

**q -deformed binomial
coefficients of words**

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Joint work with Michel Rigo & Markus A.
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Overview

1. Definition and examples
2. Combinatorial interpretation
3. p -group languages
4. q -Parikh matrices
5. What's next?

The background consists of two overlapping geometric shapes: a teal triangle in the top-left corner and a light gray triangle in the bottom-left corner, both pointing towards the center. The rest of the background is white.

Definition and examples



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q -deformations

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Ex:

- ▶ q -natural numbers: $[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \dots + q^{n-1}$,
- ▶ q -factorial: $[n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q$,
- ▶ q -binomial coefficients: $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$.



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But also: q -rational numbers, q -derivative, Gaussian q -distribution, etc.



Definition and examples

Basics from COW

In this talk, a *finite word* is a sequence of *letters* belonging to a finite *alphabet* A .

The free monoid A^* is the set of all finite words over the alphabet A . The *empty word* is denoted by ε and is the identity element for the *concatenation product*.

The *length* of a word u is the number of letters of u and is denoted by $|u|$.

A *subword* v of a finite word u is a subsequence of u , *i.e.* a word made of (not necessarily consecutive) letters of u . A *factor* of u is a subword made of consecutive letters of u .

Ex: *ate* is a subword of *taste* ; *bar* is a subword of *combinatorics*.
jewel is a factor of *bejeweled* ; *arm* is a factor of *karma*.



Definition and examples

Binomial coefficients

The *binomial coefficient* $\binom{u}{v}$ of two words counts the number of occurrences of v as a subword of u .

Ex:

$$\binom{abbab}{ab} = 4$$

abbab *abbab*
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Gaussian binomial coefficient $\binom{n}{k}_q$ of two positive integers:

$$\binom{n}{k}_q = \frac{(1 - q^n) \cdots (1 - q^{n-k+1})}{(1 - q^k) \cdots (1 - q)}$$

→ counts the number of subspaces of dimension k in a vector space of dimension n over \mathbb{F}_q .



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→ ***What if we merge these two objects?***

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Coefficients on words:

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}$$

$$u, v \in A^*, a, b \in A$$

Gaussian coefficients:

$$\binom{n+1}{k+1}_q = \binom{n}{k+1}_q \cdot q^{k+1} + \binom{n}{k}_q$$

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→ ***We are going to mix these two!***



Definition and examples

q -deformed binomial coefficients of words

The *q -deformation* $\binom{\cdot}{\cdot}_q$ of binomial coefficients of words is a polynomial in $\mathbb{N}[q]$ defined as follows: for all $u, v \in A^*$ and $a, b \in A$ (where A is a finite alphabet),

$$\binom{u}{\varepsilon}_q = 1, \quad \binom{\varepsilon}{v}_q = 0 \text{ if } v \neq \varepsilon,$$

$$\binom{ua}{vb}_q = \binom{u}{vb}_q \cdot q^{|vb|} + \delta_{a,b} \binom{u}{v}_q.$$



Definition and examples

Basic properties

Directly from definition, we can show that $\forall u, v \in A^*$,

- ▶ $\binom{u}{u}_q = 1$,
- ▶ $\binom{u}{v}_q = 0 \Leftrightarrow v$ does not occur as a subword of u .

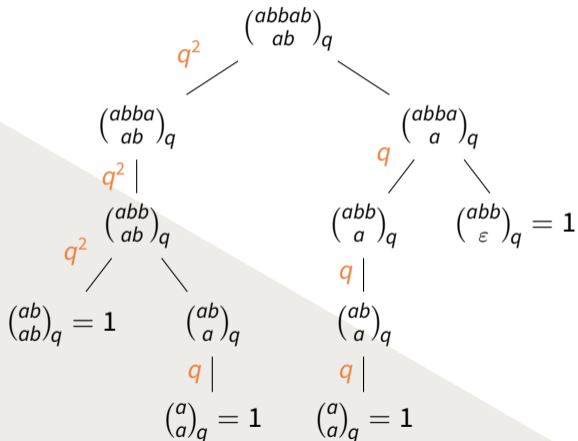
We can also link our q -coefficients to the classical ones: $\forall u, v \in A^*, a \in A$ and $k, \ell \in \mathbb{N}$,

- ▶ $\binom{u}{v}_q(1) = \binom{u}{v}$,
- ▶ $\binom{a^k}{a^\ell}_q = \binom{k}{\ell}_q$.



