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Singular networks and ultrasensitive terminal behaviors

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***Abstract*— Negative conductance elements are key in shaping the input-output behavior at the terminals of a network through localized positive feedback amplification. The balance of positive and negative differential conductances creates singularities at which rich, intrinsically nonlinear, and ultrasensitive terminal behaviors emerge. Motivated by neuromorphic engineering applications, in this note we extend a recently introduced nonlinear network graphical modeling framework to include negative conductance elements. We use this extended framework to define the class of singular networks and to characterize their ultrasensitive input-output behaviors at given terminals. Our results are grounded in the Lyapunov-Schmidt reduction method, which is shown to fully characterize the singularities and bifurcations of the input-output behavior at the network terminals, including when the underlying input-output relation is not explicitly computable through other reduction methods.**

I. Introduction

Characterizing and synthesizing the behavior of resistive networks at given terminals is a fundamental problem of network theory [1] with disparate applications [2], [3], [4]. When resistors are linear and their conductance is positive, graph theoretical methods provide a constructive solution to this problem in terms of the network Kron reduction [5], [6]. The generalization of the Kron reduction method to networks of nonlinear resistors with positive differential conductance was recently developed [7].

The goal of this paper is to further generalize, and provide solutions to, the problem of characterizing and synthesizing the terminal behavior of a network by allowing some of the network edges to have negative differential conductance. This generalization is motivated by the broader objective of developing a physical network theory of excitable behaviors [8], [9] suitable for neuromorphic engineering applications [10], [11], [12]. Whereas the design of neuromorphic excitable signal-processing units is a mature field [13], [14], [15], the design of excitable networks is still a largely unexplored one. The possibility of rigorously synthesizing large physical networks with tunable excitable behaviors, emerging from the balance of positive and negative conductance elements, could provide new strategies for the design of embodied neuromorphic systems. To this aim, in this paper we introduce the class of *signed* nonlinear networks as well as tools from singularity and bifurcation theory to analyze their terminal behaviors.

The paper contributions and its structure are the following. Section II presents preliminaries on signed Laplacian matrices. In this context, we prove a result on the relation between the (co)rank of a signed Laplacian matrix and that of its Kron reduction. We further introduce the class of singular Laplacian matrices and prove a constructive result to synthesize singular Laplacians with a single negative edge. Section III leverages the nonlinear network framework of [7] to introduce and define signed nonlinear networks. Section IV defines the class of singular networks and highlights how network parameters shape their singular behavior at given terminals. Section V shows how Lyapunov-Schmidt reduction and singularity theory methods fully characterize the terminal behavior of a singular network. Leveraging these results, we introduce in Section VI the notions of selectively ultrasensitive terminal behaviors and network bifurcations, and study the bifurcations of networks with a single negative conductance. Section VII applies the developed framework to the signed nonlinear network example used throughout the paper. Conclusions and perspectives are provided in Section VIII.

II. Preliminaries on signed Laplacians

A. Signed graphs, Laplacians, and their Kron reduction

A signed graph[1] with n nodes and m edges is defined by an incidence matrix $D \in \mathbb{R}^{n\times m}$ and a (signed) weight matrix $W = \mathrm{diag}(w_1, \ldots, w_m) \in \mathbb{R}^{m\times m}$, where w_j can now be both positive or negative. The network *signed Laplacian* L is defined analogously to the unsigned case by

$$L = DWD^\top. \tag{1}$$

As in the unsigned case, we have $L\mathbf{1}_n = 0$ and $L^\dagger$ is defined as the Moore-Penrose pseudoinverse of L [16], [17]. Again in analogy with the unsigned case, oberve that L is independent of the chosen graph orientation.

Consider the partition of the network nodes into n_B boundary (terminal) nodes and n_C central (internal) nodes, with $n_C + n_B = n$. Then, modulo an index reordering,

$$L = \begin{bmatrix} L_{BB} & L_{BC} \\ L_{CB} & L_{CC} \end{bmatrix}, \tag{2}$$

where $L_{BB} \in \mathbb{R}^{n_B\times n_B}$ and $L_{CC} \in \mathbb{R}^{n_C\times n_C}$. Suppose that L_{CC} is nonsingular. The $n_B \times n_B$ Schur complement

$$\widehat{L} = L/L_{CC} = L_{BB} - L_{BC}\, L_{CC}^{-1} L_{CB} \tag{3}$$

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[1]We refer the reader to [6] for precise definitions of the various graph-theoretical notions introduced throughout this section.

of the Laplacian matrix L is again a signed Laplacian matrix [5] and we can factorize $\widehat{L} = \widehat{D}\widehat{W}\widehat{D}^\top$ to define a reduced signed graph with n_B nodes, $\hat{m}$ edges, incidence matrix $\widehat{D}$ and weight matrix $\widehat{W}$. In analogy with the unsigned case, the Laplacian L can be interpreted as the linear relation $Lz = u$ between the vector of node potentials $z = (z_B, z_C) \in \mathbb{R}^n$ and the vector of nodal currents $u = (u_B, u_C) \in \mathbb{R}^n$ of a signed resistive network in which the conductance of edge j between nodes i and k is $w_j \in \mathbb{R}$. The Schur complement L/L_{CC} is the network *signed Kron reduction* associated with the constraint $u_C = 0$. Recall that, for a matrix M, $\operatorname{corank} M = \dim \ker M$.

Lemma 1. Assume that L_{CC} is nonsingular. The Laplacian L and its Kron reduction $\widehat{L} = L/L_{CC}$ satisfy

$$\operatorname{corank} L = \operatorname{corank} \widehat{L}. \tag{4}$$

Proof. Note that $v \in \ker L$ if and only if

$$L_{BB}v_B + L_{BC}v_C = 0, \tag{5a}$$

$$L_{CB}v_B + L_{CC}v_C = 0. \tag{5b}$$

Hence, if $v = (v_B, v_C) \in \ker L$, then $\widehat{L}v_B = [L_{BB} - L_{BC}L_{CC}^{-1}L_{CB}]v_B = L_{BB}v_B + L_{BC}v_C = 0$ and $v_B \in \ker \widehat{L}$, where the first equality follows from (5b) and the second equality follows from (5a). Conversely, let $v_B \in \ker \widehat{L}$, i.e., $[L_{BB} - L_{BC}L_{CC}^{-1}L_{CB}]v_B = 0$. Then $L_{BB}v_B + L_{BC}v_C = 0$, where $v_C = L_{CC}^{-1}L_{CB}v_B$ is uniquely determined by (5b). Hence, $v = (v_B, v_C) \in \ker L$.
Thus, for any $v = (v_B, v_C) \in \ker L$, it follows that $v_B \in \ker \widehat{L}$, and, conversely, for any $v_B \in \ker \widehat{L}$ there exists unique v_C such that $v = (v_B, v_C) \in \ker L$. Hence $\dim \ker L = \dim \ker \widehat{L}$. □

