

Formalizing Neuromorphic Control Systems A General Proposal and A Rhythmic Case Study

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Abstract—Neuromorphic control is receiving growing attention due to the advantages it brings over more classical control approaches, including: sparse and on-demand sensing, information transmission, and actuation; energy-efficient designs and realizations in neuromorphic hardware; event-based signal processing and control signal computation. In this note, we suggest a possible path toward formalizing neuromorphic control systems. We apply the proposed framework to a rhythmic control case study and apply mature control theoretical approaches like describing function analysis, fast-slow analysis, discrete and hybrid systems, and robust optimization.

I. INTRODUCTION

The original aim of neuromorphic engineering (NE) was to design large-scale analog electronic circuits that mimic the architecture and functions of biological nervous systems [1]. Across the years, NE diversified in a variety of different approaches and goals, like the design of low-power sensors [2] and neural processors [3], memristive materials [4], and many blends and interactions with classical (digital) machine learning [5]. But all recent and less recent neuromorphic approaches share a fundamental neuro-inspired property: the “spiking” or “event-based” nature of communication and computation. The asynchronous, on-demand nature of event-based signal processing and computing ensures speed, parsimonious power consumption, and scalability by overcoming the Von Neumann digital dichotomy between memory and logic units: like neuron spikes in a brain, each event is both a memory and a unit of computation.

Although control theory was close to neuromorphic engineering at its origin [6], the two fields have remained relatively apart until recently, with the appearance of both control-theoretical approaches to neuromorphic systems design [7], [8] and the use of neuromorphic event-based sensors in fast feedback control loops [9]. The potential of control theoretical approaches to and from neuromorphic engineering is more and more clear [10], [11], [12], [13], [14], [15], [16], [17]. Because events are *discrete* entities evolving in *continuous* time, neuromorphic control systems have the potential to inherit the best of both analog/continuous-time and digital/discrete-time control approaches [18], [10]. Furthermore, many existing control strategies (like event-based [19] and event-triggered control [20], maximum hands-off control [21], sliding-mode control [22], and hybrid sys-

tems theory [23]) share at least some similarities with the neuromorphic approach and provide sets of mature tools for the analysis and the design of neuromorphic control systems.

However, what a neuromorphic control system is, remains unclear. As opposed to continuous-time, digital, event-based, or hybrid systems, we still miss a general definition of “neuromorphic control systems”. The lack of such a definition can be detrimental to neuromorphic control theory by encouraging piecemeal and isolated, instead of more holistic and collective, rigorous theoretical developments and approaches to this new discipline and class of systems.

Motivated by these observations, we build upon recent efforts aiming at defining and characterizing neuromorphic control systems [24], [14] to propose a modeling architecture for neuromorphic control systems. We rigorously define all the elements of the proposed architecture in such a way that they respect the constraints imposed by neuromorphic event-based hardware. We restrict our attention to a class of rhythmic neuromorphic control problems and show how, in this setting, the proposed modeling framework can make the analysis of the resulting neuromorphic closed-loop systems amenable to a number of mature control-theoretical tools. Similarly to [24], [14], the proposed methods are applied and developed in detail on the neuromorphic pendulum control problem, for which we rigorously establish error bounds depending on the system and control parameters. Interestingly, simulations of the closed-loop system exhibit a discrete-time Hopf-like bifurcation. We formulate a robust optimization problem on the error and show that, remarkably, the optimal solution lies precisely at the bifurcation point. Proofs are provided in the preprint [25].

The complementary paper [26] engages the same problem from the hybrid system perspective. It develops a hybrid model of neuromorphic pendulum control and shows the existence and uniqueness of a hybrid limit cycle. Both papers provide complementary insights and results that are of independent interest.

II. NEUROMORPHIC ARCHITECTURE

We propose the feedback control architecture in Fig. 1 as a basic representative of a neuromorphic control system. The plant is assumed to be a single-input and single-output linear time-invariant system, characterized by a transfer function P in the field of rational functions with real coefficients $\mathbb{R}(s)$.