Definition and examples

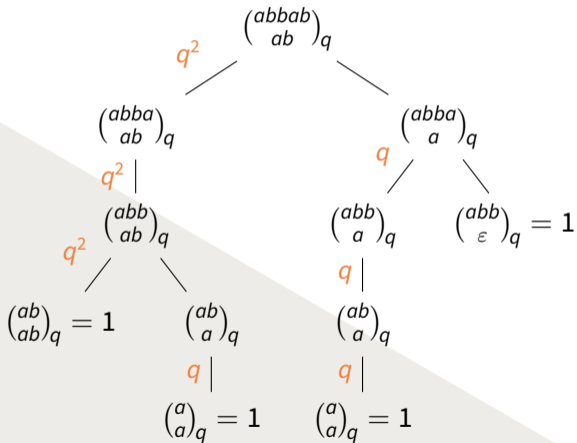
Example: an easy way to compute q -binomials





Definition and examples

Example: an easy way to compute q -binomials



$$\begin{pmatrix} abbab \\ ab \end{pmatrix}_q = q^6 + q^5 + q^3 + 1$$

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Combinatorial interpretation

Combinatorial interpretation



Main result

Theorem (R., Rigo, Whiteland, 2025)

Let u be a word over A , $k \geq 0$, and $a_1, \dots, a_k \in A$. Then

$$\binom{u}{a_1 \cdots a_k}_q = \sum_{\substack{u_0, u_1, \dots, u_k \in A^* \\ u = u_0 a_1 \cdots u_{k-1} a_k u_k}} q^{\sum_{i=1}^k i|u_i|}.$$



Combinatorial interpretation

Main result

If we take back our favourite example

$$\binom{abbab}{ab}_q = q^6 + q^5 + q^3 + 1,$$

we have to consider all factorisations of $abbab$ of the form $u_0 a u_1 b u_2$:

	$abbab$	$abbab$	$abbab$	$abbab$
(u_1, u_2)	(ε, bab)	(b, ab)	(bba, ε)	$(\varepsilon, \varepsilon)$
$ u_1 + 2 u_2 $	6	5	3	0

Combinatorial interpretation



Main result

In other words, for a fixed occurrence of v in u , one has to count, for each letter of v , the number of letters of u that are at its right and not part of this specific occurrence.

Summing these numbers gives the corresponding power of q .



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What if I told you it's back:

$$\begin{array}{cccc} \overline{abbab} & \overline{abbab} & \overline{abbab} & \overline{abbab} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ q^{3+3} & + & q^{3+2} & + & q^{3+0} & + & q^{0+0} & = & \binom{abbab}{ab}_q \end{array}$$



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→ the powers of q encode the positions of the letters of v in its occurrences in u



Another example

Of course, other coefficients may be different than 1. For instance,

$$\binom{aabaaa}{aa}_q = q^8 + q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1.$$

Here, the 2's mean that there exist two occurrences of aa in $aabaaa$ such that the sum of positions equals 4 (resp. 5):

$$aabaaa \text{ and } aabaaa \longrightarrow 2q^4$$

$$aabaaa \text{ and } aabaaa \longrightarrow 2q^5$$



Combinatorial interpretation

Information within the coefficients

From the combinatorial interpretation, we get some information about coefficients.