B. Effective resistance and singular signed Laplacians

The effective resistance r_{ij} between two nodes i and j is determined by the voltage $z_i - z_j$ between the two nodes when a unitary current is injected at node i and extracted at node j. If L is the network Laplacian, then r_{ij} is obtained by solving $Lv = e_{ij}$ for v and, if such a solution exists, by letting $r_{ij} = z_i - z_j = v_i - v_j$, where $e_{ij} \in \mathbb{R}^n$ is the vector with 1 at position i, -1 at position j, and 0 elsewhere. If $\operatorname{corank} L = 1$, then

$$v = L^\dagger e_{ij}$$

is uniquely determined on $\mathbf{1}_n^\perp$ and $z_i - z_j$ is well-defined. This leads directly to the classical characterization [18] of effective resistance for unsigned graphs

$$r_{ij} = L^\dagger_{ii} + L^\dagger_{jj} - 2L^\dagger_{ij},$$

and its (identical) generalization to signed graphs

$$\rho_{ij} = L^\dagger_{ii} + L^\dagger_{jj} - 2L^\dagger_{ij},$$

where now L is a corank-1 signed Laplacian.

Consider a signed graph with n nodes, m edges, incidence matrix D, weight matrix $W = \operatorname{diag}(w_1, \ldots, w_m)$, Laplacian $L = DWD^\top$, and a single negative edge l_- between node i and node j with weight $w_{l_-} = -k < 0$. In particular, $w_l > 0$ for all $l \neq l_-$. By removing the edge l_- we obtain an unsigned graph with n nodes, $m-1$ edges, incidence matrix D_+, weight matrix W_+, and Laplacian $L_+ = D_+W_+D_+^\top$. The following result extends and provides a more elementary proof of [19, Theorem III.3].

Theorem 1. Suppose that the graph defined by L_+ is connected and let $r_{ij} = (L_+^\dagger)_{ii} + (L_+^\dagger)_{jj} - 2(L_+^\dagger)_{ij} > 0$.

1) If $k < r_{ij}^{-1}$, then L is positive semidefinite and $\operatorname{corank} L = 1$. The restriction of L to $\mathbf{1}_n^\perp$ is positive definite.

2) If $k = r_{ij}^{-1}$, then L is positive semidefinite, $\operatorname{corank} L = 2$, and $Lv = 0$, where $v = L_+^\dagger e_{ij} \in \mathbf{1}_n^\perp$.

3) If $k > r_{ik}^{-1}$, then L is indefinite and $\operatorname{corank} L = 1$.

Proof. Observe that $L = L_+ - ke_{ij}e_{ij}^\top$. Let $v = L_+^\dagger e_{ij} \in \mathbf{1}_n^\perp$, so that $r_{ij} = v_i - v_j$. Consider $w \in \mathbf{1}_n^\perp$ such that $Lw = 0$. Then, $L_+w = ke_{ij}e_{ij}^\top w = k(w_i - w_j)e_{ij}$. If $w_i - w_j = 0$, then $L_+w = 0$, which contradicts the hypothesis that the graph defined by L_+ is connected [20]. If $w_i - w_j \neq 0$, then $w = k(w_i - w_j)L_+^\dagger e_{ij} = k(w_i - w_j)v$. Thus, either[2] $\ker L = \mathbb{R}\{\mathbf{1}_n\}$ and $\operatorname{corank} L = 1$, or $\ker L = \mathbb{R}\{\mathbf{1}_n, v\}$ and $\operatorname{corank} L = 2$. Observe that $Lv = L_+v - kr_{ij}^{-1}e_{ij} = e_{ij}(1 - kr_{ij}^{-1})$. Thus, if $k = r_{ij}^{-1}$, then $Lv = 0$ and $\operatorname{corank} L = 2$, while if $k < r_{ij}^{-1}$ or $k > r_{ij}^{-1}$, then $Lv \neq 0$ and $\operatorname{corank} L = 1$. Furthermore, by [21, Theorem 3.2], decreasing (increasing) the negative conductance weight can only increase (decrease) the eigenvalues of L, from which the result follows. □

We call a Laplacian as in point 2) of Theorem 1 a *singular signed Laplacian*. Singular signed Laplacians are associated with connected graphs, but their second smallest (Fiedler) eigenvalue (equal to the graph algebraic connectivity in the unsigned case) vanishes.

III. Signed nonlinear networks and their terminal behavior

We build upon and generalize the nonlinear network framework introduced in [7] to nonlinear networks with positive and negative differential conductance edges and characterize their input-output behavior at given terminals. The network in Figure 1 is used throughout the paper to illustrate the introduced concepts and results. Its negative conductance I-V characteristic is defined as

$$I = S(ky, \beta) = \frac{\tanh(ky - \beta) + \tanh(\beta)}{1 - \tanh(\beta)^2}, \tag{6}$$

where k is the negative conductance gain and $\beta \in \mathbb{R}$ is a parameter.

A. Signed nonlinear networks

A *signed nonlinear network* is defined as a connected signed graph with n vertices, or "nodes", m edges, an $n \times m$ incidence matrix D, and vectors $z \in \mathbb{R}^n$ and $y = D^\top z \in \mathbb{R}^m$. The components of the vectors z and y are

[2]Given n-dimensional vectors $v_1, \ldots, v_p$, we use $\mathbb{R}\{v_1, \ldots, v_p\} = \{\alpha_1 v_1 + \cdots + \alpha_p v_p \mid \alpha_1, \ldots, \alpha_p \in \mathbb{R}\}$ to denote their real "span".