A. Neuromorphic Sensors

Neuromorphic sensors differ from analog and clocked digital sensors. The analog signal of a neuromorphic sensor

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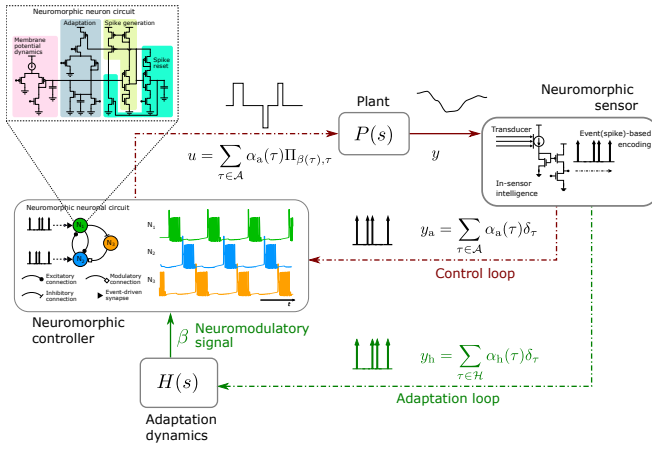


Fig. 1. Abstraction of a neuromorphic-control architecture.

transducer is processed at the sensor edge through analog computations implemented in ultra-low power subthreshold CMOS circuits. The processed sensory signal is transmitted as few-bit asynchronous events generated when sensor-specific threshold conditions are met, akin to spikes of biological sensory neurons.

We model a sensory event generated at time τ by the, possibly weighted, Dirac delta distribution δ_τ , i.e., $\int_{-\infty}^{\infty} f(t)\delta_\tau(t)dt = f(\tau)$ for all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

In the simple architecture of Fig. 1 there are two kinds of sensory events: *actuation events*, meant to trigger the activation of the neuromorphic control loop actuator, and *adaptation events*, meant to trigger an impulsive response in the adaptation block, as determined by its transfer function $H \in \mathbb{R}(s)$.

Let $\mathcal{A} \subset \mathbb{R}$ be the set of instants of time at which the actuation events are generated. At each actuation event, a Dirac delta is sent from the sensor to the controller. The event-based input from the neuromorphic sensor to the controller is

$$y_a(t) = \sum_{\tau \in \mathcal{A}} \alpha_a(\tau) \delta_\tau(t).$$

The function $\alpha_a : \mathcal{A} \rightarrow \{+1, -1\}$ is used to attach to each event a 1-bit information content that can be used to define and distinguish *positive* and *negative* actuation events.

Similarly, let $\mathcal{H} \subset \mathbb{R}$ be the set of instants of time at which the adaptation events are generated. At each adaptation event, a Dirac delta is sent from the sensor to the adaptation block. The event-based input from the adaptation sensor to the adaptation block is

$$y_h(t) = \sum_{\tau \in \mathcal{H}} \alpha_h(\tau) \delta_\tau(t),$$

where, as before, $\alpha_h : \mathcal{H} \rightarrow \{+1, -1\}$ serves to distinguish *positive* and *negative* events.

B. Neuromorphic control and actuation

Upon reception of a sensory spike at time $\tau \in \mathcal{A}$, the controller fires a burst of spikes that lasts $\beta(\tau) > 0$ units

of time and of the same polarity as the received sensory spike. This actuation burst is generated by suitable analog neural circuits implemented in subthreshold CMOS circuits. Since the inter-burst period (i.e., the dwell time between two bursts) is designed to be much shorter than the dominant time constant of P , single spikes are smoothed out by the plant low-pass filter dynamics and a burst can reasonably be modeled as a finite pulse

$$\Pi_{\beta, \tau}(t) = \begin{cases} 1 & \text{if } \tau \leq t \leq \tau + \beta \\ 0 & \text{otherwise} \end{cases}.$$

The actuation control signal then reads

$$u(t) = \sum_{\tau \in \mathcal{A}} \alpha_a(\tau) \Pi_{\beta(\tau), \tau}(t). \quad (1)$$

The choice of using bursting, instead of spiking, neurons for the actuation stage stems mainly from two reasons. Single, isolated spikes, are mostly filtered out by the low-pass nature of the actuators. Conversely, a burst of spikes received in rapid succession is integrated by the actuator dynamics, leading to reliable responses. Also, as opposed to single all-or-none spikes, bursts also possess a graded nature. Increasing (decreasing) the burst duration or the number of spikes per burst leads to stronger (weaker) actuation, which enables finely tuned control actions, in line with the neurobiological principle of neuromodulation [27].

