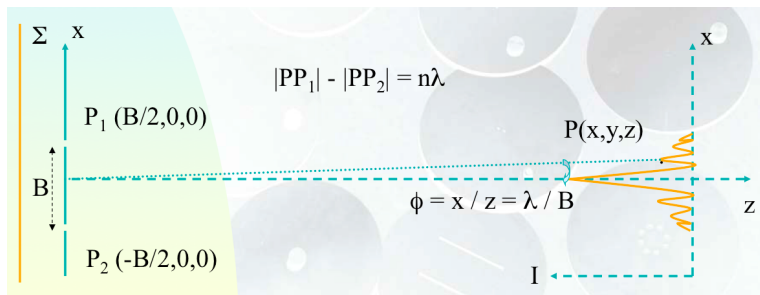


Figure: The double hole experiment of Young (see text).

## Brief history about the measurements of stellar diameters: Fizeau-type interferometry

The locus of points  $P(x, y, z)$  with cartesian coordinates  $x, y, z$  (see Fig. 15) where there will be a constructive interference is thus given by

$$|P_1P| - |P_2P| = n\lambda \quad (11)$$

with  $n = 0, \pm 1, \pm 2$ , etc.

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Let the points  $P_i(x_i, y_i, 0)$  in the screen plane and  $P(x, y, z)$  in the observer plane be such that  $|x_i|, |y_i|, |x|, |y| \ll |z|$ . We then find that

$$|P_iP| = \{(x - x_i)^2 + (y - y_i)^2 + z^2\}^{1/2} \quad (12)$$

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which can be simplified at first order (given the above conditions) as follows:

$$|P_iP| = z \left\{ 1 + \frac{(x - x_i)^2 + (y - y_i)^2}{2z^2} \right\}. \quad (13)$$

## Brief history about the measurements of stellar diameters: Fizeau-type interferometry

Considering the two points  $P_1$  and  $P_2$  in the Young's screen, Eq. (11) reduces to

$$z\left\{1 + \frac{(x + B/2)^2 + y^2}{2z^2}\right\} - z\left\{1 + \frac{(x - B/2)^2 + y^2}{2z^2}\right\} = n\lambda \quad (14)$$

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and finally

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or else

$$\Phi = \frac{x}{z} = n\frac{\lambda}{B} = \frac{n}{u} = \frac{c\tau}{B}, \quad (16)$$

where  $u = \frac{B}{\lambda}$  represents a space frequency and  $\tau$  a geometric time delay.

## Brief history about the measurements of stellar diameters: Fizeau-type interferometry

The resulting intensity  $I_T$  at the point  $P$  in the observer plane due to the interfering secondary waves coming from  $P_1$  and  $P_2$  is successively given by

$$\begin{aligned} I_T &= [A + A \exp(i2\pi\nu\tau)][A^* + A^* \exp(-i2\pi\nu\tau)], \\ &= AA^* + AA^*[\exp(-i2\pi\nu\tau) + \exp(i2\pi\nu\tau)] + AA^*, \\ &= 2I[1 + \cos(2\pi\nu\tau)], \\ &= 2I[1 + \cos(2\pi \frac{\varphi}{\lambda/B})], \quad (\text{a}) \\ &= 2I[1 + \cos(2\pi u\varphi)], \quad (\text{b}) \\ &= 2I[1 + \cos(2\pi \frac{ux}{z})], \quad (\text{c}) \end{aligned} \tag{17}$$

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry

If  $\Delta \geq \phi/2 = \lambda / (2B)$ ,

fringe disappearance!

Fringe visibility:

$$v = \left( \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \right)$$

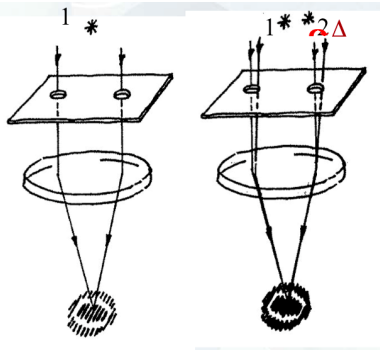


Figure: Fizeau, the father of stellar interferometry (1868; see text).

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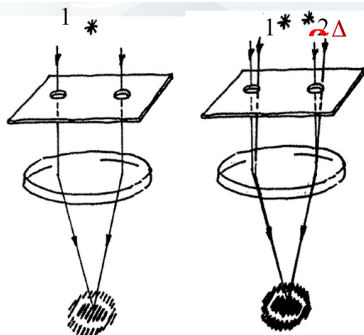


Figure: Fizeau, the father of stellar interferometry (1868; see text).

From Fig. 16, it is clear that the visibility of the fringes will significantly decrease whenever the following condition takes place

$$\Delta > \frac{\Phi}{2} = \frac{\lambda}{2B}.$$

## Brief history about the measurements of stellar diameters: Fizeau-type interferometry

A quantity that objectively measures the contrast of the fringes is called the visibility.

## Brief history about the measurements of stellar diameters: Fizeau-type interferometry

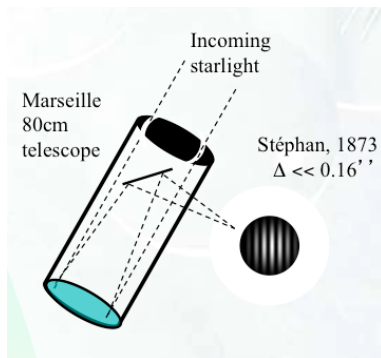
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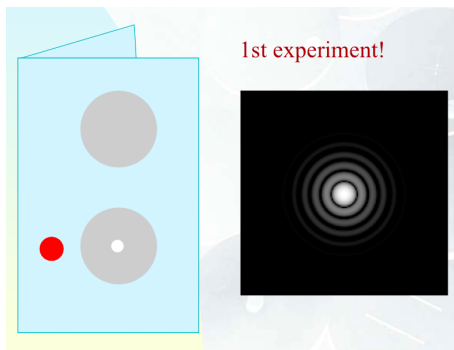
**Figure:** Diagram illustrating the way Fizeau and Stephan proceeded in order to measure the angular diameters of stars with the interferometric technique.

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry



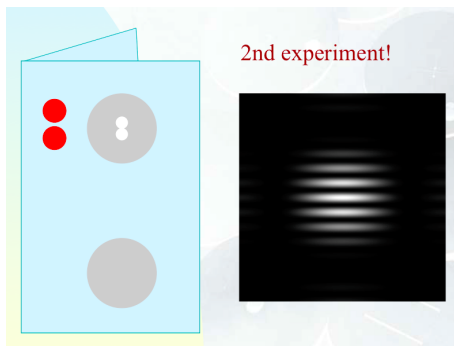
Figure: The 80cm Marseille telescope used by Fizeau and Stephan. © Michel Marcelin.

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry



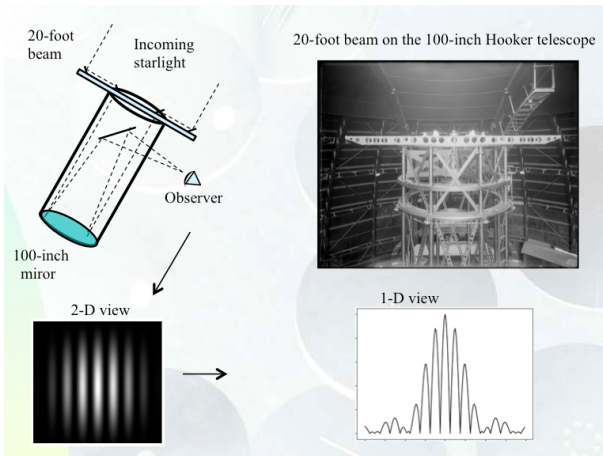
**Figure:** The one hole screen experiment: the small circular hole drilled in the aluminium paper is visible inside the lower bigger hole perforated in the cartoon screen. When looking through this hole at a distant light bulb, you perceive a nice Airy disk (cf. right image).

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry



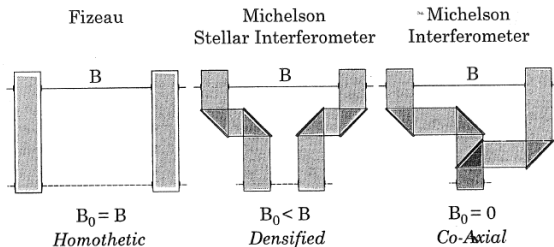
**Figure:** The two hole screen experiment: the two small circular holes drilled in the aluminium paper are visible inside the upper bigger hole perforated in the cartoon screen. When looking through these two holes at a distant light bulb, you perceive a nice Airy disk superimposed by a pattern of bright and dark fringes (cf. right image).

# Brief history about the measurements of stellar diameters: Fizeau-type interferometry



**Figure:** The stellar interferometer of Michelson-Pease set on top of the 2.5m Mount Wilson telescope. © The Observatories of the Carnegie Institution.

# Different possible recombination schemes



**Figure:** This illustration represents different possibilities for exit pupil placement, for a general exit baseline  $B_0$ .

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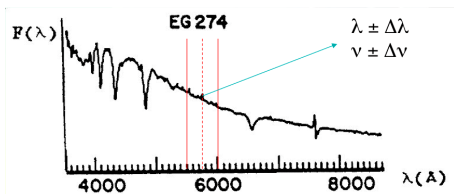
This theory consists essentially in a statistical description of the properties of the radiation field in terms of the correlation between electromagnetic vibrations at different points in the field.

## Light coherence: quasi-monochromatic light

This theory consists essentially in a statistical description of the properties of the radiation field in terms of the correlation between electromagnetic vibrations at different points in the field.

The light emitted by a real source (see Fig. 23) is of course not monochromatic. As in the case of a monochromatic wave, the intensity of such a radiation field at any point in space is defined by

$$I = \langle V(t)V(t)^* \rangle. \quad (20)$$



**Figure:** Stars do not emit monochromatic light. Quasi monochromatic light is assumed to be emitted at the wavelength  $\lambda$  (resp. the frequency  $\nu$ ) within the bandwidth  $\pm\Delta\lambda$  (resp.  $\pm\Delta\nu$ ).

In order to determine the electric field created by such a source, emitting within a certain frequency range  $\pm\Delta\nu$ , we must sum up the fields due to all the individual monochromatic components such that the resulting electric field  $V(z, t)$  is given by the real part of the following expression:

$$V(z, t) = \int_{\nu-\Delta\nu}^{\nu+\Delta\nu} a(\nu') \exp[i2\pi(\nu' t - z/\lambda')] d\nu'. \quad (21)$$

In order to determine the electric field created by such a source, emitting within a certain frequency range  $\pm\Delta\nu$ , we must sum up the fields due to all the individual monochromatic components such that the resulting electric field  $V(z, t)$  is given by the real part of the following expression:

$$V(z, t) = \int_{\nu-\Delta\nu}^{\nu+\Delta\nu} a(\nu') \exp[i2\pi(\nu' t - z/\lambda')] d\nu'. \quad (21)$$

While a monochromatic beam of radiation corresponds to an infinitely long wave train, it can easily be shown that the superposition of multiple infinitely long wave trains, with nearly similar frequencies, results in the formation of wave groups.

In order to determine the electric field created by such a source, emitting within a certain frequency range  $\pm\Delta\nu$ , we must sum up the fields due to all the individual monochromatic components such that the resulting electric field  $V(z, t)$  is given by the real part of the following expression:

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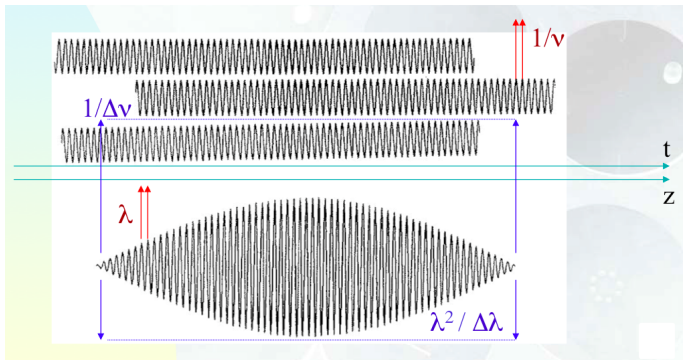
Indeed, we find that Eq. (21) may be rewritten as

$$V(z, t) = A(z, t) \exp[i2\pi(\nu t - z/\lambda)] \quad (22)$$

where

$$A(z, t) = \int_{\nu-\Delta\nu}^{\nu+\Delta\nu} a(\nu') \exp\{i2\pi[(\nu' - \nu)t - z(1/\lambda' - 1/\lambda)]\} d\nu'. \quad (23)$$

# Light coherence: quasi-monochromatic light

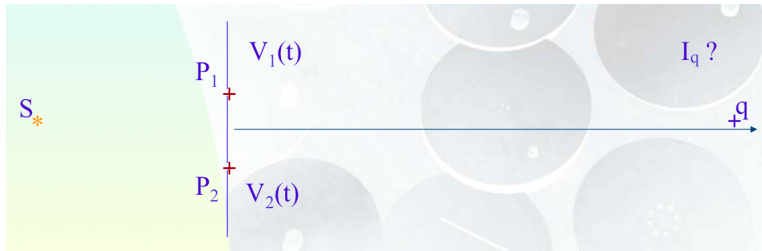


**Figure:** Superposition of long wave trains having quite similar frequencies  $\nu'$  in the range  $\nu \pm \Delta\nu$  (resp. wavelengths  $\lambda'$  in the range  $\lambda \pm \Delta\lambda$ ) results in the propagation of a long wave train with the frequency  $\nu$  (resp. wavelength  $\lambda$ ) but which amplitude  $A(z, t)$  is varying with a lower frequency  $\Delta\nu$  (resp. longer wavelength  $\lambda^2/\Delta\lambda$ ).

What becomes the visibility of the interference fringes in the Young's hole experiment for the case of a quasi-monochromatic source having a finite dimension?

## Light coherence: visibility of the interference fringes

What becomes the visibility of the interference fringes in the Young's hole experiment for the case of a quasi-monochromatic source having a finite dimension?



**Figure:** Assuming an extended source  $S$  which quasi monochromatic light passes through the two holes  $P_1$  and  $P_2$ ,  $I_q$  represents the intensity distribution at the point  $q$  which accounts for the formation of the interference fringes.

We can re-write the expression of the intensity  $I_q$  at point  $q$  as indicated below (see Eqs. (24)-(27)). It is assumed that the holes placed at the points  $P_1$ ,  $P_2$  in the Young plane have the same aperture size (i.e.  $V_1(t) = V_2(t)$ ) and that the propagation times of the light between  $P_1$  (resp.  $P_2$ ) and  $q$  are  $t_{1q}$  (resp.  $t_{2q}$ , see Fig. 25) :

$$I_q = \langle V_q^*(t) V_q(t) \rangle, \quad (24)$$

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and after a mere change of the time origin

$$V_q(t) = V_1(t) + V_2(t - \tau) \quad (26)$$

where we have defined

$$\tau = t_{2q} - t_{1q}. \quad (27)$$

It follows that Eq. (24) can be easily transformed into (28) where (29) represents the complex degree of mutual coherence, and the intensity  $I = \langle V_1 V_1^* \rangle = \langle V_2 V_2^* \rangle$ .

$$I_q = I + I + 2I \operatorname{Re}[\gamma_{12}(\tau)], \quad (28)$$

$$\gamma_{12}(\tau) = \langle V_1^*(t) V_2(t - \tau) \rangle / I. \quad (29)$$

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By means of (22), this function  $\gamma_{12}(\tau)$  can still be expressed as (30), and if  $\tau \ll 1/\Delta\nu$  (i.e. the difference between the arrival times of the two light rays is less than the beat period  $1/\Delta\nu$ ), we can give it the form (31):

$$\gamma_{12}(\tau) = \langle A_1^*(z, t) A_2(z, t - \tau) \rangle \exp(-i2\pi\nu\tau) / I, \quad (30)$$

and if  $\tau \ll 1/\Delta\nu$

$$\gamma_{12}(\tau) = |\gamma_{12}(\tau = 0)| \exp(i\beta_{12} - i2\pi\nu\tau). \quad (31)$$

Equation (28) can then be rewritten as (32) and in this case the visibility  $v$  of the interference fringes is  $|\gamma_{12}(\tau = 0)|$  (see Eq. (33)),  $I_{max}$  and  $I_{min}$  representing the brightest and weakest fringe intensities.

$$I_q = I + I + 2I |\gamma_{12}(\tau = 0)| \cos(\beta_{12} - 2\pi\nu\tau) \quad (32)$$

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and

$$v = \left( \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \right) = |\gamma_{12}(\tau = 0)|. \quad (33)$$

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We will see in the next section that the module of  $\gamma_{12}(\tau = 0)$  is directly related to the structure of the source that we are observing.

