A new energy-dissipative momentum-Conserving time integration scheme using the variational updates framework

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Scope of the presentation

1. Scientific motivations
2. Conserving scheme in the non-linear range
3. New formulation using the variational updates approach
4. Numerical examples
5. Conclusions

1. Scientific motivations Dynamic simulations

- Spatial discretization into finite elements
- Temporal integration of the balance equations $M\ddot{\vec{x}} + \vec{F}^{\text{int}} = \vec{F}^{\text{ext}}$
- 2 methods
  - Explicit method

$$\left.\begin{matrix}\vec{x}_n\\ \dot{\vec{x}}_n\\ \ddot{\vec{x}}_n\end{matrix}\right\} \xrightarrow{\text{approximation}} \ddot{\vec{x}}_{n+1} = M^{-1}\left(\vec{F}_n^{\text{ext}} - \vec{F}_n^{\text{int}}\right) \xrightarrow{\text{deduction}} \vec{x}_{n+1}, \dot{\vec{x}}_{n+1}$$

    - Non iterative
    - Small memory requirement
    - Conditionally stable (small time step)

    } Very fast dynamics

  - Implicit method

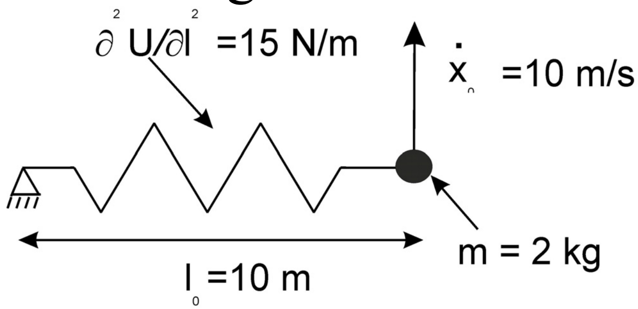
$$\left.\begin{matrix}\vec{x}_n\\ \dot{\vec{x}}_n\\ \ddot{\vec{x}}_n\end{matrix}\right\} \xrightarrow{\text{extrapolation}} \left\{\begin{matrix}\vec{x}_{n+1}\\ \dot{\vec{x}}_{n+1}\\ \ddot{\vec{x}}_{n+1}\end{matrix}\right\} \xleftrightarrow{\text{iterations}} \left\{\begin{matrix} M\ddot{\vec{x}} + \vec{F}^{\text{int}} = \vec{F}^{\text{ext}}\\ \vec{x}_{n+1} = f(\vec{x}_n, \dot{\vec{x}}_n, \dot{\vec{x}}_{n+1}, \ddot{\vec{x}}_n, \ddot{\vec{x}}_{n+1})\\ \dot{\vec{x}}_{n+1} = f(\dot{\vec{x}}_n, \vec{x}_n, \vec{x}_{n+1}, \ddot{\vec{x}}_n, \ddot{\vec{x}}_{n+1})\end{matrix}\right.$$

    - Iterative
    - Larger memory requirement
    - Unconditionally stable (large time step)

    } Slower dynamics

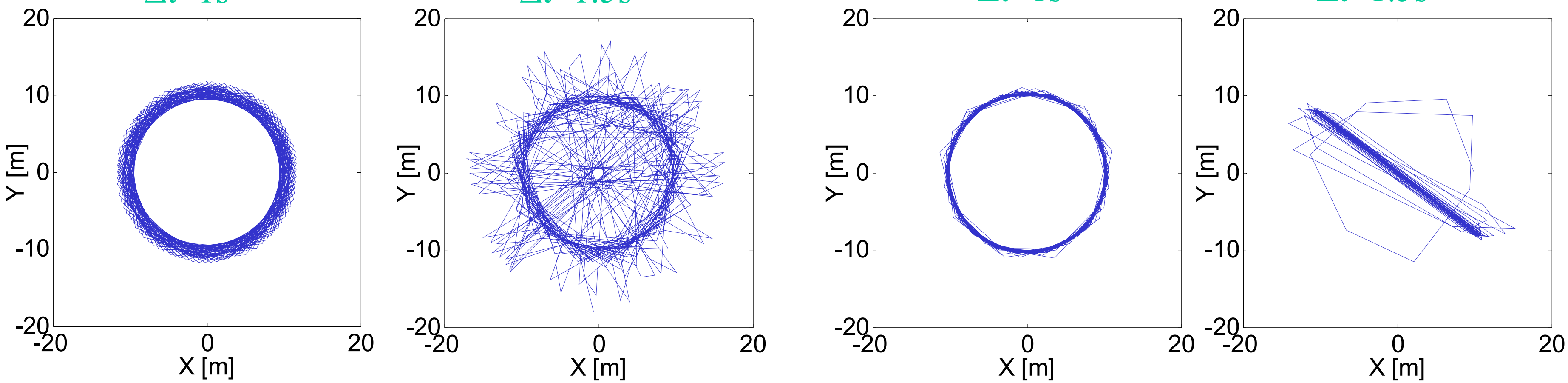
1. Scientific motivations
Non-linear numerical example

- Traditional methods can be unstable in the non-linear range
- Example (Armero & Romero [CMAME, 1999])
  - Mass spring system with geometrical non-linearity
  - 1 revolution ~ 4s



Newmark implicit scheme (no numerical damping) | HHT implicit scheme (numerical damping ρ_∞= 0.9)

Δt=1s | Δt=1.5s | Δt=1s | Δt=1.5s



- Conservation laws are not satisfied leading to divergence, and motivating the development of conserving algorithms

2. Conserving scheme in the non-linear range
Conservation laws

- Conservation of linear momentum (Newton's law $\frac{\partial M\dot{\vec{x}}}{\partial t} = \vec{F}^{\text{ext}}$)
  - Time discretization: $\sum_{\text{nodes}} M\dot{\vec{x}}_{n+1} - M\dot{\vec{x}}_n = \Delta t \sum_{\text{nodes}} \vec{F}^{\text{ext}}_{n+1/2}$ & $\sum_{\text{nodes}} \vec{F}^{\text{int}}_{n+1/2} = 0$
- Conservation of angular momentum ($\frac{\partial \vec{x} \wedge M\dot{\vec{x}}}{\partial t} = \vec{x} \wedge \vec{F}^{\text{ext}}$)
  - Time discretization: $\sum_{\text{nodes}} \sum_{\text{nodes}} \vec{x}_{n+1} \wedge M\dot{\vec{x}}_{n+1} - \vec{x}_n \wedge M\dot{\vec{x}}_n = \Delta t \sum_{\text{nodes}} \vec{x}_{n+1/2} \wedge \vec{F}^{\text{ext}}_{n+1/2}$ & $\sum_{\text{nodes}} \vec{x}_{n+1/2} \wedge \vec{F}^{\text{int}}_{n+1/2} = 0$
- Conservation of energy ($\frac{\partial}{\partial t} K + \frac{\partial}{\partial t} W^{\text{int}} = \frac{\partial}{\partial t} W^{\text{ext}} - \dot{D}^{\text{int}}$)
  - Time discretization: $W^{\text{int}}_{n+1} - W^{\text{int}}_n + \Delta D^{\text{int}} = \sum_{\text{nodes}} \vec{F}^{\text{int}}_{n+1/2} \bullet [\vec{x}_{n+1} - \vec{x}_n]$

$$\& \sum_{\text{nodes}} \frac{1}{2} M\dot{\vec{x}}_{n+1} \bullet \dot{\vec{x}}_{n+1} - \frac{1}{2} M\dot{\vec{x}}_n \bullet \dot{\vec{x}}_n + W^{\text{int}}_{n+1} - W^{\text{int}}_n + \Delta D^{\text{int}} = \sum_{\text{nodes}} \vec{F}^{\text{ext}}_{n+1/2} \bullet [\vec{x}_{n+1} - \vec{x}_n]$$

W^{int}: internal energy; W^{ext}: external energy; D^{int}: dissipation (plasticity …)

- First Energy-Momentum Conserving Algorithm (EMCA)

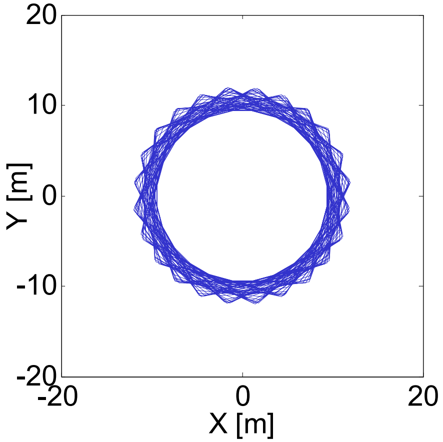
  (Simo and Tarnow [ZAMP, 1992])

  – Based on the mid point scheme:
$$\begin{cases} \dfrac{\dot{\vec{x}}_{n+1}+\dot{\vec{x}}_{n}}{2}=\dfrac{\vec{x}_{n+1}-\vec{x}_{n}}{\Delta t} \\ M\,\dfrac{\dot{\vec{x}}_{n+1}-\dot{\vec{x}}_{n}}{\Delta t}=\vec{F}^{\,\text{ext}}_{n+1/2}-\vec{F}^{\,\text{int}}_{n+1/2} \end{cases}$$

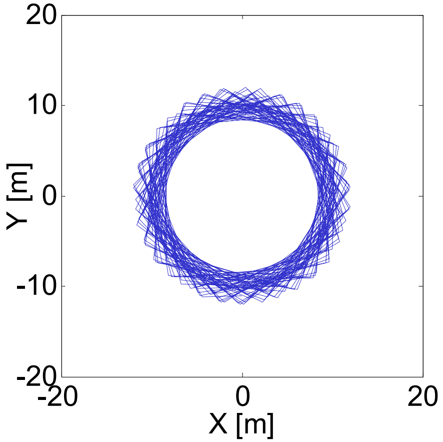
  – With $\vec{F}^{\,\text{int}}_{n+1/2}$ and $\vec{F}^{\,\text{ext}}_{n+1/2}$ designed to verify conserving equations