C. Neuromorphic adaptation

Assuming that the adaptive unit starts at rest, the burst width is given by

$$\beta(t) = \sum_{\tau \in \mathcal{H}} \alpha_h(\tau) h(t - \tau), \quad (2)$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is the impulse response of the adaptation unit, i.e., $h(t) = \mathcal{L}^{-1}\{H(s)\}$ with $\mathcal{L}$ the Laplace transform. The causality of the adaptive unit implies that $h(t - \tau) = 0$ for $t \leq \tau$. Thus, the expression (2) reduces to

$$\beta(t) = \sum_{\substack{\tau \in \mathcal{H} \\ \tau < t}} \alpha_h(\tau) h(t - \tau). \quad (3)$$

To summarize, the design parameters are the transfer function H , the event generation rule for the sets $\mathcal{A}$ and $\mathcal{H}$, and the functions α_a and α_h .

D. Neuromorphic control problem formulation

The sparse and impulsive temporal nature of neuromorphic sensors and actuators calls for new ways of formalizing control objectives in the neuromorphic setting. Consider the problem of generating a stable periodic behavior with a desired amplitude A^* in a controlled plant. A common approach [28], [29] consists in constructing a continuous reference $y^*(t)$ such that $\max_{t \geq 0} |y^*(t)| = A^*$ (via tracking or regulation). The approach does not generalize when sensors emit only sparse 1-bit events and $y(t)$ is unmeasurable.

The problem can be recast by first defining the operators

$$\mathcal{E}_i : \mathcal{L}_{qie}(\mathbb{R}_{\geq 0}, \mathbb{R}) \rightarrow \mathcal{L}_{qie}(\mathbb{R}_{\geq 0}, \mathbb{R}), \quad i \in \{1, 2\},$$

where $q_1 = 2$, $q_2 = \infty$, and

$$\mathcal{E}_1(y)(t) = y(t + 2\pi/\omega) - y(t) \quad (4a)$$

$$\mathcal{E}_2(y)(t) = A^* - \max_{\tau \geq t} |y(\tau)| \quad (4b)$$

for some $\omega > 0$. The control problem then becomes imposing

$$\limsup_{t \rightarrow \infty} |\mathcal{E}_i(y)(t)| \leq \varepsilon, \quad i = 1, 2, \quad (5)$$

for some $\varepsilon > 0$. Such formulation supports faster, adaptive decisions about when and how to sense or act, and shows how sparse 1-bit events can still carry rich information when neuromorphic sensors and controllers are suitably designed, as next illustrated in the rhythmic control problem of (4).

III. RHYTHMIC NEUROMORPHIC CONTROL

One of the main challenges in analyzing the system in Fig. 1 is that impulses in y_h cause jumps in β , effectively creating a hybrid system as in [23]. We approach the hybrid nature of the problem from a singular perturbation perspective (see, e.g., [30], [31]). By making the adaptation dynamics slow enough, a sufficient timescale separation exists so that the dynamics on the fast timescale are purely continuous, while those on the slow timescale are purely discrete. The rationale is as follows:

- 1) Assume that the adaptive unit H evolves much more slowly than the closed loop dynamics of P with the neuromorphic controller (red loop in Fig. 1), allowing β to be considered as constant on the fast timescale.
- 2) Apply the describing-function method [32], [31] to establish the existence of a limit cycle of amplitude $A > 0$ on the fast timescale.
- 3) Design H so that $A \rightarrow A^*$ on the slow timescale and such that the assumption in Item 1 holds. Because of the discontinuous nature of β , the slow dynamics are easier to analyze in discrete time.

Let us elaborate on the last two steps.