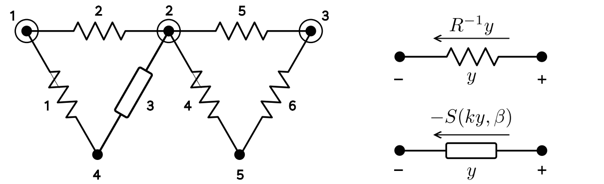

Fig. 1: A signed nonlinear network with five nodes, three terminals (encircled vertices), five resistive edges with unitary resistance ($R = 1$), and one negative conductance edge (boxed symbol).

the potentials of the network nodes and the voltages across the network edges, respectively. The relation $y = D^\top z$ expresses the network Kirchhoff's voltage law [6]. Each edge $j \in \{1, \dots, m\}$ is endowed with a conductance $g_j(y_j) = \frac{\partial G_j}{\partial y_j}(y_j)$ determining the current flowing through the edge. The conductance potential function $G_j(y_j)$ is assumed to be at least twice differentiable. The edge differential conductance is $\frac{\partial g_j}{\partial y_j} = \frac{\partial^2 G_j}{\partial y_j^2}$. A network edge j is positive if its differential conductance is positive, that is, if $g_j(y_j)$ is monotone increasing and $G_j(y_j)$ is convex. A network edge j is negative if its differential conductance is negative, that is, if $g_j(y_j)$ is monotone decreasing and $-G_j(y_j)$ is convex. Let $\mathcal{E}_+$ be the set of positive edges and $\mathcal{E}_-$ the set of negative edges. The network potential $G(y) = \sum_{j=1}^m G_j(y_j)$ is the difference of two convex functions

$$G(y) = \sum_{j\in\mathcal{E}_+} G_j(y_j) + \sum_{j\in\mathcal{E}_-} G_j(y_j).$$

For the network of Figure 1, $\mathcal{E}_+ = \{1, 2, 4, 5, 6\}$, $\mathcal{E}_- = \{3\}$, $G_j(y_j) = \frac{1}{2}y_j^2$ for $j \in \mathcal{E}_+$, and

$$G_3(y_3) = \int_0^{y_3} S(ky,\beta)dy = -\tfrac{\ln(\cosh(ky_3-\beta)+k\tanh(\beta)y_3)}{k(1-\tanh(\beta)^2)}. \tag{7}$$

Let $K : \mathbb{R}^n \to \mathbb{R}$ be defined as $K(z) = G(D^\top z)$. The gradient of K with respect to the node potentials

$$\frac{\partial K}{\partial z}(z) = D\frac{\partial G}{\partial y}(D^\top z) = u,$$

is the vector u of nodal currents at each vertex, which express the network Kirchhoff's current law [6]. The Hessian of K defines the (z-dependent) $n \times n$ *signed network Laplacian*

$$L(z) = \frac{\partial^2 K}{\partial z^2}(z) = D\frac{\partial^2 G}{\partial y^2}(D^\top z)D^\top, \tag{8}$$

where $\frac{\partial^2 G}{\partial y^2}(D^\top z)$ is the $m \times m$ diagonal matrix of edge differential conductances. Note that $K(z)$, $\frac{\partial K}{\partial z}(z)$, and $L(z)$ are invariant with respect to the shift $z \mapsto z + c\mathbf{1}_n$. For the network in Figure 1, letting $* = (k(z_2 - z_4), \beta)$, [3]

$$L(z) = \left[\begin{array}{ccc:cc} 2 & -1 & 0 & -1 & 0 \\ -1 & 3-S_y(*) & -1 & S_y(*)-1 & \\ 0 & -1 & 2 & 0 & -1 \\ \hdashline -1 & S_y(*) & 0 & 1-S_y(*) & 0 \\ 0 & -1 & -1 & 0 & 2 \end{array}\right]. \tag{9}$$

[3]For differentiable functions $f : \mathbb{R}^q \to \mathbb{R}$ and $i, j \in \{1, \dots, q\}$, we let $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Similarly for higher-order derivatives.

Remark 1 (Network Laplacian and local sensitivity analysis)**.** Let (u, z) satisfying $\frac{\partial K}{\partial z}(z) = u$ be a solution of the network behavior. The network Laplacian $L(z)$ characterizes the differential (linearized) sensitivity at (u, z) of nodal currents to infinitesimal changes δz in node potentials. The Moore-Penrose inverse $L^\dagger(z)$ of $L(z)$ characterizes the local sensitivity at (u, z) of node potentials to infinitesimal changes δu in nodal currents.

Consider now the partition of the n network nodes into n_B boundary nodes, i.e., the network terminals, and n_C central nodes, i.e., the network internal nodes, $n_C + n_B = n$. Let $z_B \in \mathbb{R}^{n_B}$ be the vector of boundary nodes' potentials and $z_C \in \mathbb{R}^{n_C}$ the vector of central nodes' potentials. Modulo reordering, $z = (z_B, z_C)$. A nonzero nodal current vector

$$\frac{\partial K}{\partial z_B}(z) = u_B \tag{10}$$

can flow at boundary nodes. Conversely, at central nodes

$$\frac{\partial K}{\partial z_C}(z) = 0\,. \tag{11}$$

At each z, the partition $z = (z_B, z_C)$ induces the block partition (2) of the network Laplacian with

$$L_{BB}(z) = \frac{\partial^2 K}{\partial z_B^2}(z), \quad L_{BC}(z) = \frac{\partial^2 K}{\partial z_B \partial z_C}(z),$$
$$L_{CB}(z) = \frac{\partial^2 K}{\partial z_C \partial z_B}(z), \quad L_{CC}(z) = \frac{\partial^2 K}{\partial z_C^2}(z).$$

For the network in Figure 1, this partition is indicated by dashed lines in (9). Assume that

$$\det L_{CC}(z) = \det \frac{\partial^2 K}{\partial z_C^2}(z) \neq 0. \tag{12}$$

Suppose that $\frac{\partial K}{\partial z_C}(z_B^*, z_C^*) = 0$ for some z_B^*, z_C^*. Then, by the implicit function theorem, (11) can be solved for z_C locally around (z_B^*, z_C^*). That is, there exists a locally defined twice differentiable function $\bar{z}_C(z_B)$, with $\bar{z}_C(z_B^*) = z_C^*$, such that $\frac{\partial K}{\partial z_C}(z_B, \bar{z}_C(z_B)) \equiv 0$.

Remark 2 (Nonsingular central Laplacian in the unsigned and signed case)**.** If the graph is connected and the network is unsigned, then, as shown in [6], $\frac{\partial^2 K}{\partial z_C^2}(z) > 0$ and (12) is always satisfied. The same is not true in general in the signed case. For the connected network in Figure 1, $\frac{\partial^2 K}{\partial z_C^2}(z)$ becomes singular whenever $S_y(k(z_2 - z_4), \beta) = 1$ despite the underlying graph being connected.

