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Lyapunov Stability Analysis of a Two-Step Anaerobic Digestion Model

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***Abstract*—The use of renewable energy sources is a major concern nowadays, where efforts are made to achieve higher efficiencies. Among the different possibilities, Biogas production from organic waste represents an interesting technology. One difficulty arises as the system can be unstable under certain conditions, being able to systematically understand these conditions could lead to efficient control design. In this paper the stability analysis of an anaerobic biodigestor using the indirect Lyapunov's method is presented using the model (AM2).**

I. Introduction

Anaerobic digestion (AD) is a microbial fermentation in the absence of oxygen that produces a mixture of gases (the major proportion of which is methane and carbon dioxide), called "biogas" and an aqueous suspension called "sludge" that has microorganisms whose function is to degrade organic matter [1]. AD occurs naturally in some natural and cultivated ecosystems, which participates in the biogeochemical cycling of organic material [1], it has been classified as one of the most efficient and environmentally beneficial technologies for bioenergy production [2]. Biogas is a renewable energy source used to replace fossil fuels for power and heat production, it is rich in methane (biomethane) and replaces natural gas as a feedstock for producing chemicals and materials [3].

AD is applied worldwide as a source for generating biogas from organic waste. Therefore, its study is of interest to the scientific community. Some studies have analyzed biogas production from cassava waste [4], with cattle manure, maize silage, grass silage and grain silage as feedstock [5]. Anaerobic digesters have been installed in different places such as Kenya [6], Europe, Japan, China, Brazil, Mexico, Colombia, among others [7].

Among the most complete mathematical models useful for simulation is the Anaerobic Digest Model 1 (ADM1) [8] developed since 1997. ADM1 considers 7 different trophic groups and 11 substrates and products. However, its complexity makes its mathematical analysis critical [9]. In 2001, an alternative to overcome this difficulty was presented by Bernard et al. in [10], who developed a modified model under the European project called AMOCO (Acronym for Advanced MOnitoring and COntrol System for anaerobic processes, European FAIR project No.ERB-FAIR-CT96-1198). This is a simplified model of the AD process, also called AM2, which considers 2 trophic groups (acidogenic biomass X_1 and methanogenic biomass X_2) associated to 2 substrates; in particular, it serves as a basis for the design of linear and non-linear control strategies of AD processes, with the aim of improving process efficiency and Biogas production.

The AD process can be unstable because the accumulation of intermediate compounds can lead to acidification of the digester [9]. Stability studies have been performed for this class of processes, such as the development of the Lyapunov's theory. The local and global stability analysis of equilibrium in the DA model performed in 2010 demonstrates under general monotonic assumptions, relevant from an applied point of view, the global asymptotic stability of a positive equilibrium point corresponding to the coexistence of acetogenic and methanogenic bacteria [11]. In 2013, Sari et. al., [12] study the global dynamics of a chemostat model with a single nutrient and several competing species focusing on the construction of Lyapunov's functions.

Using a Lyapunov's function argument and the theory of asymptotically autonomous systems, it is shown that even in the parameter regime where there is bi-stability, there are no periodic orbits and each solution converges to one of the equilibrium points. In order to guarantee the local and even global stability of the system, some authors, starting from a certain model and using the dilution rate of the bioreactor as control action, have proposed controllers such as nonlinear adaptive control in [13], Optimal control in [14] and the references therein.

Stability in the anaerobic digestion process is a subject of ongoing study. There are studies on the synergistic effects of the addition of rice straw (RS) and rice bran (RB) on methane production and the stability of the anaerobic digestion process of food waste (FW) [15]. Further research results show the stability analysis is based on Lyapunov's functions of the dead zone, and comprises: (i) definition of the quadratic form of the dead zone for each state, and determination of its properties; (ii) determination of the time derivatives of the quadratic forms and their properties [16].

This work focuses on the stability analysis of a two-step anaerobic digestion AM2 model according to the indirect

Lyapunov's method. A procedure to systematically define the stable equilibrium points and the atraction region is proposed, being these of great interest for designing control strategies aiming to optimize biogas production [17].

II. Anaerobic biodigestor: the AM2 model

The anaerobic biodigestor here considered is modelled using the AM2 representation, where two trophic groups (acidogenic biomass X_1 and methanogenic biomass X_2) are related to two substrates S_1 y S_2, respectively [9].