A. Fast dynamics

In the fast timescale, β remains constant. Using the describing-function method [32], [31], we investigate the existence of limit cycles and estimate their amplitude and frequency as functions of β . We begin by characterizing the actuation events.

Assumption 1. We have $\mathcal{A} = \{\tau \in \mathbb{R} \mid g_a(y(\tau), \dot{y}(\tau)) = 0\}$ for some function $g_a : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$g_a(y, \dot{y}) = 0 \iff g_a(-y, -\dot{y}) = 0 \quad (6a)$$

and such that, if $y(t) = A \sin(\omega t)$, with $A, \omega > 0$, then $\mathcal{A}$ is at most countable. Also, $\alpha_a(\tau) = \hat{\alpha}_a(y(\tau), \dot{y}(\tau))$, $\tau \in \mathcal{A}$, for some function $\hat{\alpha}_a : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\hat{\alpha}_a(-y, -\dot{y}) = -\hat{\alpha}_a(y, \dot{y}). \quad (6b)$$

The conditions on g_a and $\hat{\alpha}_a$ are analog to the condition of a nonlinearity being *odd* (see [32]). They can be relaxed at the expense of additional notation and computational burden.

Proposition 1. Consider the map $y \mapsto u$ defined by (1) with β constant and $\mathcal{A}$ and α_a satisfying Assumption 1. The map has a well-defined describing function $N_\beta(A, \omega)$.

Suppose that the harmonic balance (HB) equation

$$N_\beta(A, \omega)P(j\omega) - 1 = 0 \quad (7)$$

is satisfied for some, β -dependent, frequency ω and amplitude A . Then, a limit cycle of amplitude A and frequency ω is established, provided that P has a low-pass filter characteristics with negligible output for frequencies above ω (see [32] for details). Assume such a limit cycle is stable.

B. Slow dynamics

In the slow timescale, the state trajectory is assumed to have already converged to a periodic solution whose amplitude and frequency are implicitly determined by (7). To emphasize the dependence on β , we write $A = \hat{A}(\beta)$ and $\omega = \hat{\omega}(\beta)$. The purpose of the adaptation loop is to let $\hat{A}(\beta)$ approach the desired amplitude A^* .

Let $t_k \in \mathcal{A}$ be the time instant corresponding to the k th-actuation event. Evaluating (3) at $t = t_{k+1}$ gives

$$\beta(t_{k+1}) = \sum_{\substack{\tau \in \mathcal{H} \\ \tau < t_{k+1}}} \alpha_h(\tau) h(t_{k+1} - \tau). \quad (8)$$

The difference between the actual and desired amplitude is the relevant information to be transmitted along the adaptive loop (green path in Fig. 1). Note also that the amplitude of the limit cycle can effectively be measured when $\dot{y} = 0$.

Assumption 2. We have $\mathcal{H} = \{\tau \in \mathbb{R} \mid \dot{y}(\tau) = 0\}$ and

$$\alpha_h(\tau) \in \text{Sgn}(A^* - y(\tau)), \quad \tau \in \mathcal{H}, \quad (9)$$

where Sgn is the multivalued *signum* function:

$$\text{Sgn}(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}.$$

Given an adaptation event time $\tau \in \mathcal{H}$, define ρ_τ as the preceding actuation event time, $\rho_\tau = \max\{r \in \mathcal{A} \mid r \leq \tau\}$. By substituting (9) in (8) and noting that $y(\tau) = \hat{A}(\beta(\rho_\tau))$, we obtain

$$\beta(\tau_{k+1}) \in \sum_{\substack{\tau \in \mathcal{H} \\ \tau < \tau_{k+1}}} \text{Sgn}\left(A^* - \hat{A}(\beta(\rho_\tau))\right) h(\tau_{k+1} - \tau). \quad (10)$$

We consider a first-order low-pass transfer function for the adaptive unit. This will considerably simplify (10).