Corollary (R., Rigo, Whiteland, 2025)

For all words u, v , the polynomial $\binom{u}{v}_q$ is monic, and the non-zero coefficient of the monomial of least degree is 1. In particular,

- ▶ $\binom{u}{v}_q(0) = \begin{cases} 1, & \text{if } v \text{ is a suffix of } u, \\ 0, & \text{otherwise.} \end{cases}$
- ▶ $\binom{u}{v}_q [q^{|\nu|(|u|-|v|)}] = \begin{cases} 1, & \text{if } v \text{ is a prefix of } u, \\ 0, & \text{otherwise.} \end{cases}$

Combinatorial interpretation



Reconstructing words

How much information of the form $\binom{u}{v}$, with $v \in A^*$, is needed to reconstruct the word u unambiguously?



Combinatorial interpretation

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Theorem (R., Rigo, Whiteland, 2025)

Let k be an integer, u, x words such that $|u| \geq k \geq |x|$. We have

$$\binom{|u| - |x|}{k - |x|}_q \binom{u}{x}_q = \sum_{t \in A^k} \binom{u}{t}_q \binom{t}{x}_q.$$

Corollary (R., Rigo, Whiteland, 2025)

Let $u \in A^*$ and $1 \leq k \leq |u|$. The sequence $\left(\binom{u}{x}_q \right)_{x \in A^k}$ uniquely determines the word u .

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p -group languages



p -group languages

Definitions

Recall that a language L is *recognised* by a monoid M if there exist

- ▶ a subset $S \subset M$,
- ▶ a monoid morphism $\varphi : A^* \rightarrow M$,

such that $L = \varphi^{-1}(S)$.



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N.B.: regular languages = recognisable languages



p -group languages

Characterising p -group languages

Theorem (Eilenberg, 1976)

Let p be a prime. A language is a p -group language if and only if it is a Boolean combination of languages of the form

$$L_{v,r,p} = \{u \in A^* \mid \binom{u}{v} \equiv r \pmod{p}\}.$$



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Theorem (R., Rigo, Whiteland, 2025)

Let p be a prime and $\mathfrak{M} = a(q-1)^d$ with $d \geq 1$ an integer and a non-zero $a \in \mathbb{F}_p$. A language is a p -group language if and only if it is a Boolean combination of languages of the form

$$L_{v,\mathfrak{R},\mathfrak{M}} = \left\{ u \in A^* \mid \binom{u}{v}_q \equiv \mathfrak{R} \pmod{\mathfrak{M}} \right\}.$$



p -group languages

What's behind?

- ▶ Defining a (u, \mathfrak{M}) -*binomial equivalence relation* ($u \in A^+, \mathfrak{M} \in \mathbb{F}_p[q]$):

$$w_1 \sim_{u, \mathfrak{M}} w_2 \Leftrightarrow \forall v \in \text{Fac}(u) : \binom{w_1}{v}_q \equiv \binom{w_2}{v}_q \pmod{\mathfrak{M}}.$$



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 - ▶ Case 1: q is **not** invertible in $\mathbb{F}_p[q] / \langle \mathfrak{M} \rangle \rightsquigarrow A^* / \equiv_{u, \mathfrak{M}}$ is not a group;
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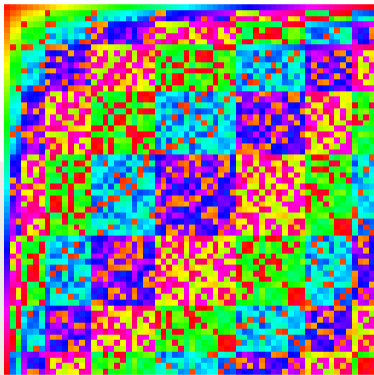
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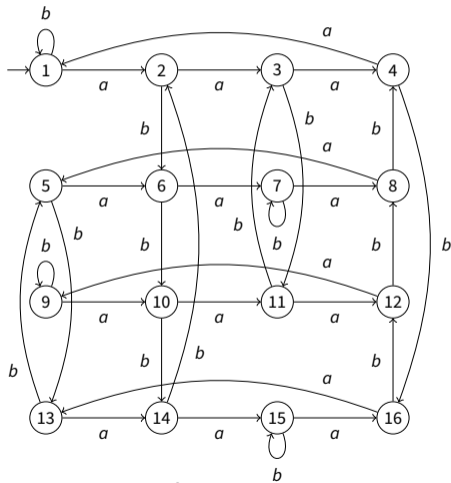
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- ▶ Conclusion using Eilenberg's original theorem.