We propose hereafter to the reader to answer the two following questions:

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And what can we say about the source when  $|\gamma_{12}(\tau = 0)| = 0$ ?

Let us now evaluate what  $\gamma_{12}(\tau = 0)$  is for the case we are interested in, namely an extended source emitting quasi-monochromatic light. This leads us directly to study the notion of the spatial coherence of light.

## Light coherence: spatial coherence

Let us thus evaluate Eq. (29) for the case  $\tau = 0$ .

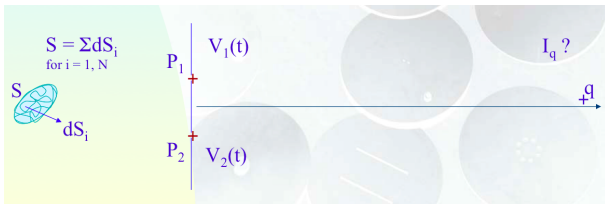
# Light coherence: spatial coherence

Let us thus evaluate Eq. (29) for the case  $\tau = 0$ .

We find

$$\gamma_{12}(\tau = 0) = \langle V_1^*(t) V_2(t) \rangle / I, \text{ with} \quad (34)$$

$$V_1(t) = \sum_{i=1}^N V_{i1}(t), V_2(t) = \sum_{i=1}^N V_{i2}(t). \quad (35)$$



**Figure:** The extended source  $S$  is assumed to be composed of a large number of infinitesimal surface elements  $dS_i$ .

It is assumed that the distinct points  $i$  of the source are separated by small distances compared to the wavelength  $\lambda$  of the light they emit in a mutually incoherent manner.

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$$\gamma_{12}(0) = \left[ \sum_{i=1}^N \langle V_{i1}^* V_{i2} \rangle + \sum_{i \neq j}^N \langle V_{i1}^* V_{j2} \rangle \right] / I. \quad (36)$$

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For an incoherent light source, the second summation appearing in (36) is obviously equal to zero. As a reminder, the contributions  $V_{ij}(t)$  can be expressed as

$$\begin{aligned} V_{i1}(t) &= \frac{a_i(t - r_{i1}/c)}{r_{i1}} \exp[i2\pi\nu(t - r_{i1}/c)], \\ V_{i2}(t) &= \frac{a_i(t - r_{i2}/c)}{r_{i2}} \exp[i2\pi\nu(t - r_{i2}/c)]. \end{aligned} \quad (37)$$

The products  $V_{i1}^*(t)V_{i2}(t)$  simplify themselves as

$$V_{i1}^*(t)V_{i2}(t) = \frac{|a_i(t - r_{i1}/c)|^2}{r_{i1}r_{i2}} \exp[-i2\pi\nu(r_{i2} - r_{i1})/c], \quad (38)$$

as long as the following condition is verified

$$|r_{i1} - r_{i2}| \leq c/\Delta\nu = \lambda^2/\Delta\lambda = \ell. \quad (39)$$

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We thus see how to naturally introduce the **coherence length**  $\ell$  which characterizes the precision with which we must obtain the equality between the optical paths in order to be able to observe interference fringes (typically 2.5 microns in the visible for  $\Delta\lambda = 1000 \text{ \AA}$ ).

To obtain the mutual intensity due to the whole source, it suffices to insert in the expression (36), the relation (38) using (40).

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The result is Eq. (41), also known as the **Zernicke-van Cittert Theorem**.

$$I(s)ds = |a_i(t - r/c)|^2, \quad (40)$$

$$\gamma_{12}(0) = \int_S \frac{I(s)}{r_1 r_2} \exp[-i2\pi(r_2 - r_1)/\lambda] ds / I. \quad (41)$$

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When the distance between the source and the screen is very large, the expression of this theorem can be simplified as follows.

## Light coherence: Zernicke-vanCittert theorem

By means of a relation analogous to (13), we find that

$$|r_2 - r_1| = |P_2 P_i - P_1 P_i| = \left| -\frac{(X^2 + Y^2)}{2Z'} + (X\zeta + Y\eta) \right| \quad (42)$$

where

$$\zeta = \frac{X'}{Z'}, \eta = \frac{Y'}{Z'} \quad (43)$$

represent the angular coordinates of the source measured from the interferometer. Using the two last relations, one can easily transform the expression (41) into (44).



**Figure:** Positions of the two elements  $P_1$  and  $P_2$  of the interferometer and of the infinitesimal element  $P_i$  of the source assuming that the distance  $Z' \gg |X'|, |Y'|, |X|$  or  $|Y|$ .

Apart from a multiplicative factor, we thus find that the visibility of the fringes (the function  $|\gamma_{12}(\tau = 0)|$ ) is simply the modulus of the Fourier transform of the normalized surface brightness  $I'$  of the source (Eq. (45)).

$$\gamma_{12}(0, X/\lambda, Y/\lambda) = \exp(-i\phi_{X,Y}) \int \int_S I'(\zeta, \eta) \exp[-i2\pi(X\zeta + Y\eta)/\lambda] d\zeta d\eta \quad (44)$$

with

$$I'(\zeta, \eta) = \frac{I(\zeta, \eta)}{\int \int_S I(\zeta', \eta') d\zeta' d\eta'}. \quad (45)$$

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Defining the angular space frequencies  $u = X/\lambda$ ,  $v = Y/\lambda$ , Eq. (44) becomes

$$\gamma_{12}(\mathbf{0}, u, v) = \exp(-i\phi_{u,v}) \int \int_S I'(\zeta, \eta) \exp[-i2\pi(u\zeta + v\eta)] d\zeta d\eta. \quad (46)$$

$$\gamma_{12}(0, u, v) = \exp(-i\phi_{u,v}) \int \int_S I'(\zeta, \eta) \exp[-i2\pi(u\zeta + v\eta)] d\zeta d\eta. \quad (46)$$

By a simple inverse Fourier transform, it is then possible to recover the surface (normalized) brightness of the source with an angular resolution equivalent to that of a telescope whose effective diameter would be equal to the baseline of the interferometer consisting of two independent telescopes

$$I'(\zeta, \eta) = \int \int \gamma_{12}(0, u, v) \exp(i\phi_{u,v}) \exp[i2\pi(\zeta u + \eta v)] dudv. \quad (47)$$

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Equations (46) and (47) thus clearly highlight the power of the complex degree of mutual coherence since they make it possible to link the visibility and the normalized intensity distribution of the source by means of the Fourier transform  $v = |\gamma_{12}(0)| = |TF_-(I')|$ , and its inverse.

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Aperture synthesis consists in observing a maximum number of visibilities of the source, thus trying to cover as well as possible the  $(u, v)$  plane from which we shall try, sometimes with some additional assumptions, to determine the structure of the source from the inverse Fourier transform (47) in which the integrand is not the visibility (i.e. the module of the complex degree of mutual coherence) but the complex degree of mutual coherence itself, within the factor  $\exp(i\phi_{x,y})$ .

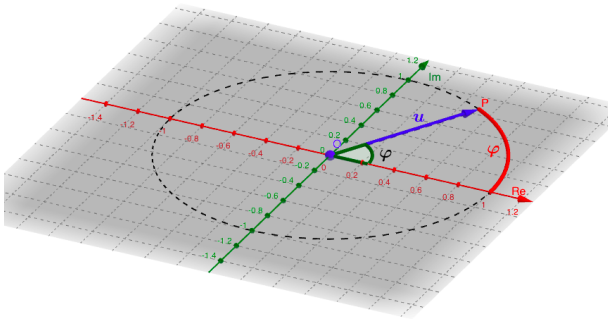
# Light coherence: some remarkable properties of the Fourier transform and applications

The expression  $e^{i\varphi}$  represents the complex exponential function of the real variable  $\varphi$ , expressed in radian:

$$e^{i\varphi} = \cos(\varphi) + i \sin(\varphi). \quad (48)$$

Its representation in the complex 2-D plane corresponds to a point  $P$  located on a circle which radius is unity, making an angle  $\varphi$  with respect to the real axis of reference (see Fig. 28). In this case, the variable  $\varphi$  does also represent the length of the subtended portion of the circular arc illustrated in red color.

# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** Illustration in the complex 2-D plane of the point  $P$ , or unit vector  $\mathbf{u}$  (in blue color), corresponding to the representation of the function  $e^{i\phi}$ .  $Re$  and  $Im$  represent the real and imaginary axes in that plane.

Similarly, the unit vector  $\mathbf{u}$  provides another representation of the function  $e^{i\phi}$ . This vector is sometimes referred in the literature as to the Fresnel vector.

# Light coherence: some remarkable properties of the Fourier transform and applications

It is interesting to visualize in the complex plane the representation of the trigonometric functions  $\cos(\varphi)$  and  $i \sin(\varphi)$  as being the summation of two Fresnels vectors (see Fig. 29).

Indeed, from the previous equation, it naturally comes that

$$\begin{aligned}\cos(\varphi) &= \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \\ i \sin(\varphi) &= \frac{e^{i\varphi} - e^{-i\varphi}}{2}.\end{aligned}\tag{49}$$

# Light coherence: some remarkable properties of the Fourier transform and applications

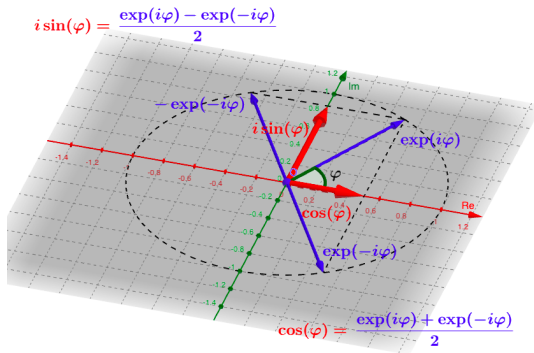


Figure: Illustration of the  $\cos(\varphi)$  (resp.  $i \sin(\varphi)$ ) function as being in the complex plane half the summation (resp. subtraction) of the two Fresnel vectors  $\exp(i\varphi)$  and  $\exp(-i\varphi)$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

In order to represent the position of the unit vector  $\mathbf{u} = \mathbf{OP}$  as a function of the time  $t$  while the point  $P$  is continuously revolving around the point  $O$  with a periodicity  $T$ , it is convenient to make use of the function

$$e^{i2\pi(t/T)} = e^{i2\pi\nu t}, \quad (50)$$

with the frequency  $\nu = 1/T$ .

Preferring to adopt the positive sense of rotation around the point  $O$  in the clockwise direction, we shall rather represent the rotation of the point  $P$  by means of the expression

$$e^{-i2\pi\nu t}. \quad (51)$$

Let us now consider a real function  $f(t)$  of the time variable  $t$  and the expression  $f(t_i) e^{-i2\pi\nu t_i} dt_i$  at the time  $t_i$  during the time interval  $dt_i$ . It is then easy to assimilate this triple factor product to a vector having the same orientation as the unit vector  $\mathbf{u}$  and which amplitude is  $f(t_i) dt_i$ .

Assuming that the function  $f(t)$  is defined in the interval  $[-\infty, \infty]$ , we could then imagine performing the summation

# Light coherence: some remarkable properties of the Fourier transform and applications

$$\sum_{t_i=-\infty}^{+\infty} f(t_i) e^{-i2\pi\nu t_i} dt_i, \quad (52)$$

corresponding to a vector which is the summation of all the previously defined individual vectors and being indicative of a kind of vectorial gravity center having the following meaning. If the function  $f(t)$  contains a feature appearing at a particular frequency  $\nu_0$ , all the individual vectors corresponding to the summation elements  $f(t_i) e^{-i2\pi\nu_0 t_i} dt_i$  will be oriented along a same direction and their summation will result in a vector which amplitude is very significant. If the function  $f(t)$  does not contain features appearing as a function of time at the frequency  $\nu_0$ , the corresponding individual vectors will be randomly oriented, also with random amplitudes, and the result of co-adding all of these will be a vector which amplitude is negligible. If the function  $f(t)$  contains features appearing at various frequencies  $\nu_j$ , there will then result several vectors which amplitudes and orientations are significant. It is then also easy to conceive that the orientations of the resulting vectors in the complex plane will be indicative of possible phase shifts of the function  $f(t)$  with respect to the corresponding functions  $e^{-i2\pi\nu_j t}$ . Considering the time interval  $dt_i \rightarrow 0$ , we may replace the summation sign in Eq. (52) by an integral sign and find out that the result of integrating all the individual elementary vectors  $f(t_i) e^{-i2\pi\nu t_i} dt_i$  results in

# Light coherence: some remarkable properties of the Fourier transform and applications

$$FT[f(t)](\nu) = \int_{-\infty}^{+\infty} f(t) e^{-i2\pi\nu t} dt, \quad (53)$$

which is the expression of the 1-D Fourier Transform of the function  $f(t)$ .

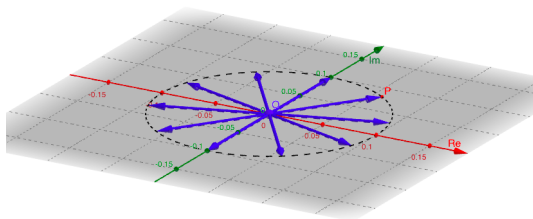
Let us illustrate this concept of vectorial gravity center by means of several examples.

For the case of the window -or door- function, i.e.  $\Pi(t) = 1$  if  $|t| \leq 0.5$  and 0 otherwise, and considering 10 values of  $t_i$  regularly spaced in the time interval  $[-0.5, 0.5]$  we obtain by means of Eq. (52)

$$\begin{aligned} \sum_{t_i=-\infty}^{+\infty} f(t_i) e^{-i2\pi\nu t_i} dt_i &= \sum_{t_i=-0.5}^{+0.5} f(t_i) e^{-i2\pi\nu t_i} dt_i \\ &= \sum_{t_i=-0.5}^{+0.5} e^{-i2\pi\nu t_i} dt_i = \frac{1}{10} [e^{i2\pi\nu 0.45} + e^{i2\pi\nu 0.35} + e^{i2\pi\nu 0.25} + e^{i2\pi\nu 0.15} + \\ &e^{i2\pi\nu 0.05} + e^{-i2\pi\nu 0.05} + e^{-i2\pi\nu 0.15} + e^{-i2\pi\nu 0.25} + e^{-i2\pi\nu 0.35} + e^{-i2\pi\nu 0.45}]. \end{aligned} \quad (54)$$

# Light coherence: some remarkable properties of the Fourier transform and applications

We have illustrated in Fig. 30, the ten individual vectors corresponding to the expression  $f(t_i) e^{-i2\pi\nu t_i} dt_i$  assuming that  $\nu = 1$ . Since for the case  $\nu = 1$  the 10 individual vectors are distributed symmetrically in the complex plane, their co-addition results is a vector which amplitude is equal to zero.