Assumption 3. The transfer function of the adaptive unit is

$$H(s) = \frac{\gamma}{s + c}, \quad \gamma, c > 0. \quad (11)$$

Note that the inverse Laplace transform of (11) is

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \gamma e^{-ct} & \text{if } t > 0 \end{cases}.$$

This impulse response has a simple but important property:

$$h(t_1 + t_2) = \frac{1}{\gamma} h(t_1) h(t_2), \quad t_1, t_2 \geq 0, \quad (12)$$

indeed,

$$\gamma e^{-c(t_1+t_2)} = \gamma e^{-ct_1} e^{-ct_2}. \quad (13)$$

Proposition 2. *Under Assumptions 2 and 3, the expression (8) can be written as the recursion*

$$\beta(t_{k+1}) \in \frac{h(\Delta t_k)}{\gamma} \beta(t_k) + \sum_{\substack{\tau \in \mathcal{H} \\ t_k \leq \tau < t_{k+1}}} \text{Sgn}(A^* - \hat{A}(\beta(\rho_\tau))) h(t_{k+1} - \tau). \quad (14)$$

Corollary 1. *Suppose that, in addition to the conditions of Proposition 2, the function $\hat{A}$ is monotonically increasing and there exists a burst width $\beta^* \geq 0$ such that $\hat{A}(\beta^*) = A^*$. Then, the recursion (14) simplifies to*

$$\beta(t_{k+1}) \in \frac{h(\Delta t_k)}{\gamma} \beta(t_k) + \sum_{\substack{\tau \in \mathcal{H} \\ t_k \leq \tau < t_{k+1}}} \text{Sgn}(\beta^* - \beta(\rho_\tau)) h(t_{k+1} - \tau). \quad (15)$$

Under the assumptions of the corollary, our problem boils down to showing that, in some sense to be clarified below, $\beta \rightarrow \beta^*$. The only remaining degrees of freedom to achieve this are the constants c and γ , and the functions $\hat{a}_a$ and g_a , whose choice depends in general on the plant P .

IV. CASE STUDY: OSCILLATIONS OF DESIRED AMPLITUDE ON A PENDULUM

As in [14], [24], we now focus on the controlled pendulum

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 \sin(y(t)) = \lambda u(t), \quad (16)$$

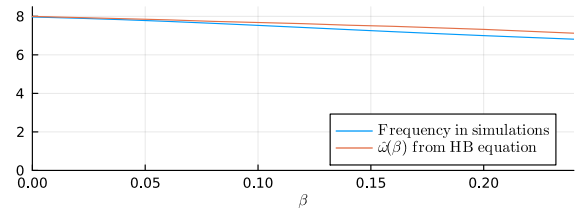
where $\xi \in [0, 1]$ is the damping ratio, $\omega_n > 0$ the undamped natural frequency, and $\lambda > 0$ the input gain. The transfer function of the linearization of (16) is $P(s) = \lambda/(s^2 + 2\xi\omega_n s + \omega_n^2)$.

A. Fast dynamics, self-sustained oscillations

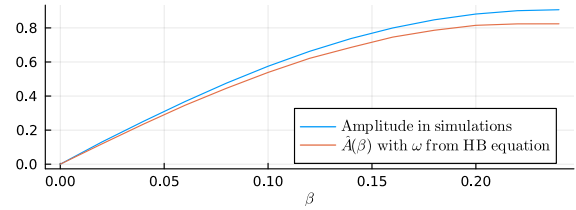
A simple choice for the functions g_a and $\hat{a}_a$, consistent with Assumption 1, is $g_a(y, \dot{y}) = y$ and $\hat{a}_a(y, \dot{y}) \in \text{Sgn}(\dot{y})$. We compute the controller describing function $N_\beta(A, \omega)$ in the standard way [32]. Assume that $y(t) = A \sin(\omega t)$, $t \geq 0$. During one period, the actuation events are $\mathcal{A} \cap [0, 2\pi/\omega) = \{0, \pi/\omega\}$. Hence, for $\beta < \pi/\omega$, the control is $u(t) = \Pi_{\beta, 0}(t) - \Pi_{\beta, \pi/\omega}(t)$, $t \in [0, 2\pi/\omega)$. The coefficients of the first harmonic are $a_1 = \frac{2}{\pi} \sin(\omega\beta)$ and $b_1 = \frac{2}{\pi} (1 - \cos(\omega\beta))$. It follows that

$$|N_\beta(A, \omega)| = \frac{\sqrt{a_1^2 + b_1^2}}{A} = \frac{4}{A\pi} \sin\left(\frac{\omega\beta}{2}\right), \quad (17a)$$

$$\angle N_\beta(A, \omega) = \arctan \frac{a_1}{b_1} = \frac{\pi - \omega\beta}{2}. \quad (17b)$$