p -group languages

Some nice pictures



$$\{a, b\}^* / \sim_{ab, q^2+1}$$



$$L_{ab, \mathfrak{A}, q^2+1}$$

q -Parikh matrices



q -Parikh matrices

“Classical” Parikh matrices

Introduced by Şerbănuță in 2004, the *Parikh matrix of u induced by a word $z = z_1 \cdots z_n$* is an upper triangular matrix of size $(n + 1) \times (n + 1)$, containing elements of the form $\binom{u}{v}$ for words v of the form $z_i z_{i+1} \cdots z_j$, $1 \leq i \leq j \leq |z|$.



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The Parikh matrix mapping is a monoid morphism $\Psi_{A,z} : A^* \rightarrow \mathbb{N}^{(n+1) \times (n+1)}$ defined by

$$[\Psi_{A,z}(a)]_{i,j} = \begin{cases} 1 & j = i, \\ \delta_{a,z_i} & j = i + 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } a \in A.$$



q -Parikh matrices

An example

Fix $z = aba$ and $u = abbaba$. We have

$$\Psi_{A,z}(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Psi_{A,z}(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $\Psi_{A,z}(u) = \Psi_{A,z}(a)\Psi_{A,z}(b)\Psi_{A,z}(b)\Psi_{A,z}(a)\Psi_{A,z}(b)\Psi_{A,z}(a)$, i.e.

$$\Psi_{A,z}(u) = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \binom{u}{a} & \binom{u}{ab} & \binom{u}{aba} \\ 0 & 1 & \binom{u}{b} & \binom{u}{ba} \\ 0 & 0 & 1 & \binom{u}{a} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$



q -Parikh matrices

q -deformed Parikh matrices

Let $z = z_1 \cdots z_n$ be a word and A be the alphabet of z , i.e. the set of letters occurring in z . For $a \in A$ and $\ell \geq 0$, we let $\mathcal{M}_{a,\ell}$ denote the upper triangular matrix defined by

$$[\mathcal{M}_{a,\ell}]_{i,j} = \begin{cases} 1 & j = i, \\ \delta_{a,z_i} q^\ell & j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

We now define the map

$$\mathcal{P}_z : A^* \rightarrow (\mathbb{N}[q])^{(n+1) \times (n+1)} : u_k u_{k-1} \cdots u_1 u_0 \mapsto \mathcal{M}_{u_k,k} \cdots \mathcal{M}_{u_0,0}.$$



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Unfortunately, **this is not a monoid morphism anymore.**

\rightsquigarrow recently improved thanks to a joint work with J.-E. Pin.



q -Parikh matrices

q -deformed Parikh matrices

Here, we have the following:

Theorem (R., Rigo, Whiteland, 2025)

Let z be a word of length $n \geq 1$ whose alphabet is A . Let $u \in A^*$. The corresponding $(n + 1) \times (n + 1)$ q -Parikh matrix is such that

- ▶ $[\mathcal{P}_z(u)]_{i,j} = 0$, for all $1 \leq j < i \leq n + 1$,
- ▶ $[\mathcal{P}_z(u)]_{i,i} = 1$, for all $1 \leq i \leq n + 1$.
- ▶ Let $r \in \{1, \dots, n\}$. For all $1 \leq i \leq n - r + 1$, $[\mathcal{P}_z(u)]_{i,i+r} = q^{s(r-1)} \binom{u}{z_i z_{i+1} \dots z_{i+r-1}}_q$,
where $s(\ell) = \sum_{k=1}^{\ell} k$.



