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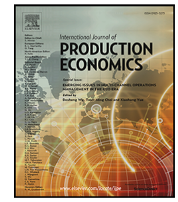

Data-driven control of a production system by using marking-dependent threshold policy

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ABSTRACT

As increasingly more shop-floor data becomes available, the performance of a production system can be improved by developing effective data-driven control methods that utilize this information. We focus on the following research questions: *how can the decision to produce or not to produce at any time be given depending on the real-time information about a production system?; how can the collected data be used directly in optimizing the policy parameters?; and what is the effect of using different information sources on the performance of the system?* In order to answer these questions, a production/inventory system that consists of a production stage that produces to stock to meet random demand is considered. The system is not fully observable but partial production and demand information, referred to as *markings* is available. We propose using the marking-dependent threshold policy to decide whether to produce or not based on the observed markings in addition to the inventory and production status at any given time. An analytical method that uses a matrix geometric approach is developed to analyze a production system controlled with the marking-dependent threshold policy when the production, demand, and information arrivals are modeled as Marked Markovian Arrival Processes. A mixed integer programming formulation is presented to determine the optimal thresholds. Then a mathematical programming formulation that uses the real-time shop floor data for joint simulation and optimization (JSO) of the system is presented. Using numerical experiments, we compare the performance of the JSO approach to the analytical solutions. We show that using the marking-dependent control policy where the policy parameters are determined from the data works effectively as a data-driven control method for manufacturing.

1. Introduction

Increasing usage of industrial data collection hardware and software has allowed manufacturers to collect extensive shop-floor data. Developing data-driven production control methods that utilize this abundant collected data in an effective way can bring significant benefits. Our study is motivated by the need of developing implementable data-driven control policies that use the selected information to match supply and demand in an effective way. We aim at answering the following research questions: *how can the decision to produce or not to produce at any time be given depending on the real-time information about a production/inventory system?; how can the collected data be used directly in optimizing the policy parameters?; and what is the effect of using different information sources on the performance of the system?*

In order to investigate these research questions, we consider a production/inventory system that consists of a production stage that produces to stock to meet random demand. The system is not fully observable but partial production and demand information, referred to as the *markings* arrive together with demand or production, or separately. The production is controlled with the marking-dependent threshold policy where the decision to produce or not to produce is given depending on the observed markings in addition to the inventory and production status at any given time. The marking-dependent threshold policy is an easily implementable policy: if the production stage is idle and the inventory position is less than the threshold determined for the last observed values of the markings, production of a new item is triggered with the release of the material into the production stage. Fig. 1 illustrates this production/inventory system.

We first present an analytical method that uses a matrix analytical approach to analyze a production/inventory system controlled with the marking-dependent threshold policy where the demand and information arrivals and the production times are modeled as Marked Markovian Arrival Processes (MMAP). The MMAP framework allows us to model correlated arrivals and general inter-arrival time distributions. A mixed integer programming (MIP) formulation is developed to

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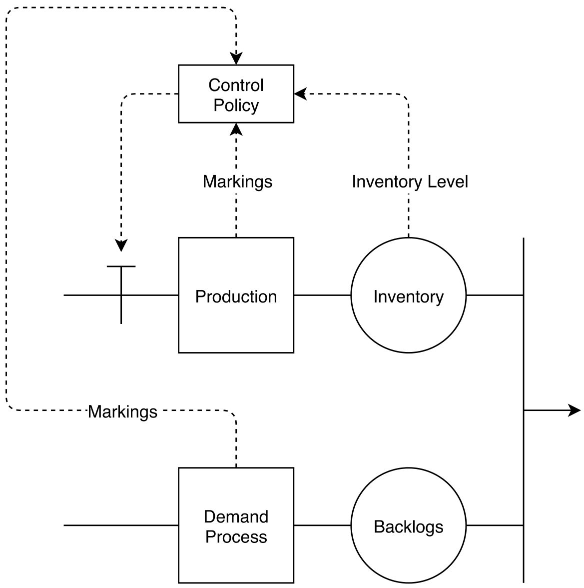

Fig. 1. A production/inventory system controlled with the marking-dependent threshold policy.

determine the optimal thresholds for the marking-dependent threshold policy depending on the markings selected to control the system.

We then propose using the data-driven joint simulation and optimization (JSO) approach as an online control method for manufacturing systems. We develop the JSO approach by formulating a mixed integer program to determine the parameters of the threshold policy by using the shop-floor data together with the existing statistical information about the processes. The data-driven JSO uses the available data directly for optimization and does not impose any distributional assumptions for the processes and does not require independence of the inter-arrival times.

The analytical method presented in this study allows us to evaluate the performance of the data-driven approach under different information scenarios by comparing the results obtained from different-length traces with the analytical results. An experimental setup for the release control of a feeder line where the demand for the feeder is generated by an unreliable production line with two stations separated by a finite buffer is used. The release into the feeder work station is controlled to minimize the expected holding and backlog costs. The effects of using different markings related to the availability of information on the machine status and the buffer status are investigated. This system is analyzed by using the joint simulation and optimization approach with different markings and with the different-length inter-departure and processing time data collected from the shop-floor and the available statistical information. The results are compared with the analytical results.

There are two main contributions of this study. The first one is modeling and analysis of a production/inventory system where the production, demand, and information processes are modeled as Marked Markovian Arrival Processes and controlled with a multiple-threshold policy. With the multiple-threshold policy, production is allowed at a given time when the current inventory position is below the target threshold level corresponding to the latest observed markings. The optimal thresholds for this system are determined by using a mathematical programming formulation that uses the embedded matrix geometric solution of the quasi birth and death process. To the best of our knowledge, this is the first study in the literature that presents a method to determine the optimal thresholds in a stochastic model of a controlled queue by using a mathematical programming formulation. The second one is developing a joint simulation and optimization method to determine the control parameters based on the shop-floor data collected from the system. Although there are studies in the literature that use JSO to design production systems, this study proposes using JSO as an online control method for the first time. The mixed integer program presented in this study extends the range of discrete event models that can be simulated and optimized by using the JSO approach. Our analytical and numerical results show that the marking-dependent control policy together with a JSO approach that determines the policy parameters works effectively as a data-driven control method for manufacturing.

The remainder of this paper is organized as follows. We review the pertinent literature in Section 2. Section 3 describes the problem under study and the control method. Section 4 describes the experimental setup with the specific production/inventory system used for the numerical experiments. The method for the exact analysis of the system is given in Section 5. The data-driven joint simulation and optimization method for this system is given in Section 6. The performance of the data-driven JSO approach is evaluated by comparing with the analytical results in Section 7. Finally, the conclusions are given in Section 8.

2. Literature review

This work is related to two streams of literature: stochastic modeling and control of production/inventory systems and joint simulation and optimization.

2.1. Stochastic modeling and control of production/inventory systems

Controlling production to match random supply with random demand has been the subject of numerous studies in the last 40 years. For a production/inventory system with i.i.d. exponential inter-arrival and service times, it has been shown that a base-stock policy is optimal e.g., (Gavish and Graves, 1980; Sobel, 1982). That is, production continues until the inventory level reaches a threshold. In this study, we consider demand and service processes that can have correlated inter-event times. Empirical studies and analytical models of production systems have shown that the autocorrelation of the inter-event times in these systems is not negligible (Wein, 1988; Schömig and Mittler, 1995; Tan and Lagershausen, 2017; Wein, 1988; Manafzadeh Dizbin and Tan, 2019). Manafzadeh Dizbin and Tan (2019) investigate how correlated inter-event times affect the steady-state behavior of a production/inventory system and show, via numerical experiments, how ignoring autocorrelation can lead to considerable losses.

In order to capture possible inter-dependency in the inter-event times, we use the framework of Markov arrival processes (MAP) introduced by Neuts (1979). Markov arrival processes are generalizations of phase-type distribution in the sense that the inter-event times of a Markov arrival process has phase-type distribution. However, unlike the phase-type distribution, after an arrival, the information about the phase of the system is not lost, hence the inter-arrival times might be dependent. In order to analyze MAPs, quasi birth and death processes (QBD) are commonly used in the literature. We mainly use the results of Ost (2013) and Horváth et al. (2010) for QBDs. In order to model a production system controlled with a marking-dependent threshold policy that uses partial information about the system status, we use the Marked MAPs (MMAP) (He and Neuts, 1998). MMAPs have been mostly used for modeling arrival of different classes of customers. In this work, we contribute to this literature by proposing using the MMAP framework to capture different sources of information about the state of the arrival process in different markings in a control setting.

The number of studies on the production control of systems with correlated demand and service processes is limited. These studies suggest that state-dependent base-stock policy is an effective control policy for these systems. Song and Zipkin (1993) prove the optimality of a state-dependent base-stock policy when production times are i.i.d.

and the demand is a Markov modulated Poisson process. For systems with correlated processes that can be modeled with MAPs, Manafzadeh Dizbin and Tan (2019) evaluate the performance of the state-dependent base-stock policy and discuss the optimality of the state-dependent base-stock policy. Similarly, the studies on the production control for systems with partially observable processes also suggest that state-dependent base-stock policy can be used as a control mechanism (Bertsimas and Paschalidis, 2001; Treharne and Sox, 2002; Karaesmen et al., 2004; Arifoğlu and Özekici, 2011; Bayraktar and Ludkovski, 2010).

Our contribution to the literature on stochastic modeling and control of production/inventory systems is modeling and analysis of a partially observable production/inventory system controlled with a threshold policy that uses limited information about the system and proposing a solution method that is based on mathematical programming to determine the optimal thresholds.

2.2. Joint simulation and optimization as a data-driven control method

In this study, we propose using joint simulation and optimization as a data-driven control method. Data-driven control can be defined as a control scheme that does not make use of explicit information about the mathematical model of the input data streams (Akcay et al., 2011; Hou and Wang, 2013). Data-driven optimization methods have been developed in close relation to robust optimization (Bertsimas and Thiele, 2006; Rudin and Vahn, 2018; Gallego and Moon, 1993).

The joint simulation and optimization method we propose uses the empirical data together with the available statistical information. This allows using the JSO approach as a data-driven control method based on the real-time data collected from the shop-floor combined with the simulated data by using the bootstrapping approach. Schruben (2000) was the first to introduce a joint simulation and optimization formulation that was able to replicate a simulation run of a system.

Most of the joint simulation and optimization formulations make use of objective functions that ensure that events start and finish at the correct time instances. The literature that followed this method was for the most part concerned with scheduling (Helber et al., 2011; Alfieri and Matta, 2012b). This is because scheduling also aims at making sure that every event happens as soon as possible. Whereas minimizing the average cost in an inventory system does not necessarily schedule events to happen as soon as possible. Chan and Schruben (2004) argue that finding the exact event times without making use of the objective function is only possible through adding a large number of integer variables. Since the exact event times are needed to model the marking-dependent base-stock policy in a model with both backlog and inventory costs, our formulation is a mixed integer programming formulation.

The work of Schruben (2000) has been extended to design and control of production systems. Alfieri and Matta (2012b) give the mathematical programming formulation for simulation of different types of production systems. Alfieri and Matta (2012b) and Pedrielli et al. (2015) give formulations for pull control policies however when the control parameters are variables in the problem, they do not consider the backlog costs. Alfieri and Matta (2012a) give approximate mathematical programming formulations for the aforementioned problems by using the concept of time buffers. Tan (2015) use joint simulation and optimization approach to model and analyze flow rate control problems for production systems with continuous and discrete state space. Hosseini and Tan (2017) extend these results to model a two-stage continuous flow productions systems with a finite buffer.

We contribute to this stream of literature by proposing JSO as a real-time control method and providing a joint simulation and optimization formulation for controlling a system with multiple thresholds. This formulation can be used to determine the parameters of the control policy by using the collected data with the available statistical information about the processes.

3. Problem description

We consider a production/inventory system that consists of a production stage that produces to stock to meet random demand. Each demand arrival is demanding one item. If there are available items in the inventory, the demand is satisfied and leaves the system. Otherwise, the demand is backlogged until an item is available. The release of the material into the production process is controlled by a policy that determines whether or not to produce depending on the information available at a given time. Fig. 1 depicts the system.

3.1. Model

3.1.1. Production

The production time process can be in one of w different states. At an arbitrary time t, the state of the production time process is denoted by $\eta_W(t) \in \{1, \dots, w\}$. When the production starts, it cannot be preempted. $M(t) \in \{0, 1\}$ indicates if the production stage is working (1) or idle (0). The production times can be correlated. This general representation of the production process allows analyzing various settings where the release of materials into the production stage is controlled to match the output of the production stage with the demand.