(a) Frequency



(b) Amplitude

Fig. 2. Frequency and amplitude for $\beta \in [0, 0.2]$ obtained using the HB equation (orange line) and by simulation (blue line). The pendulum parameters are $\lambda = 15$, $\xi = 0.1$, and $\omega_n = 8$.

Proposition 3. *For every $\beta > 0$, there exists a unique amplitude A and frequency ω satisfying the HB equation (7). Writing $\omega = \hat{\omega}$ and $A = \hat{A}(\beta)$, the amplitude is given by*

$$A = \frac{4}{\pi} |P(j\hat{\omega}(\beta))| \sin\left(\frac{\hat{\omega}(\beta)\beta}{2}\right). \quad (18)$$

Fig. 2 compares simulated amplitude and frequency with the HB predictions

$$|N_\beta(A, \omega)| = \frac{1}{|P(j\omega)|}, \quad \angle N_\beta(A, \omega) = -\angle P(j\omega). \quad (19)$$

Intuitively, oscillations period and amplitude increases with pulse width, while frequency decreases.

B. Slow dynamics, discrete-time behavior

We can see from Fig. 2 that $\hat{A}$ is monotonically increasing, so Corollary 1 holds and (15) reduces to

$$\beta(t_{k+1}) \in \frac{h(\Delta t_k)}{\gamma} \beta(t_k) + \text{Sgn}(\beta^* - \beta(\rho_\tau)) h(t_{k+1} - \tau) \quad (20)$$

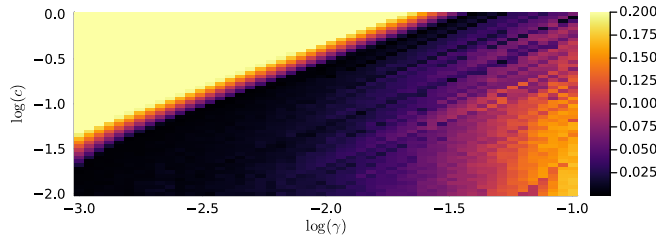
with τ the unique time in $\mathcal{H}$ such that $t_k \leq \tau < t_{k+1}$. Moreover, the time difference between two consecutive actuation events is half a period. In contrast, the time interval between an actuation and the following adaptation event is one-quarter of the period. We will approximate the period by $2\pi/\omega^*$ with $\omega^* = \hat{\omega}(\beta^*)$, that is, $\Delta t_k = \frac{\pi}{\omega^*}$, so that $h(\Delta t_k) = \gamma e^{-c\frac{\pi}{\omega^*}}$ and $h(t_k - \tau) = \gamma e^{-c\frac{\pi}{2\omega^*}}$. Finally, note that $\rho_\tau = t_k$ for $t_k \leq \tau < t_{k+1}$, so that

$$\beta_{k+1} \in e^{-c\frac{\pi}{\omega^*}} \beta_k - \gamma \text{Sgn}(\beta_k - \beta^*) e^{-c\frac{\pi}{2\omega^*}}, \quad (21)$$

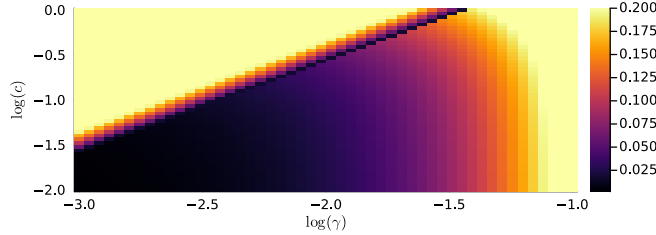
where, for notational simplicity, we let $\beta_k = \beta(t_k)$.