The system of differential equations obtained by mass balance of the continuous process is:

$$\frac{dS_1}{dt} = D(S_{1in} - S_1) - k_1\mu_1(S_1)X_1 \quad (1)$$
$$\frac{dX_1}{dt} = (\mu_1(S_1) - \alpha D)X_1 \quad (2)$$
$$\frac{dS_2}{dt} = D(S_{2in} - S_2) + k_2\mu_1(S_1)X_1 - k_3\mu_2(S_2)X_2 \quad (3)$$
$$\frac{dX_2}{dt} = (\mu_2(S_2) - \alpha D)X_2 \quad (4)$$
$$\frac{dZ}{dt} = D(Z_{in} - Z) \quad (5)$$
$$\frac{dC}{dt} = D(C_{in} - C) - qc(\xi) + k_4\mu_1(S_1)X_1 + k_5\mu_2(S_2)X_2 \quad (6)$$

where,

X_1 : concentration of acidogenic bacteria [g/L]
X_2 : concentration of methanogenic bacteria [g/L]
S_1 : organic substrate concentration [g/L]
S_2 : volatile fatty acids concentration [mmol/L]
D : dilution rate [1/day]
Z : total alkalinity [mmol/L]
C : total inorganic carbon concentration [mmol/L]
S_{1in} : organic substrate influent concentration [g/L]
S_{2in} : volatile fatty acids influent concentration [mmol/L]
C_{in} : inorganic carbon influent concentration [mmol/L]
Z_{in} : alkalinity influent [mmol/L]
k_1 : yield for substrate degradation
k_2 : yield for VFA production [mmol/g]
k_3 : yield for VFA consumption [mmol/g]
k_4, k_5 : yield for CO_2 production [mmol/g]
k_La : liquid-gas transfer constant [1/day]

The model described dynamically, how the organic substrate S_1 is degraded in volatile fatty acids S_2 by an acidogenic bacteria X_1. Whilst the volatile fatty acids S_2 are degraded in methane CH_4 by a methanogenic bacteria X_2. The growing dynamics of degrading biomasses X_1 and X_2 are due to the consumption of their corresponding substrates. The adimensional parameter $\alpha \in [0, 1]$ is a fraction of the biomass not affected by the dilution rate D [18]. Moreover, S_{1in},S_{2in}, C_{in} and Z_{in} are the input concentrations of the fed waste.

The AM2 model in (6) is represented by the following models:

The Monod model is considered to describe the growth of the acidogenic bacteria [10]

$$\mu_1(S_1) = \mu_{1max}\frac{S_1}{S_1 + K_{S1}}$$

where,

μ_{1max} : maximum acidogenic bacteria growth rate [1/day]
K_{S1} : half-saturation constant [g/L]

The methanogenic microorganism kinetics is described by the Haldane model as in [10].

$$\mu_2(S_2) = \mu_{2max}\frac{S_2}{S_2 + K_{S2} + \frac{S_2^2}{K_{I2}}}$$

where,

μ_{2max} : maximum methanogenic bacteria growth rate [1/day]
K_{S2} : half-saturation constant [mmol/L]
K_{I2} : inhibition constant [mmol/L]

Considering model variables in vector form $\xi = [S_1, X_1, S_2, X_2]^T$, the CO_2 gas flow rate and partial pressure, qc and P_c respectively are obtained:

$$qc(\xi) = k_La[C + S_2 - Z - K_HP_c(\xi)]$$
$$P_c(\xi) = \frac{\phi - \sqrt{\phi^2 - 4K_HP_T(C + S_2 - Z)}}{2K_H} \quad (7)$$

where

$$\phi = C + S_2 - Z + k_HP_T + \frac{k_6}{k_La}\mu_2(S_2)X_2$$

with,

K_H : Henry's constant [mmol/L per atm]
P_T : total pressure [atm]
k_6 : yield for CH_4 production [mmol/g]

The positive root in (7) is not used because is not a physically admissible [10].