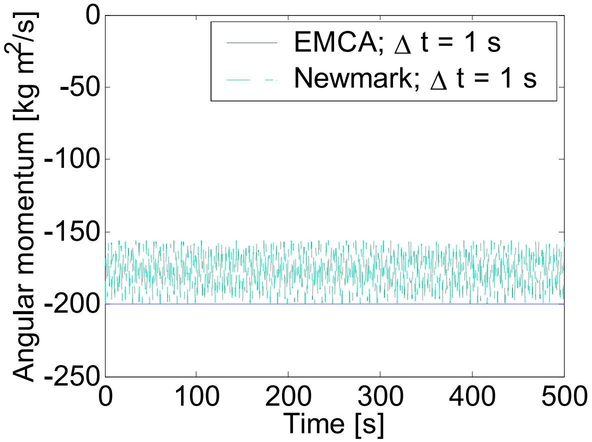
  – Example of the mass-spring system $\vec{F}^{\,\text{int}}_{n+1/2}=\dfrac{V(l_{n+1})-V(l_n)}{l_{n+1}^2-l_n^2}\left[\vec{x}_{n+1}+\vec{x}_n\right]$

EMCA, Δt=1s



EMCA, Δt=1.5s

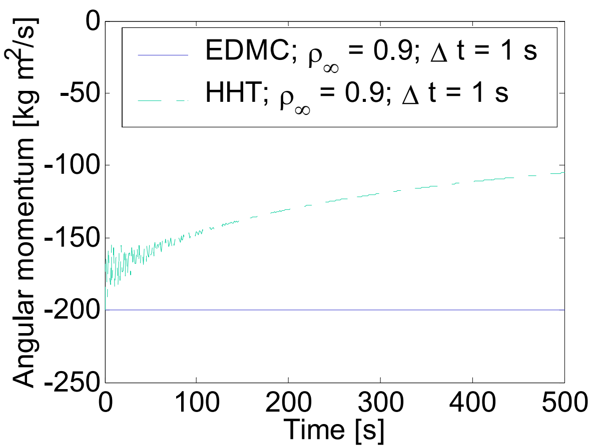


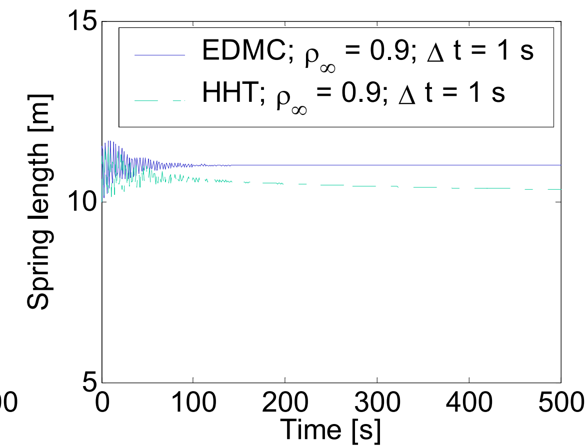


2. Conserving scheme in the non-linear range EDMC

- Introduction of numerical dissipation: Energy-Dissipative Momentum Conserving scheme (EDMC) (Armero & Romero [CMAME, 2001])
  – Introduction of new terms in the EMCA: $\left\{\begin{array}{l} M\dfrac{\dot{\vec{x}}_{n+1}+\dot{\vec{x}}_n}{\Delta t}=\vec{F}^{\text{ext}}_{n+1/2}-\vec{F}^{\text{int}}_{n+1/2}-\vec{F}^{\text{diss}}_{n+1/2} \\ \dfrac{\dot{\vec{x}}_{n+1}+\dot{\vec{x}}_n}{2}+\dot{\vec{x}}^{\text{diss}}_{n+1}=\dfrac{\vec{x}_{n+1}-\vec{x}_n}{\Delta t} \end{array}\right.$
  – Same internal $\vec{F}^{\text{int}}_{n+1/2}$ and external forces $\vec{F}^{\text{ext}}_{n+1/2}$ as in the EMCA
  – With $\dot{\vec{x}}^{\text{diss}}_{n+1}$ and $\vec{F}^{\text{diss}}_{n+1/2}$ designed to achieve positive numerical dissipation without spectral bifurcation $\sum_{\text{nodes}}\frac{1}{2}M\dot{\vec{x}}_{n+1}\bullet\dot{\vec{x}}_{n+1}-\frac{1}{2}M\dot{\vec{x}}_n\bullet\dot{\vec{x}}_n+W^{\text{int}}_{n+1}-W^{\text{int}}_n+\Delta D^{\text{int}}+\Delta D^{\text{num}}=\sum_{\text{nodes}}\vec{F}^{\text{ext}}_{n+1/2}\bullet[\vec{x}_{n+1}-\vec{x}_n]$
  – Example of the mass-spring system for first order accurate EDMC

$$\left\{\begin{array}{l} \vec{F}^{\text{diss}}_{n+1/2}=\chi\dfrac{V''\left(\frac{l_{n+1}+l_n}{2}\right)}{l_{n+1}+l_n}\dfrac{[l_{n+1}-l_n]\,[\vec{x}_{n+1}+\vec{x}_n]}{2} \\ \dot{\vec{x}}^{\text{diss}}_{n+1}=\chi\dfrac{\left\|\dot{\vec{x}}_{n+1}\right\|-\left\|\dot{\vec{x}}_n\right\|}{\left\|\dot{\vec{x}}_{n+1}\right\|+\left\|\dot{\vec{x}}_n\right\|}\dfrac{[\vec{x}_{n+1}+\vec{x}_n]}{2} \end{array}\right.$$



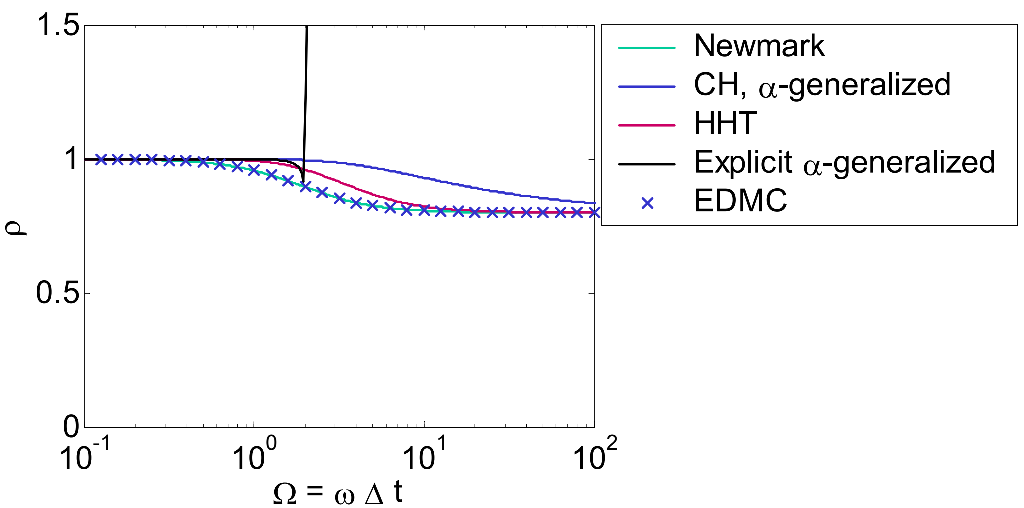


- Comparison of the spectral radius
  - Integration of a linear oscillator $\vec{x}_{n+1} = \exp\left(\frac{\ln \rho^2}{\Delta t} t_{n+1}\right)\cos\left(\omega \quad t_{n+1}\right)$

ρ: spectral radius; ω: pulsation

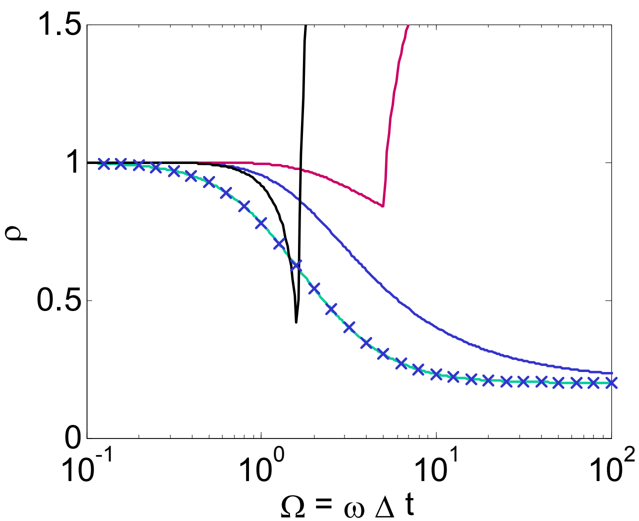
Low numerical dissipation

$\rho_\infty = 0.8$



High numerical dissipation

$\rho_\infty = 0.2$



2. Conserving scheme in the non-linear range
Formulations in the literature

- Elastic formulation
  - Saint Venant-Kirchhoff hyperelastic model (Simo et al. [ZAMP, 1992])
  - General formulation for hyperelasticity (stress derived from a potential V) (Gonzalez [CMAME, 2000])

$$\vec{F}^{\text{int}}_{n+1/2} = \int_{V_0} \frac{\mathbf{F}_n + \mathbf{F}_{n+1}}{2} \mathbf{S}_{n+1/2} \vec{D} dV \quad \text{with} \quad \mathbf{S}_{n+1/2} = 2\frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) + 2\underbrace{\left[\frac{V(\mathbf{C}_{n+1}) - V(\mathbf{C}_n) - \frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) : \Delta\mathbf{C}}{\|\Delta \mathbf{C}\|^2}\right]\Delta\mathbf{C}}_{O\left(\|\mathbf{C}_{n+1} - \mathbf{C}_n\|^2\right)}$$