Our task now is to study the dynamics of the error $\tilde{\beta}_k = \beta_k - \beta^*$. It follows from (21) that

$$\tilde{\beta}_{k+1} \in -g_0 + g_1 \tilde{\beta}_k - g_2 \text{Sgn}(\tilde{\beta}_k) \quad (22)$$



(a) Full dynamics. Error (25), computed by simulation of the nonlinear system (16) in feedback loop with the control (1) and the adaptive unit (11).



(b) Slow dynamics. Error (26), computed using Propositions 5 and 6.

Fig. 3. Amplitude error. Comparison between the full and slow dynamics. Plant parameters $\lambda = 15$, $\xi = 0.1$, and $\omega_n = 8$, and adaptive parameters $\gamma \in [0.01, 1]$ and $c \in [0.01, 1]$ (in logarithmic scale).

with

$$g_0 = (1 - e^{-c \frac{\pi}{\omega^*}}) \beta^* > 0, \quad g_1 = e^{-c \frac{\pi}{\omega^*}} \in (0, 1), \quad \text{and} \quad g_2 = \gamma e^{-c \frac{\pi}{2\omega^*}} > 0. \quad (23)$$

Let us first compute the fixed points of (22).

Proposition 4. *The difference inclusion (22) has a unique fixed point*

$$\tilde{\beta}_{\text{fix}} = \min \left\{ 0, \frac{g_2 - g_0}{1 - g_1} \right\} \leq 0. \quad (24)$$

There are two qualitatively different asymptotic behaviors of (22). The first one is characterized by the following.

Proposition 5. *Let $g_0 \geq g_2 > 0$ and $g_1 \in (0, 1)$. The fixed point $\tilde{\beta}_{\text{fix}} = -(g_0 - g_2)/(1 - g_1)$ of the difference inclusion (22) is globally asymptotically stable.*

The other asymptotic behavior is characterized next.

Proposition 6. *Let $g_2 > g_0 > 0$ and $g_1 \in (0, 1)$. The solutions of the difference inclusion (22) satisfy the ultimate bound*

$$\limsup_{k \rightarrow \infty} |\tilde{\beta}_k| \leq \frac{g_0 + g_2}{1 + g_1}.$$

To illustrate the accuracy of the slow model, we simulate the full nonlinear system (16) with the control (1) and the adaptive unit (11). Using the simulated data, we compute the ultimate amplitude error

$$\limsup_{t \rightarrow \infty} |A(t) - A^*| \quad (25)$$

and compare it to the value predicted by (18),

$$\limsup_{k \rightarrow \infty} |\hat{A}(\beta_k) - A^*|. \quad (26)$$

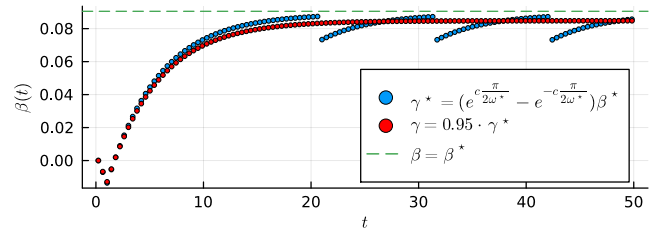


Fig. 4. Adaptation dynamics obtained via full model simulation with $A^* = 0.5$, $\beta^* = 0.0915$, and values of γ close to its bifurcation value $\gamma^* = 0.0075$.

The ultimate bounds on β_k in (26) are obtained using Propositions 5 and 6. Figure 3 shows a comparison of (25) and (26) for several values of c and γ . The slow model reproduces the full model with reasonable accuracy, provided γ is not too large.

We can see from Propositions 5 and 6 that (22) undergoes a bifurcation at $g_2 = g_0$. By the definition of g_0 and g_2 , we can see that the bifurcation occurs approximately at

$$\gamma^* = (e^{c \frac{\pi}{2\omega^*}} - e^{-c \frac{\pi}{2\omega^*}}) \beta^*.$$

The simulations show a discrete-time bifurcation resembling that of a continuous-time Hopf bifurcation: a transition occurs from a stable equilibrium to an unstable equilibrium around which oscillations develop.