3.1.2. Demand

The demand arrival process can be in one of d different states. The state of the demand process is denoted by $\eta_D(t) \in \{1, \dots, d\}$. The demand inter-arrival times can be correlated. The demand for the production stage can be generated by another stage in production. For example, in the example described in Section 4, the output from an unreliable production line generates the demand for the production/inventory system.

3.1.3. Inventory position

The difference between the cumulative production and cumulative demand at time t is referred to as the inventory position and denoted by $X(t)$. The inventory level and the backlog level are denoted by $X^+(t) = \max\{X(t), 0\}$ and $X^-(t) = \max\{-X(t), 0\}$ respectively. Every item in the inventory generates a cost of c^+ per unit time, and every demand waiting in the line generates a backorder cost of c^- per unit time.

3.1.4. Markings

The information signals referred as the *markings* arrive as attached to the demand arrivals or production completions, or arrive separately. There are C_D markings for the demand process and C_W markings for the production process. The index of the last observed marking from the information and demand process at time t is denoted by $c_D(t) \in \{1, \dots, C_D\}$. The index of the last observed marking from the production time process is denoted by $c_W(t) \in \{1, \dots, C_W\}$. The markings are uniquely determined by the demand and production states. The arrival of information signals can be correlated. The number of markings is much smaller than the number of states, i.e., $C_D \ll d$ and $C_W \ll w$.

Fig. 2 depicts the sample path of the marked trace for the system described in Section 4. In this example, the output from an unreliable production line generates the demand for the production/inventory system. The controller can use up to 4 markings for the demand process: 2 markings for the unreliable station (up or down) or 2 markings for the buffer level (empty or not empty) or 4 markings for both (down and empty, up and empty, down and not empty, and up and not empty).

3.1.5. State–space model

The state of the system at time t is $(X(t), \eta_D(t), \eta_W(t))$. We consider the case where the demand and production states $\eta_D(t)$ and $\eta_W(t)$ are not fully observable. Partial information about the system is available through the inventory status, the production status, and the marking processes, $(X(t), M(t), c_D(t), c_W(t))$. We model the system as a continuous-time discrete state–space process $\{(X(t), M(t), c_D(t), c_W(t)), t \geq 0\}$.

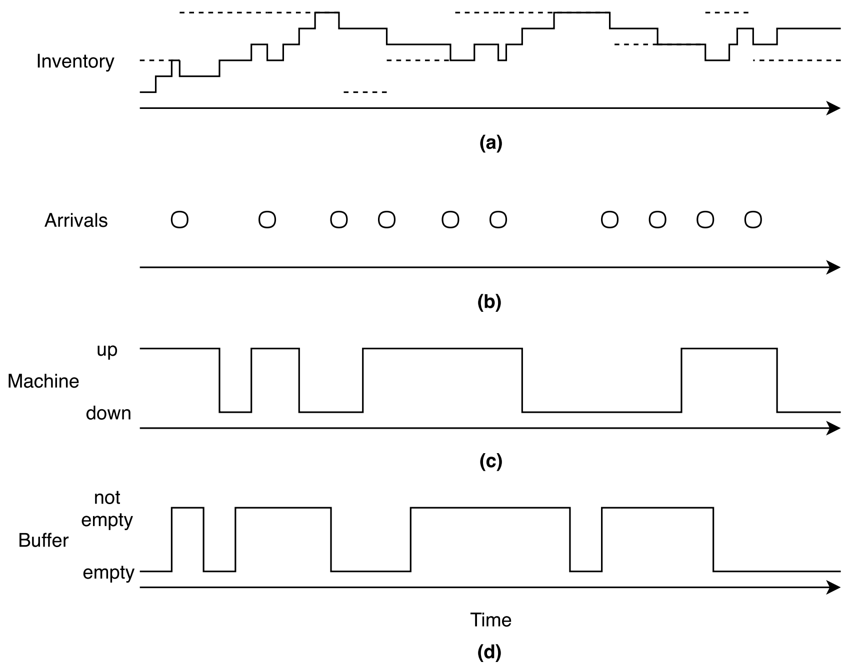

Fig. 2. A sample path of the marked trace for the system described in Section 4.

3.2. Production control problem

The optimization problem we consider is deciding on whether to produce or not depending on the inventory level, the production status, the marking of the last demand and information arrival, and the marking of the last production time in order to minimize the average inventory holding and backlog cost of the system in the long run.

3.2.1. Decision

Let $u^l(X(t), M(t), c_D(t), c_W(t))$ denote the decision to produce ($u^l = 1$) or not to produce ($u^l = 0$) depending on the inventory position, the machine status, and the last observed markings for demand and production at time t under policy l.

3.2.2. Objective

The problem is finding the optimal policy that minimizes the steady-state average cost π in the long run:

$$\pi^* = \min_l J^l$$
$$= \mathbb{E}\left[\frac{1}{T}\lim_{T\to\infty}\int_{t=0}^{T}\left(c^+X^+(t) + c^-X^-(t)\right)dt|X(0), \eta_D(0), \eta_W(0)\right]. \quad (1)$$

3.2.3. Optimal control policy

If the production times are i.i.d. exponential random variables and the inter-arrival times are i.i.d exponential random variables, then a base-stock policy that allows production when the inventory position is less than a threshold would be the optimal policy when the production and demand states are fully observable (Gavish and Graves, 1980; Sobel, 1982). Extensions of the basic base-stock model show that a state-dependent threshold policy is optimal for various systems under different assumptions (Song and Zipkin, 1993; Treharne and Sox, 2002; Arifoğlu and Özekici, 2011). For a system with correlated production and demand processes modeled as MAPs, Manafzadeh Dizbin and Tan (2019) present the state-dependent threshold policy as the optimal policy.

The model presented in this study can be considered as a modified version of this problem where the controller cannot take action at certain states that are not observable, and the state-space is extended to include additional states corresponding to the markings. In this modified system, the state-dependent threshold policy would be the optimal policy. Since the state space includes the states corresponding to the markings, the marking-dependent threshold policy is a candidate for the optimal policy. However, a formal proof of the optimality of the marking-dependent control policy is not given in this paper.

3.2.4. Marking-dependent threshold policy

We refer to a control policy where the decision to produce or not depends on the inventory position and a threshold that depends on the last observed pair of markings as the *marking-dependent* threshold policy. Using the marked processes, the marking of each arrival and the marking of each production time bring some information about the state of the system. The control policy assigns a threshold to each pair of markings as opposed to assigning a threshold to the states of the processes, and production will continue until this threshold is reached.

The marking-dependent threshold policy compares the inventory position at time t with the threshold level set for the last observed markings of the demand and production processes to trigger production. Namely, let S_{c_D,c_W} be the threshold level for the marking pair (c_D, c_W), $c_D \in \{1, 2, \ldots, C_D\}$, $c_W \in \{1, 2, \ldots, C_W\}$. The arrival of a demand while the last observed marking pair is (c_D, c_W) triggers production if there is no item being produced at that moment and the inventory level is less than or equal to S_{c_D,c_W}. Upon completion of a part, production is allowed to continue if the inventory level is less than S_{c_D,c_W}. If the inventory level reaches S_{c_D,c_W}, the production stops until a new demand or information signal arrives. The production will not start if the threshold that corresponds to the newly observed marking pair is lower than the inventory level. The control policy can be expressed as

$$u(X(t), M(t), c_D(t), c_W(t)) = \begin{cases} 1 & \text{if } X(t) < S_{c_D,c_W} \text{ and } M(t) = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (2)$$

3.2.5. Parameters of the marking-dependent threshold policy

The parameters of the marking-dependent threshold policy are the thresholds corresponding to different markings. The threshold set $S = \{S_{c_D,c_W}\}$, $c_D \in \{1, 2, \ldots, C_D\}$, $c_W \in \{1, 2, \ldots, C_W\}$ includes $C_D \times C_W$ thresholds of the control policy. We define the lowest and the highest threshold levels as $\underline{S} = \min_{c_D,c_W} S_{c_D,c_W}$ and $\overline{S} = \max_{c_D,c_W} S_{c_D,c_W}$ respectively.

With much fewer parameters compared to a state-dependent threshold policy and as a policy that does not require full observability of the system, the marking-dependent threshold policy can easily be implemented in a production system. The implementation requires determining the markings that will be collected from the shop floor and setting the threshold for each marking. Once the thresholds are determined, the real-time production decision can be given by comparing the inventory level with the thresholds corresponding to the last observed markings.

For given marked traces from a system, there are two alternatives for determining the best threshold levels. The first alternative is fitting stochastic processes to the traces and then using the fitted processes in the analytical method presented in Section 5. The second alternative is using the data-driven JSO that directly uses the data in optimization presented in Section 6. While we make the assumption that the process $\{(X(t), M(t), c_D(t), c_W(t)), t \geq 0\}$ can be modeled as a Marked Markovian Arrival Process in Section 5, we do not impose any assumptions for this process for the data-driven JSO method presented in Section 6. The analytical model presented in Section 5 also allows us to evaluate the performance of the data-driven method presented in Section 6. We use the specific system given in Section 4 to illustrate the application of these two different approaches.

4. Example

Before describing the general model and the solution methodologies in detail, we first introduce a specific system to show the operation of the marking-dependent threshold policy in a production system. Note that the analytical method presented in Section 5 and the data-driven method described in Section 6 can be applied to a wider range of models and are not restricted to the assumptions of the specific system presented in this section.

We consider a setting where the downstream of a production system is fed by two upstream flows and the release into one flow is controlled to minimize the holding and backlog to synchronize the flows. This is similar to a setting where the release of the material into a feeder line is synchronized with the flow of the main assembly line. Fig. 3 illustrates this production line together with the workstation that forms the production/inventory system.

The output of the upstream flow of the main assembly line is synchronized with the parts supplied by the feeder station denoted by WS_1. WS_1 has exponential processing time with rate μ_1. The release of material to WS_1 is controlled with a marking-dependent threshold policy.

The demand stream to the work station WS_1 is generated by a two-machine line with one unreliable machine WS_2, and one reliable machine WS_3 and a finite inter-station buffer of size q. The processing times of WS_2 and WS_3 are exponentially distributed random variables with rates μ_2 and μ_3. The throughput of the line is denoted with TP.

The unreliable machine WS_2 has operation-dependent failures and the failure time is exponentially distributed with rate $\gamma\mu_2$. If a breakdown occurs, the repair process starts. The repair time follows an Erlang distribution with r phases each with rate λ. The phase of the repair process is not observable.

The markings collected from the shop-floor are related to the repair status of WS_2, whether the repair process is in progress or not, and the buffer status, whether the buffer is empty or not at the time of a demand arrival.

The inventory status of the buffer is recorded as *empty* or *not empty* as a marking. Fig. 2 depicts a sample realization for this case. With each arrival (Fig. 2b), the controller receives a marking that can include the corresponding machine availability status (Fig. 2c) and/or the buffer status (Fig. 2d). According to the production completion times and demand arrivals, the corresponding inventory sample path is shown in Fig. 2a, where the dashed lines represent the threshold levels and the solid line represents the inventory level. In Fig. 2, if both the machine availability and the buffer status is included in the marking, then there are $C_D = 4$ markings. If only the machine availability or the buffer status is used, then $C_D = 2$. If only the inter-arrival time is used by the controller and no additional information is used, then $C_D = 1$.

In Section 7, we use this experimental setup to evaluate the data-driven JSO and compare its performance with the analytical solution. We will also discuss the effect of using different markings for control.

5. Exact analysis of the production/inventory system controlled by using marking-dependent threshold policy

In order to analyze the production/inventory system, we first develop the quasi birth and death process representation of the system with MMAP production times and MMAP demand arrivals for the given values of the threshold for each marking pair. We then use the matrix geometric method to determine the steady-state probabilities. The steady-state probabilities give us the expected cost of the system. Finally, by using a mathematical programming formulation, we determine the set of the threshold levels for each marking pair that minimizes the expected cost.

5.1. Marked MAP representation for arrival processes with partial information

In order to model an arrival process along with the associated partial information, we use the Marked MAP (MMAP) representation. A MMAP describes a MAP whose arrivals are marked.

5.1.1. Marked Markovian arrival processes

A MAP A is described by the matrices $(\mathbf{A0}, \mathbf{A1})$, where $\mathbf{A0}$ records the transitions that do not result in an arrival and $\mathbf{A1}$ records the transitions that result in an arrival. The infinitesimal generator matrix of the process A is $\mathbf{A0} + \mathbf{A1}$.