The function $\widehat{K}(z_B) = K(z_B, \bar{z}_C(z_B))$ defines a reduced model of the network terminal behavior. Indeed,

$$\begin{aligned} \frac{\partial \widehat{K}}{\partial z_B}(z_B) &= \frac{\partial K}{\partial z_B}(z_B, \bar{z}_C(z_B)) + \frac{\partial K}{\partial z_C}(z_B, \bar{z}_C(z_B))\frac{\partial \bar{z}_C}{\partial z_B}(z_B) \\ &= \frac{\partial K}{\partial z_B}(z_B, \bar{z}_C(z_B)) = u_B, \end{aligned} \tag{13}$$

where we used that $\frac{\partial K}{\partial z_C}(z_B, \bar{z}_C(z_B)) = 0$ from (11). Furthermore, for fixed z_B, one can verify as in [7, Equation (11)] that $\frac{\partial^2 \widehat{K}}{\partial z_B^2}(z_B)$ is the Schur complement of the Laplacian

$L(z_B, z_C(z_B))$ with respect to $\frac{\partial^2 K}{\partial z_C^2}(z_B, z_C)$, and hence a Laplacian itself [6, Theorem 3.1]. The graph associated with the reduced Laplacian $\widehat{L}(z_B) = \frac{\partial^2 \widehat{K}}{\partial z_B^2}(z_B)$ defines, under certain conditions, a global (independent of z_B) Kron-reduced network with only boundary nodes [7].

Remark 3 (Non-computability of signed nonlinear Kron-reduced models.)**.** The function $\bar{z}_C(z_B)$ is in general hard to compute. Despite this, in the unsigned case it is still possible to characterize globally the network terminal behavior [7]. In the signed case, the same global approach fails in general, motivating the use of singularity theory.

IV. Singular networks and externally singular behaviors

In this section we introduce and start to develop singularity theory ideas for nonlinear signed networks.

A. Singular networks

The network input-output behavior (III-A) can be rewritten in the relational form

$$F(z, u, \alpha) = \frac{\partial K}{\partial z}(z) - u = 0, \tag{14}$$

where α is a vector of network parameters, like linear conductance values and nonlinear conductance gains. For the network of Figure 1, $\alpha = (k, \beta)$. Given the partition $z = (z_B, z_C)$ and $u = (u_B, u_C)$, $u_C = 0$, relation (14) reads

$$0 = F(z, u, \alpha) = \begin{pmatrix} F_B(z_B, z_C, u_B, \alpha) \\ F_C(z_B, z_C, 0, \alpha) \end{pmatrix}, \tag{15}$$

where

$$F_i(z_B, z_C, u_i, \alpha) = \frac{\partial K}{\partial z_i}(z_B, z_C, \alpha) - u_i, \quad i = B, C.$$

Let $L(z) = \frac{\partial F}{\partial z}(z, u, \alpha) = \frac{\partial^2 K}{\partial z^2}$ be the Jacobian of (15), i.e., the network Laplacian (8). Assume that, for $\alpha = \alpha^*$, $L(z)$ loses rank at (z^*, u^*), that is,

$$F(z^*, u^*, \alpha^*) = 0, \tag{16a}$$
$$\operatorname{corank} L(z^*) = 2. \tag{16b}$$

Hence, there exists $0 \neq v \in \mathbf{1}_n^\perp$, such that $\ker L(z^*) = \mathbb{R}\{\mathbf{1}_n, v\}$. A network satisfying (16a) is called *singular* and (z^*, u^*) is called a *network singularity*.

Remark 4 (Connection with Thom's Catastrophe Theory)**.** Network singularities define degenerate critical points z^* of $K(z)$. The potential function K is locally flat along v at z^*. Hence, even infinitesimal perturbations can induce changes in the convexity of K at z^*, letting critical points appear or disappear. The importance, for modeling, of (perturbations of) degenerate critical points was first highlighted in Thom's Catastrophe Theory [22].

In the following, we will assume, without loss of generality, that $(z^*, u^*, \alpha^*) = 0$ and write L instead of $L(0)$.

B. Externally singular behaviors

Assume that

$$\det \frac{\partial F_C}{\partial z_C}(0, 0, 0) \neq 0. \tag{17}$$

As in Section III-A, the implicit function theorem then implies that locally around $z_B = 0$ there exists a differentiable function $\bar{z}_C(z_B, \alpha)$ such that $\bar{z}_C(0, 0) = 0$ and $F_C(z_B, \bar{z}_C(z_B, \alpha), 0, \alpha) \equiv 0$. Plugging this solution in the first component of (15) gives the reduced terminal behavior

$$0 = F_B(z_B, \bar{z}_C(z_B, \alpha), u_B, \alpha) = \widehat{F}(z_B, u_B, \alpha). \tag{18}$$

By Lemma 1, it follows that the reduced Laplacian $\widehat{L} = \frac{\partial \widehat{F}}{\partial z_B}(0, u_B, \alpha) = L_{BB} - L_{BC} L_{CC}^{-1} L_{CB}$ of (18) also satisfies $\operatorname{corank} \widehat{L} = 2$ at singularity. In particular, there exists $\hat{v} \in \hat{\mathbf{1}}^\perp$ such that $\widehat{L}\hat{v} = 0$ and $\ker \widehat{L} = \mathbb{R}\{\hat{\mathbf{1}}, \hat{v}\}$, where we denoted $\hat{\mathbf{1}} = \mathbf{1}_{n_B}$. Hence, the input-output behavior at the network terminals (13) is also singular at a network singularity. A singular network satisfying (17) is said to exhibit an *externally singular behavior*.

Remark 5 (Network topology and singularities)**.** By determining the spectral properties of L and its Kron reduction $\widehat{L}$, the network topology directly determines the singular behavior at its terminals.

V. Lyapunov-Schmidt reduction of singular physical networks

We compute the Lyapunov-Schmidt (LS) reduction [23, Section I.3] of a singular network's input-output behavior using either the full network equations (15) or the reduced network equations (18). We show that the two reductions coincide (up to a suitable notion of equivalence). Hence, Kron and LS reductions must be mutually consistent, confirming the physical interpretation of LS reduction and resulting bifurcation diagrams thoroughly stressed in [23].