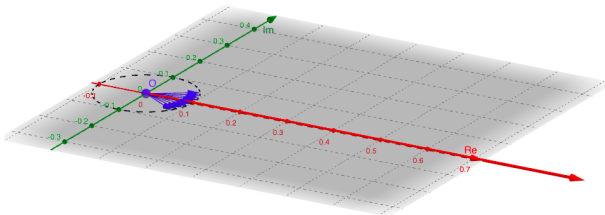
**Figure:** Illustration in the complex plane of the 10 individual vectors corresponding to the expression  $\Pi(t_i) e^{-i2\pi\nu t_i} dt_i$  for the case  $\nu = 1$ . The co-addition of these individual vectors results in the zero null vector.

# Light coherence: some remarkable properties of the Fourier transform and applications

In Figs. 31 & 32, we have illustrated the situations for the cases  $\nu = 0.2$  and  $\nu = 0$ . Later in the course, we shall very precisely establish that the analytical expression of the Fourier transform of  $\Pi(t)$  is

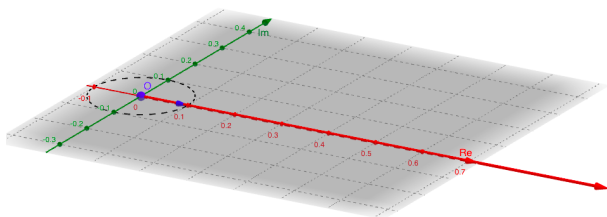
$$FT[\Pi(t)](\nu) = \text{sinc}(\nu) = \frac{\sin(\pi\nu)}{\pi\nu}. \quad (55)$$

From this expression, we straightforwardly find that  $FT[\Pi(t)](\nu) = 0, 0.94, 1$  and  $-0.21$ , respectively for the cases  $\nu = 1, 0.2, 0$  and  $1.5$ , in perfect agreement with our previous simulations.



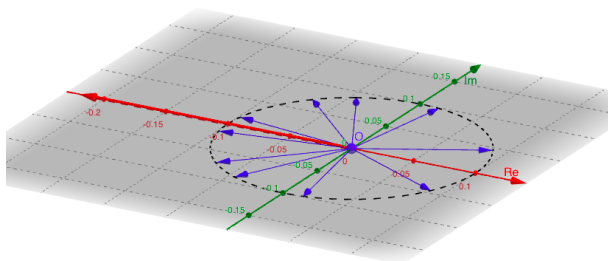
**Figure:** Illustration in the complex plane of the 10 individual vectors corresponding to the expression  $\Pi(t_i) e^{-i2\pi\nu t_i} dt_i$  for the case  $\nu = 0.2$ . Co-addition of these individual vectors results in the red vector which amplitude is equal to 0.94.

# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** Illustration in the complex plane of the 10 individual vectors corresponding to the expression  $\prod(t_i) e^{-i2\pi\nu t_i} dt_i$  for the case  $\nu = 0$ . Co-addition of these individual vectors results in the red vector which amplitude is equal to 1.

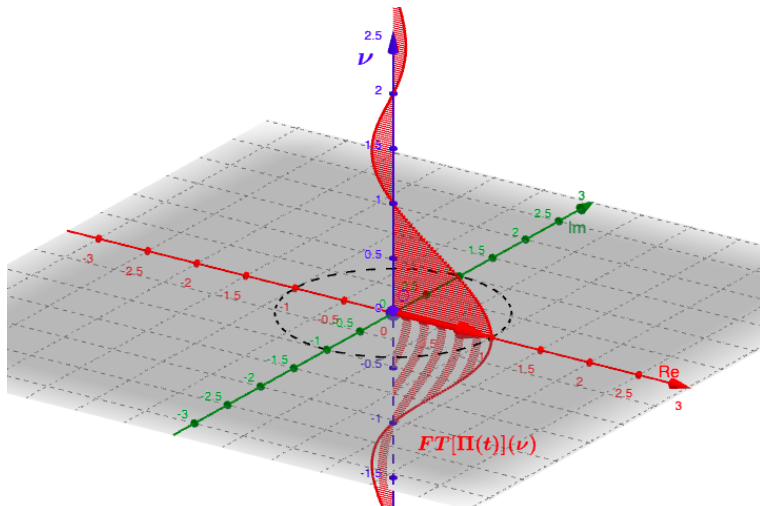
# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** Illustration in the complex plane of the 10 individual vectors corresponding to the expression  $\Pi(t_i) e^{-i2\pi\nu t_i} dt_i$  for the case  $\nu = 1.5$ . Co-addition of these individual vectors results in the red vector which amplitude is equal to -0.21, i.e. oriented along the negative side of the real axis.

# Light coherence: some remarkable properties of the Fourier transform and applications

Fig. 34 represents the Fourier transform  $FT[\Pi(t)](\nu)$  of the window function  $\Pi(t)$  as a function of the frequency  $\nu \in [-3, 3]$ . Because of the symmetry of the function  $\Pi(t)$ , its Fourier transform is real.



# Light coherence: some remarkable properties of the Fourier transform and applications

It is now good to remind some specific properties of the Fourier transform. The Fourier transform of the function  $f(x)$ , denoted  $FT_-f(s)$ , where  $x \in \mathfrak{R}$ , is the function

$$FT_-f(s) = \int_{-\infty}^{\infty} f(x) \exp(-2i\pi sx) dx \quad (56)$$

where  $s \in \mathfrak{R}$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

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The functions  $f$  and  $FT_{-}f$  form a Fourier pair. The function  $FT_{-}f(s)$  exists if the function  $f(x)$  is bounded, summable and has a finite number of extrema and discontinuities.

# Light coherence: some remarkable properties of the Fourier transform and applications

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The functions  $f$  and  $FT_-f$  form a Fourier pair. The function  $FT_-f(s)$  exists if the function  $f(x)$  is bounded, summable and has a finite number of extrema and discontinuities.

This does not necessarily imply that the inverse Fourier transform, denoted  $FT_-^{-1}FT_-f$  transform is  $f$ . For the Fourier transformation to be reciprocal,

$$f(x) = \int_{-\infty}^{\infty} FT_-f(s) \exp(2i\pi xs) ds, \quad (57)$$

it suffices  $f$  to be of summable square, i.e. that the following integral exists

$$\int_{-\infty}^{\infty} |f(x)|^2 dx.$$



# Light coherence: some remarkable properties of the Fourier transform and applications

We can generalize the  $FT$  to several dimensions, by defining  $f$  on the space  $\mathfrak{R}^n$ . Let  $\mathbf{r}$ ,  $\mathbf{w} \in \mathfrak{R}^n$ , we then have

$$FT_{-}f(\mathbf{w}) = \int_{-\infty}^{\infty} f(\mathbf{r}) \exp(-2i\pi\mathbf{w}\mathbf{r})d\mathbf{r}. \quad (59)$$

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As a reminder, if  $f(t)$  designates a function of time,  $FT_-f(s)$  represents its content as a function of time frequencies.

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Among the interesting properties of the Fourier transform, let us first remind the **linearity properties**:

$$FT_-(af) = aFT_-f, \quad (60)$$

with the constant  $a \in \mathfrak{R}$ ,

$$FT_-(f + g) = FT_-f + FT_-g.$$

# Light coherence: some remarkable properties of the Fourier transform and applications

## **The symmetry and parity properties:**

The considerations of symmetry are very useful during the study of the Fourier transform.

# Light coherence: some remarkable properties of the Fourier transform and applications

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The considerations of symmetry are very useful during the study of the Fourier transform. Let  $P(x)$  and  $I(x)$  be the even and odd parts of  $f(x)$  such that

$$f(x) = P(x) + I(x), \quad (62)$$

we find that

$$FT_{-}f(s) = 2 \int_0^{\infty} P(x) \cos(2\pi xs) dx - 2i \int_0^{\infty} I(x) \sin(2\pi xs) dx. \quad (63)$$

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From this result, we can deduce for instance that if  $f(x)$  is real, the real part of  $FT_{-}f(s)$  will be even and its imaginary part will be odd whereas if  $f(x)$  is complex, the imaginary part of  $FT_{-}f(s)$  will be even and its real part will be odd.

# Light coherence: some remarkable properties of the Fourier transform and applications

**The similarity property:** the relationship of similarity is the following one

$$FT_-(f(x/a))(s) = |a| FT_-(f(x))(as) \quad (64)$$

where  $a \in \mathfrak{R}$ , is a constant.

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**The translation property:** the translation relation is written as

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**The translation property:** the translation relation is written as

$$FT_-(f(x - a))(s) = \exp(-2i\pi as) FT_-(f(x))(s). \quad (65)$$

A translation of the function in its original space corresponds to a phase rotation of its Fourier transform in the transformed space.

# Light coherence: some remarkable properties of the Fourier transform and applications

The **door function**, denoted  $\Pi(x)$ , is defined by (see Fig. 35)

$$\Pi(x) = 1 \quad \text{if } x \in [-0.5, 0.5], \quad \text{and} \quad \Pi(x) = 0 \quad \text{otherwise.} \quad (66)$$

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$$\Pi(x) = 1 \quad \text{if } x \in [-0.5, 0.5], \quad \text{and } \Pi(x) = 0 \quad \text{otherwise.} \quad (66)$$

It is easy to find that

$$FT_{-}(\Pi(x))(s) = \text{sinc}(s) = \frac{\sin(\pi s)}{\pi s}. \quad (67)$$

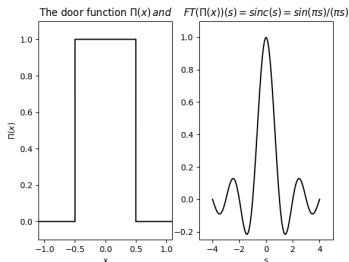


Figure: The door function and its Fourier transform (cardinal sine).

# Light coherence: some remarkable properties of the Fourier transform and applications

Applying the similarity relation, we also find that

$$FT_{-}(\Pi(x/a))(s) = |a| \operatorname{sinc}(as) = |a| \frac{\sin(\pi as)}{\pi as}. \quad (68)$$

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The door function is also sometimes called the window function or simply window. The **Dirac distribution**, also called Dirac peak, is noted  $\delta(x)$ . It is defined by the following integral, which exists only in the sense of the distributions

$$\delta(x) = \int_{-\infty}^{\infty} \exp(2i\pi xs) ds. \quad (69)$$

Its Fourier transform is therefore 1 in the interval  $]-\infty, +\infty[$  since  $\delta(x)$  appears above as the inverse Fourier transform of 1.

# Light coherence: some remarkable properties of the Fourier transform and applications

**Some applications:** we propose hereafter several astrophysical applications that make use of the previous remarkable properties of the Fourier transform.

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Let us first consider the case of a double star which two point-like components are equally bright and separated by an angle  $2\zeta_0$ . Making use of Eqs. (33), (45) and (46), one may easily establish using the properties (65) and (69) that the normalized intensity  $I'(\zeta)$  takes the simple form

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$$I'(\zeta) = \frac{\delta(\zeta - \zeta_0) + \delta(\zeta + \zeta_0)}{2} \quad (70)$$

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$$I'(\zeta) = \frac{\delta(\zeta - \zeta_0) + \delta(\zeta + \zeta_0)}{2} \quad (70)$$

and that the visibility  $v$  measured with an interferometer composed of 2 telescopes separated by the baseline  $X$  is given by the expression

$$v = |\gamma_{12}(u, 0)| = |\cos(2\pi\zeta_0 u)| \quad (71)$$

where  $u = X/\lambda$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

A second nice application consists in deriving the visibility of the interference fringes measured with the same interferometer of a 1-D Gaussian star which intensity  $I(\zeta)$  distribution is given by the following expression

$$I(\zeta) = \exp\left(\frac{-4 \ln(2)\zeta^2}{FWHM^2}\right) \quad (72)$$

where *FWHM* represents the angular full width at half maximum of the 1-D Gaussian star.

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The expression of the corresponding visibility is then easily found to be

$$v = |\gamma_{12}(0, u)| = \exp\left(\frac{-\pi^2 u^2 FWHM^2}{4 \ln(2)}\right), \quad (73)$$

and we notice that narrower is the angular size of the star, broader is its visibility content in angular space frequencies.

# Light coherence: some remarkable properties of the Fourier transform and applications

In the third proposed application, we ask to establish the expression of the visibility of a  $2-D$  uniformly bright square star which each angular side is  $\zeta_0$ ,

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The expression to be derived is the following one

$$v = |\gamma_{12}(0, u, v)| = \left| \frac{\sin(\pi\zeta_0 u)}{\pi\zeta_0 u} \frac{\sin(\pi\zeta_0 v)}{\pi\zeta_0 v} \right|. \quad (74)$$

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## Remark:

To check whether your original function  $I(\zeta, (and/or \eta))$  has been properly normalized, see whether its Fourier Transform for  $u=0$  (and/or  $v=0$ ) is equal to unity! Why?

# Light coherence: some remarkable properties of the Fourier transform and applications

Finally, a generalization of the previous application consists in deriving the visibility of a star which is seen as a projected  $2-D$  uniform circular disk which angular radius is  $\rho_{UD}$  and its angular diameter  $\theta_{UD}$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

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Due to the circular symmetry of the problem, it is convenient to make use of polar coordinates in Eq. (44) as follows:

$$\begin{aligned}u &= X/\lambda = R \cos(\psi)/\lambda \\v &= Y/\lambda = R \sin(\psi)/\lambda,\end{aligned}\tag{75}$$

where  $R$  denotes the baseline between the two telescopes of the interferometer, and

$$\begin{aligned}\zeta &= \theta \cos(\phi) \\ \eta &= \theta \sin(\phi).\end{aligned}\tag{76}$$

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Eq. (44) then transforms into

# Light coherence: some remarkable properties of the Fourier transform and applications

$$|\gamma_{12}(0, R/\lambda, \psi)| = \left| \frac{1}{\pi \rho_{UD}^2} \int_0^{\rho_{UD}} \theta \int_0^{2\pi} \exp[-i2\pi\theta R/\lambda(\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi))] d\phi d\theta \right| \quad (77)$$

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Making use of the additional changes of variables

$$\begin{aligned} z &= 2\pi\theta R/\lambda \\ \Phi &= \pi/2 - \phi + \psi, \end{aligned} \quad (78)$$

Eq. (77) becomes

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Reminding the definition of the zero order Bessel function  $J_0(x)$

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos[x \sin(\theta)] d\theta,$$



# Light coherence: some remarkable properties of the Fourier transform and applications

and the relation existing between  $J_0(x)$  and the first order Bessel function  $J_1(x)$ , namely

$$xJ_1(x) = \int x' J_0(x') dx', \quad (81)$$

Eq. (79) successively reduces to

$$|\gamma_{12}(0, R/\lambda)| = \left| \left( \frac{\lambda}{2\pi R} \right)^2 \frac{1}{\pi \rho_{UD}^2} 2\pi \int_0^{2\pi \rho_{UD} R/\lambda} z J_0(z) dz \right| \quad (82)$$

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and

$$|\gamma_{12}(0, R/\lambda)| = \left| 2 \frac{J_1(2\pi \rho_{UD} R/\lambda)}{2\pi \rho_{UD} R/\lambda} \right|. \quad (83)$$

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and

$$|\gamma_{12}(0, R/\lambda)| = \left| 2 \frac{J_1(2\pi \rho_{UD} R/\lambda)}{2\pi \rho_{UD} R/\lambda} \right|. \quad (83)$$

We thus find that the expression (33) of the fringe visibility for the case of a star seen as a projected 2-D uniform circular disk with an angular diameter  $\theta_{UD} = 2\rho_{UD}$  is

$$v = \left( \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \right) = |\gamma_{12}(0, u)| = \left| \frac{2J_1(\pi \theta_{UD} u)}{\pi \theta_{UD} u} \right|, \quad (84)$$

where we have set  $u = R/\lambda$ .