  - Classical formulation $\vec{F}^{\text{int}} = 2\int_{V_0} \mathbf{F} \frac{\partial V}{\partial \mathbf{C}} \vec{D} dV$

F: deformation gradient; **C**: right Cauchy-Green strain; V: potential; φ: shape functions; $\vec{D} = \partial\varphi / \partial\vec{x}_0$

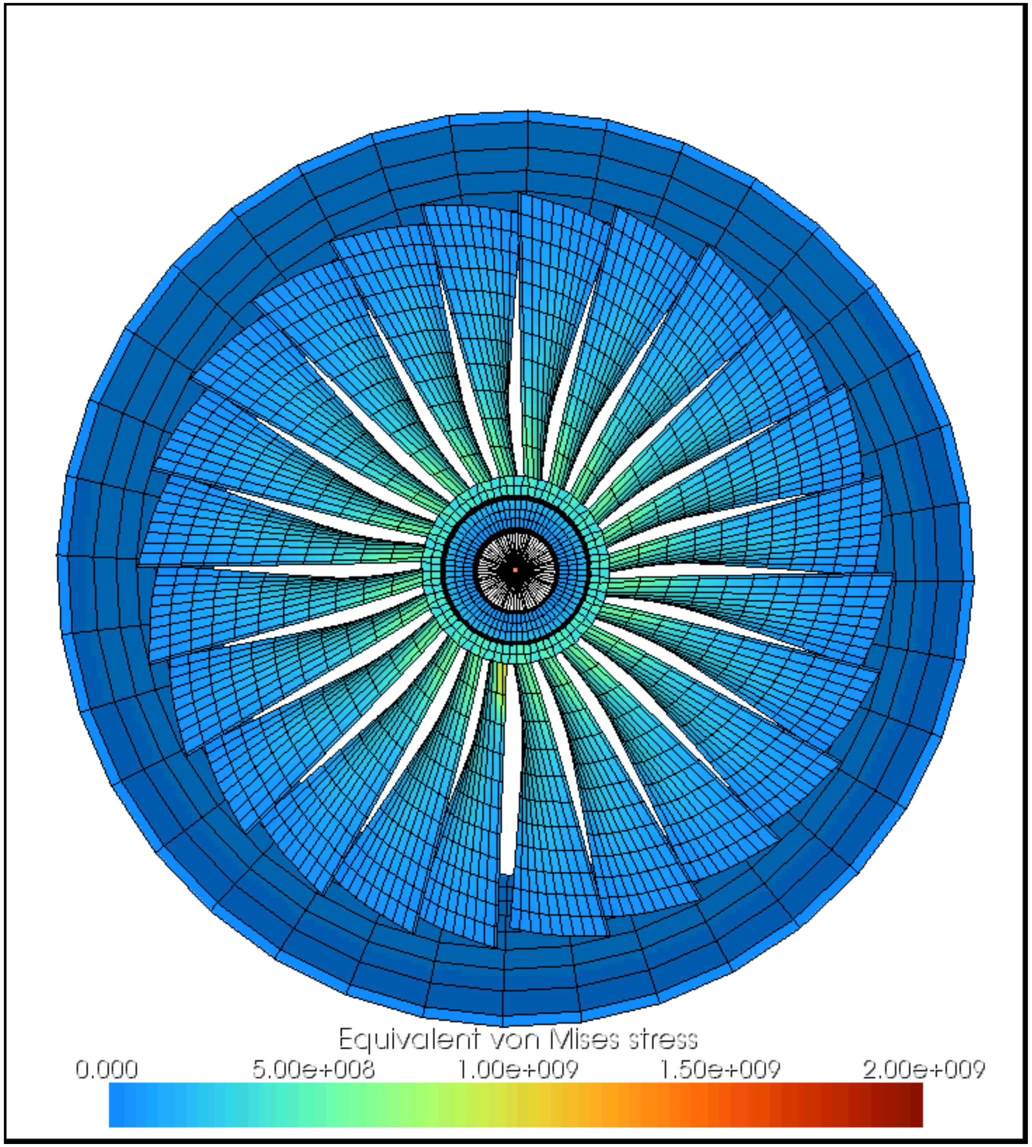
- Penalty contact formulation (Armero & Petöcz [CMAME, 1998-1999])

$$\begin{cases} \vec{F}^{\text{cont}}_{n+1/2} = \dfrac{V\left(g^{\text{d}}_{n+1}\right) - V\left(g^{\text{d}}_n\right)}{g^{\text{d}}_{n+1} - g^{\text{d}}_n} \vec{n}_{n+1/2} \\ g^{\text{d}}_{n+1} = g^{\text{d}}_n + \vec{n}_{n+1/2} \bullet \left[\vec{x}_{n+1} - \vec{x}_n - \vec{y}_{n+1}(u_{n+1/2}) + \vec{y}_n(u_{n+1/2})\right] \end{cases}$$

- Elasto-plastic materials
  - Hyperelasticity with elasto-plastic behavior (Meng & Laursen [CMAME, 2001])
    - energy dissipation of the algorithm corresponds to the internal dissipation of the material
    - Isotropic hardening only
  - Hyperelasticity with elasto-plastic behavior (Armero [CMAME, 2006])
    - Energy dissipation from the internal forces corresponds to the plastic dissipation
    - Modification of the radial return mapping
    - Yield criterion satisfied at the end of the time-step
  - Hypoelastic formulation
    - Stress obtained incrementally from a hardening law
    - No possible definition of an internal potential!
    - Idea: the internal forces are established to be consistent on a loading/unloading cycle
    - Assumption made on the Hooke tensor
    - Energy dissipation from the internal forces corresponds to the plastic dissipation

2. Conserving scheme in the non-linear range
Numerical results

- Blade off
  - Rotation velocity 5,000rpm
  - EDMC algorithm
  - Hypoelastic formulation
  - 29,000 dof's
  - One revolution simulation
  - 9,000 time steps
  - 50,000 iterations (only 9,000 with stiffness matrix updating)
- Demonstrates the robustness and efficiency of the conserving schemes



3. Variational updates approach
Purpose of the work

- Development of a general approach leading to conserving/dissipating algorithms for any material behavior!
- What we want
  - No assumption on the material behavior
  - Material model unchanged compared to the standard approach:
    - From a given strain tensor, the outputs of the model are the same
    - Use of the same material libraries
  - Expression of the internal forces for the conserving algorithm remains the same as in the elastic case
  - Yield criterion satisfied at the end of the time step
- Solution derives from the variational formulation of visco-plastic updates

  [Ortiz & Stainier, CMAME 1999]
  - Definition of an energy, even for complex material behaviors $\mathbf{S}_{n+1} = 2\frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}_{n+1}}\left(\mathbf{C}_{n+1},\mathbf{C}_{n}\right)$

    $\mathbf{C}$: right Cauchy-Green strain; $\mathbf{S}$: second Piola-Kirchhoff stress; ΔD^{eff}: incremental potential
- Introduction of numerical dissipation via ΔD^{eff}

3. Variational updates approach
Use of an incremental potential

- Description of the variational updates for elasto-plasticity
  - Multiplicative plasticity $\mathbf{F} = \mathbf{F}^{\mathrm{el}}\mathbf{F}^{\mathrm{pl}}$
  - Plastic flow $\mathbf{F}^{\mathrm{pl}}_{n+1} = \exp(\Delta\varepsilon^{\mathrm{pl}}\mathbf{N})\mathbf{F}^{\mathrm{pl}}_{n}$ **N**: plastic flow direction
  - Functional increment $$\Delta D(\mathbf{F}_{n+1},\mathbf{F}_n,\varepsilon^{\mathrm{pl}}_{n+1},\varepsilon^{\mathrm{pl}}_{n},\mathbf{N}) = W^{\mathrm{el}}\left(\mathbf{F}_{n+1}\mathbf{F}^{\mathrm{pl}\,-1}_{n+1}(\varepsilon^{\mathrm{pl}}_{n+1},\mathbf{N})\right) - W^{\mathrm{el}}\left(\mathbf{F}_{n}\mathbf{F}^{\mathrm{pl}\,-1}_{n}(\varepsilon^{\mathrm{pl}}_{n},\mathbf{N})\right) + W^{\mathrm{pl}}\left(\varepsilon^{\mathrm{pl}}_{n+1}\right) - W^{\mathrm{pl}}\left(\varepsilon^{\mathrm{pl}}_{n}\right) + \Delta t\Psi^{*}\left(\frac{\varepsilon^{\mathrm{pl}}_{n+1}-\varepsilon^{\mathrm{pl}}_{n}}{\Delta t}\right)$$

    W^{el}: reversible potential; W^{pl}: dissipation by plasticity; Ψ^{*}: dissipation by viscosity
  - Effective potential: $$\Delta D^{\mathrm{eff}}\left(\mathbf{F}_{n+1}\right) = \min_{\varepsilon^{\mathrm{pl}}_{n+1},\mathbf{N}} \Delta D(\mathbf{F}_{n+1},\mathbf{F}_n,\varepsilon^{\mathrm{pl}}_{n+1},\varepsilon^{\mathrm{pl}}_{n+1},\mathbf{N})$$
    - Minimization with respect to $\varepsilon^{\mathrm{pl}}$ satisfies yield criterion
    - Minimization with respect to **N** satisfies radial return mapping
  - Usual stress derivation: $$\mathbf{S}_{n+1} = 2\frac{\partial \Delta D^{\mathrm{eff}}\left(\mathbf{F}_{n+1}\right)}{\partial \mathbf{C}_{n+1}}$$
- Introduction of numerical dissipation
  - Modification of potential $$\Delta D^{\mathrm{eff}}\left(\mathbf{F}_{n+1}\right) = \min_{\varepsilon^{\mathrm{pl}}_{n+1},\mathbf{N}} \Delta D(\mathbf{F}_{n+1},\mathbf{F}_n,\varepsilon^{\mathrm{pl}}_{n+1},\varepsilon^{\mathrm{pl}}_{n+1},\mathbf{N}) + \Delta t\Psi^{\mathrm{diss}}\left(\frac{\mathbf{C}^{\mathrm{el}}_{n+1}-\mathbf{C}^{\mathrm{el}}_{n}}{\Delta t}\right)$$
  - Dissipation function $$\Psi^{\mathrm{diss}}\left(\frac{\mathbf{C}^{\mathrm{el}}_{n+1}-\mathbf{C}^{\mathrm{el}}_{n}}{\Delta t}\right) = \frac{\chi}{2\Delta t}\Delta\mathbf{C}^{\mathrm{el}} : \frac{\partial^2 V^{\mathrm{el}}}{\partial \mathbf{C}^{\mathrm{el}}\partial \mathbf{C}^{\mathrm{el}}} : \Delta\mathbf{C}^{\mathrm{el}}$$