A. Projecting out global shift invariance

The null eigenvalue/eigenvector pair $L\mathbf{1} = 0$ reflects the network invariance to uniform shifts of currents and potentials, not a network singularity. We therefore restrict (15), as well as the potential and current vectors z and u to the subspace $\mathbf{1}^\perp$ through the projection $\Pi = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$. Because of shift invariance, it is immediate that the projected equations have exactly the same form as (15). Similarly, $\Pi L \Pi = L$. We therefore let

$$z, u \in \mathbf{1}^\perp, \quad F(\cdot, \cdot, \alpha) : \mathbf{1}^\perp \times \mathbf{1}^\perp \to \mathbf{1}^\perp, \tag{19a}$$
$$F_i((\cdot, \cdot), \cdot, \alpha) : \mathbf{1}^\perp \times \mathbf{1}^\perp \to \mathbf{1}^\perp, \ i = B, C, \tag{19b}$$
$$L : \mathbf{1}^\perp \to \mathbf{1}^\perp, \quad \operatorname{corank} L = 1, \quad \ker L = \mathbb{R}\{v\}. \tag{19c}$$

Similarly, since the reduced equation (18) is also shift invariant [7], we let

$$z_B, u_B \in \mathbf{1}^\perp, \quad \widehat{F}(\cdot, \cdot, \alpha) : \hat{\mathbf{1}}^\perp \times \hat{\mathbf{1}}^\perp \to \hat{\mathbf{1}}^\perp, \tag{20a}$$
$$\widehat{L} : \hat{\mathbf{1}}^\perp \to \hat{\mathbf{1}}^\perp, \quad \operatorname{corank} \widehat{L} = 1, \quad \ker \widehat{L} = \mathbb{R}\{\hat{v}\}. \tag{20b}$$

B. Critical eigenvectors of full and reduced networks

We now characterize the boundary component v_B of the full critical eigenvector v, obtained from (15), and link it to the reduced critical eigenvector $\hat{v}$, obtained from (18).

Lemma 2. Let $v=(v_B,v_C)\in\mathbf{1}^\perp$. If, $0\neq v\in\ker L$ then $v_B\notin\mathbb{R}\{\hat{\mathbf{1}}\}$.

Proof. Assume $v_B=a\hat{\mathbf{1}}$. Then v can be written as

$$\mathbf{1}^\perp\ni v=a\mathbf{1}+\begin{pmatrix}0\\ \tilde{v}_C\end{pmatrix}=\begin{pmatrix}0\\ \tilde{v}_C\end{pmatrix},$$

for some $\tilde{v}_C$, and therefore $a=0$. Since

$$Lv=L\begin{bmatrix}0\\ \tilde{v}_C\end{bmatrix}=0,$$

using (5b) we obtain $\tilde{v}_C=-L_{CC}^{-1}L_{CB}0=0$ and therefore $v=0$, which contradicts $v\neq 0$. $\square$

It follows from Lemma 2 that v_B takes the form

$$v_B=a\hat{\mathbf{1}}+\tilde{v}_B,\quad 0\neq\tilde{v}_B\in\hat{\mathbf{1}}^\perp. \tag{21}$$

Furthermore, invoking Lemma 1, if $0\neq\hat{v}\in\ker\widehat{L}$, then

$$\tilde{v}_B=a_B\hat{v},\quad 0\neq a_B\in\mathbb{R}. \tag{22}$$

C. Lyapunov-Schmidt reduction of the full network equations

The Lyapunov-Schmidt reduction aims to solve (14), restricted to $\mathbf{1}_n^\perp$ as in (19a), on a complement of $\ker L$ where L is full rank and the implicit function theorem applies. The resulting implicit solution defines a reduced equation on $\ker L$ amenable to singularity theory methods. Let[4]

$$E^c=\begin{bmatrix}v_Bv_B^\top & v_Bv_C^\top\\ v_Cv_B^\top & v_Cv_C^\top\end{bmatrix}\quad\text{and}\quad E=I-E^c.$$

Observe that $E^c:\mathbf{1}_n^\perp\to\mathbf{1}_n^\perp$ is the projection onto $\ker L$, $E:\mathbf{1}_n^\perp\to\mathbf{1}_n^\perp$ is the projection onto $\operatorname{range}L$, and $\mathbf{1}_n^\perp=\operatorname{range}E\oplus\operatorname{range}E^c=\operatorname{range}L\oplus\ker L$. Consider the $n-2$, because of the restriction (19a), independent equations

$$0=EF(z,u,\alpha) \tag{23}$$

obtained by projecting (15) on $\operatorname{range}L$. Because the linearization $EL=L$ of $EF(z,u,\alpha)$ is invertible on $\operatorname{range}L$, we can split $z=xv+w$, where $xv\in\ker L$, $x\in\mathbb{R}$, and $w=Ez\in\operatorname{range}L$, to solve (23) for w as a function of x, $u=(u_B,0)$, and α. Let the resulting implicit function be $W(x,u_B,\alpha)$. In particular, $EF(xv+W(x,u_B,\alpha),u,\alpha)\equiv 0$.

On network solutions, the implicit function $W(x,u_B,\alpha)$ must be consistent with the network internal behavior $z_C=\bar{z}_C(z_B)$. In particular, if $F(z,u,\alpha)=0$ then

$$\begin{aligned}\begin{bmatrix}z_B\\ z_C\end{bmatrix}&=\begin{bmatrix}xv_B+W_B(x,u_B,\alpha)\\ xv_C+W_C(x,u_B,\alpha)\end{bmatrix}\\ &=\begin{bmatrix}xv_B+W_B(x,u_B,\alpha)\\ \bar{z}_C(xv_B+W_B(x,u_B,\alpha))\end{bmatrix},\end{aligned}$$

[4]With a little abuse of notation, in what follows all vectors are assumed to be normalized.

leading to the *LS and Kron reduction consistency condition*

$$xv_C+W_C(x,u_B,\alpha)=\bar{z}_C(xv_B+W_B(x,u_B,\alpha)). \tag{25}$$

Inserting (25) into (23) and using (18) leads to

$$E\begin{bmatrix}\widehat{F}(*)\\ 0\end{bmatrix}=\begin{bmatrix}\widehat{F}(*)-v_Bv_B^\top\widehat{F}(*)\\ -v_Cv_C^\top\widehat{F}(*)\end{bmatrix}, \tag{26}$$

where $*=(xv_B+W_B(x,u_B,\alpha),u_B,\alpha)$. Solving the first component of (26) for $\widehat{F}(*)$ and replacing in the second gives $-v_Cv_C^\top\widehat{F}(*)=\|v_B\|v_Cv_C^\top\widehat{F}(*)$, that is, $v_Cv_C^\top\widehat{F}(*)=0$. Hence, using (22) and (20a), the first component of (26) shows that $W_B(x,u,\alpha)$ satisfies the defining condition

$$\begin{aligned}0&=(I-v_Bv_B^\top)\widehat{F}(xv_B+W_B(x,u_B,\alpha),u_B,\alpha)\\ &=(I-a_B^2\hat{v}\hat{v}^\top)\widehat{F}(a_Bx\hat{v}+W_B(x,u_B,\alpha),u_B,\alpha).\end{aligned} \tag{27}$$