MMAPs are extensions of MAPs that allow for different types of arrivals (He and Neuts, 1998). A MMAP A is described by the matrices $(\mathbf{A0}, \mathbf{A1}_1, \ldots, \mathbf{A1}_C)$, where $\mathbf{A0}$ records the transitions that do not result in an arrival and $\mathbf{A1}_c$ records the transitions that result in an arrival marked c. The infinitesimal generator matrix of the process A is $\mathbf{A0} + \sum_{c=1}^{C} \mathbf{A1}_c$.

5.1.2. Model of part arrivals with markings

We consider information arrivals at different times not necessarily coupled with part arrival times. Let $L = (\mathbf{L0}, \mathbf{L1}_1, \ldots, \mathbf{L1}_C, \mathbf{L2}_1, \ldots, \mathbf{L2}_C)$ denote the MMAP that captures the dynamics of the information and part arrivals. The transition rates corresponding to a part arrival coupled with the information marked i are captured in $\mathbf{L1}_i$. The transition rates corresponding to an arrival of the information marked i without a part arrival are captured in $\mathbf{L2}_i$.

5.1.3. Model of production

Production happens one item at a time with a production time that evolves according to a MMAP. Once the production of an item starts, it cannot be interrupted until completion. We assume that the state of the production time process does not change during the idle periods.

The MMAP describing the production time process is denoted by $W = \left(\mathbf{W0}, \mathbf{W1}_1, \ldots, \mathbf{W1}_{C_W}\right)$. The number of states in W is w. Since a production cannot be interrupted, arrival of information about the production time process without completion of a production cannot be used for the control of the system. For this reason, entries of $\mathbf{W2}_1, \ldots, \mathbf{W2}_{C_W}$ are zero without loss of generality. For notational convenience, $\mathbf{W2}_1, \ldots, \mathbf{W2}_{C_W}$ have been dropped from the definition of W.

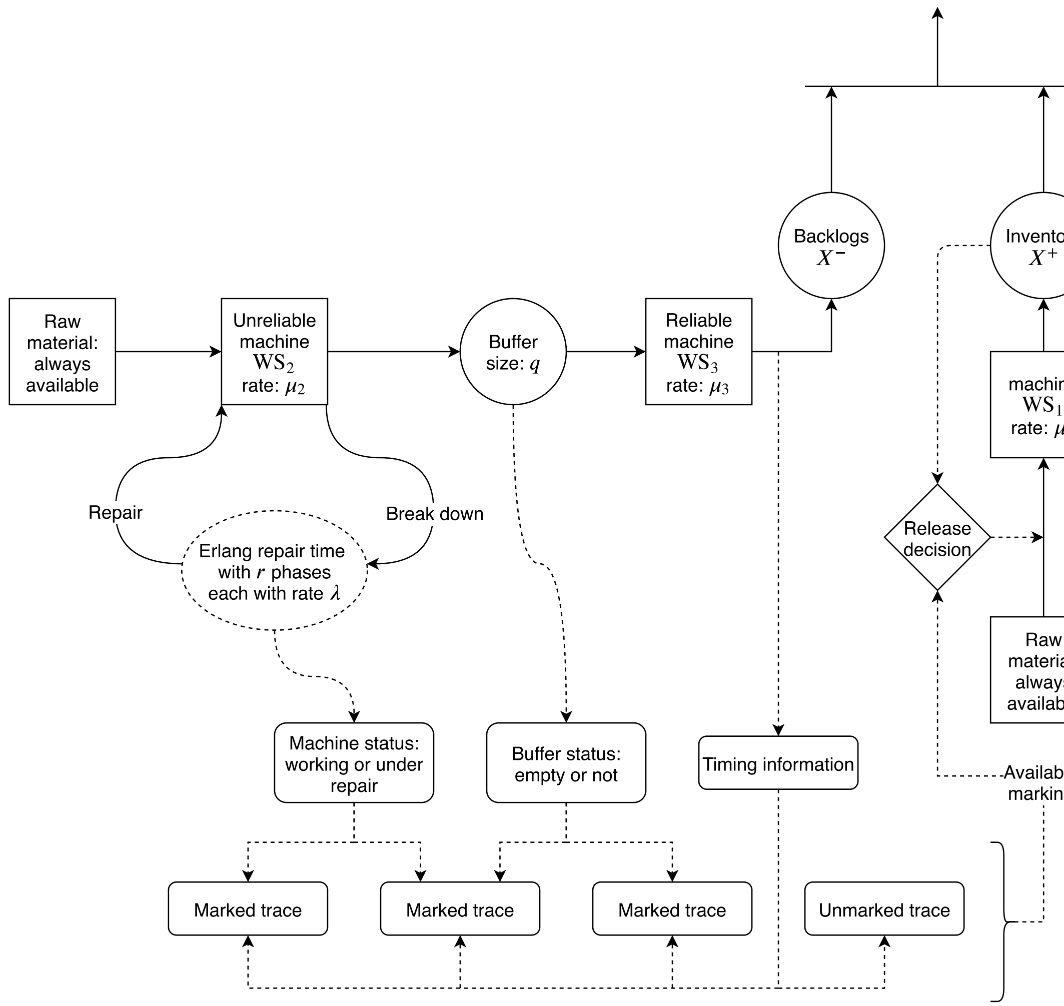

Fig. 3. A production line with an unreliable machine. The output of this line is used as the information and demand process in the numerical experiments.

5.1.4. Model of demand arrivals with markings

Similarly, we also assume that a unit of demand arrives according to a MMAP. Information about the demand process arrive with or without the arrival of a demand. Let $D = \left(\mathbf{D0}, \mathbf{D1}_1, \ldots, \mathbf{D1}_{C_D}, \mathbf{D2}_1, \ldots, \mathbf{D2}_{C_D}\right)$ denote the MMAP of information and demand. The transition rates corresponding to a demand arrival marked i are captured in $\mathbf{D1}_i$. The transition rates corresponding to arrival of demand information marked i without a demand arrival are captured in $\mathbf{D2}_i$.

5.2. Evaluation of the production/inventory system for given thresholds

The process of shortfalls is not affected if all the thresholds are increased by a constant value. Therefore, in order to analyze the process $\{(X(t), M(t), c_D(t), c_W(t)), t \geq 0\}$ for given thresholds, we focus on the shortfall from the largest threshold $\overline{S}$. The shortfall increases when a part is produced and decreases when a demand arrives and stays the same for all other transitions that do not change the inventory position. We define the number of shortfalls from the largest threshold value as $Y(t) = \overline{S} - X(t)$ and analyze the process $\{(Y(t), M(t), c_D(t), c_W(t)), t \geq 0\}$ as a QBD.

5.2.1. Model for the shortfalls

The state space structure of the process $\{(Y(t), M(t), c_D(t), c_W(t)), t \geq 0\}$ is given in Fig. 4. Let $\mathbf{Q}$ denote the CTMC generator matrix of the system, and in addition, let $\bar{\mathbf{Q}}_{Y,Y'}$ denote the sub-block of $\mathbf{Q}$ corresponding to transitions from shortfall level Y to shortfall level Y'.

Table 1
Description of the notation used in the matrix geometric analysis.

Notation	Description
$\mathbf{I}_n$	Identity matrix of size $n \times n$
$\mathbf{e}_{i,n}$	Row vector of length n with 1 for ith entry and 0 elsewhere
$\mathbf{J}_{i,j,n}$	Square matrix of size $n \times n$, with a single entry with value one in row i and column j
$\mathbf{1}_{n,m}$	$n \times m$ matrix of ones
$\mathbf{0}_{n,m}$	$n \times m$ matrix of zeros
$\delta_{\{x\}}$	Binary indicator function specifying if statement x is correct or not
x^T	Transpose of x
$\otimes$	Kronecker product

With these submatrices, $\mathbf{Q}$ has the structure given in Eq. (3). In this Quasi Birth and Death process, the size of each block of the corresponding CTMC for a given level of the shortfall will be $n \times n$ where $n = 2C_D C_W dw$.

$$\mathbf{Q} = \begin{bmatrix} \bar{\mathbf{Q}}_{0,0} & \bar{\mathbf{Q}}_{0,1} & & & \\ \bar{\mathbf{Q}}_{1,0} & \bar{\mathbf{Q}}_{1,1} & \bar{\mathbf{Q}}_{1,2} & & \\ & \bar{\mathbf{Q}}_{2,1} & \bar{\mathbf{Q}}_{2,2} & \bar{\mathbf{Q}}_{2,3} & \\ & & \ddots & \ddots & \ddots \\ & & & & \\ & & & & \end{bmatrix}. \tag{3}$$

The sub-blocks of $\mathbf{Q}$ can be specified for the cases where the shortfall increases by one, stays the same, or decreases by one. The notation

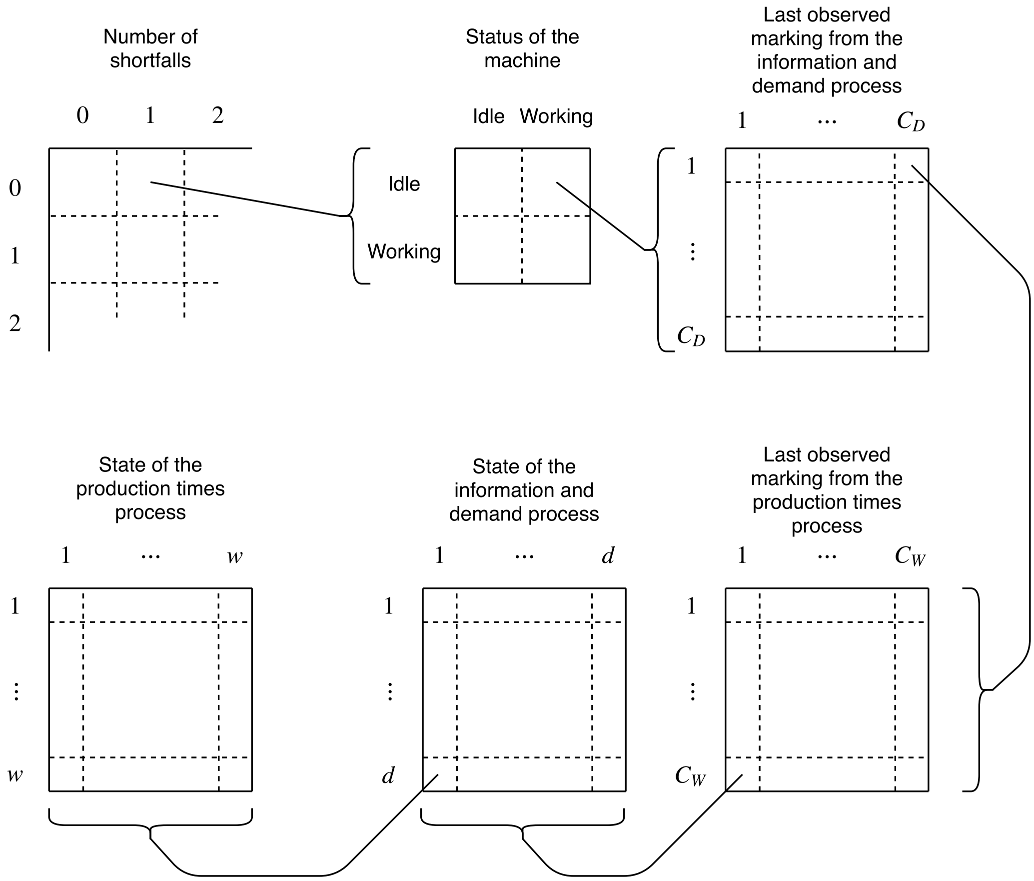

Fig. 4. State-space structure of the process of shortfalls.

used in the determination of the sub-matrices for each case is given in Table 1.