A. Equilibrium points

The equilibrium points E_1, E_2, E_3 and E_4 are trivial. E_1 corresponds to the washout of X_1 and X_2, whilst E_2 and E_3 to the washout of X_1. The equilibrium E_4 is related to the washout of X_2. The equilibrium points E_5 and E_6 are positive and non trivial. E_5, is the nominal operating point, as it is locally stable, while E_6, is always unstable as shown in [19].

$$E_1 = (\ S_{1in}\ ,\ 0\ ,\ S_{2in}\ ,\ 0) \quad (8)$$

$$E_2 = \left(\ S_{1in}\ ,\ 0\ ,\ \lambda_{S_2}^{(1)}\ ,\ \frac{S_{2in} - \lambda_{S_2}^{(1)}}{\alpha k_3}\right) \quad (9)$$

$$E_3 = \left(\ S_{1in}\ ,\ 0\ ,\ \lambda_{S_2}^{(2)}\ ,\ \frac{S_{2in} - \lambda_{S_2}^{(2)}}{\alpha k_3}\right) \quad (10)$$

$$E_4 = \left(\ \lambda_{S_1},\ \frac{S_{1in} - \lambda_{S_1}}{\alpha k_1}\ ,\ S_2^{\star}\ ,\ 0\right) \quad (11)$$

$$E_5 = \left(\ \lambda_{S_1},\ \frac{S_{1in} - \lambda_{S_1}}{\alpha k_1},\ \lambda_{S_2}^{(1)},\ \frac{S_2^{\star} - \lambda_{S_2}^{(1)}}{\alpha k_3}\right) \quad (12)$$

$$E_6 = \left(\ \lambda_{S_1},\ \frac{S_{1in} - \lambda_{S_1}}{\alpha k_1},\ \lambda_{S_2}^{(2)}\ ,\ \frac{S_2^{\star} - \lambda_{S_2}^{(2)}}{\alpha k_3}\right) \quad (13)$$

where,

$$\lambda_{S_1} = \frac{K_{S1}\alpha D}{\mu_{1max} - \alpha D},\ \ \mu_{1max} > \alpha D.$$

$$\lambda_{S_2}^{(1)} = -\ \frac{r_{S_2} - K_{I2}\mu_{2_{max}} + \alpha D K_{I2}}{2\alpha D}$$

$$\lambda_{S_2}^{(2)} = \frac{r_{S_2} + K_{I2}\mu_{2_{max}} - \alpha D K_{I2}}{2\alpha D}$$

$$S_2^{\star} = S_{2in} + \frac{k_2}{k_1}\left(S_{1in} - \lambda_{S_1}\right)$$

with $r_{S_2} = \sqrt{(K_{I2}\mu_{2_{max}} - \alpha D K_{I2})^2 - 4\alpha^2 D^2 K_{I2} K_{S2}}$.

The AM2 model is represented by a set of first-order, autonomous and non-linear ordinary differential equations

$$\dot{x} = f(x) \quad (14)$$

with hyperbolic and non-hyperbolic equilibrium and $x = (S_1,\ X_1,\ S_2,\ X_2)^T$.

1) Hyperbolic Equilibrium :

Definition 1: An equilibrium point $x^{\star}$ is called a hyperbolic equilibrium point of (14) if all the eigenvalues of the matrix $A = J_f(x^{\star})$, that is, the eingenvalues of the Jacobian matrix have real parts other than zero. Where, $J_f(x^{\star})$ is the Jacobian of the function f evaluated in the equilibrium $x^{\star}$.

Let f be a C^1 vector field on $\mathbb{R}^n$ such that $f(0) = 0$. If the linearization A of f at the origin is infinitesimally hyperbolic, then f is locally topologically conjugate to A at the origin.

The Theorem Hartman-Grobman in [20] stays that the behaviour of the nonlinear system (14) close to an hyperbolic equilibrium point $x^{\star}$ is determined qualitatively by the linear behavior of the system.

A complete report on the classification of the equilibirum points as stable or unstable according the variations of dilution rate D using the linearized system can be found in Benyahia et.al., in [19].

2) Non-hyperbolic Equilibrium: When two of the values of $\lambda_{S_2}^{(1)}$, $\lambda_{S_2}^{(2)}$, S_{2in} or $S_2^{\star}$ are equal, lead to a situation where at least one of the eigenvalues of the Jacobian matrix has real part zero and the equilibriums are fusioned and non-hyperbolic. The complete characterization as hyperbolic or non-hyperbolic can be found in [19].