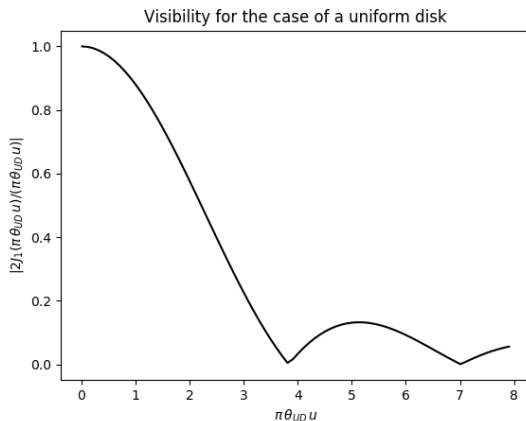
# Light coherence: some remarkable properties of the Fourier transform and applications

As a reminder, the Bessel function has the following properties

$$\begin{aligned} J_1(x = 3.8317\dots) &= 0 \\ \lim_{x \rightarrow 0} \frac{J_1(x)}{x} &= 1/2, \end{aligned} \tag{85}$$

which allow us to easily understand the behavior of the visibility function illustrated in Fig. 36.

# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** Visibility function expected for a star consisting of a uniformly bright circular disk with an angular diameter  $\theta_{UD}$ .

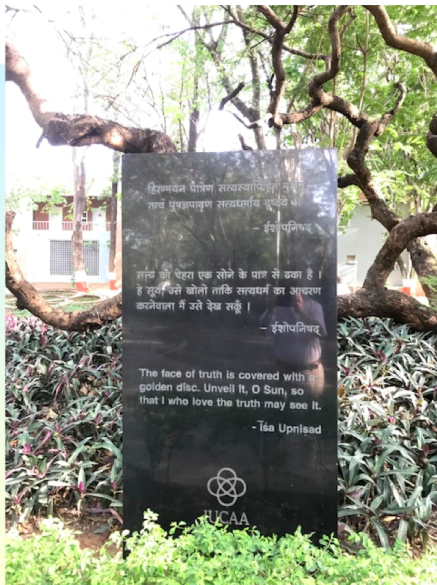


Figure: Face of the Truth and the Golden Disk.

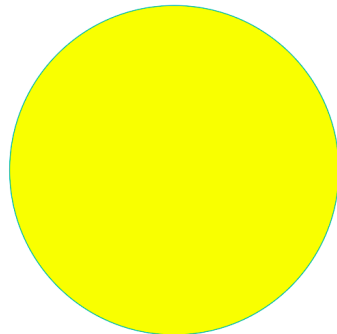
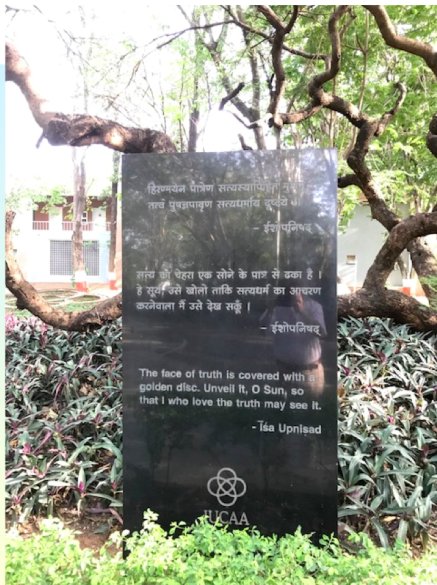
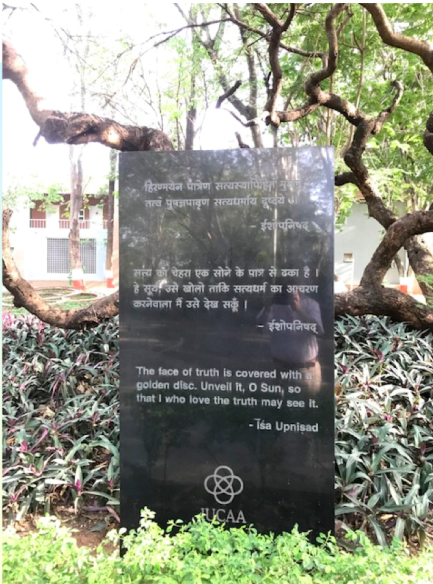


Figure: Face of the Truth and the Golden Disk.



$$v = |\gamma_{12}| = \frac{2|J_1(\pi\theta_{UD}u)|}{(\pi\theta_{UD}u)}$$

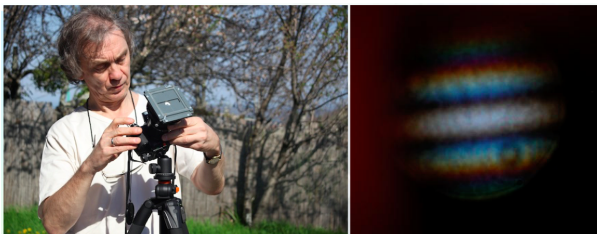
Figure: Face of the Truth and the Golden Disk.

# Light coherence: some remarkable properties of the Fourier transform and applications

One could then wonder whether it is possible to observe interferometric fringes from our nearest star, i.e. the Sun?

# Light coherence: some remarkable properties of the Fourier transform and applications

One could then wonder whether it is possible to observe interferometric fringes from our nearest star, i.e. the Sun? Figure 40 illustrates such fringes in white light obtained on 9th of April 2010 using a micro interferometer consisting of 2 holes with a diameter of  $11.8 \mu$  separated by a baseline of  $29.4 \mu$ . This micro-interferometer was placed in front of the objective of an EOS 5D Canon camera. Since the picture was taken in white light, it is possible to see the effects due to color dispersion. It is then easy to get an estimate of the fringe visibility, using Eq. (84), assuming that the Sun is a uniform disk with an angular diameter of  $30'$ .



**Figure:** Solar fringes photographed with an EOS 5D Canon camera in front of which was set a micro-interferometer consisting of two holes having a diameter of  $11.8 \mu$  separated by a baseline of  $29.4 \mu$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

## The general case of a double star

# Light coherence: some remarkable properties of the Fourier transform and applications

## The general case of a double star

Let us derive the expressions of the module and phase of the complex degree of mutual coherence  $\gamma_{12}(\tau = 0, u, v)$  for the case of a binary star consisting of two unequally bright point-like components with an unknown orientation projected on the plane of the sky. Assuming that the angular coordinates of the vector connecting the two unresolved stars are  $(\zeta_0, \eta_0)$  with the angular coordinates  $\zeta, \eta$  oriented along the East-West and North-South directions, respectively, just alike the  $u, v$  space frequencies, and adopting  $I_1, I_2$  for the intensities of the two individual unresolved stars, we easily find for  $I'(\zeta, \eta)$

$$I'(\zeta, \eta) = \frac{I_1 \delta(\zeta - \frac{\zeta_0}{2}) \delta(\eta - \frac{\eta_0}{2}) + I_2 \delta(\zeta + \frac{\zeta_0}{2}) \delta(\eta + \frac{\eta_0}{2})}{I_1 + I_2}, \quad (86)$$

where  $\delta$  represents the Dirac function. We immediately verify that the integration of  $I'(\zeta, \eta)$  over the coordinates  $\zeta, \eta$  is correctly normalized (i.e. = 1).

# Light coherence: some remarkable properties of the Fourier transform and applications

## The general case of a double star

Let us derive the expressions of the module and phase of the complex degree of mutual coherence  $\gamma_{12}(\tau = 0, u, v)$  for the case of a binary star consisting of two unequally bright point-like components with an unknown orientation projected on the plane of the sky. Assuming that the angular coordinates of the vector connecting the two unresolved stars are  $(\zeta_0, \eta_0)$  with the angular coordinates  $\zeta, \eta$  oriented along the East-West and North-South directions, respectively, just alike the  $u, v$  space frequencies, and adopting  $I_1, I_2$  for the intensities of the two individual unresolved stars, we easily find for  $I'(\zeta, \eta)$

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where  $\delta$  represents the Dirac function. We immediately verify that the integration of  $I'(\zeta, \eta)$  over the coordinates  $\zeta, \eta$  is correctly normalized (i.e. = 1).

Inserting Eq. (86) into Eq. (44), and making use of the properties of variable separation, Dirac function and translation of the Fourier transform, we successively obtain:

# Light coherence: some remarkable properties of the Fourier transform and applications

$$\begin{aligned}\gamma_{12}(\tau = 0, u, v) &= \frac{1}{l_1 + l_2} \left\{ l_1 FT\left[\delta\left(\zeta - \frac{\zeta_0}{2}\right)\delta\left(\eta - \frac{\eta_0}{2}\right)\right](u, v) + \right. \\ &\quad \left. l_2 FT\left[\delta\left(\zeta + \frac{\zeta_0}{2}\right)\delta\left(\eta + \frac{\eta_0}{2}\right)\right](u, v) \right\}, \\ &= \frac{1}{l_1 + l_2} \left\{ l_1 FT\left[\delta\left(\zeta - \frac{\zeta_0}{2}\right)\right](u) FT\left[\delta\left(\eta - \frac{\eta_0}{2}\right)\right](v) + \right. \\ &\quad \left. l_2 FT\left[\delta\left(\zeta + \frac{\zeta_0}{2}\right)\right](u) FT\left[\delta\left(\eta + \frac{\eta_0}{2}\right)\right](v) \right\}, \\ &= \frac{1}{l_1 + l_2} \left\{ l_1 \exp[-i\pi(\zeta_0 u + \eta_0 v)] + l_2 \exp[i\pi(\zeta_0 u + \eta_0 v)] \right\}.\end{aligned}\tag{87}$$

# Light coherence: some remarkable properties of the Fourier transform and applications

$$\begin{aligned}\gamma_{12}(\tau = 0, u, v) &= \frac{1}{I_1 + I_2} \left\{ I_1 FT[\delta(\zeta - \frac{\zeta_0}{2}) \delta(\eta - \frac{\eta_0}{2})](u, v) + \right. \\ &\quad \left. I_2 FT[\delta(\zeta + \frac{\zeta_0}{2}) \delta(\eta + \frac{\eta_0}{2})](u, v) \right\}, \\ &= \frac{1}{I_1 + I_2} \left\{ I_1 FT[\delta(\zeta - \frac{\zeta_0}{2})](u) FT[\delta(\eta - \frac{\eta_0}{2})](v) + \right. \\ &\quad \left. I_2 FT[\delta(\zeta + \frac{\zeta_0}{2})](u) FT[\delta(\eta + \frac{\eta_0}{2})](v) \right\}, \\ &= \frac{1}{I_1 + I_2} \left\{ I_1 \exp[-i\pi(\zeta_0 u + \eta_0 v)] + I_2 \exp[i\pi(\zeta_0 u + \eta_0 v)] \right\}.\end{aligned}\tag{87}$$

Defining the relative intensities of the two stars as being

$$\begin{aligned}\epsilon &= \frac{I_1}{I_1 + I_2}, \\ 1 - \epsilon &= \frac{I_2}{I_1 + I_2},\end{aligned}\tag{88}$$

# Light coherence: some remarkable properties of the Fourier transform and applications

the expression (87) of  $\gamma_{12}(\tau = 0, u, v)$  can be successively rewritten as

$$\begin{aligned}\gamma_{12}(\tau = 0, u, v) &= \epsilon \exp[-i\pi(\zeta_0 u + \eta_0 v)] + (1 - \epsilon) \exp[i\pi(\zeta_0 u + \eta_0 v)], \\ &= \cos[\pi(\zeta_0 u + \eta_0 v)] + i(1 - 2\epsilon) \sin[\pi(\zeta_0 u + \eta_0 v)].\end{aligned}\tag{89}$$

# Light coherence: some remarkable properties of the Fourier transform and applications

the expression (87) of  $\gamma_{12}(\tau = 0, u, v)$  can be successively rewritten as

$$\begin{aligned}\gamma_{12}(\tau = 0, u, v) &= \epsilon \exp[-i\pi(\zeta_0 u + \eta_0 v)] + (1 - \epsilon) \exp[i\pi(\zeta_0 u + \eta_0 v)], \\ &= \cos[\pi(\zeta_0 u + \eta_0 v)] + i(1 - 2\epsilon) \sin[\pi(\zeta_0 u + \eta_0 v)].\end{aligned}\quad (89)$$

Still better! Adopting for the expression of  $\gamma_{12}(\tau = 0, u, v)$  the following form (cf. Eq. (31))

$$\gamma_{12}(\tau = 0, u, v) = |\gamma_{12}(\tau = 0, u, v)| \exp(i\beta_{12}), \quad (90)$$

we successively find for the module of the complex degree of mutual coherence (i.e. the visibility  $V$  of the fringes)

$$\begin{aligned}V = |\gamma_{12}(\tau = 0, u, v)| &= \sqrt{\cos^2[\pi(\zeta_0 u + \eta_0 v)] + (1 - 2\epsilon)^2 \sin^2[\pi(\zeta_0 u + \eta_0 v)]} \\ &= \sqrt{\epsilon^2 + (1 - \epsilon)^2 + 2\epsilon(1 - \epsilon) \cos[2\pi(\zeta_0 u + \eta_0 v)]}\end{aligned}\quad (91)$$

and for its phase  $\beta_{12}$

$$\beta_{12} = \arctan [(1 - 2\epsilon) \tan[\pi(\zeta_0 u + \eta_0 v)]], \text{ or also}$$

$$\beta_{12} = \arcsin \left[ \frac{(1 - 2\epsilon) \sin[\pi(\zeta_0 u + \eta_0 v)]}{|\gamma_{12}(\tau = 0, u, v)|} \right].$$

# Light coherence: some remarkable properties of the Fourier transform and applications

Some astronomers prefer to represent the previous relations in terms of the flux ratio between the two stars, i.e. the quantity  $I_{21} = I_2/I_1$ .

# Light coherence: some remarkable properties of the Fourier transform and applications

Some astronomers prefer to represent the previous relations in terms of the flux ratio between the two stars, i.e. the quantity  $l_{21} = I_2/I_1$ .