The first sub-block $\bar{\mathbf{Q}}_{Y,Y+1}$ is related to the transitions that increase the level of shortfalls. Therefore, only demand arrivals appear in this sub-matrix:

$$\begin{aligned}\bar{\mathbf{Q}}_{Y,Y+1} = {} & \begin{bmatrix}1&0\\0&0\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\bar{S}-(Y+1)\ge S_{i_2,j_2}\right\}} \mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ & \otimes \left(\delta_{\{j_1=j_2\}}\mathbf{D1}_{i_2} \otimes \mathbf{I}_w\right) \\ & + \begin{bmatrix}0&1\\0&0\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\bar{S}-(Y+1)< S_{i_2,j_2}\right\}} \mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ & \otimes \left(\delta_{\{j_1=j_2\}}\mathbf{D1}_{i_2} \otimes \mathbf{I}_w\right) \\ & + \begin{bmatrix}0&0\\1&0\end{bmatrix} \otimes \mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ & + \begin{bmatrix}0&0\\0&1\end{bmatrix} \otimes \left(\sum_{i=1}^{C_D}\left(\left(\mathbf{e}_{i,C_D}\otimes \mathbf{1}_{C_D}\right)\otimes \mathbf{I}_{C_W}\otimes \mathbf{D1}_i \otimes \mathbf{I}_w\right)\right), \\ & Y \ge 0.\end{aligned}$$

The first term in the above equation refers to transitions from states where the machine is *idle* to states where the machine is still *idle*. (i_1, j_1) denotes the marking pair before the arrival, and (i_2, j_2) refers to the marking pair after the arrival. Hence, making a transition to an *idle* state depends on the inventory level $\bar{S} - (Y+1)$ being greater than or equal to the threshold for the current marking pair S_{i_2,j_2}. The second term is the counterpart of the first term for the transition to a *working* state. The third term is zero because if the machine is *working*, an increase in the shortfall level cannot make it *idle*. The fourth term records the transitions that only affect the last observed marking pair. They change the last observed marking from the demand process.

The second sub-block $\bar{\mathbf{Q}}_{Y,Y}$ includes the transitions that do not affect the number of shortfalls:

$$\begin{aligned}\bar{\mathbf{Q}}_{Y,Y} = {} & \delta_{\{Y>0\}}\left(\begin{bmatrix}0&0\\0&1\end{bmatrix} \otimes \mathbf{I}_{C_D} \otimes \mathbf{I}_{C_W} \otimes \mathbf{I}_d \otimes \mathbf{W0}\right) \\ & + \left(\mathbf{I}_2 \otimes \mathbf{I}_{C_D} \otimes \mathbf{I}_{C_W} \otimes \mathbf{D0} \otimes \mathbf{I}_w\right) \\ & + \begin{bmatrix}1&0\\0&0\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\bar{S}-Y\ge S_{i_2,j_2}\right\}} \mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ & \otimes \left(\delta_{\{j_1=j_2\}}\mathbf{D2}_{i_2} \otimes \mathbf{I}_w\right) \\ & + \begin{bmatrix}0&1\\0&0\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\bar{S}-Y< S_{i_2,j_2}\right\}} \mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ & \otimes \left(\delta_{\{j_1=j_2\}}\mathbf{D2}_{i_2} \otimes \mathbf{I}_w\right) \\ & + \begin{bmatrix}0&0\\1&0\end{bmatrix} \otimes \mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ & + \begin{bmatrix}0&0\\0&1\end{bmatrix} \otimes \left(\sum_{i=1}^{C_D}\left(\left(\mathbf{e}_{i,C_D}\otimes \mathbf{1}_{C_D}\right)\otimes \mathbf{I}_{C_W}\otimes \mathbf{D2}_i \otimes \mathbf{I}_w\right)\right), \\ & Y \ge 0.\end{aligned}$$

These transitions include the phase changes in the processes that do not result in any arrivals and also the information arrivals received without a demand arrival. The first set of transitions in the above equation are recorded in the first two terms. The other terms are similar to the previous equation, i.e. the arrival of an information signal can cause an *idle* machine to start working but cannot stop a *working* machine.

The last sub-block $\bar{\mathbf{Q}}_{Y,Y-1}$ includes the transitions that take place when a part is produced:

$$\begin{aligned}\bar{\mathbf{Q}}_{Y,Y-1} &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \otimes \mathbf{0}_{C_D C_W dw, C_D C_W dw}\\ &+ \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \otimes \mathbf{0}_{C_D C_W dw, C_D C_W dw}\\ &+ \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\overline{S}-(Y-1)\geq S_{i_2,j_2}\right\}} \mathbf{J}_{i_1 j_1, i_2 j_2, C_D C_W}\\ &\otimes \left(\delta_{\{i_1=i_2\}} \mathbf{I}_d \otimes \mathbf{W1}_{j_2}\right)\\ &+ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \otimes \sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W} \delta_{\left\{\overline{S}-(Y-1)< S_{i_2,j_2}\right\}} \mathbf{J}_{i_1 j_1, i_2 j_2, C_D C_W}\\ &\otimes \left(\delta_{\{i_1=i_2\}} \mathbf{I}_d \otimes \mathbf{W1}_{j_2}\right), \qquad Y \geq 1.\end{aligned}$$

In the above equation, the first two terms are zero because production cannot be completed if the machine is not *working*. The third term records the transitions to an *idle* state, depending on the threshold for the last observed marking pair S_{i_2,j_2}. The fourth term records transitions to a *working* state, from a *working* state, following the same pattern.

5.2.2. The matrix geometric method

We determine the steady-state probabilities of the shortfall process by using the matrix geometric method (Ost, 2013). In order to use the matrix geometric method, the repeating levels and the boundary levels are specified, and $\mathbf{Q}$ is represented in the format given in Eq. (4).

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & & & \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} & \mathbf{A}_0 & & \\ & \mathbf{B}_{2,1} & \mathbf{A}_1 & \mathbf{A}_0 & \\ & & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{4}$$

When $Y > \overline{S} - \underline{S} + 1$, the sub-blocks are not dependent on Y. For these values of the shortfall, the boundary and repeating sub-matrices can be specified as given in Eqs. (5)–(10).

$$\mathbf{B}_{0,0} = \begin{bmatrix} \bar{\mathbf{Q}}_{0,0} & \bar{\mathbf{Q}}_{0,1} & & & & \\ \bar{\mathbf{Q}}_{1,0} & \bar{\mathbf{Q}}_{1,1} & \bar{\mathbf{Q}}_{1,2} & & & \\ & \bar{\mathbf{Q}}_{2,1} & \bar{\mathbf{Q}}_{2,2} & \bar{\mathbf{Q}}_{2,3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+1,\overline{S}-\underline{S}} & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+1,\overline{S}-\underline{S}+1} & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+1,\overline{S}-\underline{S}+2} \\ & & & & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+2,\overline{S}-\underline{S}+1} & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+2,\overline{S}-\underline{S}+2} \end{bmatrix} \tag{5}$$

$$\mathbf{B}_{0,1} = \begin{bmatrix} \mathbf{0}_n \\ \vdots \\ \mathbf{0}_n \\ \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+2,\overline{S}-\underline{S}+3} \end{bmatrix} \tag{6}$$

$$\mathbf{B}_{1,0} = \begin{bmatrix} \mathbf{0}_n & \cdots & \mathbf{0}_n & \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+3,\overline{S}-\underline{S}+2} \end{bmatrix} \tag{7}$$

$$\mathbf{B}_{1,1} = \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+3,\overline{S}-\underline{S}+3} \tag{8}$$

$$\mathbf{B}_{2,1} = \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+4,\overline{S}-\underline{S}+3} \tag{9}$$

$$\mathbf{A}_0 = \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+5,\overline{S}-\underline{S}+6}, \mathbf{A}_1 = \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+5,\overline{S}-\underline{S}+5}, \mathbf{A}_2 = \bar{\mathbf{Q}}_{\overline{S}-\underline{S}+5,\overline{S}-\underline{S}+4} \tag{10}$$

Since $\mathbf{B}_{0,0}$ depends on the threshold levels, the equations related to $\mathbf{B}_{0,0}$ must be solved for a given set of threshold levels.

5.2.3. Determining the steady-state probabilities

Given the aforementioned sub-matrices, the steady-state distribution can be calculated by using the successive substitution method. Let the steady-state probabilities form the vector $\mathbf{p}$. The elements of $\mathbf{p}$ are organized in sub-vectors according to the representation of $\mathbf{Q}$ in Eq. (4): $\mathbf{p} = \left(\mathbf{b}, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots\right)$.

The balance equations where $\mathbf{pQ} = 0$ and $\mathbf{p1} = 1$ can be written according to the partitioning of $\mathbf{Q}$ as

$$\mathbf{bB}_{0,0} + \mathbf{v}_0\mathbf{B}_{1,0} = \mathbf{0}, \tag{11}$$

$$\mathbf{bB}_{0,1} + \mathbf{v}_0\mathbf{B}_{1,1} + \mathbf{v}_1\mathbf{B}_{2,1} = \mathbf{0}, \tag{12}$$

$$\mathbf{v}_j\mathbf{A}_0 + \mathbf{v}_{j+1}\mathbf{A}_1 + \mathbf{v}_{j+2}\mathbf{A}_2 = \mathbf{0}, \qquad j = 0, 1, 2, \ldots, \tag{13}$$

$$\mathbf{b1} + \sum_{j=1}^{\infty} \mathbf{v}_j \mathbf{1} = 1. \tag{14}$$

The solution to the matrix difference equation given in Eq. (13) is obtained by the matrix geometric structure as

$$\mathbf{v}_j = \mathbf{v}_0\mathbf{R}^j, \quad \mathbf{R} \in \mathbb{R}^{n\times n}. \tag{15}$$

where matrix $\mathbf{R}$ can be calculated independently of $\mathbf{v}_0$ by successive iterations with

$$\mathbf{R} = -\left(\mathbf{A}_0 + \mathbf{R}^2\mathbf{A}_2\right)\mathbf{A}_1^{-1}, \tag{16}$$

starting from $\mathbf{R} = \mathbf{0}_n$, until convergence (Ost, 2013). Given matrix $\mathbf{R}$, $\mathbf{p}$ is obtained by solving the linear equations (11)–(14). Finally, when the steady-state distribution $\mathbf{p}$ is obtained, the average cost of the system can be calculated as

$$\pi(S) = \sum_{x=0}^{\infty} \sum_{i=1+nx}^{n(x+1)} p_i \left(c^+\left[\overline{S} - x\right]^+ + c^-\left[x - \overline{S}\right]^+\right), \tag{17}$$

where p_i is the ith element of $\mathbf{p}$ and $\mathbf{prob}(X = x) = \sum_{i=1+nx}^{n(x+1)} p_i$.

5.3. Mathematical programming formulation to determine the optimal thresholds

The method described in the preceding section yields the steady-state performance measures of the system for the *given* threshold values. In this section, we present a mathematical programming formulation to determine the *optimal* thresholds for the marking-dependent threshold policy. At every given inventory level and the last observed marking pair, if the machine is idle, there is a choice of starting production or doing nothing. Since the thresholds are non-negative and the transition rates between the repeating levels do not depend on the threshold levels, this problem can be restated as choosing among the possible options for the boundary state transitions of a QBD in order to minimize the steady-state cost of the system.

In other words, depending on the choice, some transitions will be added or eliminated from $\mathbf{B}_{0,0}$. Hence, the possible options for the boundary state transitions recorded in $\mathbf{B}_{0,0}$ are a function of a set of actions that the controller can take. In order to determine the thresholds in a computationally efficient way, we formulate the problem as a MIP problem.

We first introduce the MIP formulation for a more general problem where the objective is deciding on adding or eliminating transitions between the boundary states of a QBD in order to minimize the steady-state cost generated by the QBD.

Let $y_k : k \in \{1, \ldots, K\}$ denote the binary actions that the controller can take. In the case of the marking-dependent threshold policy, y_k values correspond to the decision to continue production for different inventory levels and marking pairs. If $y_k = 1$, certain transitions are allowed and if $y_k = 0$, these transition are prohibited. Hence, $\mathbf{B}_{0,0} = \mathbf{B}^0_{0,0} + \sum_{k=1}^{K} y_k \mathbf{B}^k_{0,0}$, where $\mathbf{B}^0_{0,0}$ is the boundary transition matrix if $y_k = 0, \forall k$ and $\mathbf{B}^k_{0,0}$ denotes the transitions that depend on $y_k = 1$.