The analysis of the behavior of hyperbolic equilibrium is thus based on the Theorem Hartman-Grobman, later characterized in [19]. Aiming to perform an stability analysis for control purposes, this study focuses only on the hyperbolic equilibrium points and does not take into account the cases in which $S_{1in} = \lambda_{S_1}$, because in these cases the system presents instability.

B. Analysis of the AM2 model

Let $S_{1in} > \lambda_{S_1}$, then the condition

$$\lambda_{S_2}^{(1)} < S_{2in} < \lambda_{S_2}^{(2)} < S_2^{\star} \quad (15)$$

leads to the stable equilibrium points E_4, E_5 and unstable equilibrium E_1, E_2, E_6.

The trivial solution E_4, for the case described above is stable, this corresponds to acidification of the digester is called acidification in steady state. It is besides characterized by having a null bacterial biomass, which results in the non-production of biogas.

The two stable states E_4 and E_5 have positive biomass. In addition, E_5 has the equilibrium with the highest biomass,so it will be called the operating point [21].The behavior of the system mainly depends on the initial conditions. For some conditions the trajectories converge to E_4 and for others to E_5. In order to evaluate the global stability of the system the set of initial conditions that lead to E_5, as characterized in [21] are employed.

The equilibrium E_6 is a saddle point whose separation separates the phase plane in the basins of attraction of E_4 and E_5 as illustrated in Fig. 1.

The separatrix in Fig. 1 can be calculated numerically, starting near the unstable equilibrium towards the stable direction. It is important to clarify that the larger the basin of attraction at the point of operation, the less likely there is that after the disturbances the new initial conditions will be found in the basin of attraction of the point of acidification, and that eventually the system will evolve towards acidification [21]. In order to establish the inhibition and convergence zones, the relative size of the attraction basin of the operating point, known as the risk index [9], together with an analysis of the system location regarding the separatirx can be employed.

This risk index associated with the minimum number of zones that the system has to cross (that is, the number of transitions) before reaching the convergence regions) [9].

III. Systematic procedure for calculating the Lyapunov's function

A procedure to determine the Lyapunov function for a hyperbolic equilibrium point $x^{\star}$ for an autonomous system, is:

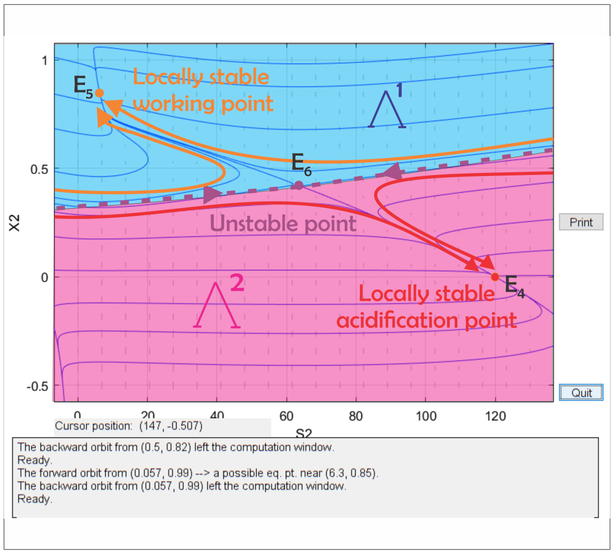

Fig. 1. Phase Plane of (S_2, X_2)

Step 1. Compute the matrix

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=0} \quad (16)$$

Step 2. Verify that A is Hurwitz ($\forall Re\ \lambda_i < 0$). If not then change the equilibrium point.

Step 3. Choose the Q matrix to be symmetric positive definite. Commonly the identity matrix is chosen $Q = I$.

Step 4. Obtain P from

$$PA + A^T P = -Q. \quad (17)$$

Step 5. If P matrix is symmetric positive definite (P is symmetric and it has all positive eigenvalues), then $x^\star$ is asymptotically stable and can continue to the next step. Otherwise, a different Q matrix must be chosen or change the equilibrium point.