Plugging the implicit solution $W_B(x,u_B,\alpha)$ into the complementary rank 1 system of equations $E^cF(z,u,\alpha)=0$ on $\ker L$ leads to

$$\begin{aligned}E^cF(z,u,\alpha)&=\begin{pmatrix}v_Bv_B^\top\widehat{F}(xv_B+W_B(x,u_B,\alpha),u_B,\alpha)\\ v_Cv_C^\top\widehat{F}(xv_B+W_B(x,u_B,\alpha),u_B,\alpha)\end{pmatrix}.\\ &=\begin{pmatrix}v_Bv_B^\top\widehat{F}(xv_B+W_B(x,u_B,\alpha),u_B,\alpha)\\ 0\end{pmatrix}.\end{aligned}$$

Using again (22) and (20a), the LS reduction of (15) is

$$\begin{aligned}f(x,u_B,\alpha)&=v^\top E^cF(z,u,\alpha)\\ &=v_B^\top\widehat{F}\big(xv_B+W_B(x,u_B,\alpha),u_B,\alpha\big)\\ &=a_B\hat{v}^\top\widehat{F}\big(a_Bx\hat{v}+W_B(x,u_B,\alpha),u_B,\alpha\big),\end{aligned} \tag{28}$$

where $W_B(x,u_B,\alpha)$ satisfies (27).

D. Lyapunov-Schmidt reduction of the reduced equations

Let $\widehat{E}^c=\hat{v}\hat{v}^\top:\hat{\mathbf{1}}^\perp\to\hat{\mathbf{1}}^\perp$ and $\widehat{E}=I-\widehat{E}^c:\hat{\mathbf{1}}^\perp\to\hat{\mathbf{1}}^\perp$ be the projection onto $\ker\widehat{L}$ and $\operatorname{range}\widehat{L}$, respectively. Consider the splitting $z_B=x\hat{v}+\hat{w}$, where $x\hat{v}\in\ker\widehat{L}$, $x\in\mathbb{R}$, and $\hat{w}\in\operatorname{range}\widehat{L}$. From the implicit function theorem, the n_B-2 independent equations $\widehat{E}\widehat{F}(z_B,u_B,\alpha)=0$ are uniquely solved by $\hat{w}=\widehat{W}(x,u_B,\alpha)$, where $\widehat{W}(x,u_B,\alpha)$ satisfies

$$(I-\hat{v}\hat{v}^\top)\widehat{F}\big(x\hat{v}+\widehat{W}(x,u_B,\alpha),u_B,\alpha\big)=0. \tag{29}$$

The rank 1 system of equations $\widehat{E}^c\widehat{F}(z_B,u_B,\alpha)=0$ on $\ker\widehat{L}$, then leads to the Lyapunov-Schmidt reduction

$$\begin{aligned}\hat{f}(x,u_B,\alpha)&=\hat{v}^\top\widehat{E}^c\widehat{F}(z_B,u_B,\alpha)\\ &=\hat{v}^\top\widehat{F}\big(x\hat{v}+\widehat{W}(x,u_B,\alpha),u_B,\alpha\big),\end{aligned} \tag{30}$$

of (18), where $\widehat{W}(x,u_B,\alpha)$ satisfies (29).

E. Consistency of the Lyapunov-Schmidt reduction of full and reduced network behaviors

Theorem 2. Let a_B be defined in (22). Then

$$\hat{f}(x, u_B, \alpha) = a_B^{-1} f(x, u_B, \alpha).$$

Proof. Using the change of variables $\tilde{x} = a_B x$, (27) and (28) can be rewritten as

$$0 = (I - a_B^2 \hat{v}\hat{v}^\top)\widehat{F}(\tilde{x}\hat{v} + W_B(\tilde{x}/a_B, u_B, \alpha), u_B, \alpha)$$

and

$$f(\tilde{x}, u_B, \alpha) = a_B \hat{v}^\top \widehat{F}(\tilde{x}\hat{v} + W_B(\tilde{x}/a_B, u_B, \alpha), u_B, \alpha).$$

Comparing with (29) and (30), it follows immediately that

$$\widehat{W}(x, u_B, \alpha) = W_1(a_B^{-1}x, u_B, \alpha),$$
$$\hat{f}(x, u_B, \alpha) = a_B^{-1} f(x, u_B, \alpha). \qquad \square$$

Theorem 2 implies that $f(x, u_B, \alpha)$ and $\hat{f}(x, u_B, \alpha)$ define *strongly equivalent* bifurcation problems [23]. The nonlinear terminal behavior of a network close to a singularity can equivalently be studied through the Lyapunov-Schmidt reduction of its full (15) or reduced (18) models.

Remark 6 (Application of Theorem 2)**.** The function $\bar{z}_C(z_B)$ and therefore the reduced network equations $\widehat{F}(z_B, u_B, \alpha) = 0$ are hard to compute in general. Theorem 2 ensures that the network boundary behavior, its singularities, and bifurcations can nonetheless be characterized by the Lyapunov-Schmidt reduction of the full network equations, at least close to an externally singular behavior.

VI. Ultrasensitive input-output behaviors and network bifurcations

A. Ultrasensitive input-output behaviors

Unpacking the definition of $\widehat{F}$ in (28) gives

$$\begin{aligned} f(x, u_B, \alpha) &= \hat{v}^\top \frac{\partial K_B}{\partial z_B}(z_B, \bar{z}_C(z_B)) + \hat{v}^\top u_B \\ &= \tilde{f}(x, \alpha) + \hat{v}^\top u_B, \end{aligned}$$

where $\tilde{f}(x, \alpha) = \hat{v}^\top \frac{\partial K_B}{\partial z_B}(z_B, \bar{z}_C(z_B))$. As shown in [23, Section I.3], the network singularity condition implies that

$$\frac{\partial f}{\partial x}(0,0,0) = \frac{\partial \tilde{f}}{\partial x}(0,0) = 0.$$

The linearization of the equation $0 = \tilde{f}(x, \alpha) + \hat{v}^\top u_B$ at $(x, u_B, \alpha) = (0,0,0)$ then reads

$$0 = 0 \cdot x + \hat{v}^\top u_B,$$

which reveals that, close to a singularity, the linearized input-to-state sensitivity blows up. The behavior is *ultrasensitive*. Furthermore, this ultrasensitive behavior is *selective*, in the sense that it is not triggered by any input vectors u_B orthogonal to the critical eigenvector $\hat{v}$.