Any admissible action must correspond to a well-defined transition matrix. Note that $\mathbf{B}_{0,0}^{k}$ can have negative off-diagonal values. These negative values represent the transitions that are prohibited when $y_k = 1$. However, $\mathbf{B}_{0,0}$ cannot have off-diagonal negative values at any feasible point. This must be reflected in the constraints on y_k values. These constraints along with other operational constraints on the binary variables can be represented as $\mathbf{Ey} \geq \mathbf{g}$, where $\mathbf{y}$ denotes the vector of binary variables, $\mathbf{E}$ is a matrix with K columns, and $\mathbf{g}$ is a column vector of appropriate size. Then the problem can be expressed as given in Eqs. (18)–(24):

$$\min \mathbf{f}\mathbf{p}^{T} \tag{18}$$

$$\mathbf{p} = \begin{bmatrix} \mathbf{b} & \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots \end{bmatrix} \tag{19}$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_{\mathbf{b}} & \mathbf{f}_{\mathbf{v}_0} & \mathbf{f}_{\mathbf{v}_1} & \mathbf{f}_{\mathbf{v}_2} & \cdots \end{bmatrix} \tag{20}$$

$$\mathbf{b}\mathbf{1}_{m,1} + \mathbf{v}_0 (\mathbf{I}_n - \mathbf{R})^{-1} \mathbf{1}_{n,1} = 1 \tag{21}$$

$$\begin{bmatrix} \mathbf{b} & \mathbf{v}_0 \end{bmatrix} \begin{bmatrix} \mathbf{B}_{0,0}^{0} + \sum_{k=1}^{K} y_k \mathbf{B}_{0,0}^{k} & \mathbf{B}_{0,1} \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} + \mathbf{R}\mathbf{A}_2 \end{bmatrix} = \mathbf{0}_{1,m+n} \tag{22}$$

$$y_k \in \{0,1\}^{K} \tag{23}$$

$$\mathbf{Ey} \geq \mathbf{g}, \tag{24}$$

where the size of $\mathbf{B}_{0,0}^{0}$ is denoted by m and the size of $\mathbf{A}_0$ (referred to as the block size) is denoted by n. The vector of cost rates for each state of the CTMC is denoted by $\mathbf{f}$. The cost rates for the boundary states is denoted by $\mathbf{f}_{\mathbf{b}}$, and $\mathbf{f}_{\mathbf{v}_0}, \mathbf{f}_{\mathbf{v}_1}, \ldots$ denote the cost rates for the repeating states. Since $\mathbf{R}$ does not depend on the actions of the controller, it is calculated prior to solving the problem and treated as a parameter in this formulation.

This formulation contains quadratic constraints in Eqs. (22) and infinite number of variables. Given that $\mathbf{R}$ is calculated before solving the problem, the infinite number of variables can be reduced to finite variables using the matrix geometric relations. In addition, the quadratic constraints can be linearized using additional variables. Let $(\mathbf{u}_k^{+} - \mathbf{u}_k^{-})^{T} = y_k \left(\mathbf{B}_{0,0}^{k}\right)^{T} \mathbf{b}^{T}$ denote the contribution of y_k to the steady-state Eqs. (22). Hence, the problem can be reformulated as the MIP given in Eqs. (25)–(34).

$$\min \begin{bmatrix} \mathbf{f}_{\mathbf{b}} & \left(\sum_{j=0}^{\infty} \mathbf{R}^{j} \left(\mathbf{f}_{\mathbf{v}_j}\right)^{T} \right)^{T} \end{bmatrix} \begin{bmatrix} \mathbf{b}^{T} \\ \mathbf{v}_0{}^{T} \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} \mathbf{1}_{1,m} & \left((\mathbf{I}_n - \mathbf{R})^{-1} \mathbf{1}_{n,1} \right)^{T} \end{bmatrix} \begin{bmatrix} \mathbf{b}^{T} \\ \mathbf{v}_0{}^{T} \end{bmatrix} = 1 \tag{26}$$

$$\mathbf{B}_{0,0}^{0}{}^{T} \mathbf{b}^{T} + \sum_{k=1}^{K} (\mathbf{u}_k^{+} - \mathbf{u}_k^{-})^{T} + \mathbf{B}_{1,0}{}^{T} \mathbf{v}_0{}^{T} = \mathbf{0}_{m,1} \tag{27}$$

$$\begin{bmatrix} \mathbf{B}_{0,1}{}^{T} & \mathbf{B}_{1,1}{}^{T} + (\mathbf{R}\mathbf{A}_2)^{T} \end{bmatrix} \begin{bmatrix} \mathbf{b}^{T} \\ \mathbf{v}_0{}^{T} \end{bmatrix} = \mathbf{0}_{n,1} \tag{28}$$

$$\left(\mathbf{B}_{0,0}^{k}\right)^{T} \mathbf{b}^{T} \leq (\mathbf{u}_k^{+} - \mathbf{u}_k^{-})^{T} + (1 - y_k)\,\mathbf{M} \qquad 1 \leq k \leq K \tag{29}$$

$$\left(\mathbf{B}_{0,0}^{k}\right)^{T} \mathbf{b}^{T} \geq (\mathbf{u}_k^{+} - \mathbf{u}_k^{-})^{T} - (1 - y_k)\,\mathbf{M} \qquad 1 \leq k \leq K \tag{30}$$

$$\mathbf{u}_k^{+T} \leq y_k \mathbf{M} \qquad 1 \leq k \leq K \tag{31}$$

$$\mathbf{u}_k^{-T} \leq y_k \mathbf{M} \qquad 1 \leq k \leq K \tag{32}$$

$$y_k \in \{0,1\}^{K} \tag{33}$$

$$\mathbf{Ey} \geq \mathbf{g} \tag{34}$$

The big-M values in the $\mathbf{M}$ vectors in Eqs. (29) and (31) can be set to $\mathbf{M} = \sum_{i=1}^{m} \mathbf{e}_{i,m}{}^{T} \max \left\{ \mathbf{B}_{0,0}^{k} \mathbf{e}_{i,m}{}^{T} \right\}$ and in Eqs. (30) and (32) to $\mathbf{M} = -\sum_{i=1}^{m} \mathbf{e}_{i,m}{}^{T} \min \left\{ \mathbf{B}_{0,0}^{k} \mathbf{e}_{i,m}{}^{T} \right\}$. When $\mathbf{B}_{0,0}^{k}$ matrices are sparse, this setting eliminates a considerable chunk of the variables prior to solving the model using their respective big-M values. In addition, after eliminating these variables, a large number of constraints will be eliminated as well because their relevance depended on the ability of the eliminated variables to take positive values. The number of variables and constraints in this formulation is given in Table 2.

Table 2
Number of variables and constraints of the MIP.

Variables		Constraints
Continuous	Binary	
$(2K+1)m+n$	K	$(4K+1)m+n+1$ (in addition to $\mathbf{Ey} \geq \mathbf{g}$)

Given the general formulation (25)–(34), in order to solve the problem of choosing the optimal thresholds for the marking-dependent threshold policy, the relevant binary variables have to be defined and $\mathbf{B}_{0,0}^{0}, \mathbf{B}_{0,0}^{k}$ must be specified accordingly. The steps for forming these matrices are given in Appendix A.

In summary, given the MMAP representations of the demand and production D, W, the costs c^{-}, c^{+}, we first determine the matrix $\mathbf{R}$ using successive substitution using Eq. (16). Then, we use an estimate for the upper bound for the thresholds, denoted by $\bar{X}$ to build $\mathbf{B}_{0,0}^{0}, \mathbf{B}_{0,0}^{k}, \mathbf{B}_{1,0}, \mathbf{B}_{0,1}, \mathbf{B}_{1,1}, \mathbf{A}_2$ using the method given in this subsection and Section 5 using Eqs. (6)–(10) and (73)–(74) in Appendix A. Finally, all this information is given as parameters to the MIP formulation given in Eqs. (25)–(34). Then, the solution of the MIP problem is converted to the optimal threshold levels using Eq. (71) in Appendix A.

6. Data-driven control of the production/inventory system by using marking-dependent threshold policy

Implementing the analytical method presented in Section 5 as a real-time control policy requires estimating the MMAP representations for the production, demand, and information inter-event times by using the historical data.

Alternatively, the parameters for the marking-dependent threshold policy can be determined by simulating a system controlled with the marking-dependent threshold policy and then using a simulation optimization approach. We give the outline of the discrete event simulation algorithm for the production inventory system presented in this study in Appendix B.

In this section, we present a mathematical programming approach for joint simulation and optimization of a production system that is controlled with the marking-dependent threshold policy.

6.1. Data

The data-driven JSO approach uses a trace of information and demand arrival times, their markings, the indicators for demand and information, a trace of the production times and their markings gathered from the facility together with the available statistical information about the random variables. The collected traces can also be used to generate additional replications by using the bootstrapping methods. In this study, the details of combining collected and generated traces with the simulated data are not given and we use a general definition of a trace that combines collected, generated, and simulated arrival times.

The information and demand inter-arrival times are denoted as $\mathcal{T} = \{\tau_1, \tau_2, \ldots\}$. The time of the ith arrival is denoted with $a_i = \sum_{k=1}^{i} \tau_i$ and the first arrival happens at time τ_1. The corresponding markings at the moment of these arrivals are denoted by $\alpha_i \in \{1, \ldots, C_D\}$. An indicator vector $\Xi = \{\xi_i\}$ is used to distinguish a demand arrival coupled with information ($\xi_i = 1$) and an information arrival without the demand ($\xi_i = 0$) for the ith arrival. The trace of the observed production times is denoted as $G = \{g_1, g_2, \ldots\}$ and their markings are denoted by $\beta_k \in \{1, \ldots, C_W\}$.

Table 3
Description of parameters and variables.

		Domain	Description
Parameter	c^+	$\mathbb{R}_+$	Inventory cost for one item in inventory in unit time
	c^-	$\mathbb{R}_+$	Backlog cost for one customer waiting for a unit time
	a_i	$\mathbb{R}_+$	The time of the i'th arrival of information
	ξ_i	$\{0,1\}$	Indicator for arrivals of demand alongside information
	g_k	$\mathbb{R}_+$	The k'th process time
	α_i	$\{1,\dots,C_D\}$	The marking of the information and demand process observed at a_i
	β_k	$\{1,\dots,C_W\}$	The marking of the production time process observed at F_k
	M	–	A big enough value
Variable	Δ_i^+	$\mathbb{R}_+$	The amount of time that product i is produced earlier than customer i arriving
	Δ_i^-	$\mathbb{R}_+$	The amount of time that product i is produced later than customer i arriving
	X_i^1	$\mathbb{Z}$	Inventory position exactly after a_i
	X_i^2	$\mathbb{Z}$	Inventory position exactly after F_k
	R_k	$\mathbb{R}_+$	Beginning of k'th production
	F_k	$\mathbb{R}_+$	Completion of k'th production
	U_i^a	$\mathbb{B}$	Indicates if production starts at a_i
	U_k^f	$\mathbb{B}$	Indicates if production resumes at F_k
	M_i	$\mathbb{B}$	Indicates if the machine is working exactly before a_i
	S_{c_D,c_W}	$\mathbb{N}$	Threshold level for the marking pair (c_D, c_W)
	$O_{i,k}^1$	$\mathbb{B}$	$a_i > F_k \rightarrow O_{i,k}^1 = 1$
	$O_{i,k}^2$	$\mathbb{B}$	$a_i < F_k \rightarrow O_{i,k}^2 = 1$
	$O_{i,k}^3$	$\mathbb{B}$	$O_{i,k}^3 = 1 \rightarrow a_i = F_k$
	$O_{i,k}^4$	$\mathbb{B}$	$a_i > R_k \rightarrow O_{i,k}^4 = 1$
	$O_{i,k}^5$	$\mathbb{B}$	$a_i < R_k \rightarrow O_{i,k}^5 = 1$
	$O_{i,k}^6$	$\mathbb{B}$	$O_{i,k}^6 = 1 \rightarrow a_i = R_k$
	O_k^7	$\mathbb{B}$	$O_k^7 = 1 \rightarrow R_k = F_{k-1}$
	O_k^8	$\mathbb{B}$	$R_k > f_{k-1} \rightarrow O_k^8 = 1$

6.2. Joint simulation and optimization representation

In this section, we give the joint simulation and optimization formulation for the problem of determining the optimal marking-dependent thresholds S_{c_D,c_W} for each marking pair that minimize the average total cost for given traces. Table 3 gives a short description of the variables and parameters of the problem.

The full mathematical programming formulation of the problem for one replication is given in Eqs. (35)–(66).