Step 6. Calculate the positive definite Lyapunov's function $V(x)$ for the equilibrium point $x^\star$, which is in quadratic form in terms of $(x - x^\star)$.

$$V(x) = (x - x^\star)^T P (x - x^\star)$$

IV. Results and Discussion

In order to characterize the equilibrium points of the AM2 model, the system is translated to the origin, since it does not have null equilibrium points. Following procedure is in general equivalent to make a variable change as recommended in [22] on each of the equilibria, namely

$$z_1 = S_1 - E_{(i)} \quad (18)$$
$$z_2 = X_1 - E_{(i)} \quad (19)$$
$$z_3 = S_2 - E_{(i)} \quad (20)$$
$$z_4 = X_2 - E_{(i)} \quad (21)$$

with $i = 1, 2, .., 6$. Where $E_{(i)}(j)$ corresponds to the j-th component of the equilibrium i. The Jacobian $\frac{\partial f}{\partial z}(z)$ is evaluated in $z = 0$.

Subsequently, the system transferred to the origin is obtained according to the equilibrium where the change is made. Six Jacobians are obtained, one for each equilibrium, the eigenvalues are calculated for each of them according to the parameters from Table I and reported in Table II.

TABLE I
Nominal values of the model parameters (Source:[19])

$k_1 = 25$	$K_{I2} = 40$	$\alpha = 0.5$	$D = 0.8$
$k_2 = 250$	$K_{S1} = 2$	$\mu_{1_{max}} = 1.2$	$S_{1in} = 8$
$k_3 = 268$	$K_{S2} = 10$	$\mu_{2_{max}} = 1.1$	$S_{2in} = 50$

TABLE II
Eigenvalues according to equilibrium E_i

Equilibrio	λ_1	λ_2	λ_3	λ_4
E_1	0.049	0.56	−0.8	
E_2	−3.36	−0.35	0.56	−0.8
E_3	−23.977	−0.393	0.56	−0.8
E_4	−4.176	−0.358	−0.131	−0.8
E_5	−7.990	−0.379	−4.176	−0.35
E_6	−0.656	0.225	−4.176	−0.358

Remark 1: Both the eigenvalues of the Jacobian and the Jacobian evaluated in the translated system are the same as in the untranslated system. As the analysis is made for control design purposes, in this work we consider the system without transferring and the quadratic form is calculated in $(x - E_i)$ for $i = 4, 5$.

In the cases where there is at least one positive eigenvalue, an unstable equilibrium is obtained. Therefore, the stable equilibria are E_4 and E_5, which are of total interest to find a Lyapunov's function. The nominal values of Table I are under the conditions of the case in equation 15 and are evaluated in the mentioned equilibria:

$$E_4 = (1,\ \ 0.5600,\ \ 120,\ \ 0)$$
$$E_5 = (1,\ \ 0.5600,\ \ 6.2772,\ \ 0.8487)$$

The Jacobians evaluated in E_4 and E_5 are the matrices A_4 and A_5, respectively. Moreover, it is shown that A_i (for $i = 4, 5$) is Hurwitz when computing $V(x)$, thus fulfilling the hypothesis from theorem Lyapunov's stability [22].

1) Lyapunov's function for equilibrium E_4: Systematic procedure to obtain the Lyapunov's function is described as follows:

Step 1. Matrix A_4 is computed

$$A_4 = \frac{\partial f}{\partial x}(x)\Big|_{x=E_4} = \begin{pmatrix} a_{411} & a_{412} & 0 & 0 \\ a_{421} & a_{422} & 0 & 0 \\ a_{431} & a_{432} & -D & a_{434} \\ 0 & 0 & 0 & a_{444} \end{pmatrix}$$

considering the components

$$a_{411} = \frac{(S_{1in} - \lambda_{S1})\,\mu_{1_{max}}}{(\lambda_{S1} + K_{S1})\,\alpha}\left[-1 + \frac{\lambda_{S1}}{\lambda_{S1} + K_{S1}}\right] - D$$

$$a_{412} = -\frac{\lambda_{S1}\,k_1\,\mu_{1_{max}}}{\lambda_{S1} + K_{S1}}$$

$$a_{421} = \frac{(S_{1in} - \lambda_{S1})\left(\frac{\mu_{1_{max}}}{\lambda_{S1}+K_{S1}} - \frac{\lambda_{S1}\,\mu_{1_{max}}}{(\lambda_{S1}+K_{S1})^2}\right)}{\alpha\,k_1}$$

$$a_{422} = \frac{\lambda_{S1}\,\mu_{1_{max}}}{\lambda_{S1} + K_{S1}} - D\,\alpha$$