3. Variational updates approach
Use of an incremental potential

- Internal forces directly obtained from Gonzalez elastic formulation [CMAME 2000]

$$\vec{F}^{\text{int}}_{n+1/2} = \int_{V_0} \frac{\mathbf{F}_n + \mathbf{F}_{n+1}}{2} \mathbf{S}_{n+1/2} \vec{D} dV$$

with $$\mathbf{S}_{n+1/2} = 2\frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) + \underbrace{2\left[\frac{V(\mathbf{C}_{n+1}) - V(\mathbf{C}_n) - \frac{\partial V}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) : \Delta\mathbf{C}}{\|\Delta\mathbf{C}\|^2}\right]\Delta\mathbf{C}}_{O\left(\|\mathbf{C}_{n+1} - \mathbf{C}_n\|^2\right)}$$

$$\boxed{\mathbf{S}_{n+1/2} = 2\frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) + 2\left[\frac{\Delta D^{\text{eff}}(\mathbf{C}_{n+1}, \mathbf{C}_n) - \frac{\partial \Delta D^{\text{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_n + \mathbf{C}_{n+1}}{2}\right) : \Delta\mathbf{C}}{\|\Delta\mathbf{C}\|^2}\right]\Delta\mathbf{C}}$$

- Dissipation forces $\vec{F}^{\text{diss}}_{n+1/2}$ are part of $\vec{F}^{\text{int}}_{n+1/2}$

- Modification to quasi-incompressible form
  - Definition of a mean volumetric deformation on the element θ^e.
  - Definition of distortion gradient $\hat{\mathbf{F}} = J^{-1/3}\mathbf{F}$
  - Split of the stress tensor $\mathbf{S}_{n+1/2} = \mathbf{S}^{\text{dev}}_{n+1/2} + 2p_{n+1/2}\mathbf{dG}$
  - Volume part

$$\begin{cases} p_{n+1/2} = \dfrac{\partial \Delta D^{\text{eff}}\left(\theta^e_{n+1}/2+\theta^e_n/2\right)}{\partial \theta^e} + \dfrac{\Delta D^{\text{eff}}\left(\theta^e_{n+1},\theta^e_n\right) - \dfrac{\partial \Delta D^{\text{eff}}\left(\theta^e_{n+1}/2+\theta^e_n/2\right)}{\partial \theta^e}\Delta\theta^e}{\Delta {\theta^e}^2}\Delta\theta^e \\ \mathbf{dG} = \dfrac{\partial J_{n+1/2}}{\partial \mathbf{C}} + \dfrac{\Delta J - \dfrac{\partial J_{n+1/2}}{\partial \mathbf{C}} : \Delta\mathbf{C}}{\left\|\Delta\mathbf{C}\right\|^2}\Delta\mathbf{C} \end{cases}$$

  - Deviatoric part

$$\begin{cases} \mathbf{S}^{\text{dev}}_{n+1/2} = 2\left.\dfrac{\partial \Delta D^{\text{eff}}\left(\hat{\mathbf{C}}^{\text{el}}_{n+1/2}\right)}{\partial \mathbf{C}}\right|_{\theta^e} + 2\dfrac{\Delta D^{\text{eff}}\left(\hat{\mathbf{C}}^{\text{el}}_{n+1},\hat{\mathbf{C}}^{\text{el}}_n\right) - \left.\dfrac{\partial \Delta D^{\text{eff}}\left(\hat{\mathbf{C}}^{\text{el}}_{n+1/2}\right)}{\partial \mathbf{C}}\right|_{\theta^e}\Delta\mathbf{C}}{\left\|\Delta\mathbf{C}\right\|^2}\Delta\mathbf{C} \\ \left.\dfrac{\partial \Delta D^{\text{eff}}\left(\hat{\mathbf{C}}^{\text{el}}_{n+1/2}\right)}{\partial \mathbf{C}}\right|_{\theta^e} = J^{-2/3}_{n+1/2}\text{DEV}\left({\mathbf{F}^{\text{pl}}_{n+1/2}}^{-1}\dfrac{\partial \Delta D^{\text{eff}}\left(\hat{\mathbf{C}}^{\text{el}}_{n+1/2},\hat{\mathbf{C}}^{\text{el}}_n\right)}{\partial \hat{\mathbf{C}}^e}{\mathbf{F}^{\text{pl}}_{n+1/2}}^{-T}\right) \end{cases}$$

with $J_{n+1/2}, \mathbf{F}^{\text{pl}}_{n+1/2}$ and $\hat{\mathbf{C}}^{\text{el}}_{n+1/2}$ resulting from the minimization of $\Delta D\left(\dfrac{\mathbf{C}_{n+1}+\mathbf{C}_n}{2}, {\varepsilon^{\text{pl}}}^{n+1/2}, \mathbf{N}, \mathbf{C}_n, {\varepsilon^{\text{pl}}}^{n}\right)$

- Properties:
  - 2 material configurations computed
    - Mid configuration trough $\frac{\partial \Delta D^{\mathrm{eff}}}{\partial \mathbf{C}}\left(\frac{\mathbf{C}_{n+1}+\mathbf{C}_n}{2}\right)$
    - Final configuration trough $\Delta D^{\mathrm{eff}}\left(\mathbf{C}_{n+1},\mathbf{C}_n\right)$
  - Material model unchanged
  - Yield criterion verified at configuration *n+1*
  - Conservation of linear and angular momentum
  - Numerical dissipation of energy (first order accurate)

$$\sum_{nodes}\vec{F}^{\mathrm{int}}_{n+1/2}\bullet[\vec{x}_{n+1}-\vec{x}_n]=\int_{V_0}\left(W^{\mathrm{el}}_{n+1}-W^{\mathrm{el}}_n+W^{\mathrm{pl}}_{n+1}-W^{\mathrm{pl}}_n+\Delta t\Psi^*+\Delta t\Psi^{\mathrm{diss}}\right)dV$$

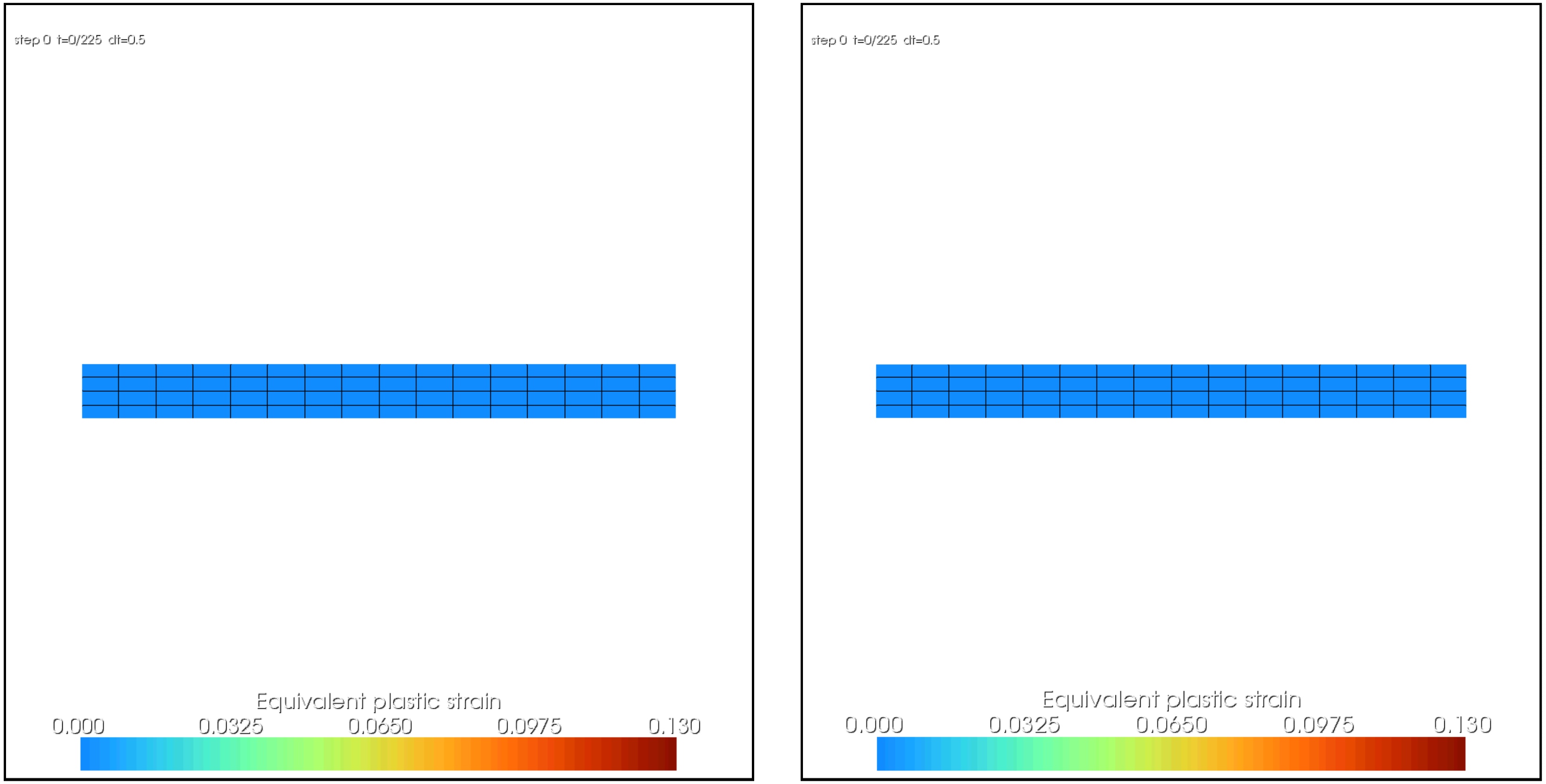
$$\begin{cases} W^{\mathrm{int}}=\int_{V_0}W^{\mathrm{el}}dV \\ \Delta D^{\mathrm{int}}=\int_{V_0}\left(W^{\mathrm{pl}}_{n+1}-W^{\mathrm{pl}}_n+\Delta t\Psi^*\right)dV \\ \Delta D^{\mathrm{num}}=\int_{V_0}\Delta t\Psi^{\mathrm{diss}}dV+M\left(\vec{x}_{n+1}-\vec{x}_n\right)\bullet\vec{x}^{\mathrm{diss}}_{n+1}=\frac{\chi}{2}m\left(\|\vec{x}_{n+1}\|-\|\vec{x}_n\|\right)^2+\frac{\chi}{2}\int_{V_0}\Delta\mathbf{C}^{\mathrm{el}}:\frac{\partial^2 V^{\mathrm{el}}}{\partial\mathbf{C}^{\mathrm{el}}\partial\mathbf{C}^{\mathrm{el}}}:\Delta\mathbf{C}^{\mathrm{el}}dV \end{cases}$$

