Sensitivity analysis for linear changes of the constraint matrix of an LP/MIP

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Sensitivity analysis

Consider a Linear Program

$$
\begin{aligned}
\min \quad & c_1 x_1 + \cdots + c_n x_n \\
\text{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n = b_1 \\
& \vdots \\
& a_{m1} x_1 + \cdots + a_{mn} x_n = b_m \\
& x_1, \ \ldots, \ x_n \geq 0
\end{aligned}
$$

Sensitivity analysis

Consider a Linear Program

$$\begin{array}{rl} \min & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n = b_1 + \lambda \\ & \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \\ & x_1, \ \ldots, \ x_n \geq 0 \end{array}$$

- One coefficient in the right-hand-side

Sensitivity analysis

Consider a Linear Program

$$\begin{array}{rl} \min & c_1 x_1 + \cdots + (c_n + \lambda) x_n \\ \text{subject to} & a_{11} x_1 + \cdots + \quad a_{1n} x_n = b_1 \\ & \vdots \\ & a_{m1} x_1 + \cdots + \quad a_{mn} x_n = b_m \\ & x_1, \ldots, \quad x_n \geq 0 \end{array}$$

- One coefficient in the right-hand-side
- One coefficient in the cost

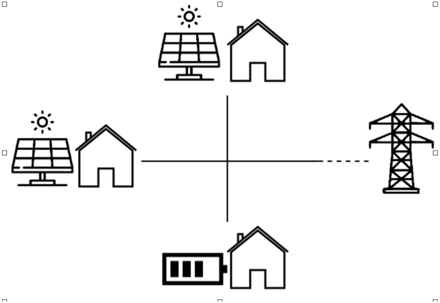
Sensitivity analysis

Consider a Linear Program

$$\begin{aligned} \min \quad & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n = b_1 \\ & \vdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n = b_m \\ & x_1, \ \ldots, \ x_n \geq 0 \end{aligned}$$

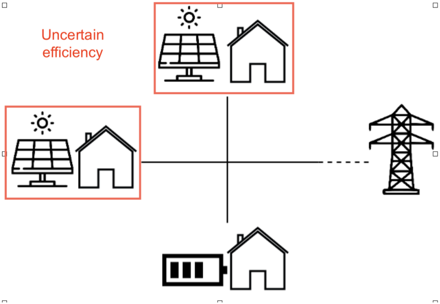
- One coefficient in the right-hand-side
- One coefficient in the cost
- Matrix coefficients : only coefficients of nonbasic variables

Uncertainty



$$\begin{pmatrix} \cdots & & \cdots \\ & \ddots & \\ \vdots & & \vdots \\ & \cdots & \end{pmatrix}$$

Uncertainty



$$\begin{pmatrix} \cdots & & & \cdots & \\ & & \ddots & & \\ & \vdots & & & \vdots \\ \square & & \square & & \\ & \square & \cdots & \square & \\ & & \square & & \square \end{pmatrix}$$

The formalization

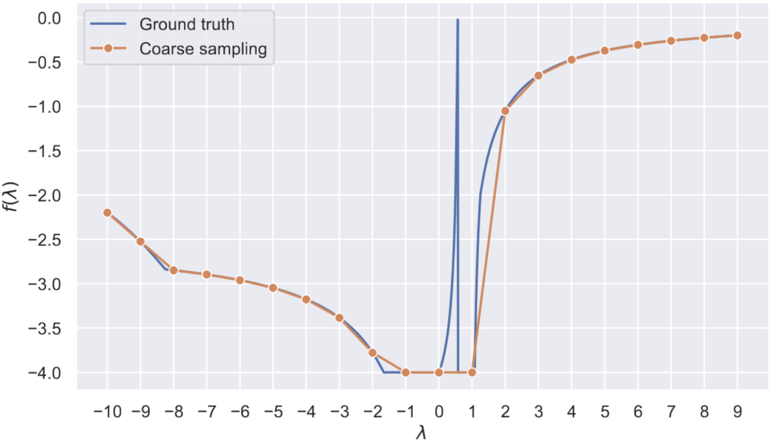
$$\begin{aligned} f(\lambda) = \min\ & c^T x \\ \text{subject to}\ & A_1 x \leq b_1 \\ & A_2 x + \lambda D x \leq b_2 \\ & x \geq 0 \end{aligned}$$

for $\lambda \in [\lambda_1, \lambda_2]$.