Noticing that

$$\epsilon = \frac{1}{1 + l_{21}}, \quad (93)$$

we obtain

$$V = |\gamma_{12}(\tau = 0, u, v)| = \sqrt{\frac{1 + l_{21}^2 + 2l_{21} \cos[2\pi(\zeta_0 u + \eta_0 v)]}{(1 + l_{21})^2}}, \text{ and}$$
$$\beta_{12} = \arctan \left[ \frac{l_{21} - 1}{l_{21} + 1} \tan[\pi(\zeta_0 u + \eta_0 v)] \right], \text{ or still} \quad (94)$$
$$\beta_{12} = \arcsin \left[ \frac{\frac{l_{21}-1}{l_{21}+1} \sin[\pi(\zeta_0 u + \eta_0 v)]}{|\gamma_{12}(\tau = 0, u, v)|} \right].$$

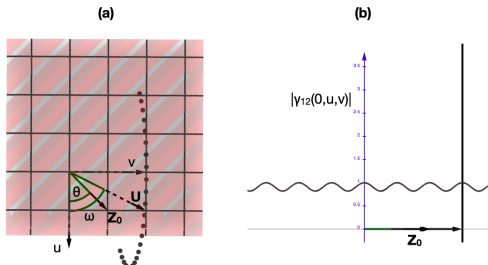
# Light coherence: some remarkable properties of the Fourier transform and applications

Noticing that the quantity  $\zeta_0 u + \eta_0 v$  is merely the scalar product between the angular vector  $\mathbf{Z}_0(\zeta_0, \eta_0)$  joining the two stars and the baseline vector  $\mathbf{U}(u, v)$ , i.e.  $\mathbf{Z}_0 \mathbf{U}$ , we may replace in all the previous relations the former quantity by

$$\zeta_0 u + \eta_0 v = \mathbf{Z}_0 \mathbf{U} = \rho \frac{B}{\lambda} \cos(\theta - \omega), \quad (95)$$

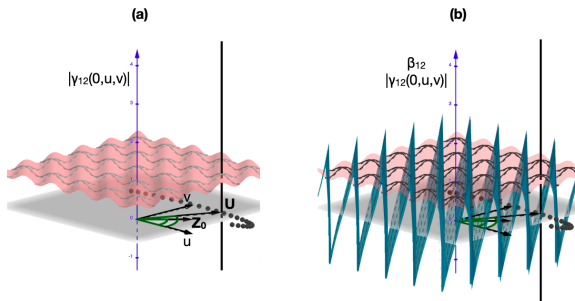
where  $\rho (= \sqrt{\zeta_0^2 + \eta_0^2})$  represents the module of the angular vector  $\mathbf{Z}_0(\zeta_0, \eta_0)$ , i.e. the angular separation between the two components of the binary star,  $B (= \sqrt{u^2 + v^2})$  the module of the projected baseline, and finally  $\theta$  and  $\omega$  the position angles of the projected line joining the two stars and of the baseline projected on the plane of the sky, respectively (see Figures 41 & 42).

# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** (a) Top view of the  $(u, v)$  plane and representation of a given vector  $\mathbf{U}$  with components  $(u=1, v=2)$  and of the angular vector  $\mathbf{Z}_0$  ( $\zeta_0 = 1, \eta_0 = 1$ ) corresponding to the separation between the two components of a double star. The dots projected on the  $(u, v)$  plane represent the positions of a fixed baseline on Earth as seen every 30 minutes from the binary star starting 6 hours before and ending 6 hours after passing the meridian (see the text). The angles  $\theta$  and  $\omega$  refer to the position angles of the  $\mathbf{Z}_0$  and  $\mathbf{U}$  vectors as measured from the  $u$  axis. The grey lines seen at  $45^\circ$  refer to positions in the  $(u, v)$  plane for which the visibility of the interferometric fringes of the double star characterized by a flux ratio of 10 is maximum. (b) Expected visibility ( $V = |\gamma_{12}(0, u, v)|$ ) of the fringes caused by the same double star along any axis parallel to the angular vector  $\mathbf{Z}_0$ .

# Light coherence: some remarkable properties of the Fourier transform and applications



**Figure:** (a) 3-D view of the visibility ( $V = |\gamma_{12}(0, u, v)|$ ) of the fringes caused by the double star described in the previous figure caption. (b) Same as (a) but with in addition the superimposition of the angle  $\beta_{12}$  defined in Eq. (94) illustrating the fact that when an asymmetric object is observed with an interferometer, the complex degree of mutual coherence  $\gamma_{12}(0, u, v)$  is a complex (and not any longer a real) quantity (cf. Eq. (90)). In that case both the observed visibility  $V = |\gamma_{12}(0, u, v)|$  and the phase angle  $\beta_{12}$  contain information relative to the observed source.

# Light coherence: some remarkable properties of the Fourier transform and applications

To be complete, let us mention that the dots plotted in Figs. 41 & 42 refer to the changing  $u, v$  coordinates of a projected baseline at positions  $u_0 = 1.4, v_0 = 2.6$  on the horizontal plane (latitude  $\varphi = 52^\circ$ ) seen during the rotation of the Earth from a binary star ( $\zeta_0 = 1, \eta_0 = 1$ ) with declination  $\delta = 10^\circ$ . A time interval of 30 minutes separates each dot, starting 6 hours before and ending 6 hours after passing the local meridian. Simulations illustrating the fringe patterns expected for the case of a binary star observed with different baselines have been prepared using the Geogebra open source applications (<https://www.geogebra.org>). They can be viewed on the video `Interf_Binary_Star.mp4` to be downloaded via the URL: <https://orbi.uliege.be/handle/2268/253241>. In the remainder, we shall explain how to calculate the  $(u, v)$  coverage corresponding to a selected baseline of an interferometer located on Earth, seen from a distant celestial source during the Earth's rotation.

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## Some examples of interferometers

One of the most respected sanctuaries of optical interferometry is located on the plateau of Caussols, north of Grasse, in the south of France. The I2T (in French, "Interféromètre à 2 Télescopes"), made of 2 telescopes with an aperture of 26cm each and separated by a baseline of up to 144m was characterized by an angular resolution  $\Phi \sim 0.001''$  attainable for objects with an apparent magnitude brighter than  $V_{lim} \sim 6$  (see Figure 43, left).

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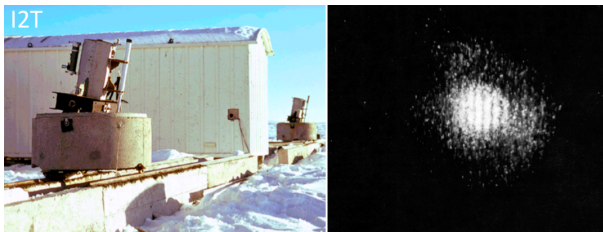
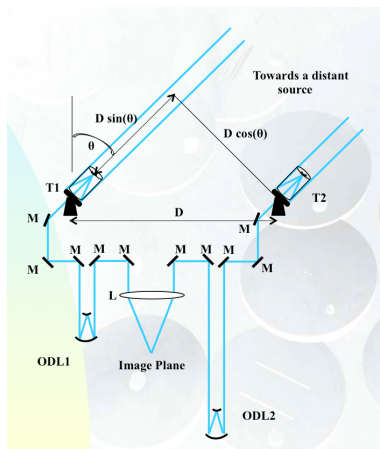


Figure: First fringes obtained with the I2T on Vega (Labeyrie et al. 1975, © Observatoire de la Côte d'Azur ).

## Some examples of interferometers

In order to equalize the light paths collected from the stars passing through the two telescopes, optical delay lines are mandatory. These have been successfully used for the first time in 1975 (see Fig. 44 for an illustration of how delay lines work).



**Figure:** Use of optical delay lines to compensate for the continuous change in the lengths of the two light paths as the Earth rotates.



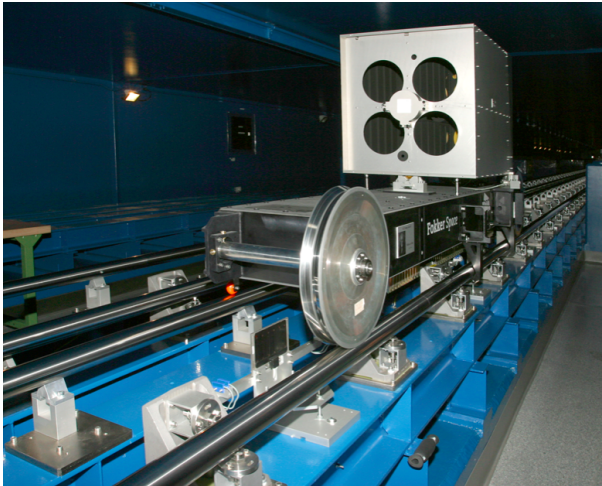


Figure: Optical delay lines at VLT. © ESO.

## Some examples of interferometers

The GI2T (in French, "Grand Interféromètre à 2 Télescopes") composed of two 1.5m telescopes was subsequently used by the same team. The two big telescopes could in principle be set 2 km apart, corresponding to an angular resolution  $\Phi \sim 0.0001''$  for  $V_{lim} = 15-17$  (see Fig. 47). Prof. A. Labeyrie also proposed to build a large network of ( $\sim 27$ ) optical telescopes whose individual diameters would be of the order of 10m.



**Figure:** The GI2T constructed by Antoine Labeyrie and his close collaborators on the plateau of Caussols, north of Grasse, near Nice (France, © Observatoire de la Côte d'Azur).

## Some examples of interferometers

Since the beginning of the 21<sup>st</sup> century, the modern sanctuary of stellar interferometry and aperture synthesis is undoubtedly the Very Large Telescope Interferometer (VLTI) of ESO (Southern European Observatory), located in Chile on Mount Paranal (see Fig. 48).

## Some examples of interferometers

Since the beginning of the 21<sup>st</sup> century, the modern sanctuary of stellar interferometry and aperture synthesis is undoubtedly the Very Large Telescope Interferometer (VLTI) of ESO (Southern European Observatory), located in Chile on Mount Paranal (see Fig. 48). The VLTI is a European interferometer that can re-combine the signal from 2, 3 or 4 telescopes depending on the instrument used. It has 4 telescopes of 8.2m and 4 mobile telescopes of 1.8m. Only telescopes of the same size can be re-combined together. The auxiliary telescopes of 1.8m can be easily moved allowing a better coverage of the  $u, v$  plane. The maximum base length of this interferometer is about 200m.

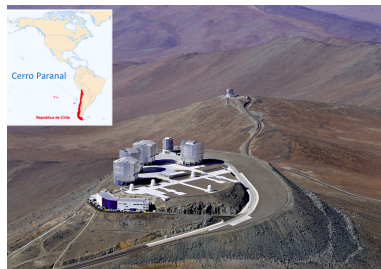



Figure: The Very Large Telescope Interferometer (VLTI) at the top of Paranal (Chile,  ESO université

## Some examples of interferometers

CHARA is another very performing interferometer located on the heights of Los Angeles, California (see Fig. 49). It is installed on the historic observatory of Mount Wilson. Remember that it was with the 2.5m telescope of this observatory that the first measurement of a stellar diameter was made by Michelson and Pease by installing a beam of 7m at the top of the telescope.

## Some examples of interferometers

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The CHARA interferometric array, operational since 1999 is composed of 6 telescopes of 1m in diameter. These 6 telescopes can be either re-combined by 2, by 3 since 2008 and very soon the 6 together. The maximum base length of this interferometer is 330m allowing to achieve an angular resolution of  $200\mu\text{arcsec}$ .



Figure: The CHARA interferometer composed of six 1m telescopes at Mount Wilson Observatory (California, USA). © The Observatories of the Carnegie Institution.

## Some examples of interferometers

It is mainly used for angular diameter measurements but also for the detection and characterization of tight binary stars as well as for the detection of exo-zodiacal clouds (clouds of dust gravitating around the stars).

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Another famous optical/IR interferometer is the Keck Interferometer made of two 10m telescopes separated by a fixed baseline of 85m (see Fig. 50) on top of Mauna Kea (Hawaii, USA).



Figure: The Keck interferometer on top of Mauna Kea (Hawaii, USA). © Ethan Tweedie.

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When we previously established the relation existing between the structure of a celestial source and the visibility of the fringes observed with an interferometer (Sections 3.3 and 4.4), we implicitly assumed that the size of the apertures was infinitely small (pinhole apertures). Use of the fundamental theorem allows one to calculate the response function of an interferometer equipped with finite size apertures. This theorem actually formalizes, in mathematical terms, the physical connection existing between the focal plane and the pupil plane of an optical instrument (telescope, interferometer, grating, etc.). Use of the convolution theorem will then enable us to establish the relation between a celestial source that is extended and its observed image in the focal plane of an optical instrument. Finally, the Wiener-Khinchin theorem establishes the relation between the frequency content of the point spread function of an optical instrument and its pupil plane characteristics.

The fundamental theorem that we shall demonstrate here merely stipulates that given a converging optical system which can be assimilated to the lens or to the mirror of a telescope, or of an optical interferometer, the complex amplitude distribution  $a(p, q)$  of the electromagnetic field of radiation in the focal plane is the Fourier transform of the complex amplitude distribution  $A(x, y)$  of the electromagnetic field in the pupil plane, i.e.

$$a(p, q) = \int_{R^2} A(x, y) \exp[-i2\pi(px + qy)] dx dy, \quad (96)$$

or in a more compact form

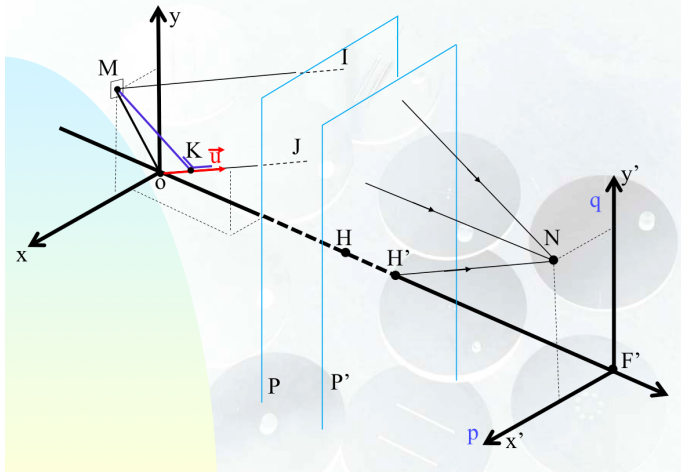
$$a(p, q) = FT[A(x, y)](p, q), \quad (97)$$

with

$$p = \frac{x'}{\lambda f} \quad \text{and} \quad q = \frac{y'}{\lambda f}, \quad (98)$$

where  $x'$ ,  $y'$  refer to the Cartesian coordinates in the focal plane,  $\lambda$  to the wavelength of the monochromatic light under consideration and  $f$  to the effective focal length of the converging system.

Figure 51 represents a convergent optical system, its focal point  $F'$ , its principal planes  $P$ ,  $P'$  and its principal points  $H$  and  $H'$ . The latter degenerate with the optical center in the case of a thin lens or with the bottom of the dish in the case of a single mirror. The two orthonormal coordinate systems  $(O, x, y, z)$  and  $(F', x', y', z')$  make it possible to locate the input pupil plane and the image focal plane of the optical system. The term 'pupil plane' serves as the support for the definition of the vibration state at the entrance of the collector while the 'focal plane' serves as the support for the definition of the image that the collector gives of the source located at infinity. Defining the action of the collector is thus to establish the transformation that it operates on the radiation between these two planes.



**Figure:** Fourier transform by a focusing optical system represented by its main planes  $P$  and  $P'$ . For the case of a thin lens, the latter would be degenerated into a single plane passing through its center.

The **hypotheses** underlying this theorem are:

H1. The optical system is free from any geometric aberration.

H2. The edges of the diaphragm do not disturb the electromagnetic field of radiation, 

that is to say that the diaphragm behaves as an "all (1) or nothing (0)" function with respect to this field. This is equivalent to assume that the dimensions of the collecting aperture(s) are large with respect to the wavelength of the light.

H3. No disturbance, other than those imposed by the optical system, intervenes between the pupil and the focal planes. The optical elements are thus assumed to be perfectly transparent or reflective.

H4. The light source is located at an infinite distance from the optical system and can thus be considered to be point-like.

H5. The disturbances occurring between the source and the pupil plane are weak and have very long evolution times relative to the period (i.e.  $T = 1/\nu = \lambda/c$ ) of the radiation.

H6. The radiation is monochromatic and has a fixed polarization plane.

### Theorem statement:

Within a multiplicative coefficient of the variables, the amplitude distribution in the focal plane is the Fourier transform of the amplitude distribution in the pupil plane.

### Demonstration:

Consider the different points  $(x, y)$  of the pupil plane. H6 (i.e. the previous hypothesis 6) makes it possible to represent the electrical component of the electromagnetic field by the real part of the vibration distribution

$$A(x, y) \exp(i2\pi\nu t),$$

with the very general representation of the expression of the complex amplitude  $A(x, y)$

$$A(x, y) = \mathcal{A}(x, y) \exp [i\Phi(x, y)] P_0(x, y), \quad (100)$$

where  $\mathcal{A}(x, y)$  and  $\Phi(x, y)$  represent the amplitude and phase of the electric field and  $P_0(x, y)$  the input pupil function which is 1 inside the pupil and 0 outside (in agreement with H2 and H3).