$$\sum_{\mathrm{nodes}}\frac{1}{2}M\dot{\vec{x}}_{n+1}\bullet\dot{\vec{x}}_{n+1}-\frac{1}{2}M\dot{\vec{x}}_n\bullet\dot{\vec{x}}_n+W^{\mathrm{int}}_{n+1}-W^{\mathrm{int}}_n+\Delta D^{\mathrm{int}}+\Delta D^{\mathrm{num}}=\sum_{\mathrm{nodes}}\vec{F}^{\mathrm{ext}}_{n+1/2}\bullet[\vec{x}_{n+1}-\vec{x}_n]$$

- Tumbling beam
  - Initial symmetrical loads ($t < 10s$)
  - Elasto-perfectly-plastic hyperelastic material

EMCA, Δt=0.5 s

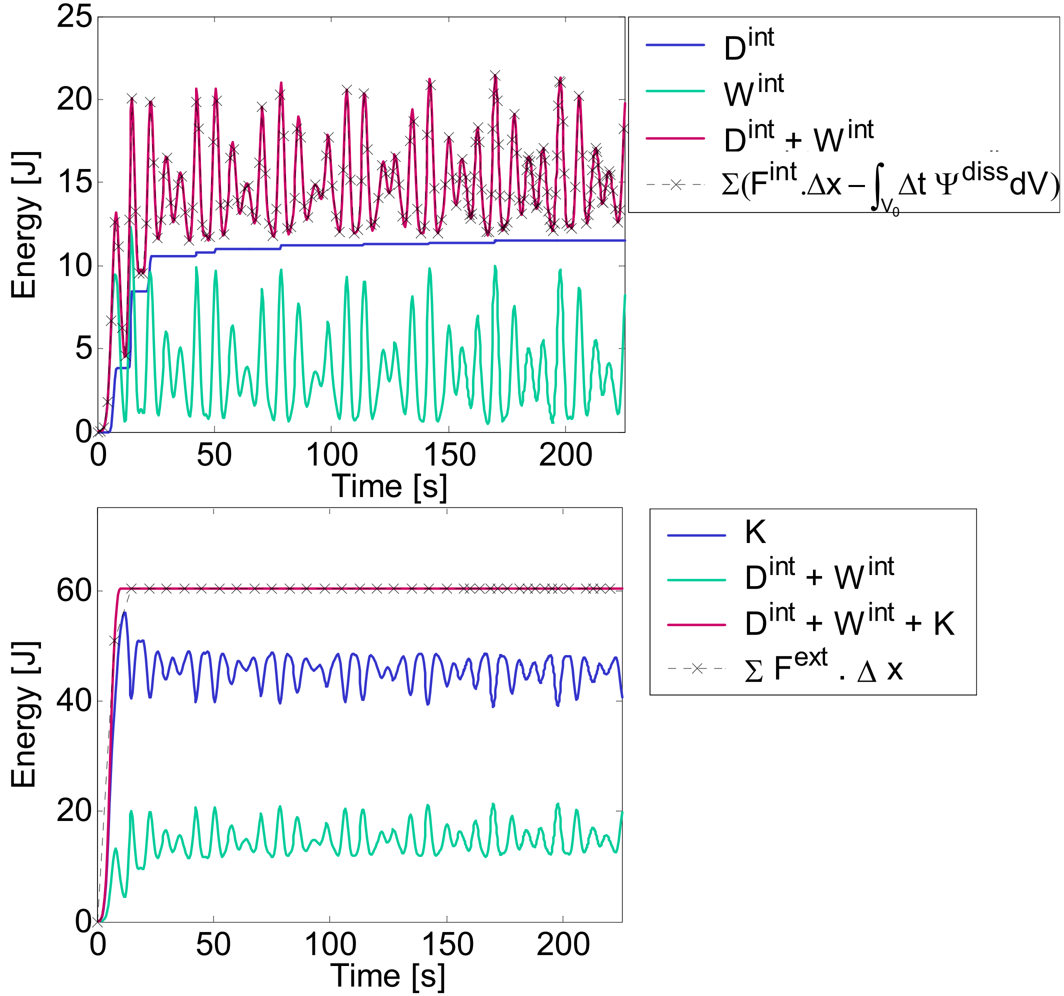
EDMC, Δt=0.5s, ρ_∞= 0.7



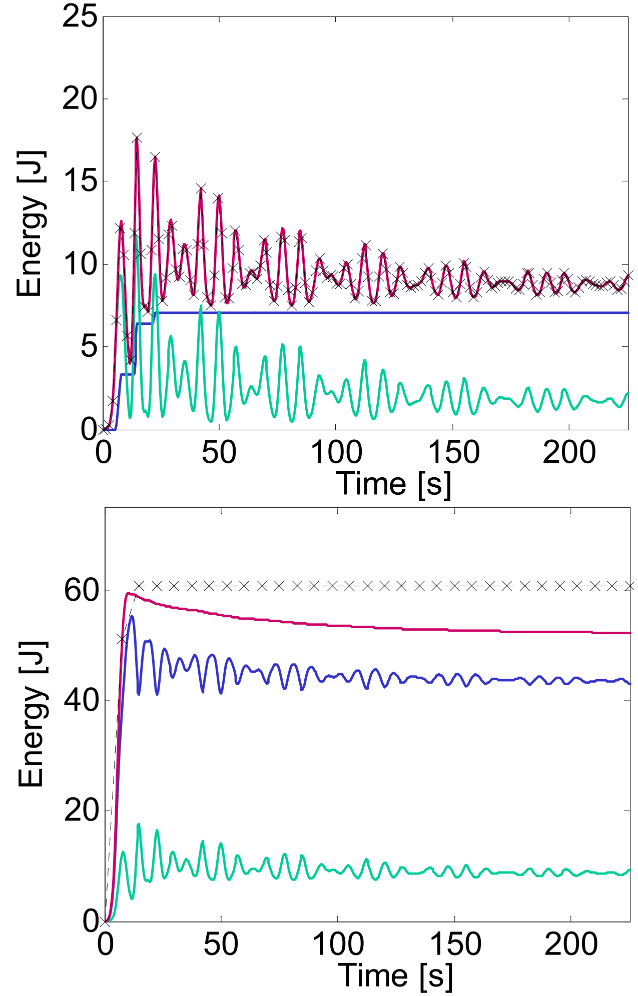
4. Numerical Examples
Simulation of a tumbling beam