q -Parikh matrices

An example (again)

Back with our friends $z = aba$ and $u = abbaba$, we have

$$\mathcal{M}_{a,\ell} = \begin{pmatrix} 1 & q^\ell & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q^\ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_{b,\ell} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q^\ell & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$



q -Parikh matrices

An example (again)

Thus,

$$\begin{aligned} \mathcal{P}_z(u) &= \mathcal{M}_{a,5} \mathcal{M}_{b,4} \mathcal{M}_{b,3} \mathcal{M}_{a,2} \mathcal{M}_{b,1} \mathcal{M}_{a,0} \\ &= \begin{pmatrix} 1 & q^5 + q^2 + 1 & q^9 + q^8 + q^6 + q^3 & q^{11} + q^{10} + q^9 + q^8 + q^6 + q^3 \\ 0 & 1 & q^4 + q^3 + q & q^6 + q^5 + q^4 + q^3 + q \\ 0 & 0 & 1 & q^5 + q^2 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \binom{u}{a}_q & q \binom{u}{ab}_q & q^3 \binom{u}{aba}_q \\ 0 & 1 & \binom{u}{b}_q & q \binom{u}{ba}_q \\ 0 & 0 & 1 & \binom{u}{a}_q \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$



q -Parikh matrices

Convergence to formal power series

Theorem (R., Rigo, Whiteland, 2025)

The q -binomial $\binom{u^n}{z}_q$ can be expressed as

$$\frac{1}{q^{\binom{|z|}{2}}} \sum_{k=1}^m R_k(q) \frac{1 - q^{c_k n |u|}}{1 - q^{c_k |u|}},$$

where m and c_k are positive integers, and R_k are rational functions whose denominators only have factors of the form $(1 - q^{t|u|})$ for some integer t . Moreover, these quantities c_k and R_k can be effectively computed. In particular, the sequence $(\binom{u^n}{z}_q)_{n \geq 0}$ converges in $\mathbb{N}[[q]]$ to the formal power series $\mathfrak{s}_{\mathbf{u},z}(q)$ expressed by the rational function

$$\frac{1}{q^{s(|z|-1)}} \sum_{k=1}^m R_k(q) \frac{1}{1 - q^{c_k |u|}}.$$



q -Parikh matrices

Convergence to formal power series

Ex:

The sequence $\left(\binom{(abba)^n}{ab}_q \right)_{n \geq 0}$ converges to the series

$$q^3 + 2q^4 + q^5 + q^7 + 2q^8 + q^9 + 2q^{11} + 4q^{12} + 2q^{13} + 2q^{15} + 4q^{16} + 2q^{17} + 3q^{19} + 6q^{20} + \dots,$$

which corresponds to the rational function

$$R(q) = \frac{q^3}{(q-1)^2 (q^2+1)^2 (q^4+1)}.$$



q -Parikh matrices

Convergence to formal power series

We generalise a result from Salomaa (2008), which we can recover by taking $q = 1$:

Corollary (R., Rigo, Whiteland, 2025)

The sequence of q -binomials $\left(\binom{u^n}{z}\right)_{n \geq 0}$ satisfies a linear recurrence relation with polynomial coefficients. In particular, the sequence of binomials $\left(\binom{u^n}{z}\right)_{n \geq 0}$ satisfies a linear recurrence relation with constant coefficients.



q -Parikh matrices

Convergence to formal power series

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Ex: The sequence $\left(\binom{((abba)^n)}{ab}\right)_{n \geq 0}$ satisfies the relation

$$p_{n+3} = (1 + q^4 + q^8)p_{n+2} - (q^4 + q^8 + q^{12})p_{n+1} + q^{12}p_n,$$

so that the integer sequence $\left(\binom{((abba)^n)}{ab}\right)_{n \geq 0}$ satisfies $p_{n+3} = 3p_{n+2} - 3p_{n+1} + p_n$.



q -Parikh matrices


Other consequences

- ▶ Alternative way to compute q -binomials;
- ▶ New identities, e.g.

$$\sum_{\substack{z=xy \\ x,y \in A^*}} (-1)^{|y|} q^{s(|x|-1)+s(|y|-1)} \binom{u}{x}_q \binom{u}{\tilde{y}}_q = 0$$

for words z with no equal consecutive letters;

- ▶ etc.