- In the literature:
  - Usually rely on heavy computations
  - or hypotheses on D (e.g. low rank)

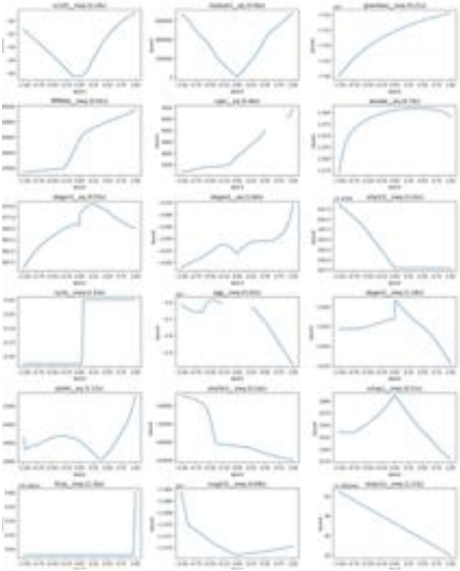
Brute force solution

Sample λ and recompute from scratch



May be very chaotic!

A sampling of problems



Main idea

Try to bound $f(\lambda)$

$$
\begin{aligned}
lb(\lambda) \leq f(\lambda) = \min\ & c^T x & \leq ub(\lambda) \\
\text{s.t.}\ & A_1 x \leq b_1 \\
& A_2 x + \lambda D x \leq b_2 \\
& x \geq 0
\end{aligned}
$$

We present the techniques on the primal but we can apply them to the dual.

Some of the techniques may work for MIP as well.

Outline

- Constant robust bound
- Variable robust bound
- Coefficientwise relaxation
- Lagrangian relaxation
- Computational results

Idea 1 : Constant robust solution

Idea: Find a solution which is feasible for a range $\lambda \in [\lambda_1, \lambda_2]$.

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A_1 x \leq b_1 \\ & (A_2 + \lambda D)x \leq b_2 \quad \text{for all } \lambda \in [\lambda_1, \lambda_2] \end{aligned}$$

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Robust counterpart

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A_1 x \le b_1 \\ & A_2 x + \max_{\lambda \in [\lambda_1, \lambda_2]} \lambda D x \le b_2 \end{aligned}$$

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Componentwise, all rows attain their maximum either in λ_1 or λ_2, hence

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A_1 x \leq b_1 \\ & (A_2 + \lambda_1 D)x \leq b_2 \\ & (A_2 + \lambda_2 D)x \leq b_2 \end{aligned}$$

provides an upper bound to the primal.

The technique also works for MIPs and gives a primal bound.

Idea 2 : Affine robust solution

Reformulate $x = y + \lambda z$ for $\lambda \in [\lambda_1, \lambda_2]$.

$$\begin{aligned} &\min c^T(y + \lambda z) \\ &\text{s.t. } A_1(y + \lambda z) \leq b_1 \quad \text{for all } \lambda \in [\lambda_1, \lambda_2] \\ &\qquad (A_2 + \lambda D)(y + \lambda z) \leq b_2 \quad \text{for all } \lambda \in [\lambda_1, \lambda_2] \end{aligned}$$

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Robust counterpart:

$$\begin{aligned} \min\ & c^T(y + \lambda z) \\ \text{s.t. }\ & A_1 y + \max_{\lambda \in [\lambda_1, \lambda_2]} \lambda A_1 z \leq b_1 \qquad (1) \\ & \max_{\lambda \in [\lambda_1, \lambda_2]} A_2 y + \lambda(Dy + A_2 z) + \lambda^2 Dz \leq b_2 \qquad (2) \end{aligned}$$

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We can handle (1) as before.
How to handle (2)?

- If $(Dz)_i \ge 0 \to$ Convex $\to$ max attained in λ_1 or λ_2
- If $(Dz)_i < 0 \to$ Concave $\to$ max attained in $\lambda = -\frac{(Dy + A_2 z)_i}{2(Dz)_i}$

We lose the linearity (would be computable with an SOCP)!