In agreement with the Huygens-Fresnel principle, we will consider in the following that every point reached by a wave can be considered as a secondary source re-emitting a vibration with the same amplitude, the same frequency  $\nu$ , the same polarization and the same phase (within a constant phase shift of  $\pi/2$ ) as those of the incident vibration at this point. The point  $N(x', y')$  of the focal plane will thus receive vibrations emitted by all the points of the pupil plane. The laws of geometrical optics, deduced from the Fermat principle, make it possible to write that the rays which, after the optical system, converge at the point  $N$  of the image focal plane, were, before the optical system, parallel to  $H'N$ . Having assumed that the source is at infinity (in agreement with H4), the amplitude will be preserved between the pupil plane and the focal plane. From the point  $M(x, y)$  of the pupil plane, the point  $N(x', y')$  of the focal plane will thus receive the vibration

$$A(x, y) \exp (i2\pi\nu t + i\Psi). \quad (101)$$

Let us take as the zero phase shift reference that of the ray passing through the point  $O$  along the direction  $OJN$ . The phase shift  $\Psi$  can then be expressed using the difference between the optical paths

$$\delta = d(MIN) - d(OJN), \quad (102)$$

where  $d()$  refers to the distance along the specified path, and the relation

$$\Psi = 2\pi\delta/\lambda. \quad (103)$$

If the point  $K$  corresponds to the orthogonal projection of  $M$  onto  $OJ$ ,  $M$  and  $K$  belong to the same wave plane which, after the optical system, will converge at the  $N$  point of the focal plane.

The Fermat principle, according to which the optical path between a point and its image is constant (rigorous stigmatism) or extremum (approximate stigmatism) makes it possible to write that the difference in optical path ( $MIN$ )–( $KJN$ ) behaves in the neighborhood of zero as an infinitely small second order with respect to the  $d(I,J)$  and thus also with respect to  $d(O,M)$  and  $d(O,K)$ , which are of the same order as  $d(I,J)$ . As a result (see Fig. 51),

$$\delta = -d(O, K) = -|(\mathbf{OM} \mathbf{u})|, \quad (104)$$

$\mathbf{u}$  designating the unit vector along the direction  $H'N$  and  $(\mathbf{OM} \mathbf{u})$  the scalar product between the vectors  $\mathbf{OM}$  and  $\mathbf{u}$ . If the angle that  $H'N$  makes with the optical axis is small, the vector of components  $(x'/f, y'/f, 1)$  is the vector director of  $H'N$  and has a norm close to 1 (at first order because  $f \gg |x'|, |y'|$ ). Moreover,  $\mathbf{OM}$  has for components  $(x, y, 0)$ . Using Eq. (109) in (103), the expression (101) becomes

$$A(x, y) \exp(i2\pi\nu t - xx'/\lambda f - yy'/\lambda f).$$

Choosing as new variables in the focal plane those defined in (98), we get

$$A(x, y) \exp[-i2\pi(xp + yq)] \exp(i2\pi\nu t). \quad (106)$$

The resulting vibration at the point  $N$  will be the resultant of the vibrations transmitted towards  $N$  by all the points of the pupil plane.

The equi-phase wave surfaces which reach the pupil plane are not planes if the radiation has been disturbed between the source and the entrance pupil. But the hypotheses  $H4$  and  $H5$  make it possible to affirm that the pupil plane is spatially coherent, that is to say that at the time scale of the vibration periods, the relative phase shift of its different points is constant. Consequently, to calculate the resulting vibration at the point  $N(p, q)$  of the focal plane, it is necessary to sum up the amplitudes that  $N$  receives from the different points of the pupil plane. The amplitude distribution  $a(p, q)$  in the focal plane then becomes

$$a(p, q) = \int_{R^2} A(x, y) \exp[-i2\pi(xp + yq)] dx dy, \quad (107)$$

that is, the complex amplitude distribution in the focal plane  $a(p, q)$  is the Fourier transform of the complex amplitude distribution  $A(x, y)$  in the pupil plane, i.e.

$$a(p, q) = FT[A(x, y)](p, q). \quad (108)$$

Simulations illustrating all the concepts of the fundamental theorem have been prepared using the Geogebra open source applications (<https://www.geogebra.org>). They can be

viewed on the videos Fundamental\_Th\_1.mp4 and Fundamental\_Th\_2.mp4 to be downloaded via the URL: <https://orbi.uliege.be/handle/2268/253241>.

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# Three important theorems and some applications: the fundamental theorem

Choosing as new variables in the focal plane those defined in (98), we get

$$A(x, y) \exp(-i2\pi(xp + yq) \exp(i2\pi\nu t)). \quad (111)$$

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The equi-phase wave surfaces which reach the pupil plane are not planes if the radiation has been disturbed between the source and the entrance pupil. But the hypotheses  $H4$  and  $H5$  make it possible to affirm that the pupil plane is spatially coherent, that is to say that at the time scale of the vibration periods, the relative phase shift of its different points is constant.

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$$a(p, q) = TF_-[A(x, y)](p, q).$$

# Applications of the fundamental theorem: the case of a single square aperture

Considering first the case of a single square aperture as depicted in Fig. 52 (left) and a point-like source perfectly located at zenith, i.e. the plane wavefronts arrive parallel to the aperture with a constant and real amplitude  $A(x, y) = A_0$ , we find that the calculation of the amplitude in the focal plane is very simple

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Making use of the separation of the variables  $x, y$  and of the relation (68), Eq. (97) successively transforms into

$$a(p, q) = A_0 FT(\Pi(x/a)(p)\Pi(y/a))(q), \quad (115)$$

$$a(p, q) = A_0 a^2 \frac{\sin(\pi ap)}{\pi ap} \frac{\sin(\pi aq)}{\pi aq}. \quad (116)$$

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This is the impulse response, in amplitude, for a square pupil and in the absence of any external disturbance.

## Applications of the fundamental theorem: the case of a single square aperture

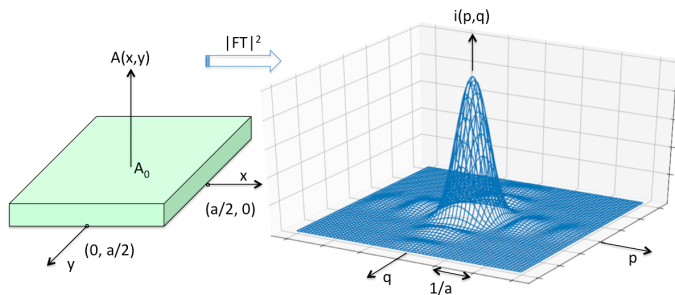
Adopting the definition (9) for the intensity of the vibrations, we find that (see Fig. 52, at right)

$$i(p, q) = a(p, q) a^*(p, q) = |a(p, q)|^2 = i_0 a^4 \left( \frac{\sin(\pi ap)}{\pi ap} \right)^2 \left( \frac{\sin(\pi aq)}{\pi aq} \right)^2. \quad (117)$$

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**Figure:** Complex amplitude distribution  $A(x, y)$  in the plane of a single square aperture (left) and resulting response function in intensity  $i(p, q)$  (right).

# Applications of the fundamental theorem: the case of a single square aperture

Defining the angular resolution  $\Phi$  of an optical system as being the angular width of the response function in intensity inside the first minima, we obtain for the values of  $\pi p a = \pm\pi$  (resp.  $\pi q a = \pm\pi$ ), i.e.  $p = \pm 1/a$  (resp.  $q = \pm 1/a$ ) and with the definition (98) for  $p, q$

$$\frac{x'}{\lambda f} = \pm \frac{1}{a} \quad (\text{resp.} \quad \frac{y'}{\lambda f} = \pm \frac{1}{a}), \quad (118)$$

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The angular resolution is thus inversely proportional to the size  $a$  of the square aperture, and proportional to the wavelength  $\lambda$ . Working at short wavelength with a big size aperture thus confers a better angular resolution.

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Up to now, we have considered that the source  $S$ , assumed to be point-like and located at an infinite distance from the optical system, was on the optical axis of the instrument.

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Up to now, we have considered that the source  $S$ , assumed to be point-like and located at an infinite distance from the optical system, was on the optical axis of the instrument. Suppose now that it is slightly moved away from the zenith direction by a small angle. Let  $(b/f, c/f, 1)$  be the unit vector representing the new direction of the source, the previous one being  $(0, 0, 1)$ . The plane wavefront falling on the square aperture will not have anymore a constant amplitude  $A_0$  because each point of the pupil touched by such a wavefront will experience a phase shift given by the angle

$$\psi = \frac{2\pi\delta}{\lambda} = \frac{2\pi(xb/f + yc/f)}{\lambda} \quad (120)$$

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and consequently the correct expression of the complex amplitude  $A(x, y)$  to be inserted in Eq. (97) becomes

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Proceeding as previously, we easily find that

# Applications of the fundamental theorem: the case of a single square aperture

$$a(p, q) = A_0 FT\left(\Pi\left(\frac{x}{a}\right)\right)\left(p - \frac{b}{\lambda f}\right) FT\left(\Pi\left(\frac{y}{a}\right)\right)\left(q - \frac{c}{\lambda f}\right), \quad (122)$$

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and finally

$$i(p, q) = a(p, q) a^*(p, q) = |a(p, q)|^2 = i_0 a^4 \left[ \frac{\sin\left(\pi a\left(p - \frac{b}{\lambda f}\right)\right)}{\pi a\left(p - \frac{b}{\lambda f}\right)} \right]^2 \left[ \frac{\sin\left(\pi a\left(q - \frac{c}{\lambda f}\right)\right)}{\pi a\left(q - \frac{c}{\lambda f}\right)} \right]^2. \quad (123)$$

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The resulting intensity response function in the focal plane is nearly the same as the one previously calculated for the case  $b = 0, c = 0$ . It is being merely translated by a linear offset  $(b, c)$  in the  $x', y'$  focal plane and implies the invariance of the response function for a reference star that is being slightly offset from the optical axis of the system.

## Three important theorems and some applications: the case of a circular aperture

Considering now a circular aperture with radius  $R$ , the complex amplitude  $A(x, y)$  in the pupil plane may be represented as a circular symmetric distribution, i.e.  $A(\rho, \varphi) = A_0$  for  $\rho < R$ ,  $\varphi \in [0, 2\pi]$  and  $A(\rho, \varphi) = 0$  for  $\rho > R$  (see Figure 53, at left).

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We naturally expect the distribution of the complex amplitude in the focal plane to be also circular symmetric, i.e.

$$a(\rho') = FT(A(\rho, \varphi))(\rho'). \quad (124)$$

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It is here interesting to note that performing the above Fourier transform is quite alike deriving the expression of the visibility of a 2-D uniform circular disk star which angular diameter is  $\theta_{UD}$  (see the last application in Section 4.5). We may just make use of the result (84) with appropriate changes of the corresponding variables. We easily find that

$$a(\rho') = A_0 \pi R^2 \left[ 2J_1 \left( \frac{2\pi R \rho' / (\lambda f)}{2\pi R \rho' / (\lambda f)} \right) \right]. \quad (125)$$

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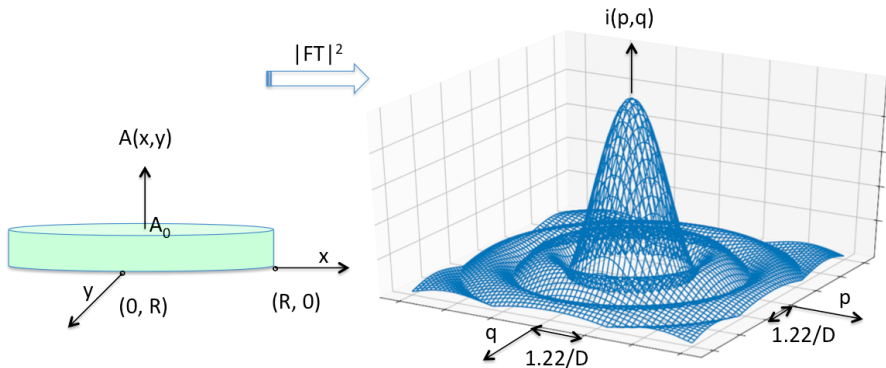
The resulting intensity response function is thus given by

$$i(\rho') = a(\rho')^2 = A_0^2 (\pi R^2)^2 \left[ 2J_1 \left( \frac{2\pi R \rho' / (\lambda f)}{2\pi R \rho' / (\lambda f)} \right) \right]^2.$$



# Three important theorems and some applications: the case of a circular aperture

This is the very expression of the famous Airy disk (see Fig. 53, at right).



**Figure:** The Airy disk: complex amplitude distribution  $A(\rho, \varphi) = A_0$  in the plane of a circular aperture (left) and the resulting response function in intensity  $i(\rho')$  (right).

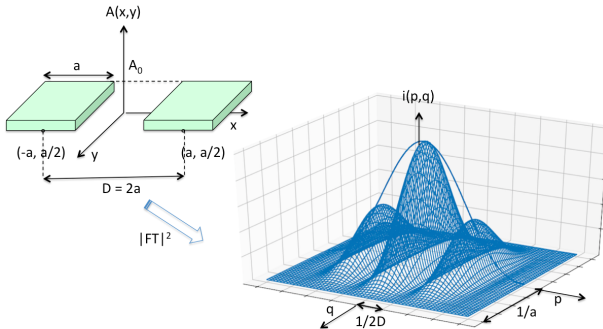
## Three important theorems and some applications: the case of a circular aperture

Knowing that the first order Bessel function  $J_1(x) = 0$  for  $x \sim 3.96$ , it is easy to deduce that the angular resolution  $\Phi$  of a telescope equipped with a circular objective which diameter is  $D = 2R$  is given by

$$\Phi = \frac{\Delta\rho'}{f} = \frac{2.44 \lambda}{D}. \quad (127)$$

# Three important theorems and some applications: the two telescope interferometer

Figure 54 (upper left) illustrates the principle of optically coupling two telescopes. Such a system is equivalent to a huge telescope in front of which would have been placed a screen pierced with two openings corresponding to the entrance pupils of the two telescopes. The pupil function  $A(x, y)$  of this system is shown in that same Figure for the case of two square apertures.



**Figure:** The two telescope interferometer: distribution of the complex amplitude for the case of two square apertures (upper left) and the corresponding impulse response function (lower right).

# Three important theorems and some applications: the two telescope interferometer

Let us now calculate the impulse response function  $a(p, q)$  of such a system.

## Three important theorems and some applications: the two telescope interferometer

Let us now calculate the impulse response function  $a(p, q)$  of such a system. Representing the distribution of the complex amplitude over each of the individual square apertures by means of the function  $A_0(x, y)$  and assuming that the distance between their optical axes is  $D$ , we find that

$$a(p, q) = FT_-[A_0(x + D/2, y) + A_0(x - D/2, y)](p, q). \quad (128)$$

## Three important theorems and some applications: the two telescope interferometer

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Making use of the relation (65), the previous equation reduces to

$$a(p, q) = [\exp(i\pi pD) + \exp(-i\pi pD)] FT_- [A_0(x, y)](p, q), \quad (129)$$

$$a(p, q) = 2 \cos(\pi pD) FT_- [A_0(x, y)](p, q) \quad (130)$$

and finally

$$i(p, q) = a(p, q)a^*(p, q) = |a(p, q)|^2 = 4 \cos^2(\pi pD) \{FT_- [A_0(x, y)](p, q)\}^2. \quad (131)$$

## Three important theorems and some applications: the two telescope interferometer

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## Three important theorems and some applications: the two telescope interferometer

Particularizing this intensity distribution to the case of two square apertures, or circular apertures, and making use of relations (117) or (126) leads to the respective results

$$i(p, q) = A_0^2 (2a^2)^2 \left[ \frac{\sin(\pi pa)}{\pi pa} \right]^2 \left[ \frac{\sin(\pi qa)}{\pi qa} \right]^2 \cos^2(\pi pD) \quad (132)$$

or

$$i(\rho, \rho') = A_0^2 (2\pi R^2)^2 \left[ \frac{2J_1(2\pi R\rho'/(\lambda f))}{2\pi R\rho'/(\lambda f)} \right]^2 \cos^2(\pi pD). \quad (133)$$

## Three important theorems and some applications: the two telescope interferometer

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Figure 54 (lower right) illustrates the response function for the former case. We see that the impulse response of each individual telescope is modulated by the  $\cos(2\pi pD)$  function and that the resulting impulse response function shows consequently a more detailed structure along the  $p$  axis, leading to a significantly improved angular resolution  $\Phi$  along that direction.