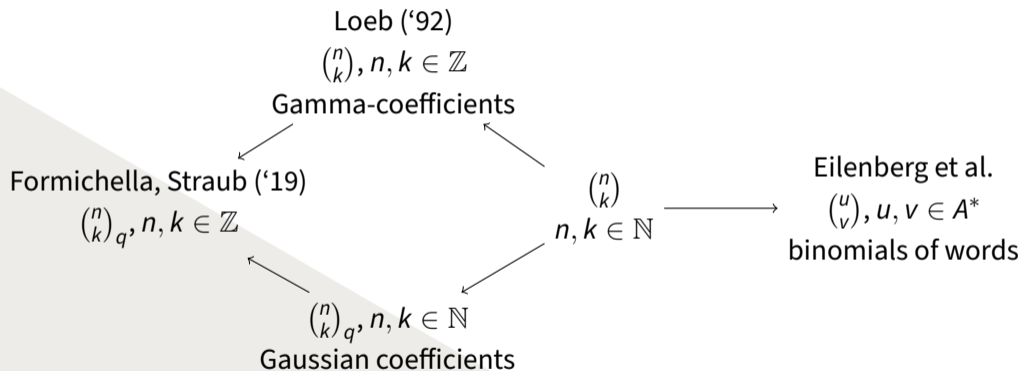
The background consists of two large, overlapping geometric shapes. A teal-colored shape is in the upper-left corner, and a light gray shape is in the lower-left corner. The rest of the background is white. The text "What's next?" is centered in the white area.

What's next?



What's next?

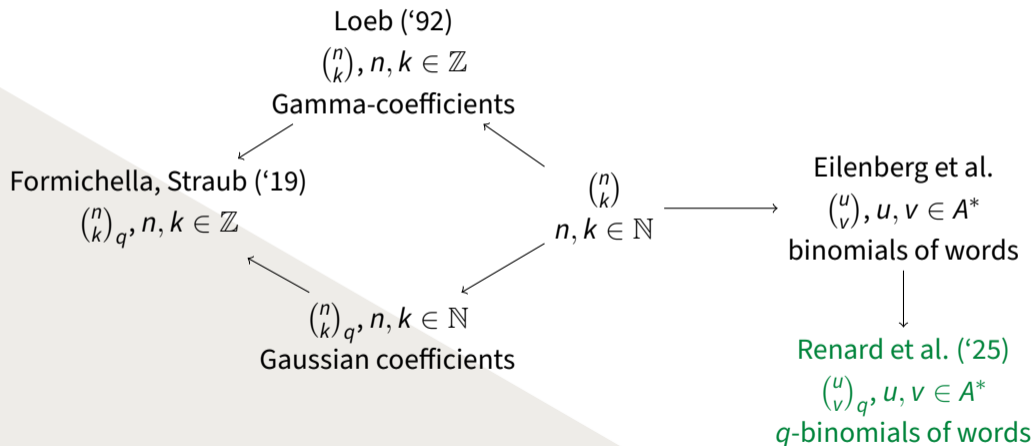
Completing the picture!





What's next?

Completing the picture!



What's next?

Completing the picture!



Magnus transform, e.g. Reutenauer ('93)

$$\binom{u}{v}, u \in F(A), v \in A^*$$

extension to the free group

Eilenberg et al.

$\binom{u}{v}, u, v \in A^*$
binomials of words

More to
come...