- Time evolution of the results

EMCA, Δt=0.5 s

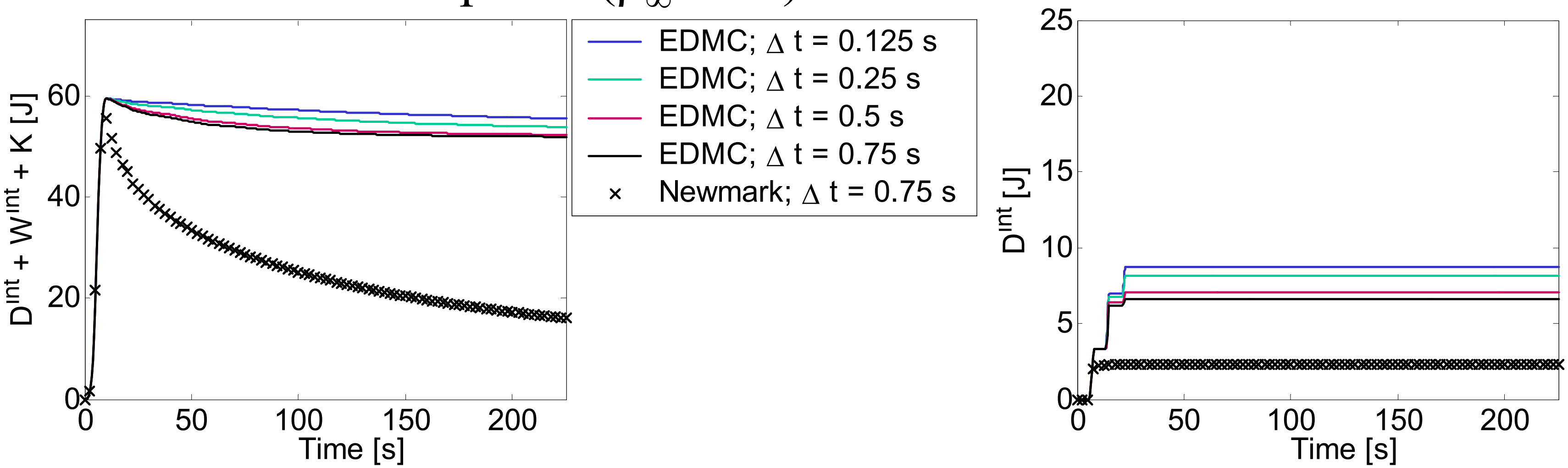


EDMC, Δt=0.5s, ρ_∞= 0.7

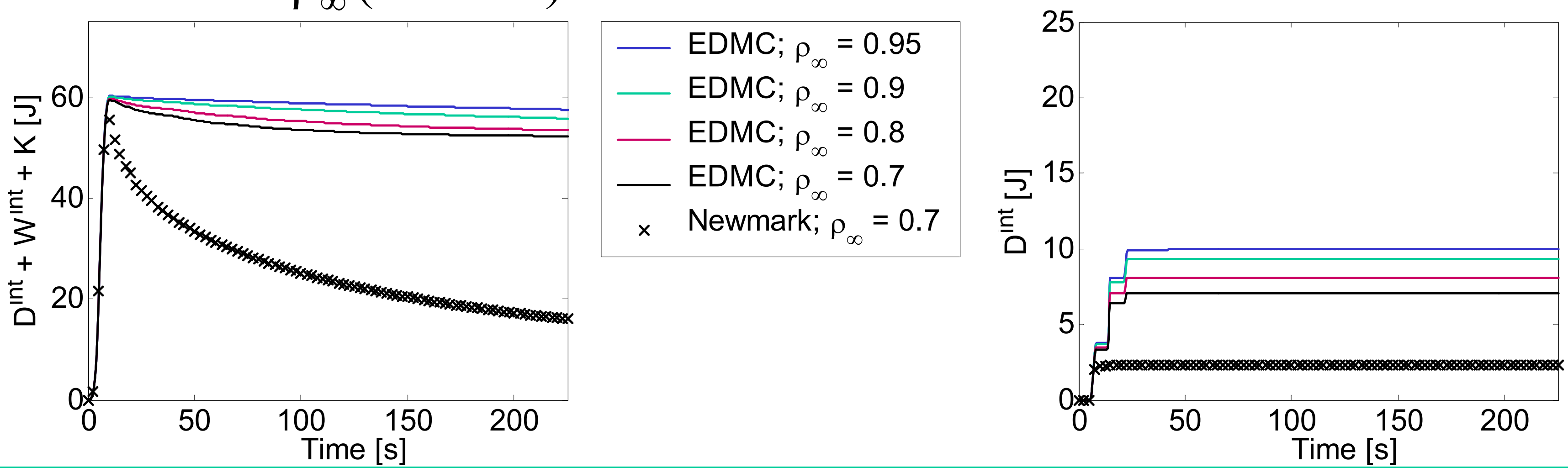


Simulation of a tumbling beam

- Influence of time step size ($\rho_\infty = 0.7$)



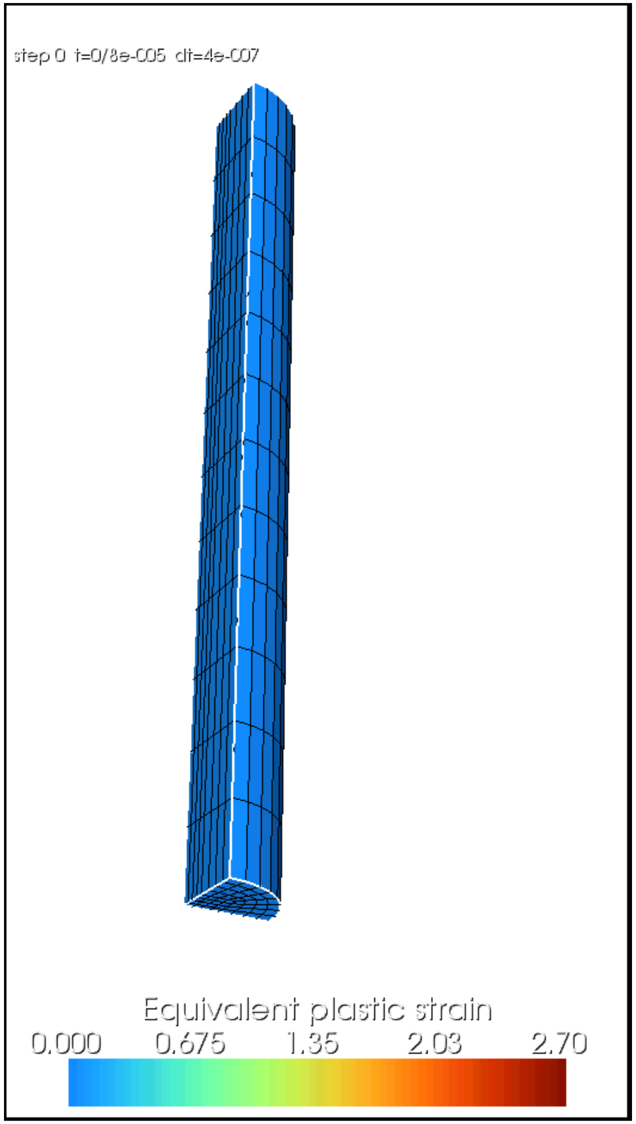
- Influence of ρ_∞ ($\Delta t = 0.5$)



Simulation of the Taylor impact

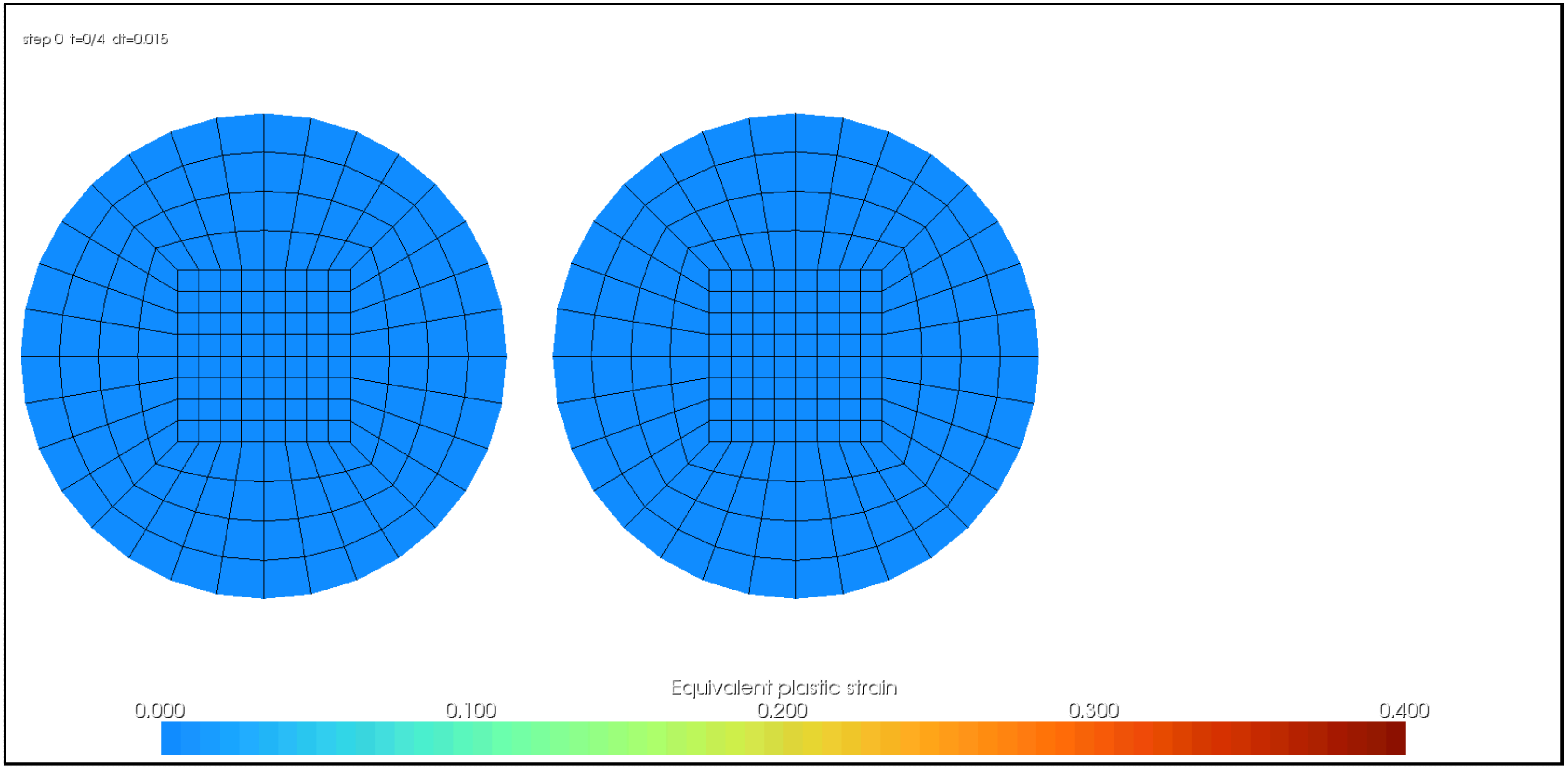
- Impact (270 m/s) of a cylinder
  - Hyperelastic model
  - Elasto-plastic hardening law
  - Simulation during 80 µs

Method	Final length [mm]	Final radius [mm]	Max ε^{pl}
EMCA; Δt = 25 ns	21.4	6.77	2.61
EMCA; Δt = 400 ns	21.4	6.81	2.61
EDMC; Δt = 400 ns; ρ_∞= 0.9	21.46	6.87	2.69
EDMC; Δt = 400 ns; ρ_∞= 0.7	21.50	6.91	2.76
Newmark; Δt = 400 ns; ρ_∞= 0.7	21.56	6.85	2.79
Chung-Hulbert; Δt = 400 ns; ρ_∞= 0.7	21.51	6.91	2.79
EMCA; Meng & Laursen	21.6	6.78	2.62



