Volume-contracting systems are entropy-reducing

Gustave Bainier
Allée de la Découverte 10, Liège, Belgium
gustave.bainier@uliege.be

Alessio Franci
Allée de la Découverte 10, Liège, Belgium
afranci@uliege.be

At the beginning of cybernetics, *homeostasis* was roughly described as an *entropy-reducing mechanism*: achieved through negative feedback loops, it actively counteracts disorder and uncertainty to maintain a system behavior. Although the concept of entropy played a central role in the early foundations of cybernetics (most notably in the work of Norbert Wiener [1]) it has largely faded from mainstream control theory, except in specialized areas such as stochastic and information-theoretic control (e.g. [2, 3]). In recent decades, control theory has focused on *contracting systems* (along with other incremental concepts) marking a shift away from classical notions of stability [4, 5, 6]. Contraction emphasizes convergence between trajectories rather than convergence to a specific equilibria. In this short note, it is demonstrated that volume-contracting dynamics induce a reduction of entropy in probability distributions. This simple result reconnects modern contraction-based control with the original entropy-based perspective of early cybernetics.

Take a n-dimensional, time-invariant, smooth dynamical system with both forward-complete and backward-complete solutions:

$$\frac{d}{dt}x(t) = f(x(t)). \tag{1}$$

Let φ_t denote the flow of (1). It is a diffeomorphism from R^n to itself. The quantity $|\det(J(\varphi_t)(x))|$, where J stands for the Jacobian, measures the change in volume of a small open neighborhood of x after it has evolved under the flow of (1) for a time t. Volume-contraction for infinitesimal volume elements implies:

$$\lim_{t\to+\infty} |\det(J(\varphi_t)(x))| = 0 \text{ for a.e. } x \in R^n. \tag{2}$$

The Liouville-Ostrogradski formula [5] yields:

$$|\det(J(\varphi_t)(x))| = \exp\Big(\int_0^t \operatorname{div}(f)(\varphi_s(x))ds\Big),$$

so (2) implies:

$$\lim_{t\to+\infty} \int_0^t \operatorname{div}(f)(\varphi_s(x))ds = -\infty \quad \text{for a.e. } x \in R^n.$$

We make the technical assumption that there exist $t^* > 0$ and $M \in R$ such that for all $t \geq t^*$:

$$\int_0^t \operatorname{div}(f)(\varphi_s(x))ds < M \text{ for a.e. } x \in R^n. \tag{3}$$

Both (2) and this assumption are easily verified for a uniformly negative divergence, $\operatorname{div}(f) \leq -\lambda < 0$.

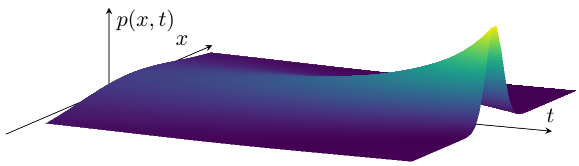

Figure 1: Volume-contracting flows cause probability distributions to compress, thus decreasing their entropy.

Now consider (1) at a random initial condition X_0, with a probability density function p. The entropy of X_0 is:

$$h(X_0) := -\int_{R^n} p(x)\ln(p(x))dx.$$

For all $t \geq 0$, $X_t := \varphi_t(X_0)$. Since φ_t is a diffeomorphism from R^n to itself, a well-known result in information theory [7] expresses the difference of entropy between X_t and X_0 in terms of the Jacobian of φ_t as:

$$h(X_t) - h(X_0) = \int_{R^n} p(x)\ln|\det(J(\varphi_t)(x))|dx.$$

Applying the Liouville-Ostrogradski formula yields:

$$\boxed{h(X_t) - h(X_0) = \int_{R^n} p(x)\Big(\int_0^t \operatorname{div}(f)(\varphi_s(x))ds\Big)dx.}$$

If (1) is such that (2) and assumption (3) hold, the reverse Fatou's lemma demonstrates that the system is entropy-reducing:

$$\lim_{t\to+\infty} h(X_t) - h(X_0) = -\infty.$$

References

[1] N. Wiener, “Cybernetics: or control and communication in the animal and the machine.”; 1948, revised ed. 1961.

[2] T. Bourdais, N. Oudjane, F. Russo. “An entropy penalized approach for stochastic control problems.”; 2025.

[3] H. Touchette, S.-Lloyd. “Information-theoretic approach to the study of control systems.”; Physica A, 2004.

[4] W. Lohmiller, J-J E. Slotine, “On contraction analysis for non-linear systems”; Automatica, 1998.

[5] C. Wu, I. Kanevskiy, M. Margaliot, “k-contraction: theory and applications.”; Automatica, 2022.

[6] F. Bullo, “Contraction theory for dynamical systems.”; 2024.

[7] S. Ihara. “Information theory for continuous systems.”; World Scientific Publishing, 1993.