Renard et al. ('25)

$\binom{u}{v}_q, u, v \in A^*$
 q -binomials of words

Loeb ('92)

$$\binom{n}{k}, n, k \in \mathbb{Z}$$

Gamma-coefficients

Formichella, Straub ('19)

$$\binom{n}{k}_q, n, k \in \mathbb{Z}$$

$$\binom{n}{k}, n, k \in \mathbb{N}$$

$$\binom{n}{k}_q, n, k \in \mathbb{N}$$

Gaussian coefficients



What's next?

Some teasers

The binomial coefficient of words can also be defined through the *Magnus transform*:

$$\mu: A^* \rightarrow U(\mathbb{Z}\langle\langle A \rangle\rangle), \quad \text{such that } \mu(a) = 1 + a, \quad \forall a \in A.$$

We then have

$$\mu(u) = \sum_{v \in A^*} \binom{u}{v} v.$$

Ex: $\mu(abbab) = 1 + 2 \cdot a + 3 \cdot b + aa + 4 \cdot ab + 2 \cdot ba + 3 \cdot bb + \dots$

In Reutenauer's book (*Free Lie Algebras*, '93), the Magnus transform is extended to a group morphism:

$$\mu(a^{-1}) = (1 + a)^{-1} = \sum_{n \geq 0} (-1)^n a^n.$$

We can thus define $\binom{u}{v}$ for $u \in F(A)$, $v \in A^*$. What can we say about it?



Thank you for your attention!



p -groups

Let p be a prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field of integers modulo p and \mathfrak{M} be a non-zero polynomial in $\mathbb{F}_p[q]$. We denote $\mathbb{K} = \mathbb{F}_p[q]/\langle \mathfrak{M} \rangle$

Quite naturally, we will consider languages of the form

$$L_{v, \mathfrak{A}, \mathfrak{M}} := \left\{ u \in A^* \mid \binom{u}{v}_q \equiv \mathfrak{A} \pmod{\mathfrak{M}} \right\}.$$

Idea: To prove the theorem, we are going to define a congruence \cong , so that A^*/\cong is a finite monoid, and thus have regular languages.



p -groups

Let $u \in A^*$ and $\mathfrak{M} \in \mathbb{F}_p[q]$. Two finite words $w_1, w_2 \in A^*$ are *(u, \mathfrak{M}) -binomially equivalent* and we write $w_1 \sim_{u, \mathfrak{M}} w_2$ whenever

$$\forall v \in \text{Fac}(u) : \binom{w_1}{v}_q \equiv \binom{w_2}{v}_q \pmod{\mathfrak{M}}.$$

→ # classes $\leq \#\mathbb{K}(\#\text{Fac}(u)-1)$

→ $A^* / \sim_{u, \mathfrak{M}}$ is finite



p -groups

Problem: $\sim_{u, \mathfrak{M}}$ is not always a congruence...

Ex: For $p = 2$, $\mathfrak{M} = q^2 + 1$, $A = \{a, b\}$ and $u = a$, we have $a \sim_{u, \mathfrak{M}} abbbaa$ and $b \sim_{u, \mathfrak{M}} bb$:

$$\binom{a}{a}_q \equiv 1 \equiv \binom{abbbaa}{a}_q \pmod{\mathfrak{M}} \quad \text{and} \quad \binom{b}{a}_q \equiv 0 \equiv \binom{bb}{a}_q \pmod{\mathfrak{M}}$$

but $ab \not\sim_{u, \mathfrak{M}} abbbaabb$:

$$\binom{ab}{a}_q \equiv q \not\equiv 1 \equiv \binom{abbbaabb}{a}_q \pmod{\mathfrak{M}}$$

We will consider a congruence \cong which is a **refinement** of $\sim_{u, \mathfrak{M}}$, i.e.