$$a_{431} = \frac{(S_{1in} - \lambda_{S1})\,\mu_{1_{max}} k_2}{(\lambda_{S1} + K_{S1})\,\alpha k_1}\left[1 - \frac{\lambda_{S1}}{\lambda_{S1} + K_{S1}}\right]$$

$$a_{432} = \frac{\lambda_{S1}\,k_2\,\mu_{1_{max}}}{\lambda_{S1} + K_{S1}}$$

$$a_{434} = -\frac{S_2^\star\,k_3\,\mu_{2_{max}}}{\frac{S_2^{\star 2}}{K_{I2}} + S_2^\star + K_{S2}}$$

$$a_{444} = \frac{S_2^\star\,\mu_{2_{max}}}{\frac{S_2^{\star 2}}{K_{I2}} + S_2^\star + K_{S2}} - D\,\alpha$$

Under the nominal parameter values presented in Table I, obtaining A_4.

Step 2. Matrix A_4 is verified to be Hurwitz. Then, according to Theorem Lyapunov's Indirect Method in [22], the equilibrium is asymptotically stable. Since A_4 is Hurwitz then for any positive definite symmetric matrix Q_4 there exists a positive definite symmetric matrix P_4 that satisfies the Lyapunov's equation (17), being P_4 the unique solution.

Step 3. Q_4 is chosen as an identity matrix of size 4x4.

Step 4. Obtain the solution for (17)

$$P_4 = \begin{pmatrix} 45.864 & 173.661 & 4.861 & -567.664 \\ 173.661 & 747.590 & 17.361 & -3257.318 \\ 4.861 & 17.361 & 0.625 & -48.487 \\ -567.664 & -3257.318 & -48.487 & 26804.9301 \end{pmatrix}$$

Step 5. It is verified that P_4 is positive definite.

Step 6. Finally, the Lyapunov's function $V_{E_4}(x)$ positive definite is calculated for the equilibrium point E_4,

$$\begin{aligned} V_{E_4}(x) = {} & (S_2 - 120)(4.861S_1 + 0.625S_2 + 17.361X_1 \\ & - 48.487X_2 - 89.583) + (X_1 - 14/25)(173.661S_1 \\ & + 17.361S_2 + 747.590X_1 - 3257.318X_2 - 2675.6451) \\ & + (S_1 - 1)(45.863S_1 + 4.861S_2 + 173.661X_1 \\ & - 567.664X_2 - 726.447) - X_2(567.664S_1 + 48.487S_2 \\ & + 3257.318X_1 - 26804.9301X_2 - 8210.183) \end{aligned}$$

with $\mathcal{D} = \left\{x \in \mathbb{R}^4 : \|x\|_2 < 488.78\right\}$, $V_{E_4}(E_4) = 0$, and $V_{E_4}(x) > 0$ in $\mathcal{D} - \{0\}$, and $\dot{V}_{E_4}(x) < 0$ in $\mathcal{D} - \{0\}$. Therefore, E_4 is asymptotically stable in a Lyapunov's sense.

2) Lyapunov's function for equilibrium E_5: . The process is described as follows:

Step 1. Matrix A_5 is obtained

$$A_5 = \frac{\partial f}{\partial x}(x)\bigg|_{x=E_5} = \begin{pmatrix} a_{511} & a_{512} & 0 & 0 \\ a_{521} & a_{522} & 0 & 0 \\ a_{531} & a_{532} & a_{533} & a_{534} \\ 0 & 0 & a_{543} & a_{544} \end{pmatrix}$$

with components

$$a_{511} = a_{411},\quad a_{512} = a_{412},\quad a_{521} = a_{421},$$
$$a_{522} = a_{422},\quad a_{531} = a_{431},\quad a_{532} = a_{432},$$

$$a_{533} = \frac{\left(S_2^\star - \lambda_{S_2}^{(1)}\right)\,\mu_{2_{max}}}{r_\lambda\,\alpha}\left[-1 + \frac{\lambda_{S_2}^{(1)}\left(\frac{2\lambda_{S_2}^{(1)}}{K_{I2}} + 1\right)}{r_\lambda}\right] - D$$

$$a_{534} = -\frac{\lambda_{S_2}^{(1)}\,k_3\,\mu_{2_{max}}}{r_\lambda}$$

$$a_{543} = \frac{\left(S_2^\star - \lambda_{S_2}^{(1)}\right)\left(\frac{\mu_{2_{max}}}{r_\lambda} - \frac{\lambda_{S_2}^{(1)}\left(\frac{2\lambda_{S_2}^{(1)}}{K_{I2}}+1\right)\mu_{2_{max}}}{{r_\lambda}^2}\right)}{\alpha\,k_3}$$