6.2.1. Objective function

The objective is minimizing the total inventory and backlog costs for all the demand arrivals. Since there are no lost sales in this model, when the system starts with no items in the inventory, the item that takes $g_{\sum_{l\le i}\xi_l}$ time units to produce will be given to the customer arriving at a_i if $\xi_i = 1$. If $\xi_i = 0$, then arrival i does not produce any inventory or backlog cost directly. Hence let R_k be the time that the kth production starts and F_k be the time that it ends, then the total cost produced by arrival i will be $\xi_i\left(c^-\left[F_k - a_i\right]^+ + c^+\left[a_i - F_k\right]^+\right)$ where $k = \sum_{l\le i}\xi_l$. As a result, the total cost of the system will be $\sum_i \xi_i\left(c^-\left[F_{\sum_{l\le i}\xi_l} - a_i\right]^+ + c^+\left[a_i - F_{\sum_{l\le i}\xi_l}\right]^+\right)$. Since this objective function is not linear, we linearize it by introducing variables $\Delta_i^+ = \left[a_i - F_{\sum_{l\le i}\xi_l}\right]^+$ and $\Delta_i^- = \left[F_{\sum_{l\le i}\xi_l} - a_i\right]^+$ that represent the earliness and lateness for each item. Then, the objective is minimizing the sum of the total earliness and lateness with respect to the cost rates c^- and c^+ as given in Eqs. (35). The equations used to linearize the objective function are expressed in Eqs. (36)–(38).

$$\min z = \sum_i \xi_i\left(c^+\Delta_i^+ + c^-\Delta_i^-\right) \tag{35}$$

subject to

$$\Delta_i^+ - \Delta_i^- = a_i - F_k \qquad \forall i,k : \sum_{l\le i}\xi_l = k \tag{36}$$

$$\Delta_i^+ \le O_{i,k}^1\mathrm{M} \qquad \forall i,k : \sum_{l\le i}\xi_l = k \tag{37}$$

$$\Delta_i^- \le \left(1 - O_{i,k}^1\right)\mathrm{M} \qquad \forall i,k : \sum_{l\le i}\xi_l = k \tag{38}$$

6.2.2. Marking-dependent threshold policy

The marking-dependent threshold policy can be enforced by using two rules. These rules make use of the inventory level and the state of the machine at each decision epoch. The first rule is that if the machine is not working at the moment of an arrival and the inventory level is less than the threshold level, production must start. Let M_i denote the status of the machine at time a_i, where $M_i = 1$ indicates that the machine has been working when ith arrival happens. Let X_i^1 denote the inventory position exactly after a_i. Let U_i^a denote the decision made at a_i, where $U_i^a = 1$ if production starts at a_i and $U_i^a = 0$ otherwise. If the inventory position exactly after a_i, is less than the threshold $X_i^1 < S_{\alpha_i}$, either the machine must be already working meaning $M_i = 1$ or it must start working meaning $U_i^a = 1$. This relation is expressed in Eq. (39). Fig. 6(c) illustrates this relation. Otherwise, if the inventory position is more than or equal to the threshold level, regardless of whether the machine is idle or not, a new production cannot be initiated. Eq. (40) expresses this relation as depicted in Fig. 6(d).

$$U_i^a + M_i \ge O_{i,k+1}^2 + O_{i,k}^1 - 2 + \frac{S_{\alpha_i,\beta_k} - X_i^1}{\mathrm{M}} \qquad \forall i,k \tag{39}$$

$$U_i^a \le 2 - (O_{i,k+1}^2 + O_{i,k}^1) + 1 - \frac{X_i^1 - S_{\alpha_i,\beta_k} + 1}{\mathrm{M}} \qquad \forall i,k \tag{40}$$

The second rule is that after the completion of a production, if the inventory level is less than the threshold level set by the last arrival, then the production must be resumed. Let X_k^2 denote the inventory position exactly after F_k. Let U_k^f denote the decision made at F_k, where $U_k^f = 1$ means production will be resumed immediately. To enforce the second rule, the last arrival before F_k must be identified. For a pair i and k, a_i is the last arrival before F_k if $a_i < F_k < a_{i+1}$. Using the dummy variables given in Table 3, this can be expressed as $O_{i,k}^2 + O_{i+1,k}^1 = 2$. Hence production must continue if $X_k^2 < S_{\alpha_i}$ and $O_{i,k}^2 + O_{i+1,k}^1 = 2$. This relation is expressed in Eq. (41). Fig. 6(f) illustrates this relation. Given $O_{i,k}^2 + O_{i+1,k}^1 = 2$, if $X_k^2 \ge S_{\alpha_i}$, production cannot be resumed. This relation is expressed in Eq. (42) and in Fig. 6(g).

$$U_k^f \ge O_{i,k}^2 + O_{i+1,k}^1 - 2 + \frac{S_{\alpha_i,\beta_k} - X_k^2}{\mathrm{M}} \qquad \forall i,k \tag{41}$$

$$U_k^f \le 2 - (O_{i,k}^2 + O_{i+1,k}^1) + 1 - \frac{X_k^2 - S_{\alpha_i,\beta_k} + 1}{\mathrm{M}} \qquad \forall i,k \tag{42}$$

6.2.3. The inventory and the machine status

Since the control policy uses the inventory position as an input to operate, we express the inventory position as a function of the arrival and production processes. Since the inventory position at each arrival together with the production completion capture the inventory process, we define the variables X_i^1 and X_k^2 as the inventory position exactly after a_i and F_k. Hence X_i^1 can be calculated as the number of items produced before the ith arrival subtracted by $\sum_{l\le i}\xi_i$. Let $O_{i,k}^1$ be a binary dummy variable that is 1 if $F_k \le a_i$. Then the relation in Eq. (43) adjusts the inventory position. Similarly, with the appropriate dummy variable, X_k^2 is adjusted in Eq. (44). Fig. 5 illustrates these equations.

$$X_i^1 = \sum_k O_{i,k}^1 - \sum_{l\le i}\xi_l \qquad \forall i \tag{43}$$

$$X_k^2 = k - \sum_i \xi_i O_{i,k}^2 \qquad \forall k \tag{44}$$

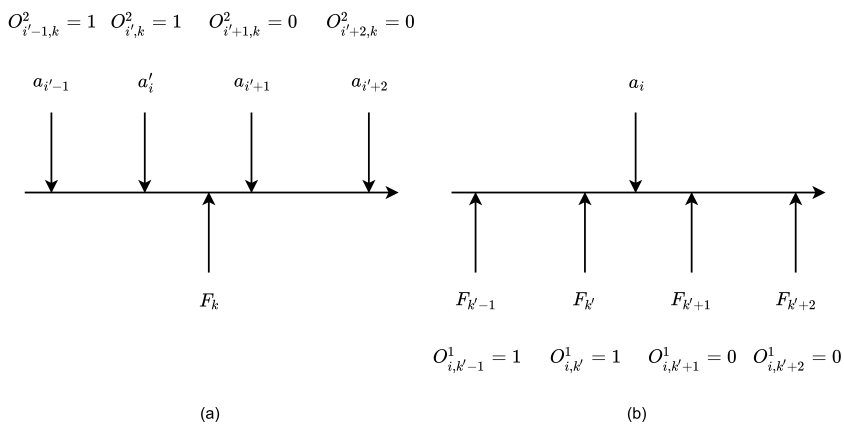

Fig. 5. (a) Inferring the inventory position exactly after completion of a production using dummy variables $O^2_{i,k}$. (b) Inferring the inventory level exactly after an arrival using dummy variables $O^1_{i,k}$.

The decision to start production also depends on whether the machine is working or not. Let M_i denote the state of the machine exactly before a_i, where $M_i = 1$ indicates that the machine is already working when ith demand arrives. For the machine to be working at a_i, there must exist a k such that $R_k < a_i < F_k$ holds. Eqs. (45)–(46) enforce this relation. Fig. 6(a)–(b) illustrates these relations.

$$M_i \geq O^2_{i,k} + O^4_{i,k} - 1 \qquad \forall i,k \tag{45}$$

$$M_i \leq 2 - O^1_{i,k} - O^5_{i,k+1} \qquad \forall i, k < |\mathcal{T}| \tag{46}$$

6.2.4. Other constraints

Eq. (47) prevents the starting of a production from appearing as if the machine has been already working. Eqs. (48) and (49) express that only one item can be produced at a time and item i takes g_i time units to produce. Eqs. (50)–(51) give the relation between the production decisions and the dummy variables. Inequalities (52)–(64) express the relation between the dummy variables and the arrival and production times a_i, R_k and F_k. Eq. (65) ensures that every production is either triggered by the completion of the previous production or the arrival of a demand. Eq. (66) ensures that no arrival triggers a production out of order.

$$M_i + \sum_k O^6_{i,k} \leq 1 \qquad \forall i \tag{47}$$

$$R_i \geq F_{i-1} \qquad \forall i > 1 \tag{48}$$

$$F_i = R_i + g_i \qquad \forall i \tag{49}$$

$$U^a_i = \sum_k O^6_{i,k} \qquad \forall i \tag{50}$$

$$U^f_k = O^7_{k+1} \qquad \forall k < |\mathcal{T}| \tag{51}$$

$$O^1_{i,k} \leq 1 + \frac{(a_i - F_k)}{\mathrm{M}} \qquad \forall i,k \tag{52}$$

$$O^2_{i,k} \leq 1 - \frac{(a_i - F_k)}{\mathrm{M}} \qquad \forall i,k \tag{53}$$

$$O^1_{i,k} \geq \frac{(a_i - F_k)}{\mathrm{M}} \qquad \forall i,k \tag{54}$$

$$O^2_{i,k} \geq -\frac{(a_i - F_k)}{\mathrm{M}} \qquad \forall i,k \tag{55}$$

$$O^1_{i,k} + O^2_{i,k} + O^3_{i,k} = 1 \qquad \forall i,k \tag{56}$$

Table 4
Number of variables and constraints of the MIP representation of data-driven JSO.

Variables			Constraints
Continuous	Binary	Integer	
$4\|\mathcal{T}\|$	$6\|\mathcal{T}\|^2 + 5\|\mathcal{T}\|$	$2\|\mathcal{T}\| + C$	$17\|\mathcal{T}\|^2 + 6\|\mathcal{T}\| + 3\sum_i \xi_i$

$$O^4_{i,k} \leq 1 + \frac{(a_i - R_k)}{\mathrm{M}} \qquad \forall i,k \tag{57}$$

$$O^5_{i,k} \leq 1 - \frac{(a_i - R_k)}{\mathrm{M}} \qquad \forall i,k \tag{58}$$

$$O^4_{i,k} \geq \frac{(a_i - R_k)}{\mathrm{M}} \qquad \forall i,k \tag{59}$$

$$O^4_{i,k} \geq -\frac{(a_i - R_k)}{\mathrm{M}} \qquad \forall i,k \tag{60}$$

$$O^4_{i,k} + O^5_{i,k} + O^6_{i,k} = 1 \qquad \forall i,k \tag{61}$$

$$O^7_k \leq 1 - \frac{(R_k - f_{k-1})}{\mathrm{M}} \qquad \forall i, k > 1 \tag{62}$$

$$O^7_k \leq 1 + \frac{(R_k - f_{k-1})}{\mathrm{M}} \qquad \forall i, k > 1 \tag{63}$$

$$O^7_1 = 0 \tag{64}$$

$$O^7_k + \sum_i O^6_{i,k} \geq 1 \qquad \forall i \tag{65}$$

$$O^6_{i,k} \leq 1 - \frac{\sum_{j<i,l>k} O^6_{j,l}}{\mathrm{M}} \qquad \forall i,k \tag{66}$$

Table 4 gives the number of variables and constraints in this formulation. The number of the constraints and binary variables is not affected by C_D and C_W and the main contributor to the complexity of the problem is the trace length. Since the number of variables increases with the square of the trace length, using the mathematical programming formulation to determine the optimal thresholds requires significant computing power and memory. For the cases where the computing power is not sufficient to implement JSO, the discrete event simulation formulation given in Appendix B with a search algorithm can be used.