B. Network bifurcations

Consider the smooth parameterization

$$(u, \alpha) = (u(\lambda), \alpha(\lambda)), \quad \lambda \in \mathbb{R}, \tag{31}$$

of the network input and parameters, with $(u(0), \alpha(0)) = (0, 0)$. As the *bifurcation parameter* λ crosses zero, the network crosses its singularity and undergoes a *bifurcation*. With a little abuse of notation, let The set

$$\mathcal{B} = \{(x, \lambda) \mid f(x, u_B(\lambda), \alpha(\lambda))) = 0\} \tag{32}$$

is called the *bifurcation diagram* of f. Since the solutions of $f(x, u, \alpha) = 0$ are in one-to-one correspondence with the solutions of (15) through the smooth mapping $z = xv + W(x, u_B, \alpha)$, $\mathcal{B}$ provides a qualitative description of how network solutions change close to a network singularity as inputs and parameters change through λ. In particular, the *recognition problem* formulation [23, Chapter II] allows one to characterize $\mathcal{B}$ via the Taylor expansion of f obtained through implicit differentiation methods [23, Section I.3.e].

C. Network bifurcations with a single negative conductance

Consider a signed nonlinear network $\mathcal{N}$ with n nodes, partitioned as above into n_B boundary nodes and n_C central nodes, m edges, and with a single negative conductance $I_{l_-} = -k\frac{\partial G_{l_-}}{\partial y_{l_-}}(y_{l_-})$, with $\frac{\partial G_{l_-}}{\partial y_{l_-}}(0) = 0$, on edge l_- between nodes i and j. $G_{l_-}(y_{l_-})$ is a convex function and k is the tunable gain of the negative conductance. Without loss of generality, let $\frac{\partial^2 G_{l_-}}{\partial y_{l_-}^2}(0) = 1$. Assume, for simplicity and conciseness of exposition, that all the positive conductances are linear and fixed, i.e., $G_l(y_l) = \frac{1}{2}R_l^{-1}y_l^2$ for all $l \in \{1, \ldots, m\} \setminus l_-$. Hence, the network $\mathcal{N}_+$ obtained by removing the negative conductance edge is a linear resistive network. Let $L(z)$ be the network Laplacian of $\mathcal{N}$ and L_+ be the Laplacian of the linear resistive network obtained by removing l_- from $\mathcal{N}$. Also let $g_-(y_{l_-}) = \frac{\partial G_{l_-}}{\partial y_{l_-}}(y_{l_-})$.

Theorem 3. Assume that the graph associated with L_+ is connected and let $r_{ij} = v_i - v_j$ with $L_+ v = e_{ij}$. Then, $\mathcal{N}$ is singular at $(z, u, k) = (0, 0, r_{ij}^{-1})$ and $\ker L(0) = \mathbb{R}\{\mathbf{1}_n, v\}$. Furthermore, if $\frac{\partial^2 K}{\partial z_C^2}$ is invertible, then the Lyapunov-Schmidt reduction (28), or equivalently (30), satisfies, at $(x, u_B, k) = (0, 0, r_{ij}^{-1})$,

$$f = f_x = f_k = 0, \quad f_{xx} = -g''(0)r_{ij}^2,$$
$$f_{kx} = -r_{ij}^2 + \mathcal{O}(g''(0)), \quad f_{xxx} = -g_-'''(0)r_{ij}^3 + \mathcal{O}(g''(0)).$$

Proof. The fact that $\mathcal{N}$ is singular at $(z, u, k) = (0, 0, r_{ij}^{-1})$ with $\ker L(0) = \mathbb{R}\{\mathbf{1}_n, v\}$ follows directly by Theorem 1. To compute the required term of the Taylor expansion of the LS reduction at the singularity, we use the first equality of (28). Since by hypothesis $g_-(0) = 0$ and all other conductances are linear, $f(0, 0, r_{ij}^{-1}) = 0$. By Theorem 1, $f_x(0, 0, r_{ij}^{-1}) = v^\top L(0) v = 0$. Using [23, Eq. (3.23.d) of Ch.I, Sec.3], $f_k(0, 0, r_{ij}^{-1}) = v^\top \widehat{D}^\top I_-$, where $I_- \in \mathbb{R}^m$ is the vector with all zero entries except for $kg_-(0)$ at the l_--th entry. Using again the hypothesis $g_-(0) = 0$, it follows that $f_k(0, 0, r_{ij}^{-1}) = 0$.

To compute the required higher-order terms, first observe that $L(z) = L_+ + kL_-(z)$, where

$$(L_-(z))_{lp} = \begin{cases} 0 & \text{if either } l, p \notin \{i, j\} \\ -1 & \text{if } l, p \in \{i, j\} \text{ and } l = p \\ 1 & \text{if } l, p \in \{i, j\} \text{ and } l \neq p \end{cases}. \tag{33}$$

Hence, if either $l, p, q \notin \{i, j\}$ then $\frac{\partial (L_-)_{lp}}{\partial z_q}(z) = 0$, whereas if $l, p, q \in \{i, j\}$ then

$$\frac{\partial (L_-)_{lp}}{\partial z_q} = \begin{cases} -g_-''(z) & \text{if } (q = i, l = p) \text{ or } (q = j, l \neq p) \\ g_-''(z) & \text{if } (q = j, l = p) \text{ or } (q = i, l \neq p) \end{cases}. \tag{34}$$

Similarly, if either $l, p, q_1, q_2 \notin \{i, j\}$ then $\frac{\partial^2 (L_-)_{lp}}{\partial z_{q_1} \partial z_{q_2}}(z) = 0$, whereas if $l, p, q_1, q_2 \in \{i, j\}$ then

$$\frac{\partial^2 (L_-)_{lp}}{\partial z_{q_1} \partial z_{q_2}} = \begin{cases} -g_-'''(z) & \text{if } (q_1 = q_2, l = p) \text{ or } (q_1 \neq q_2, l \neq p) \\ g_-'''(z) & \text{if } (q_1 = q_2, l \neq p) \text{ or } (q_1 \neq q_2, l = p)) \end{cases}. \tag{35}$$

Using [23, Eq. (3.23.b) of Ch.I, Sec.3] and (34),

$$f_{xx}(0, 0, r_{ij}^{-1}) = r_{ij}^{-1} \sum_{l,p,q=1}^{n} v_l \frac{\partial (L_-)_{lp}}{\partial z_q} v_p v_q = -g_-''(0) r_{ij}^2.$$