Relaxation of the robust counterpart

Theorem

Relaxation of the robust counterpart which is linear in y and z

$$\begin{aligned} \min\ & \textit{some function} \\ \text{s.t. } & A_1 y + A_1 \lambda_1 z \leq b_1 \qquad\qquad A_1 y + A_1 \lambda_2 z \leq b_1 \\ & (A_2 + \lambda_1 D)(y + \lambda_1 z) \leq b_2 \qquad\qquad (A_2 + \lambda_2 D)(y + \lambda_2 z) \leq b_2 \\ & A_2 y + Dz\lambda_1\lambda_2 + (Dy + A2z)\frac{\lambda_1 + \lambda_2}{2} \leq b_2 \end{aligned}$$

provides an upper bound to the primal.

Affine robust solution : what to minimize

- Minimize $c^T(y + \lambda_1 z)$ (or λ_2)
- Fix the slope (e.g. fix $c^T z = 0$)

Coefficientwise bounding method

Drawbacks of the previous methods : double or triple the number of constraints!
$\rightarrow$ Can be computationally costly !

New idea: Relax or restrict every coefficient individually.

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Working assumption: $x_i \geq 0$ for all i

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\begin{aligned}
(\mathcal{P}(\lambda)) : \min\ & c^T x \\
\text{s.t. } & A_1 x \leq b_1 \\
& \sum_{j=1}^{n} (a_{ij}^{(2)} + \lambda d_{ij}) x_{ij} \leq b_i^{(2)} \quad i = 1, \ldots, m
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Valid relaxation of every $(\mathcal{P}(\lambda))$:

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\min\ & c^T x \\
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\end{aligned}
$$

Coefficientwise bounding method : the restriction

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\text{s.t. } & A_1 x \leq b_1 \\
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\end{aligned}
$$

Valid restriction of every $(\mathcal{P}(\lambda))$:

$$
\begin{aligned}
(\mathcal{R}) : \min\ & c^T x \\
\text{s.t. } & A_1 x \leq b_1 \\
& \sum_{j=1}^{n} \max_{\lambda} (a_{ij}^{(2)} + \lambda d_{ij}) x_{ij} \leq b_i^{(2)} \quad i = 1, \ldots, m
\end{aligned}
$$

A solution x to $(\mathcal{R})$ is valid for every $(\mathcal{P}(\lambda))$ and $c^T x$ provides an upper bound.

Lagrangian relaxation

We can obtain a dual bound by Lagrangian relaxation.
We dualize the constraints that are modified by the parameter λ.

$$h(\rho, \lambda) = \min\ c^T x - \rho((A_2 + \lambda D)x - b_2)$$
$$\text{subject to } A_1 x \leq b_1$$

- Question: Which Lagrange multipliers to choose?
- Suggestion: Use the optimal dual variables of the nominal LP for both λ_1 and λ_2.

Theorem

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Theorem

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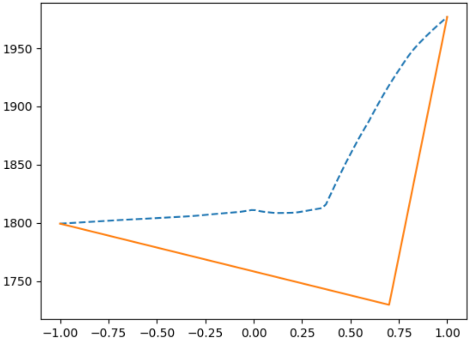
We can strengthen the relaxation by adding the coefficientwise relaxation constraints.
It avoids obtaining an unbounded problem too often.

The bisegment Lagrangian bound



Computational tests

- Instances
  - Roughly 150 LP problems essentially from netlib
  - Roughly 150 MIP problems from MIPLIB, facility locations, unit commitment problems.
- 3 criteria are considered:
  - Availability : percentage of points for which there is a finite bound
  - Error : using normalized RMSE + 1
  - Timing : how many reoptimization points are required?