4. Numerical Examples
Impact of two 2D-cylinders

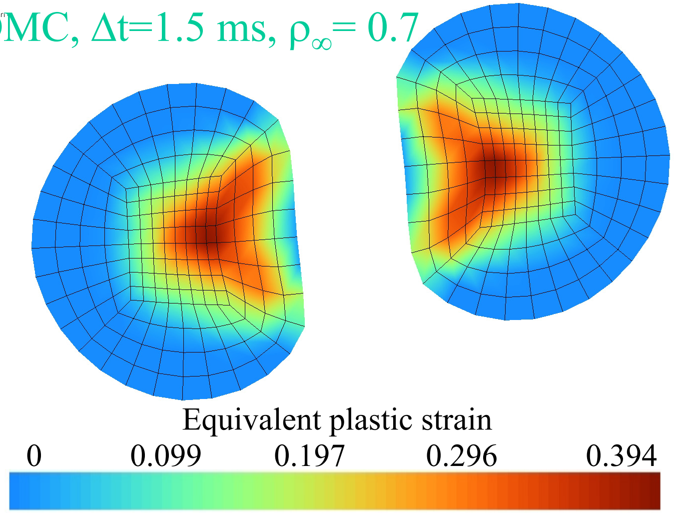
- Two cylinders (Meng & Laursen)
  - Left one has an initial velocity (initial kinetic energy 14J)
  - Elasto-perfectly-plastic hyperelastic material



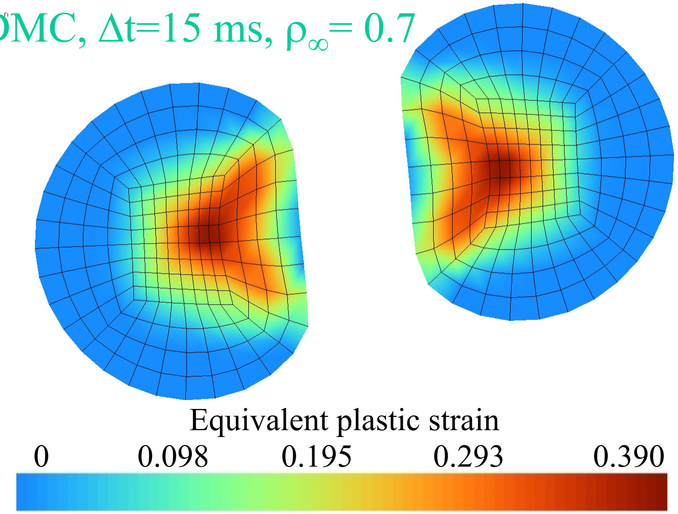
4. Numerical Examples
Impact of two 2D-cylinders