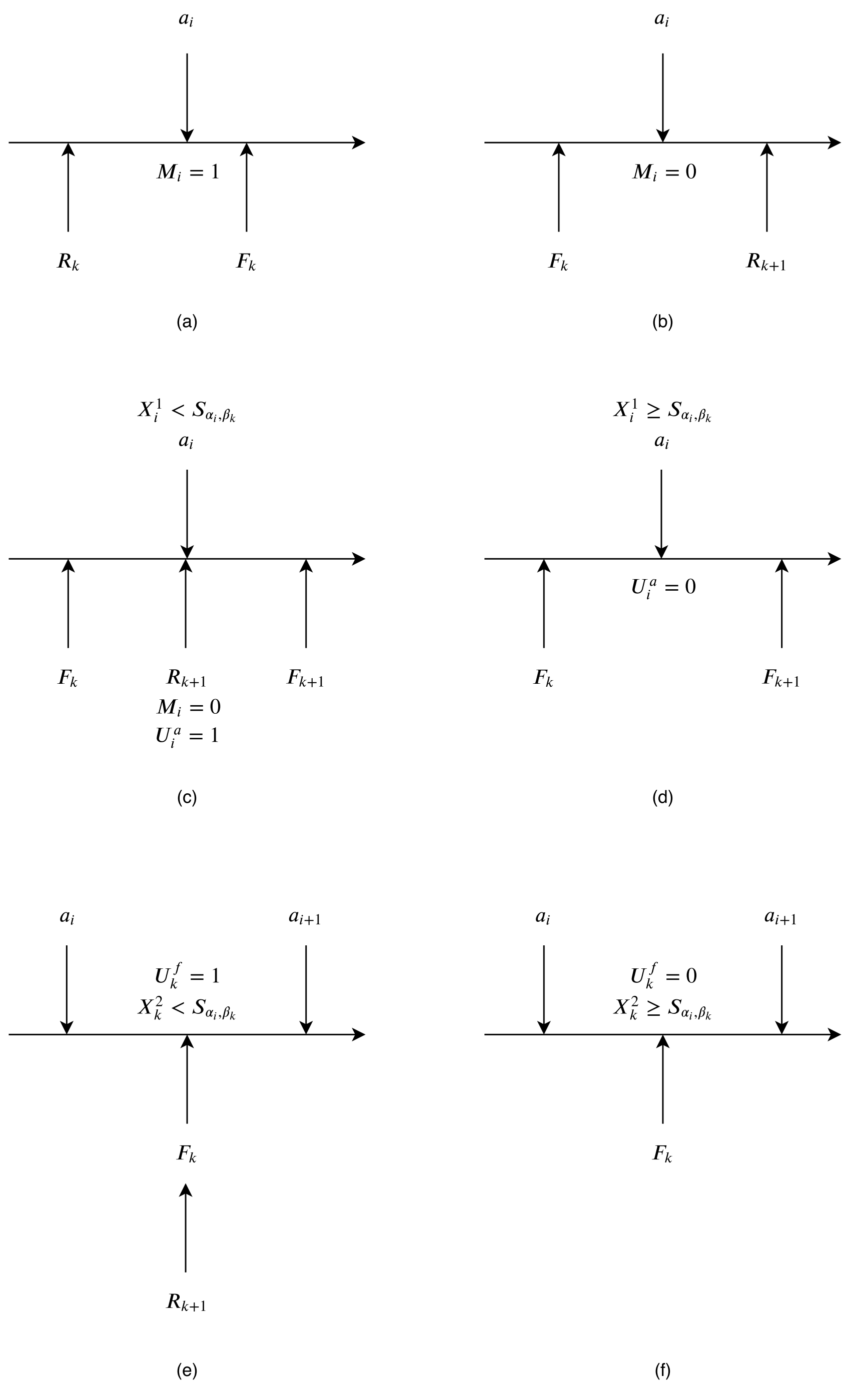

Fig. 6. Configurations of event times and their corresponding variables and decisions. (a) Since $\exists k : R_k < a_i < F_k$ the machine is working at a_i, this is reflected in $M_i = 1$. (b) Since $\exists k : F_k < a_i < R_{k+1}$ the machine is idle at a_i, this is reflected in $M_i = 0$. (c) If the machine is idle and the inventory position is less than the threshold after an arrival, a new production starts, this is reflected in $U_i^a = 1$. (d) If the inventory position is not less than the threshold after an arrival, regardless of the status of the machine new production will not start, this is reflected in $U_i^a = 0$. (e) After completion of a product, if the inventory position is less than the threshold, production continues. Arrival i is the last arrival before F_k if $a_i < F_k < a_{i+1}$. (f) After completion of a product, if the inventory position is not less than the threshold set by the last arrival, production stops..

7. Analysis of the example

In this section, we analyze the specific system introduced in Section 4 by using the methods presented in Sections 5 and 6. We evaluate the system for a range of parameter values under different information scenarios: the buffer size B, the processing rate of the unreliable machine WS_2, the processing rate of the reliable machine WS_3, the number of the repair phases r, the rate of the repair phases λ, the failure probability γ, the ratio of the production rate to the arrival rate μ_1/TP, and the ratio of the backlog cost to inventory cost c^-/c^+ are varied as shown in Table 5. 5576 different cases are used to analyze the system in four different information availability scenarios. In this experimental setup, the information about the demand is only recorded at the arrival instances.

In this system, $qr + 2r + q + 3$ states are required to model the information and demand processes. Therefore, the optimal state-dependent threshold policy that uses all the information will have up to 36 thresholds for 36 states. Even for the case where all the states are fully observable, determining all the optimal thresholds by using the information traces collected in real time is not computationally tractable.

Table 5
The range of parameters used in the numerical experiments.

Buffer size q	$\{1,2,3\}$
The processing rate of the unreliable machine μ_2	$\{1,2,3\}$
The processing rate of the reliable machine μ_3	$\{1,2,3\}$
The number of the repair phases r	$\{2,4,6\}$
The rate of each repair phase λ	$\{0.5,0.7,0.9\}$
Breakdown probability γ	$\{0.01,0.05,0.09\}$
The production rate to throughput ratio μ_1/TP	$\{1.5,2,2.5\}$
The backlog cost rate to inventory cost rate ratio c^-/c^+	$\{2,3,4,5\}$

Table 6
Percentage difference of average cost using optimal analytical solution with one or both of the sources of information with respect to using the analytical solution with none of the sources of information: $\frac{|\pi^*_{a,b}-\pi^*_{0,0}|}{\pi^*_{0,0}}$.

	Parameter level	Available information		
		Only buffer status (two thresholds) $\frac{|\pi^*_{0,1}-\pi^*_{0,0}|}{\pi^*_{0,0}}$	Only machine status (two thresholds) $\frac{|\pi^*_{1,0}-\pi^*_{0,0}|}{\pi^*_{0,0}}$	Both the machine and buffer statuses (four thresholds) $\frac{|\pi^*_{1,1}-\pi^*_{0,0}|}{\pi^*_{0,0}}$
Buffer size q	1	2.14	4.57	5.46
	2	2.81	4.24	5.68
	3	3.10	3.89	5.72
Rate of unreliable machine μ_2	1	2.39	3.23	5.10
	2	2.78	4.61	6.00
	3	2.81	4.98	5.78
Rate of reliable machine μ_3	1	2.15	3.07	3.89
	2	3.20	5.13	6.84
	3	2.94	5.41	7.38
Number of repair phases r	2	2.04	2.43	3.80
	4	2.85	4.98	6.35
	6	3.67	7.05	8.39
Rate of repair phases λ	0.5	3.18	5.65	7.01
	0.7	2.61	4.17	5.53
	0.9	2.33	3.37	4.73
Breakdown probability γ	0.01	2.05	2.56	4.01
	0.05	3.07	5.29	6.68
	0.09	3.05	5.47	6.67
Service rate to throughput rate ratio $\frac{\mu_1}{TP}$	1.5	0.94	1.31	1.73
	2	2.40	3.85	5.03
	2.5	3.68	5.97	7.93
Backlog cost rate to inventory cost rate ratio $\frac{c^-}{c^+}$	2	2.53	5.33	6.56
	3	2.51	4.04	5.54
	4	2.57	3.69	5.03
	5	3.07	3.73	5.13
Average		2.66	4.25	5.61

7.1. Evaluation methodology

In our evaluation methodology, we first use the analytical method presented in Section 5 to analyze the production/inventory system with the given production and demand processes and with the given markings selected for control. We determine the optimal marking-dependent thresholds and the optimal cost by using the MIP formulation given in Section 5.3.

In order to compare the performance of the data-driven approach with the exact evaluation, we generate different-length traces for the given system parameters and the selected markings. We then use the JSO approach described in Section 6.2 for short traces and the DES approach given in Appendix B with a search algorithm for long traces. We denote the set of the thresholds determined by the JSO formulation given the information scenario specified by a and b and trace length $|\mathcal{T}|$ as $S^{\mathrm{JSO}(a,b,|\mathcal{T}|)}$.

As another benchmark, we use the same demand and information arrival trace to fit an exponential or a phase type distribution that matches the first three moments. We set a threshold based on these fitted distribution using the analytical methods. We denote the thresholds determined after fitting an exponential distribution and a phase-type distribution to a trace of length $|\mathcal{T}|$ as $S^{\mathrm{EXP}(|\mathcal{T}|)}$ and $S^{\mathrm{PH}(|\mathcal{T}|)}$ respectively. Fig. 7 illustrates the evaluation process.

7.2. Effect of marking selection on the performance of the production/inventory system

Deciding on the markings to be used in the marking-dependent threshold policy in a right way improves the performance of the control policy. In this section, we investigate the following questions: *what will be the performance if only one marking is to be used; should we use the buffer status or the production status as the selected marking?; and what is the additional benefit of using both of the markings?*

In order to investigate these questions, each case is analyzed by using the exact method with the corresponding MMAP representations of the processes depending on the number of markings for the given parameters. When both the buffer status and the machine status are used, the marking-dependent threshold policy uses four thresholds to control the system. If only the buffer status or the machine status is used as a marking, the control policy uses two thresholds. If these markings are not used and the controller only uses the demand inter-arrival times, then only one threshold is used. The average cost of the system under an information scenario with the optimal thresholds is denoted by $\pi^*_{a,b}$ where $a \in \{0,1\}$ indicates whether the production marking is used ($a=1$) or not ($a=0$), and $b \in \{0,1\}$ indicates whether the buffer status marking is used ($b=1$) or not ($b=0$). For each case, after the optimal thresholds are determined, the minimum cost $\pi^*_{a,b}$ has been compared to the cost obtained when both of the information sources are used by the controller $\pi^*_{1,1}$.

Table 6 shows the results of these experiments. The results indicate that the marginal contribution of using both markings is lower than the contribution of using only the production status marking on average. As the expected repair time increases, the number of repair phases increases, and the breakdowns become more frequent, the contribution of using the production marking is higher than the contribution of using the buffer marking. In these cases, the effect of the breakdowns become more pronounced for the system and the disparity between the demand and production rate increases the value of information. If the production rate is much higher than the demand rate, it will be able to react to trigger or halt production depending on the markings in a faster way. Large backlog costs also decrease the value of information, as it becomes less attractive to carry backlogs based on the estimation that the demand will temporarily be low.

Table 7 shows the effect of the parameters on the optimal threshold level for each marking when both the buffer status and the production status are used as markings. In this case, the control policy uses 4 thresholds corresponding to 4 different marking pairs. On average, the threshold for the marking *machine under repair and buffer empty* is the lowest and the threshold for the marking *machine in working condition and buffer not empty* is the largest. As the number of repair phases increases, the coefficient of variation of the repair time decreases. As a result, all thresholds decrease. Similarly, as the breakdowns become more frequent, all the threshold levels increase to protect the system from the downtime. As the backlog cost increases compared to the inventory cost and as the production becomes slower compared to the demand, all thresholds increase. Furthermore, the threshold for the marking *machine in working condition and buffer not empty* is more sensitive to these parameters as opposed to the parameters related solely to the demand process. Overall, the threshold for the marking *machine under repair and buffer empty* is the most sensitive threshold to the parameters of the demand process.

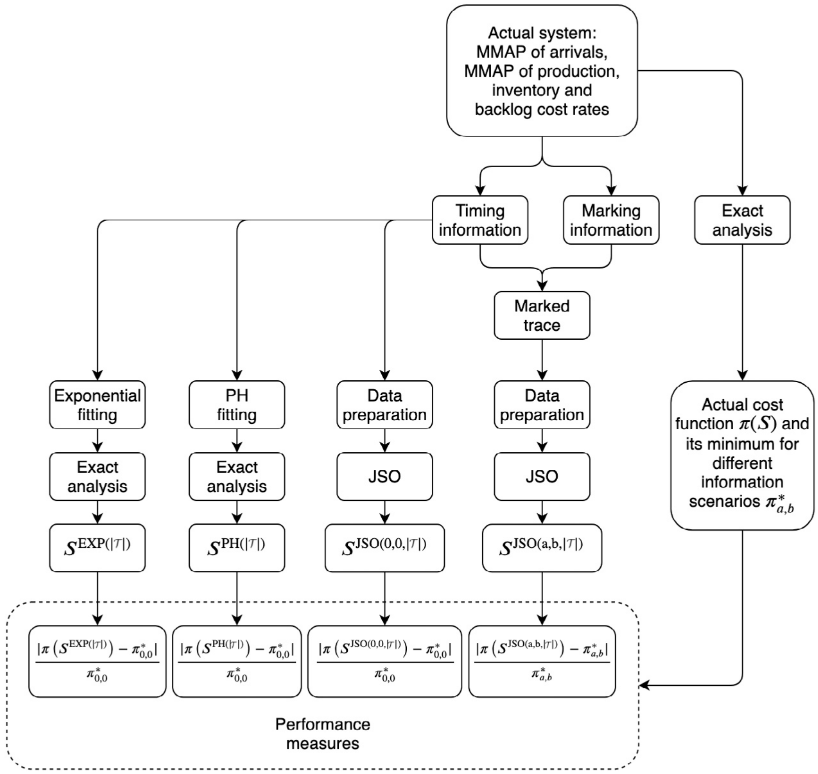

Fig. 7. Scheme of the performance evaluation for the control methods.