The bounds are (mostly) always available

	Netlib				MIPLIB				Facility location	
	ineq		eq		ineq		eq			
	lb	ub	lb	ub	lb	ub	lb	ub	lb	ub
Robust	40.0	99.5	70.9	18.2		100		75.0		99.4
Rob yz	90.2	99.7	95.6	41.8						
Coeff.wise	100	98.8	100	18.2	78.9	100	100	75.0	100	99.4
Lagrang.	82.8	89.3	100	9.1	63.2		83.3		99.4	

Average availability of bounds in percentage

Coefficientwise and Lagrangian bounds are very tight

	Netlib				MIPLIB				Facility location	
	ineq		eq		ineq		eq			
	lb	ub	lb	ub	lb	ub	lb	ub	lb	ub
Robust	24.55	0.08	4.49	2.68		0.00		0.00		0.05
Rob yz	0.53	0.06	0.12	0.13						
Coeff.wise	0.09	0.09	0.17	2.68	0.39	0.00	0.08	0.00	0.05	0.05
Lagrang.	0.10	0.00	0.01	0.04	1.69		0.78		1.99	

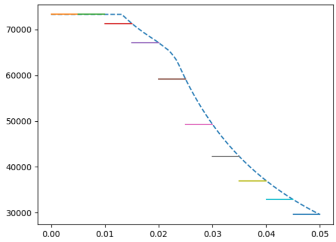
Median of RMSE computed on a sampling of 100 points

Computing times

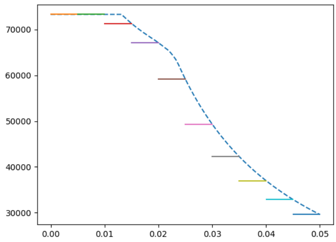
	Netlib				MIPLIB				Facility location	
	ineq		eq		ineq		eq			
	lb	ub	lb	ub	lb	ub	lb	ub	lb	ub
Robust	2.4	1.3	2.3	1.0		1.1		0.9		
Rob yz	7.5	8.0	8.7	10.4						
Coefficientwise	1.1	1.2	1.6	1.2	3.6	2.4	1.2	1.1	1.0	1.1
Lagrang.	6.0	16.5	7.6	22.7	10.4		7.0		2.1	

Time taken in order to compute the bounds (in terms of how many optimal points do we need to compute)

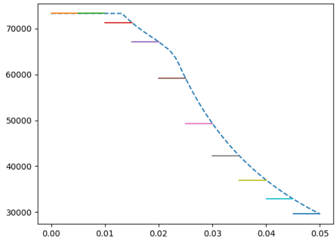
A few lower bounds on a real-life microgrid problem



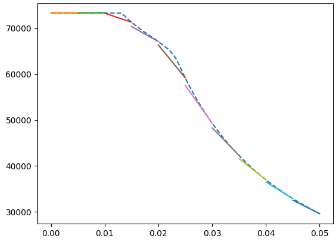
Coefficientwise



Lagrangian

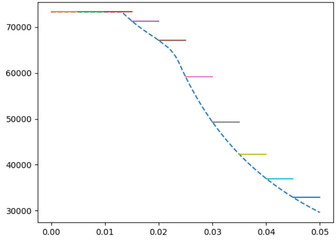


Robust left

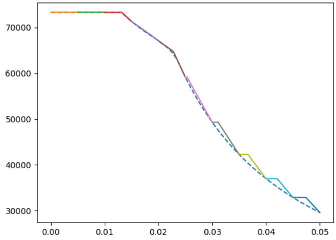


Lagrangian with coefficientwise

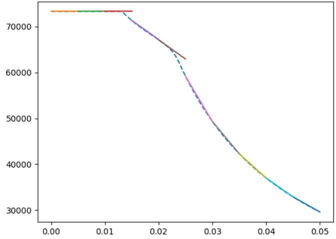
A few upper bounds on a real-life microgrid problem



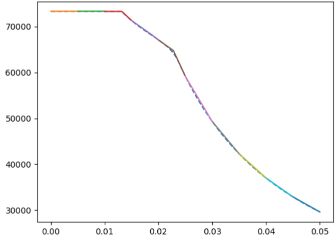
Coefficientwise



Lagrangian



Robust left



Lagrangian with coefficientwise