$$a_{544} = \frac{\lambda_{S_2}^{(1)}\,\mu_{2_{max}}}{r_\lambda} - D\,\alpha$$

with $r_\lambda = \frac{\left[\lambda_{S_2}^{(1)}\right]^2}{K_{I2}} + \lambda_{S_2}^{(1)} + K_{S2}$. Using the parameters nominal values in Table I, obtain A_5.

Step 2. Matrix A_5 is verified to be Hurwitz. Analogous to step 2 for A_4, there exists a positive definite symmetric matrix P_5 that satisfies the Lyapunov's equation (17), being P_5 the unique solution.

Step 3. The matrix Q_5 is chosen as an identity matrix.

Step 4. The solution for (17) is computed

$$P_5 = \begin{pmatrix} 1.638 & 4.439 & 0.168 & -4.658 \\ 4.439 & 22.379 & 0.439 & -22.124 \\ 0.168 & 0.439 & 0.059 & 0.005 \\ -4.658 & -22.124 & 0.005 & 228.214 \end{pmatrix}$$

Step 5. It is verified that P_5 is a positive definite matrix.

Step 6. The Lyapunov's function is calculated

$$\begin{aligned} V_{E_5}(x) = {} & (X_1 - 14/25)(4.439S_1 + 0.439S_2 \\ & + 22.379X_1 - 22.124X_2 - 0.949) \\ & - (X_2 - 0.8487)(4.658S_1 - 0.005S_2 + 22.124X_1 \\ & - 228.214X_2 + 176.662) + (S_1 - 1)(1.638S_1 \\ & + 0.168S_2 + 4.439X_1 - 4.658X_2 - 1.224) \\ & + (S_2 - 6.2772)(0.168S_1 + 0.059S_2 + 0.439X_1 \\ & + 0.0047X_2 - 0.793) \end{aligned}$$

with $V_{E_5}(E_5) = 0$ and $V_{E_5}(x)$ positive definite, and $\dot{V}_{E_5}(x) < 0$. Hence, E_5 is asymptotically stable in a Lyapunov's sense.

3) Attraction region: If A is Hurwitz, then the region of attraction of the origin can be estimated [22]. In this case A_5 is Hurwitz, therefore, E_5 is an asymptotically stable and attracting node.

According to [22] in the autonomous system (14) with $f : \mathcal{D} \to \mathbb{R}^4$. The Lyapunov's method can be used to find the region of attraction R_A or an estimate of it. The region of attraction of the origin is defined as follows:

$$R_A = \{x \in \mathcal{D}/\phi(t;x), \forall t \geqslant 0, \phi(t;x) \to 0, t \to 0\}$$

where $\phi(t;x)$ is an analytical solution of the system. In the AM2 model, there is no analytical solution. So, an estimate of R_A is sought.

From the Theorem LaSalle in [22] is seen that an estimate of R_A, is a set

$$\Omega_c = \left\{x \in \mathbb{R}^4/V_{E_5}(x) \leqslant c\right\}$$

where Ω_c is bounded and contained in $\mathcal{D}$.

For the quadratic Lyapunov's function $V_{E_5}(x) = x^T P_5 x$ y $\mathcal{D}$ it can be ensured that $\Omega_c \subset \mathcal{D}$ if $c = 122.2$. Then, the estimate of the R_A is given by

$$\Omega_c = \left\{x \in \mathbb{R}^4/V_{E_5}(x) \leqslant 122.2\right\}.$$

For more information about the calculation of R_A, the interested reader is referred to [22].

V. Conclusion

In this work, an analysis of the equilibrium points of the AM2 model is carried out by the indirect Lyapunov's method. The Lyapunov's functions $V(x)$ are constructed in quadratic form corresponding to the equilibria E_4 and E_5, allowing to establish that the two aforementioned equilibria are asymptotically stable. Since the equilibrium E_4 shows acidification, it is therefore discarded for estimating the region of attraction Ω_c through the set $\mathcal{D}$.

Future work includes the development of a robust nonlinear model based predictive controller.

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