Using [23, Eq. (3.23.b) of Ch.I, Sec.3] and (33), if $g''(0) = 0$,

$$f_{kx}(0, 0, r_{ij}^{-1}) = v^\top L_-(z) v = -r_{ij}^2.$$

Using [23, Eq. (3.23.b) of Ch.I, Sec.3] and (35), if $g''(0) = 0$,

$$f_{xxx}(0,0,r_{ij}^{-1}) = r_{ij}^{-1} \sum_{\substack{l,p \\ q_1,q_2=1}}^{n} v_l \frac{\partial^2 (L_-)_{lp}}{\partial z_{q_1} \partial z_{q_2}} v_p v_{q_1} v_{q_2} = -g_-'''(0) r_{ij}^3.$$

The $g''(0) \neq 0$ case then follows by smoothness of the LS reduction formulae. □

Corollary 1. Let $\lambda = k$ be the bifurcation parameter of the nonlinear behavior of the signed nonlinear network $\mathcal{N}$. If $g_-''(0) \neq 0$, then the input-output terminal behavior of $\mathcal{N}$ undergoes a *transcritical* bifurcation at $(x, u_B, k) = (0, 0, r_{ij}^{-1})$. If $g_-''(0) = 0$ and $g_-'''(0) \neq 0$, then the bifurcation at $(x, u_B, k) = (0, 0, r_{ij}^{-1})$ is a *pitchfork*. Furthermore, the pitchfork is supercritical if $g_-'''(0) < 0$ and subcritical if $g_-'''(0) > 0$. Finally, if $g_-''(0) = g_-'''(0) = 0$, then $\mathcal{N}$ undergoes at $(x, u_B, k) = (0, 0, r_{ij}^{-1})$ a bifurcation of codimension > 2.

Proof. The case $g_-''(0) \neq 0$ follows from Theorem 3 and [23, Proposition II.9.3]. The two cases $g_-''(0) = 0$, $g_-'''(0) \neq 0$, follow from Theorem 3 and [23, Proposition II.9.2]. The case $g_-''(0) = g_-'''(0) = 0$ follows from the Classification Theorem [23, Theorem IV.2.1]. □

Theorem 3 and Corollary 1 reveal how the nonlinear properties of the negative conductance $g_-(y_{l_-})$, and in particular its derivatives computed at $y_{l_-} = 0$, determine bifurcation behavior of the network at its singularity.

Remark 7 (Singular networks with multiple negative conductances)**.** Theorems 1 and 3 can be extended, at least locally, to the presence of m_- negative conductance edges by studying the codimension-1 algebraic variety $\mathcal{M}_-$ defined by the singularity condition $\operatorname{corank} L = 2$ in the m_--dimensional space of negative conductance gains.

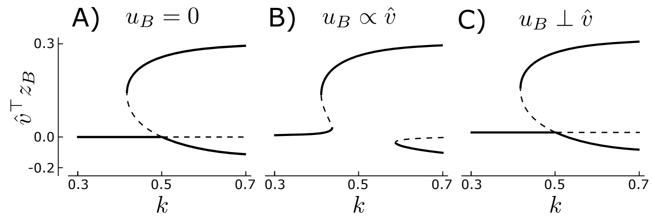

Fig. 2: Bifurcation diagrams with respect to the negative conductance gain k of the terminal potentials z_B (projected on the critical vector $\hat{v}$) for network in Figure 1 with $\beta = 0.5$ and for different terminal current vectors. Branches of stable (unstable) equilibria are indicated by solid (dashed) lines.

VII. Designing selectively ultrasensitive physical networks

Consider the network in Figure 1. A direct application of Theorem 3 and Corollary 1 reveals that, for all $\beta \in \mathbb{R}$, the network is singular at $(z, u_B, k) = (0, 0, 1/2)$. Furthermore, if $\beta \neq 0$, then $g_-''(0) = \frac{\partial^3 G_3}{\partial y_3^3}(0) \neq 0$, where $G_3(y_3)$ is defined in (7), we expect the bifurcation at $(z, u_B, k) = (0, 0, 1/2)$ to be transcritical. A final prediction is that, close to singularity, the network is insensitive to inputs $u_B \in \hat{v}^\perp$ and ultrasensitive to inputs u_B such that $\hat{v}^\top u_B \neq 0$.

Figure 2 confirms and illustrates these predictions. The bifurcation diagrams were generated in Julia BifurcationKit [24] by assuming a small parasitic capacitance at each node. Figure 2A shows that when $u_B = 0$ the network exhibits a transcritical bifurcation at $k = 0.5$, as predicted. At this bifurcation the ground network state $z = (z_B, \bar{z}_C(z_B)) = (0, 0)$ becomes unstable and, even in the absence of any boundary current, new stable network potential configurations emerge along the critical eigenvector $\hat{v}$. These new stable configurations represent "decision states" that the network can express in response to inputs. As shown in Figure 2B, even a small u_B component along $\hat{v}$ can switch the network to a new decision state. Conversely, if $u_b^\top \hat{v} = 0$ (Figure 2, bottom), the network is insensitive to input currents. This *selectively ultrasensitive* input-output behavior is at the core of excitability and excitable decision making in neurons, neural circuits, and beyond [8], [9], [25], [26]. The tools developed here pave the way to the design of selectively ultrasensitive behaviors in general physical networks.

Remark 8 (Large-scale network applications)**.** Analyzing and designing singularities amount to studying a system of polynomial (through Taylor theorem) equalities and inequalities as those appearing in the statements of Theorem 3 and Corollary 1. For large-scale networks, if an analytical approach is intractable, this algebraic geometric problem can be solved with standard nonlinear optimization methods.

VIII. Conclusions and perspectives

We introduced a class of models and a set of tools to pave the way towards the modeling, analysis, and synthesis of physical networks with intrinsically nonlinear, and in particular excitable, input-output behaviors at their terminals. The Lyapunov-Schmidt reduction, and its consistency with the Kron-reduced network terminal behavior, are the key elements supporting the resulting nonlinear network theory. Although general, the theory developed here might be particularly relevant for neuromorphic engineering applications by paving the way toward the constructive design of excitable, physical, and therefore naturally embodied, networks.

The results and ideas we presented admit various theoretical extensions. In the same sprit as [27], the generalization to networks of static conductances and grounded capacitors, as used in Figure 2, is natural and will be presented in an extended version of this work. A more involved extension, still under development, is to consider networks with dynamic edges. In this direction, the dissipative systems [28], [29], [30] and the Hankel operator theoretical approaches [31] seem particularly promising ones. The extension to multi-physical systems could naturally be tackled in the port-Hamiltonian framework [32], [33], [34].

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