$$w_1 \cong w_2 \Rightarrow w_1 \sim_{u, \mathfrak{M}} w_2.$$

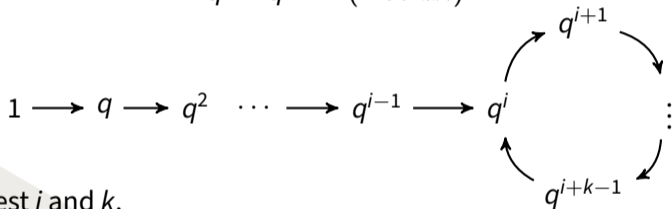
We also say that $\sim_{u, \mathfrak{M}}$ is **coarser** than \cong .



p -groups

As $\mathbb{K} = \mathbb{F}_p[q]/\langle \mathfrak{M} \rangle$ is finite, there exist $i \geq 0, k \geq 1$ such that

$$q^i \equiv q^{i+k} \pmod{\mathfrak{M}}.$$



Taking the smallest i and k ,

- ▶ i is the *index* of q ,
- ▶ k is the *period* of q .

Notice that, if q is invertible in \mathbb{K} , then $i = 0$ and $k = \text{ord}(q)$.



p -groups

If q is not a unit of \mathbb{K} , we have the following:

Proposition (R., Rigo, Whiteland, 2025)

Let $u \in A^*$, $\mathfrak{M} \in \mathbb{F}_p[q]$ and \cong be a congruence that refines $\sim_{u, \mathfrak{M}}$. If q is not a unit of \mathbb{K} , then the monoid A^*/\cong is not a group. In particular, no element except for the identity is invertible.

Why? We managed to show that for any congruence \cong refining $\sim_{u, \mathfrak{M}}$, we have

$$w_1 \cong w_2 \Rightarrow |w_1| = |w_2| \text{ or } (|w_1|, |w_2| \geq \text{ind}(q) \wedge \dots)$$

So when q is not a unit of \mathbb{K} , $\text{ind}(q) \geq 1$, and $[\varepsilon]$ is a singleton, so that $[w] \cdot [x] = [wx] \neq [\varepsilon]$ for all $w, x \in A^+$.



p -groups

If q is a unit of \mathbb{K} , we can look for the *coarsest* congruence refining $\sim_{u, \mathfrak{M}}$.

We showed that $\sim_{u, \mathfrak{M}} \cap \sim_{\text{ord}(p)}$, where

$$w_1 \sim_{\text{ord}(q)} w_2 \Leftrightarrow |w_1| \equiv |w_2| \pmod{\text{ord}(q)},$$

and denoted $\equiv_{u, \mathfrak{M}}$ is the coarsest congruence refining $\sim_{u, \mathfrak{M}}$.

Theorem (R., Rigo, Whiteland, 2025)

Let $u \in A^$, $\mathfrak{M} \in \mathbb{F}_p[q]$. If q is a unit in \mathbb{K} , $A^* / \equiv_{u, \mathfrak{M}}$ is a group whose order divides $\text{ord}(q) \cdot p^{|\mathfrak{M}|}$.*



p -groups

Corollary (R., Rigo, Whiteland, 2025)

Let v be a word and $\mathfrak{M} = a(q-1)^d$ for some integer $d \geq 1$. The language

$$L_{v, \mathfrak{R}, \mathfrak{M}} = \left\{ u \in A^* \mid \binom{u}{v}_q \equiv \mathfrak{R} \pmod{\mathfrak{M}} \right\}$$

is a p -group language.

Theorem (R., Rigo, Whiteland, 2025)

Let p be a prime and $\mathfrak{M} = a(q-1)^d$ with $d \geq 1$ an integer. A language is a p -group language if and only if it is a Boolean combination of languages of the form

$$L_{v, \mathfrak{R}, \mathfrak{M}} = \left\{ u \in A^* \mid \binom{u}{v}_q \equiv \mathfrak{R} \pmod{\mathfrak{M}} \right\}.$$