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The angular width  $\Phi$  of the bright central fringe is equal to the angular width separating the two minima located on its two sides. We thus find successively

$$\pi pD = \pm \frac{\pi}{2},$$

## Three important theorems and some applications: the two telescope interferometer

$$p = \pm \frac{1}{2D}, \quad (135)$$

$$\Delta p = \frac{1}{D} \quad (136)$$

and making use of relation (98)

$$\Phi = \frac{\Delta x'}{f} = \frac{\lambda}{D}. \quad (137)$$

## Three important theorems and some applications: the two telescope interferometer

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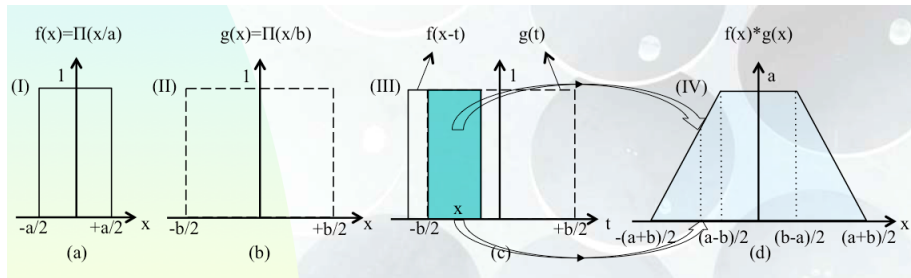
The angular resolution of the interferometer along the direction joining the two telescopes is approximately equivalent to that of a single dish telescope which diameter is equal to the baseline  $D$  separating them, and not any longer to the diameter of each single telescope (see Eqs. (119) or (127)).

# Three important theorems and some applications: the convolution theorem

The convolution theorem states that the convolution of two functions  $f(x)$  and  $g(x)$  is given by the following expression

$$f(x) * g(x) = (f * g)(x) = \int_R f(x-t)g(t)dt. \quad (138)$$

Figure 55 illustrates such a convolution product for the case of two rectangular functions  $f(x) = \Pi(x/a)$  and  $g(x) = \Pi(x/b)$  having the widths  $a$  and  $b$ , respectively.



**Figure:** Convolution product of two 1-D rectangular functions. (a)  $f(x)$ , (b)  $g(x)$ , (c)  $g(t)$  and  $f(x-t)$ . The dashed area represents the integral of the product of  $f(x-t)$  and  $g(t)$  for the given  $x$  offset, (d)  $f(x) * g(x) = (f * g)(x)$  represents the previous integral as a function of  $x$ .

## Three important theorems and some applications: the convolution theorem

Every day when the Sun is shining, it is possible to see nice illustrations of the convolution product while looking at the projected images of the Sun on the ground which are actually produced through small holes in the foliage of the trees (see the illustration in Fig. 56). It is a good exercise to establish the relation existing between the observed surface brightness of those Sun images, the shape of the holes in the foliage of the trees, their distance from the ground and the intrinsic surface brightness distribution of the Sun.



**Figure:** Projected images of the Sun on the ground actually produced through small holes in the foliage of trees (bamboo trees at IUCAA, Pune, India, June 2016). These images actually result from the convolution of the intrinsic Sun intensity distribution and the shapes of the holes in the trees.

# The convolution theorem: application to the two telescope interferometer

We have previously seen that for the case of a point-like source having an intrinsic surface brightness distribution  $O(p, q) = \delta(p)\delta(q)$ , there results the formation of an image  $e(p, q)$  in the focal plane which is the impulse response  $e(p, q) = i(p, q) = |a(p, q)|^2$  of the optical instrument (see Eqs. (117), (126), (132), (133) for the case of a single square aperture, a single circular aperture, an interferometer composed of two square or circular apertures, respectively).

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Considering now an extended source represented by its intrinsic surface brightness distribution  $O(p, q)$ , application of the convolution theorem in two dimensions directly leads to the expression of its brightness distribution  $e(p, q)$  in the focal plane of the optical system

$$e(p, q) = O(p, q) * |a(p, q)|^2 \quad (139)$$

or more explicitly

$$e(p, q) = \int_{R^2} O(r, s) |a(p - r, q - s)|^2 dr ds. \quad (140)$$

Since the Fourier transform of the convolution product of two functions is equal to the product of their Fourier transforms, we find that

$$FT_{-}[e(p, q)] = FT_{-}[O(p, q)] FT_{-}[|a(p, q)|^2] \quad (141)$$

and also that the inverse Fourier transform of  $FT_{-}(O(p, q))$  leads to the result

$$O(p, q) = FT_{-}^{-1}[FT_{-}(O(p, q))] = FT_{-}^{-1}\left[\frac{FT_{-}[e(p, q)]}{FT_{-}[|a(p, q)|^2]}\right], \quad (142)$$

namely, that it should be possible to recover interesting information on the intrinsic surface brightness distribution of the source  $O(p, q)$  at high angular resolution provided that we get sufficient information at high frequencies in the  $u, v$  plane on the object  $FT_{-}(e(p, q))$  itself as well as on a reference point-like object  $FT_{-}(|a(p, q)|^2)$ .

## The convolution theorem: interferometric observations of a circular symmetric source

Considering the case of a symmetric source around the  $Y$  axis observed by means of an interferometer composed of two square apertures which size of their sides is  $d$  separated along the  $X$  axis by the baseline  $D$ , we find by means of Eqs. (117), (132) and (139) that

$$e(p) = 2d^2 \left( \frac{\sin(\pi pd)}{\pi pd} \right)^2 [O(p) * \cos^2(\pi pD)]. \quad (143)$$

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Making use of the relation  $\cos(2x) = 2 \cos^2(x) - 1$ , Eq. (143) reduces to

$$e(p) = 2d^2 \left( \frac{\sin(\pi pd)}{\pi pd} \right)^2 \left[ \frac{1}{2} \int_R O(p) dp + \frac{1}{2} O(p) * \cos(2\pi pD) \right]. \quad (144)$$

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Since the function  $O(p)$  is real, the previous relation may be rewritten in the form

$$e(p) = A \left[ B + \frac{1}{2} \text{Re}[O(p) * \exp(i2\pi pD)] \right] \quad (145)$$

where

$$A = 2d^2 \left( \frac{\sin(\pi pd)}{\pi pd} \right)^2 \quad \text{and} \quad B = \frac{1}{2} \int_R O(p) dp.$$



# The convolution theorem: interferometric observations of a circular symmetric source

Given the definition of the convolution product (cf. Eq. (138)), relation (145) can be rewritten as

$$e(p) = A \left[ B + \frac{1}{2} \operatorname{Re} \left[ \int_R O(r) \exp(i2\pi(p-r)D) dr \right] \right], \quad (147)$$

or

$$e(p) = A \left[ B + \frac{1}{2} \cos(2\pi pD) FT_- [O(r)](D) \right], \quad (148)$$

because  $O(p)$  being real and even, its Fourier transform is also real.

# The convolution theorem: interferometric observations of a circular symmetric source

Given the definition of the convolution product (cf. Eq. (138)), relation (145) can be rewritten as

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or

$$e(p) = A \left[ B + \frac{1}{2} \cos(2\pi pD) \operatorname{FT}_-[O(r)](D) \right], \quad (148)$$

because  $O(p)$  being real and even, its Fourier transform is also real. The visibility of the fringes being defined by (see Eq. (19))

$$v = |\gamma_{12}(D)| = \left( \frac{e_{\max} - e_{\min}}{e_{\max} + e_{\min}} \right), \quad (149)$$

we obtain

$$v = |\gamma_{12}(D)| = \operatorname{FT}_-\left[\frac{O(r)}{2B}\right](D) = \operatorname{FT}_-\left[\frac{O(r)}{\int O(p)dp}\right](D).$$

# The convolution theorem: interferometric observations of a circular symmetric source

We have thus recovered the important result (46), first established for the case of two point-like apertures, according to which the visibility of the fringes is the Fourier transform of the normalized intensity distribution of the source. This result can be generalized to the case of a source that is not symmetric.

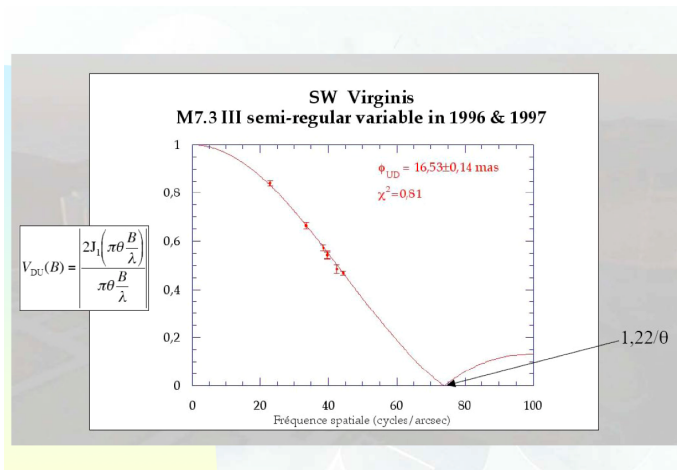


Figure: One example of stellar visibilities.

# Using interferometric masks for didactical purposes



Figure: Using interferometric masks for didactical purposes.

# Using interferometric masks for didactical purposes



Figure: Using interferometric masks for didactical purposes.

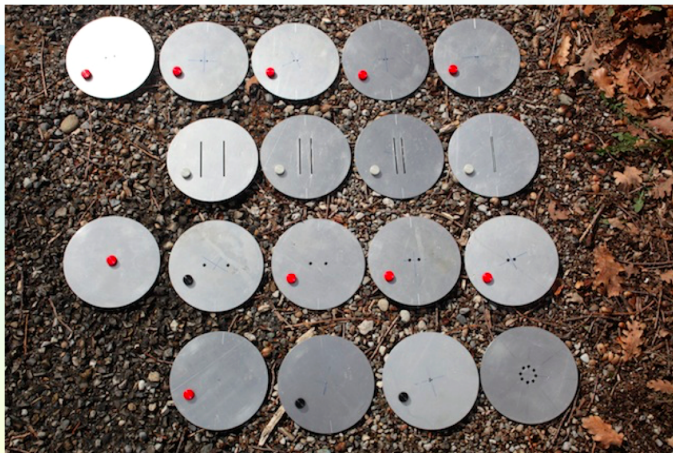


Figure: Using interferometric masks for didactical purposes.

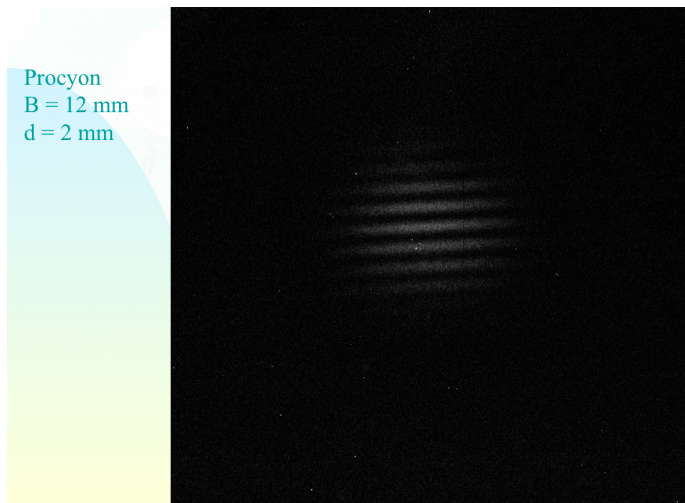


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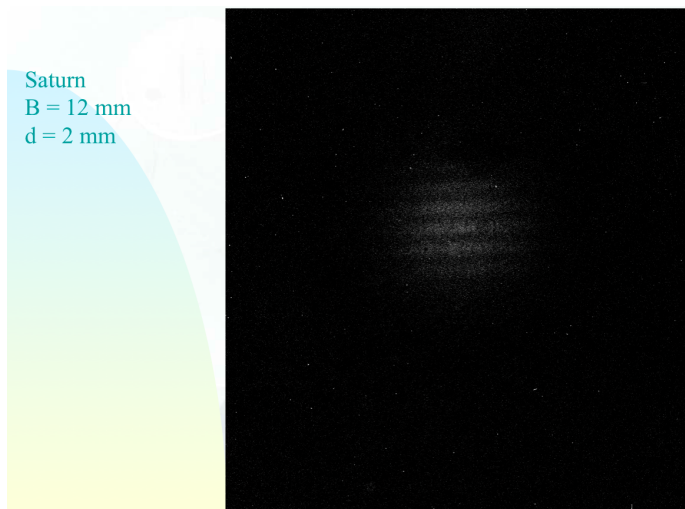


Figure: Using interferometric masks for didactical purposes.



Figure: More about VLTI.

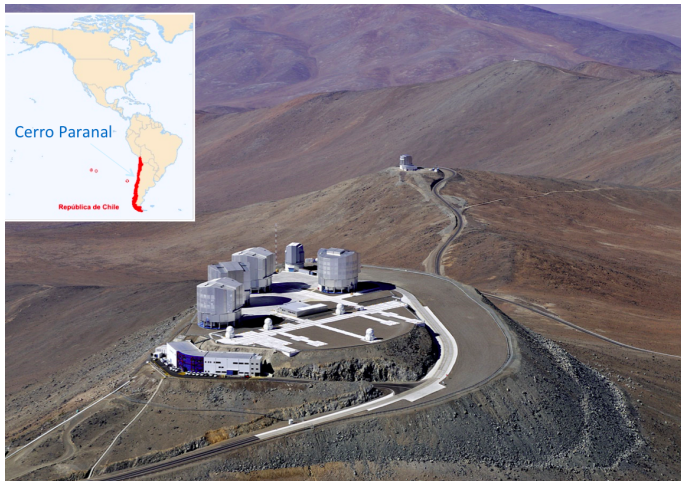


Figure: More about VLTI.