C. Parameter Optimization

Our final objective is to choose c and γ such that the error $\tilde{\beta}_k$ remains as small as possible. According to Proposition 5, we have the steady-state error

$$J_s(\gamma, \beta^*, c) = \frac{g_0 - g_2}{1 - g_1} = \beta^* - \gamma \frac{e^{-c \frac{\pi}{2\omega^*}}}{1 - e^{-c \frac{\pi}{\omega^*}}}$$

when $g_2 \leq g_0$, i.e., when $\tilde{\beta}_{\text{fix}}$ is stable. According to Proposition 6, the error is ultimately bounded by

$$J_u(\gamma, \beta^*, c) = \frac{g_0 + g_2}{1 + g_1} = \frac{(1 - e^{-c \frac{\pi}{\omega^*}}) \beta^* + \gamma e^{-c \frac{\pi}{\omega^*}}}{1 + e^{-c \frac{\pi}{\omega^*}}}$$

when $g_2 > g_0$, i.e., when $\tilde{\beta}_{\text{fix}}$ is unstable. For fixed $c > 0$, we propose the performance index

$$J(\gamma, \beta^*, c) = \begin{cases} J_s(\gamma, \beta^*, c) & \text{if } \beta^* \geq \gamma \frac{e^{-c \frac{\pi}{2\omega^*}}}{1 - e^{-c \frac{\pi}{\omega^*}}} \\ J_u(\gamma, \beta^*, c) & \text{if } \beta^* < \gamma \frac{e^{-c \frac{\pi}{2\omega^*}}}{1 - e^{-c \frac{\pi}{\omega^*}}} \end{cases} \quad (27)$$

and the robust optimization problem

$$\min_{\gamma > 0} \sup_{\beta^* \in [\underline{\beta}, \bar{\beta}]} J(\gamma, \beta^*, c), \quad (28)$$

where $\underline{\beta}$ and $\bar{\beta}$ are, respectively, lower and upper bounds on the (uncertain) burst width β^* .

Proposition 7. *Consider the optimization problem (28) with J as in (27) and g_i , $i = 0, 1, 2$, as in (23). The solution is*

$$J(\gamma_{\text{opt}}, \bar{\beta}, c) = \min_{\gamma > 0} \sup_{\beta^* \in [\underline{\beta}, \bar{\beta}]} J(\gamma, \beta^*, c)$$

with

$$\gamma_{\text{opt}} = \max \left\{ \bar{\beta} \frac{1 - e^{-c \frac{2\pi}{\omega^*}}}{3e^{-c \frac{\pi}{2\omega^*}} - e^{-c \frac{3\pi}{2\omega^*}}}, \beta \frac{1 - e^{-c \frac{\pi}{\omega^*}}}{e^{-c \frac{\pi}{2\omega^*}}} \right\}. \quad (29)$$

Allow us to summarize by providing a step-by-step algorithm to tune the neuromorphic controller:

- 1) Choose the desired amplitude value A^* .
- 2) Compute the value of the required pulse width, β^* , using the solution of the HB equation with respect to ω^* and β^* with fixed A^* .
- 3) Estimate the bounds of β^* : $\beta^* \in [\underline{\beta}, \bar{\beta}]$. The bounds $\underline{\beta}$ and $\bar{\beta}$ can be estimated for known bounded parameter uncertainties $\Delta\lambda$, $\Delta\xi$, and $\Delta\omega_n$.
- 4) Fix time constant c and compute the optimal gain (29).

V. CONCLUSIONS

We introduced a general modeling architecture for neuromorphic control systems that aligns with the event-based nature of neuromorphic hardware. By adopting an input-output perspective, we formulated control problems that respect neuromorphic sensing, computing, and actuation constraints, while remaining amenable to classical control-theoretical tools. Applied to the neuromorphic pendulum, our framework enabled the derivation of rigorous performance bounds and revealed a discrete-time Hopf-like bifurcation in the closed-loop system. Interestingly, robust optimization of the control error led to an optimal solution that coincides with the bifurcation point, uncovering a meaningful trade-off between performance and robustness.

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