- Results comparison at the end of the simulation

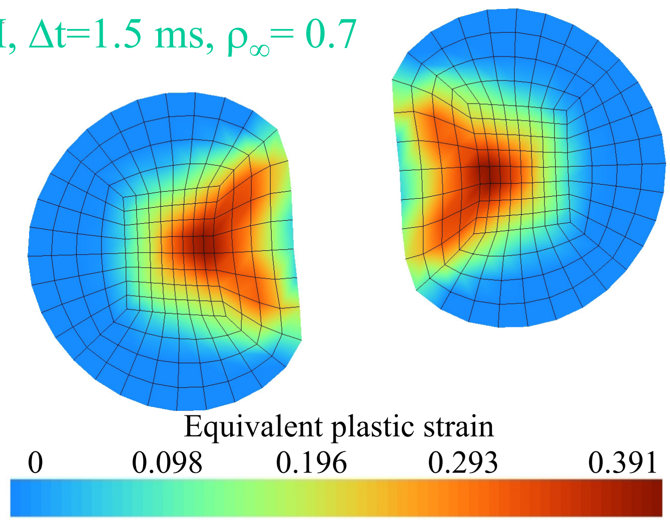
EDMC, Δt=1.5 ms, ρ_∞= 0.7



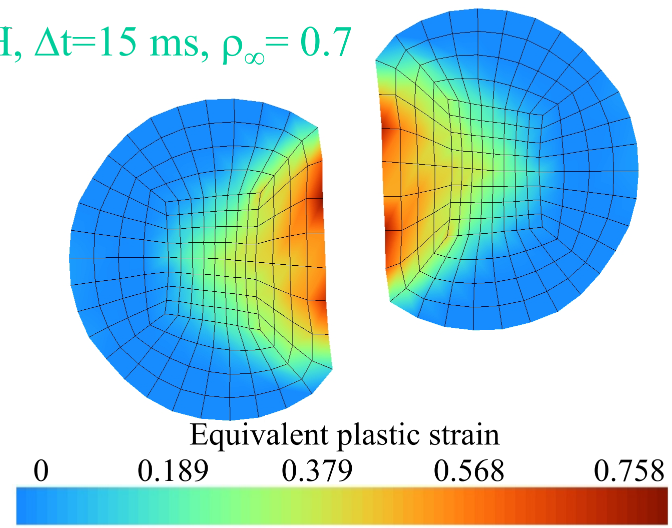
EDMC, Δt=15 ms, ρ_∞= 0.7



CH, Δt=1.5 ms, ρ_∞= 0.7

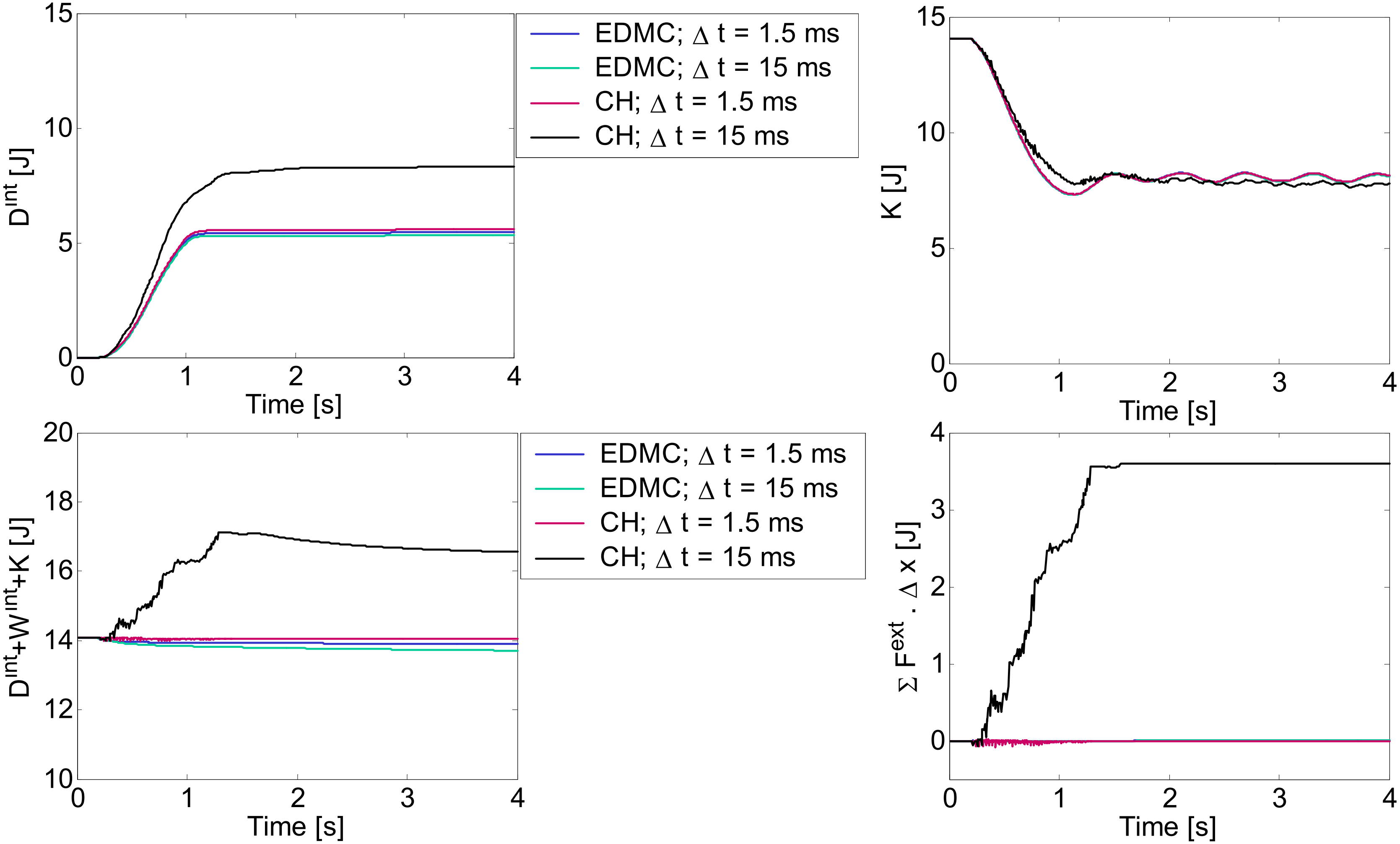


CH, Δt=15 ms, ρ_∞= 0.7



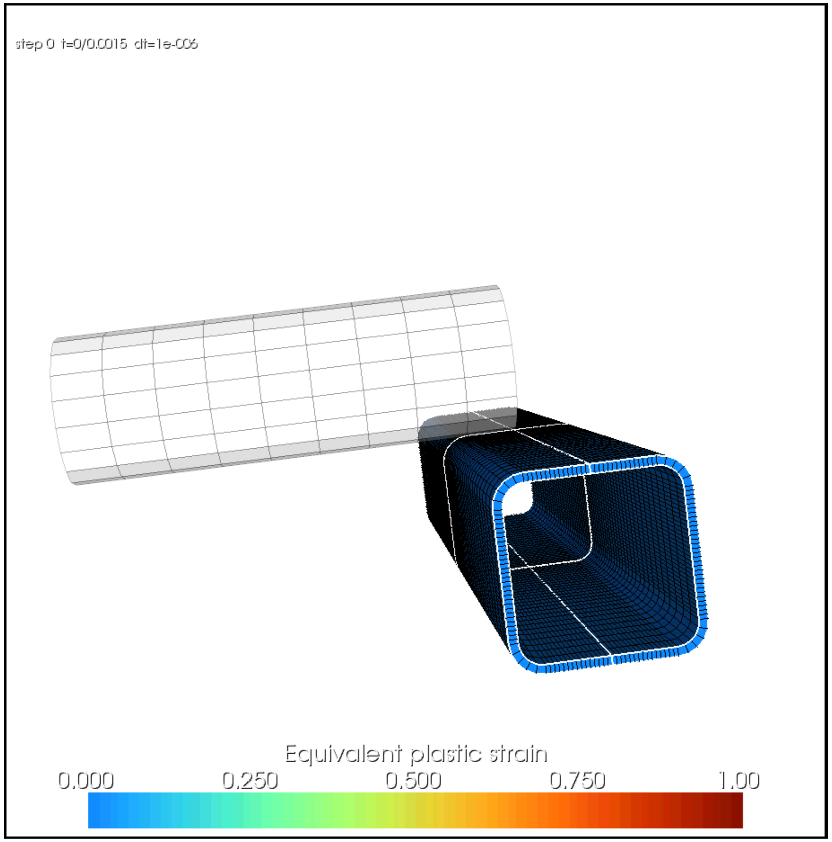
4. Numerical Examples
Impact of two 2D-cylinders

- Results evolution comparison

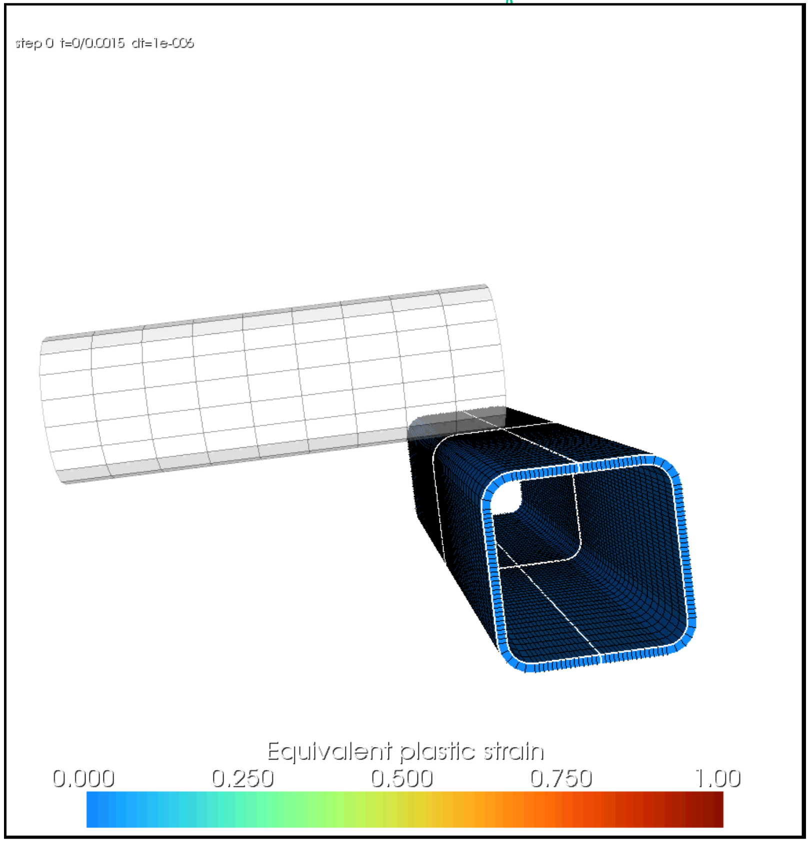


- Impact (50 m/s) of a square tubes on a rigid punch
  - Saturated hardening law (aluminum)
  - Masses added at tube extremities (to simulate structure inertia)

EDMC, $\rho_\infty = 0.8$, no point mass



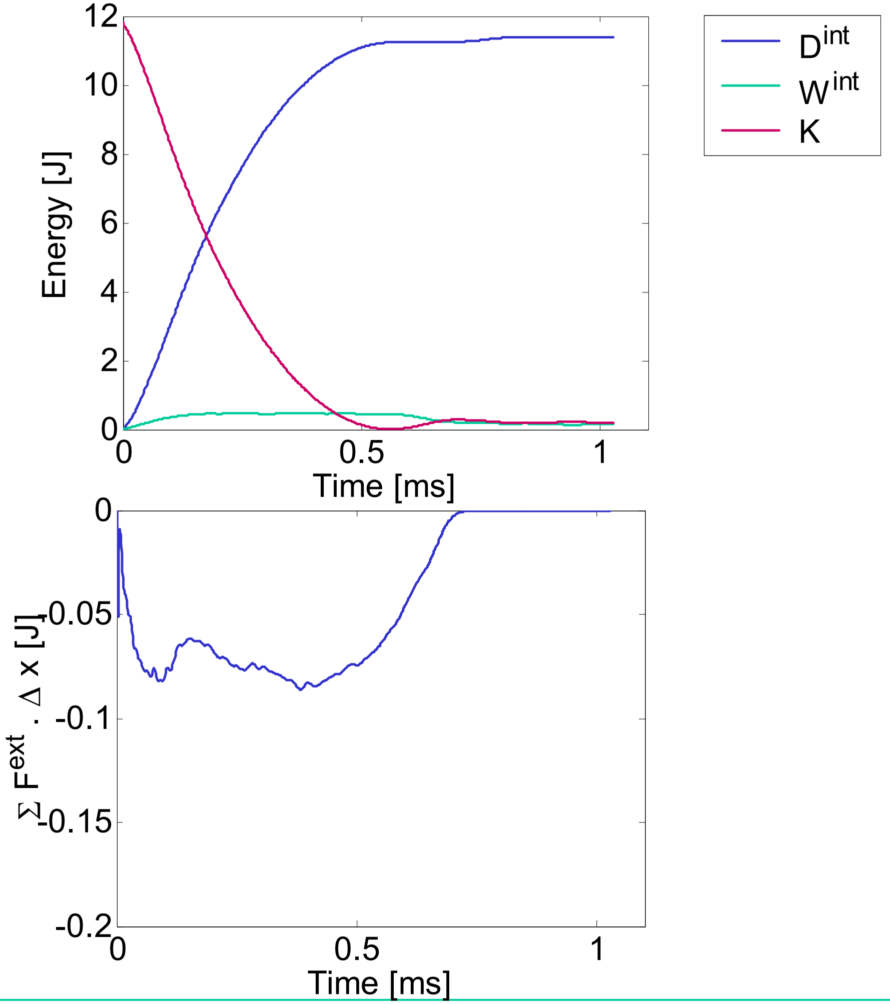
EDMC, $\rho_\infty = 0.8$,
point masses = 100% of tube mass



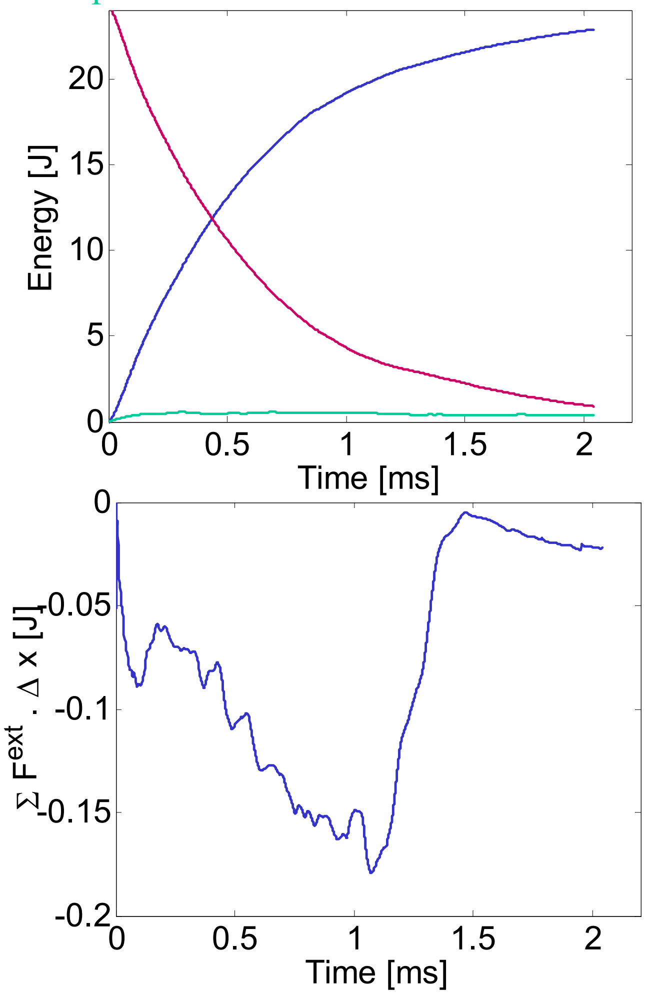
4. Numerical Examples
Impact of a square tube

- Time evolution of the results

EDMC, $\rho_\infty = 0.8$, no point mass



EDMC, $\rho_\infty = 0.8$,
point masses = 100% of tube mass



5. Conclusions

- Developed a visco-elastic formulation leading to a conserving time integration scheme
- Use of the variational updates formulation:
  - The formulation derives from a energy potential
  - The formulation is general for any material behavior
  - The formulation includes numerical dissipation
- The internal forces expression remains the same as for elasticity
- The momenta are conserved
- Numerical dissipation of energy
- The yield criterion is satisfied at the end of the time step
- Numerical examples demonstrate the robustness