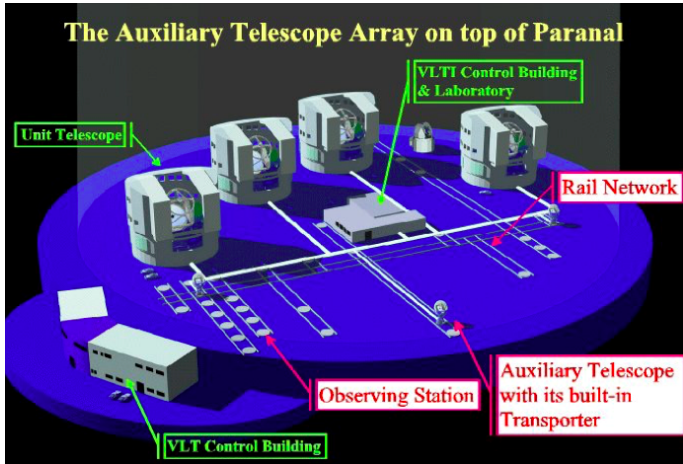
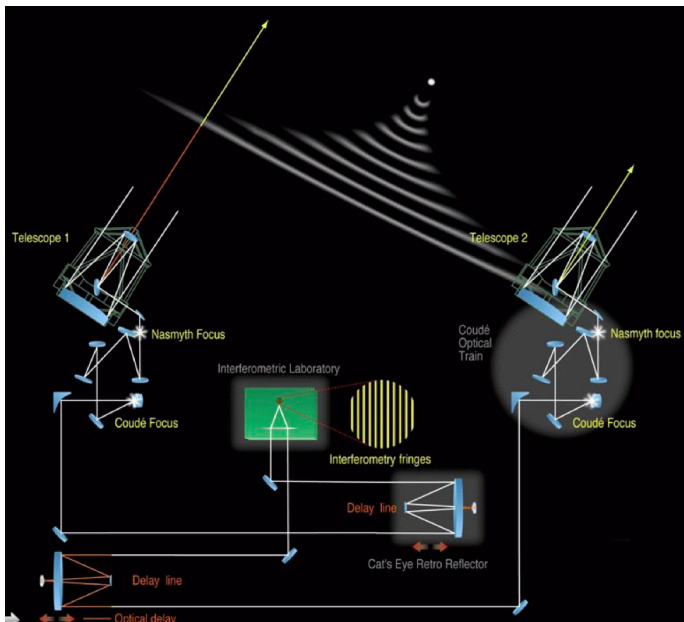


Figure: More about VLTI.

# More about VLTI



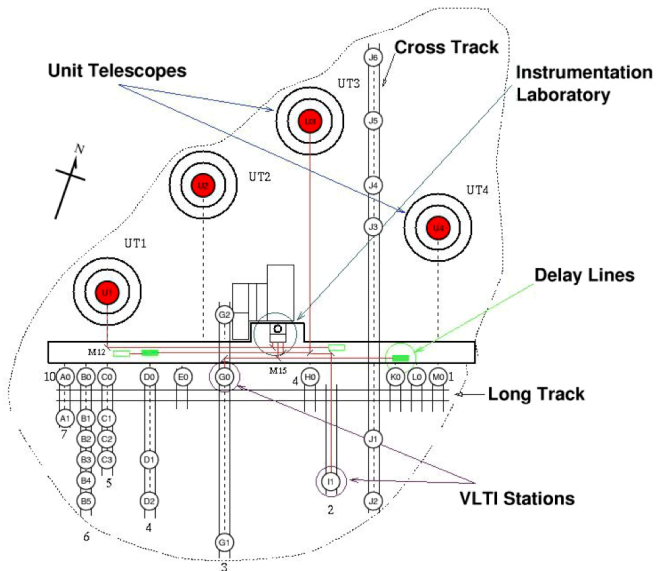


Figure: More about VLTI.

How are those locations related to the  $uv$  coverage?

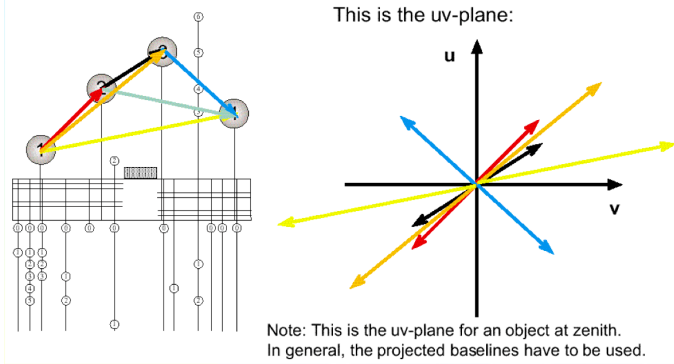
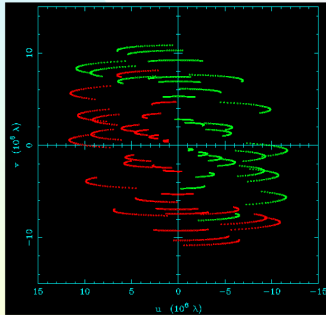


Figure: More about VLTI.

## Examples of Fourier plane coverage

Dec -15



Dec -65

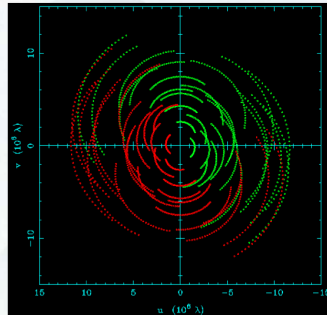


Figure: More about VLTI.

## How does the uv plane coverage impact imaging?

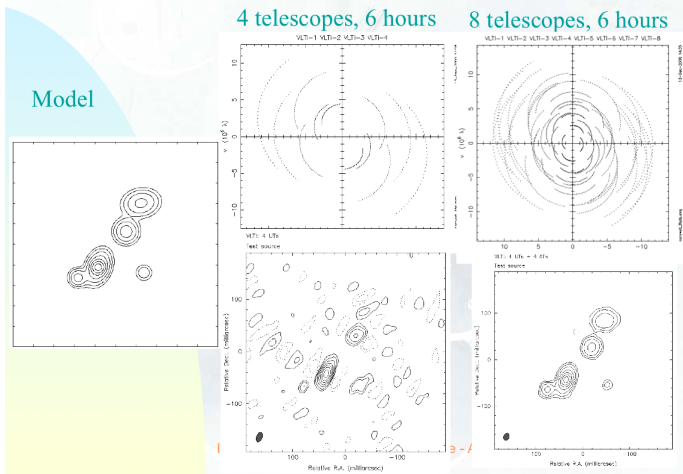


Figure: More about VLTI.

## Closure phases – what are these?

- Measure visibility phase ( $\Phi$ ) on three baselines simultaneously.

- Each is perturbed from the true phase ( $\phi$ ) in a particular manner:

$$\Phi_{12} = \phi_{12} + \varepsilon_1 - \varepsilon_2$$

$$\Phi_{23} = \phi_{23} + \varepsilon_2 - \varepsilon_3$$

$$\Phi_{31} = \phi_{31} + \varepsilon_3 - \varepsilon_1$$

- Construct the linear combination of these:

$$\Phi_{12} + \Phi_{23} + \Phi_{31} = \phi_{12} + \phi_{23} + \phi_{31}$$

The error terms are antenna dependent – they vanish in the sum.

The source information is baseline dependent – it remains.

We still have to figure out how to use it!

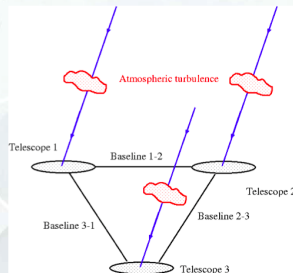


Figure: About closure phases with an optical interferometer.

Closure phase is a peculiar linear combination of the true Fourier phases:

– In fact, it is the argument of the product of the visibilities on the baselines in question, hence the name triple product (or bispectrum):

$$V_{12}V_{23}V_{31} = |V_{12}| |V_{23}| |V_{31}| \exp(i2\pi[\Phi_{12} + \Phi_{23} + \Phi_{31}]) = T_{123}$$

– So we have to use the closure phases as additional constraints  
In some nonlinear iterative inversion scheme.

Figure: About closure phases with an optical interferometer.

## Three important theorems and some applications: the Wiener-Khinchin theorem

The last theorem that we shall demonstrate during these lectures is the famous Wiener-Khinchin theorem according to which the Fourier transform of the response function of an optical system, i.e. the Fourier transform of the Point Spread Function in our case, is given by the auto-correlation of the distribution of the complex amplitude in the pupil plane. In mathematical terms, the theorem can be expressed as follows

$$FT_{-}(|a(p, q)|^2)(x, y) = FT_{-}(i(p, q))(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A^*(x'+x, y'+y)A(x', y')dx' dy'. \quad (151)$$

## Three important theorems and some applications: the Wiener-Khinchin theorem

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When establishing the expression (142), we wrote that the quantity  $FT_{-}(|a(p, q)|^2)$  appearing in its denominator could be retrieved from the observation of a point-like star. The Wiener-Khinchin theorem states that it can also be retrieved from the auto-correlation of the distribution of the complex amplitude  $A(x, y)$  in the pupil plane.

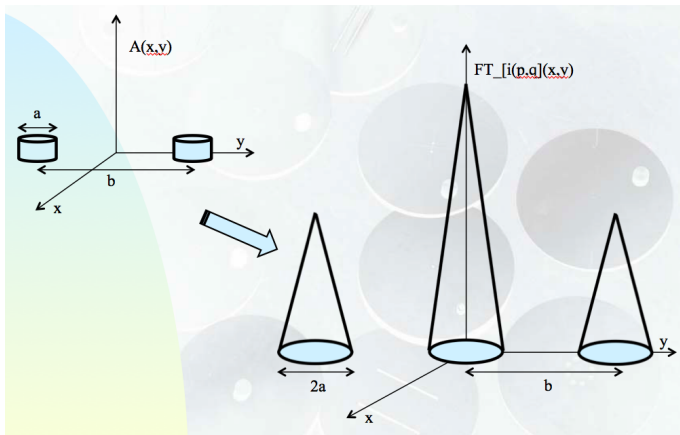
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# Three important theorems and some applications: the Wiener-Khinchin theorem



**Figure:** Diagram representing the autocorrelation function versus the space frequency, for a two telescope interferometer, each having a diameter  $a$ , separated by the baseline  $b$ .

# Three important theorems and some applications: the Wiener-Khinchin theorem

A simple demonstration of the Wiener-Khinchin theorem (151) is given below. We may successively establish that

$$FT_{-}(i(p, q))(x, y) = FT_{-}(|a(p, q)|^2)(x, y) = FT_{-}(a^{*}(p, q)a(p, q))(x, y), \quad (152)$$

$$FT_{-}[i(p, q)](x, y) = \int \int \exp[-2i\pi(px + qy)] \int \int A^{*}(x'', y'') \exp[2i\pi(px'' + qy'')] dx'' dy'' \int \int A(x', y') \exp[-2i\pi(px' + qy')] dx' dy' dpdq \quad (153)$$

# Three important theorems and some applications: the Wiener-Khinchin theorem

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$$FT_{-}[i(p, q)](x, y) = \int \int \exp[(2i\pi\{p[x'' - (x' + x)] + q[y'' - (y' + y)]\})] \int \int A^{*}(x'', y'') dx'' dy'' \int \int A(x', y') dx' dy' dp dq \quad (154)$$

and taking into account the definition (69) of the Dirac distribution

# Three important theorems and some applications: the Wiener-Khinchin theorem

$$FT_{-}[i(p, q)](x, y) = \int \int \int \int \delta[x'' - (x' + x)] \delta[y'' - (y' + y)] A^*(x'', y'') A(x', y') dx' dy' dx'' dy'' \quad (155)$$

## Three important theorems and some applications: the Wiener-Khinchin theorem

$$FT_{-}[i(p, q)](x, y) = \int \int \int \int \delta[x'' - (x' + x)] \delta[y'' - (y' + y)] A^*(x'', y'') A(x', y') dx' dy' dx'' dy'' \quad (155)$$

We finally find that

$$FT_{-}[i(p, q)](x, y) = \int \int A^*(x' + x, y' + y) A(x', y') dx' dy' dx'' dy'' = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} A^*(x' + x, y' + y) A(x', y') dx' dy' dx'' dy'', \quad (156)$$

i.e. the quoted result, namely that the Fourier transform of the impulse response function of an optical system can be represented by the autocorrelation of the distribution of the complex amplitude  $A(x, y)$  in the pupil plane.

These lecture notes are based upon lectures on the same subject delivered by the author in French at the Liège University during the past ten years (see [1]). To get deeper into the field of interferometry, we highly recommend the following books: [2], [3], [4].

1. J. Surdej, see <http://www.aeos.ulg.ac.be/teaching.php> (2018)
2. P. Léna, D. Rouan, F., Lebrun, F. Mignard, D., Pelat, D., *Observational Astrophysics* (Astronomy and Astrophysics Library, 2012)
3. A. Glindemann, *Principles of Stellar Interferometry* (Astronomy and Astrophysics Library, 2011)
4. D. Buscher, *Practical Optical Interferometry: Imaging at Visible and Infrared Wavelengths* (Cambridge University Press, 2015)

# N telescope interferometer

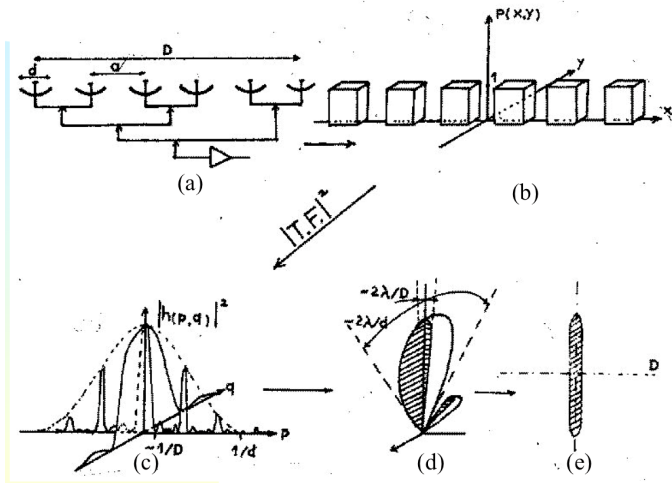


Figure: N telescope interferometer.

- **Project:** Calculate the impulse response function for the 3 cases described here below:

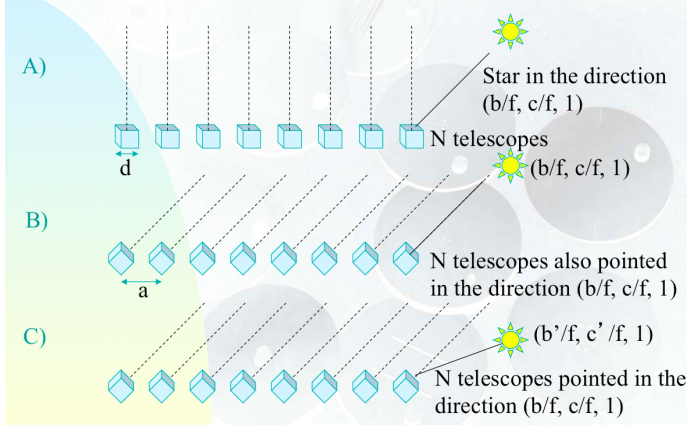


Figure: About closure phases with an optical interferometer.

- **Project:** Calculate the impulse response function for the 3 cases described here below:

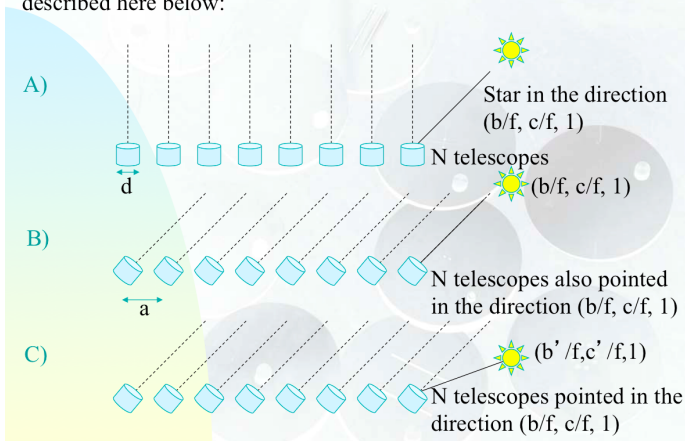


Figure: About closure phases with an optical interferometer.