Table 7
Average threshold levels for the markings.

	Parameter level	Marking			
		Machine under repair and buffer empty	Machine in working condition and buffer empty	Machine under repair and buffer not empty	Machine in working condition and buffer not empty
q	1	0.75	1.81	1.18	2.29
	2	0.75	1.73	1.28	2.32
	3	0.73	1.68	1.37	2.33
μ_2	1	0.81	1.67	1.33	2.51
	2	0.70	1.74	1.24	2.24
	3	0.72	1.82	1.25	2.17
μ_3	1	0.90	1.83	1.33	2.20
	2	0.62	1.70	1.24	2.35
	3	0.60	1.63	1.21	2.47
r	2	0.97	1.83	1.47	2.43
	4	0.63	1.71	1.14	2.25
	6	0.42	1.61	1.04	2.13
λ	0.5	0.57	1.71	1.19	2.25
	0.7	0.76	1.76	1.28	2.34
	0.9	0.84	1.75	1.33	2.33
γ	0.01	0.49	1.50	1.15	2.00
	0.05	0.77	1.79	1.29	2.38
	0.09	1.06	2.03	1.42	2.67
$\frac{\mu_1}{TP}$	1.5	2.09	3.08	2.14	3.62
	2	0.60	1.66	1.10	2.23
	2.5	0.22	1.18	1.00	1.75
$\frac{c^-}{c^+}$	2	0.31	1.32	1.08	1.81
	3	0.64	1.67	1.20	2.25
	4	0.94	1.94	1.37	2.52
	5	1.20	2.16	1.50	2.79
Average		0.74	1.74	1.27	2.31

7.3. Performance of the data-driven method

In order to capture the effect of the trace length, we conduct the experiments for traces that have 100, 1000, 10000 observed event arrivals. We use the JSO to analyze the system for a total of 66912 cases depending on the range of parameters, informational settings, and trace length. For each case, 5 traces were generated and analyzed by the JSO approach.

Table 8 reports the performance of the JSO approach with the given trace lengths compared to the optimal analytical solution for the case where both sources of information are used to control the system. The percentage deviation of the costs obtained by using the JSO compared to the analytical solution, $\frac{|\pi(\mathcal{S}^{\mathrm{JSO}(a,b,|\mathcal{T}|)})-\pi^*_{1,1}|}{\pi^*_{1,1}}$ is reported for each case.

Fig. 8 depicts the average costs obtained with the JSO approach with different levels of information and different trace lengths $\pi(\mathcal{S}^{\mathrm{JSO}(a,b,|\mathcal{T}|)})$. The results show that as the trace length increases, the cost obtained with the JSO approach gets closer to the optimal cost obtained with the analytical approach under each information scenario. That is, the JSO approach makes use of the additional information to reduce cost. However, the selection of the markings to be used in the marking-dependent threshold policy has a significant effect on the performance. When no additional markings are used and the control only uses the unmarked trace that includes the demand inter-arrival times, the performance of the JSO with the longest trace is worse than the analytical solution for the case both the production and inventory status are used as markings in addition to the inter-arrival times.

As the demand arrival pattern is interrupted by breakdowns due to longer down times, the information about the machine is expected to become more valuable. The results confirm that decreasing the repair rate has an adverse effect on the performance of the JSO when the production status information is not used in the markings.

Fig. 9 compares the performance of JSO to the methods where the analytical method is used with the parameters estimated from the unmarked traces. The percentage loss compared to the solution with

Table 8
Percentage difference of the average cost using JSO with different information levels with respect to the optimal analytical solution using both sources of information given different trace lengths.

		Available information											
		Unmarked trace $\frac{\lvert\pi(S^{JSO(0,0,\lvert\mathcal{T}\rvert)})-\pi^*_{1,1}\rvert}{\pi^*_{1,1}}$ (one threshold)			Trace with buffer status only $\frac{\lvert\pi(S^{JSO(0,1,\lvert\mathcal{T}\rvert)})-\pi^*_{1,1}\rvert}{\pi^*_{1,1}}$ (two thresholds)			Trace with machine status only $\frac{\lvert\pi(S^{JSO(1,0,\lvert\mathcal{T}\rvert)})-\pi^*_{1,1}\rvert}{\pi^*_{1,1}}$ (two thresholds)			Trace with buffer and machine status $\frac{\lvert\pi(S^{JSO(1,1,\lvert\mathcal{T}\rvert)})-\pi^*_{1,1}\rvert}{\pi^*_{1,1}}$ (four thresholds)		
		Trace Length			Trace Length			Trace Length			Trace Length		
		100	1000	10000	100	1000	10000	100	1000	10000	100	1000	10000
q	1	14.67	6.44	5.58	14.38	5.47	3.56	11.16	2.74	0.99	11.25	2.31	0.32
	2	13.18	6.96	6.08	12.23	5.08	3.40	10.48	3.24	1.53	10.18	2.43	0.37
	3	11.81	6.99	6.19	12.32	4.93	3.20	11.20	3.59	1.88	10.40	2.37	0.38
μ_2	1	12.27	6.40	5.33	11.70	4.87	3.01	10.52	3.71	1.88	9.67	2.43	0.34
	2	13.25	7.23	6.48	12.79	5.47	3.72	10.33	2.91	1.29	10.14	2.31	0.38
	3	15.01	7.10	6.29	15.30	5.32	3.43	12.48	2.64	0.89	12.65	2.42	0.40
μ_3	1	9.94	4.86	4.19	11.11	3.83	2.06	9.75	2.67	1.04	9.75	2.31	0.37
	2	19.64	7.82	6.81	14.57	5.70	3.92	12.48	3.23	1.62	11.99	2.25	0.37
	3	15.47	7.85	6.76	14.22	6.15	4.45	11.27	3.32	1.60	10.56	2.34	0.33
r	2	13.30	4.86	4.09	10.91	3.62	2.11	10.36	3.11	1.61	9.89	2.12	0.36
	4	11.98	7.40	6.34	14.51	5.91	3.76	11.79	3.48	1.31	11.65	2.71	0.36
	6	13.03	8.78	7.86	14.17	6.92	4.74	10.91	2.93	1.13	10.39	2.50	0.37
λ	0.5	15.50	8.17	7.14	14.47	6.14	4.27	11.56	3.11	1.21	11.37	2.52	0.35
	0.7	13.00	6.80	5.93	13.61	5.30	3.45	11.41	3.09	1.38	11.06	2.36	0.37
	0.9	11.34	5.96	5.10	11.83	4.74	2.85	10.27	3.22	1.47	9.82	2.32	0.36
γ	0.01	10.46	4.42	4.04	11.12	3.56	2.27	9.49	2.65	1.59	9.09	1.65	0.29
	0.05	16.12	8.34	7.32	14.10	6.20	4.23	11.78	3.17	1.36	11.28	2.53	0.40
	0.09	13.25	8.17	6.82	14.70	6.42	4.07	11.97	3.60	1.10	11.88	3.03	0.39
$\frac{\mu_1}{TP}$	1.5	16.54	3.95	2.23	15.78	4.61	1.44	14.78	3.75	0.83	14.97	3.55	0.61
	2	11.05	6.76	6.13	12.11	5.04	3.53	9.97	2.71	1.21	9.46	2.18	0.30
	2.5	12.24	10.22	9.82	12.03	6.53	5.60	8.49	2.95	2.02	7.82	1.48	0.18
$\frac{c^-}{c^+}$	2	12.92	7.88	7.19	12.77	6.12	4.60	9.43	2.67	1.27	9.37	1.95	0.24
	3	13.38	7.06	6.21	13.09	5.39	3.73	10.94	3.12	1.42	10.37	2.33	0.34
	4	13.27	6.48	5.45	13.41	4.82	3.06	11.43	3.18	1.35	11.06	2.43	0.41
	5	13.55	6.48	5.39	13.96	5.24	2.70	12.52	3.57	1.37	12.21	2.90	0.47
Average		13.28	6.98	6.06	13.31	5.39	3.52	11.08	3.14	1.35	10.75	2.40	0.36

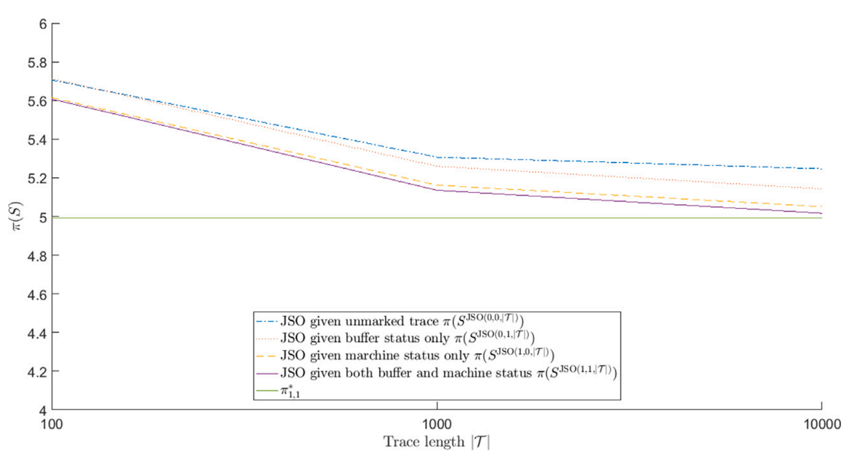

Fig. 8. The average cost of using JSO with different trace-lengths and information levels compared to the average optimal analytical cost with both sources of information.

the analytical method that uses the unmarked traces $\frac{\lvert\pi(S)-\pi^*_{0,0}\rvert}{\pi^*_{0,0}}$ is used for this comparison. With the assumption that the process times are not correlated, twenty bootstrap samples of the process times have been used in data preparation for the JSO. It is observed that using the analytical method with the fitted phase-type distribution yields results close to the JSO. When the trace length is sufficiently long, both methods yield the same results. However, when the trace length is short, parameter fitting methods cannot capture the distributions and therefore the performance is worse than the performance of the JSO approach. The average deviation with respect to the optimal analytical solution for the methods were 2.66% for JSO, 3.16% for exponential fitting and 4.15% for phase-type fitting. Fitting an exponential distribution that captures only the first moment outperforms the results obtained for the case where a phase-type distribution that matches the first three moments when the trace length is short.

Table 8 shows the effect of the parameters and information scenarios on the performance of data-driven JSO. It is worth noting that many of the patterns observed in Table 6 are present for the data-driven JSO with traces of 1000 and 10000 length. However, for short traces these patterns are not present. This indicates that with very short traces, having more information to use and having more thresholds to set might not translate to a better control of the system due to the difficulty of estimating the control policy parameters.

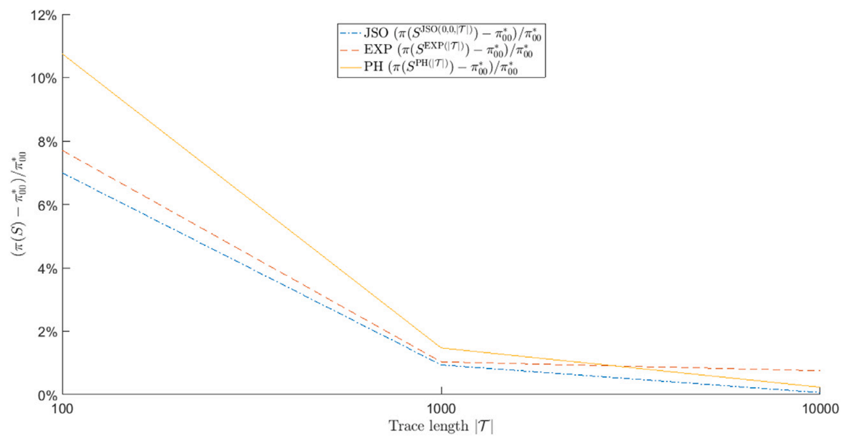

Fig. 9. Percentage difference of average cost using exponential and phase-type fitting methods and JSO with respect to the optimal analytical solution given unmarked traces.

The results from Figs. 8 and 9 and Table 8 show that the marking-dependent threshold policy implemented with the JSO approach as a real-time control policy is an effective way of matching supply and demand.

8. Conclusions

In this study, we propose using the marking-dependent threshold policy as an effective data-driven method to control production. We show that the parameters of the marking-dependent control policy can be determined with the joint simulation and optimization approach that combines the data collected from the shop floor with the available statistical information about the demand, production, and information arrival processes.

The marking-dependent threshold policy is easy to implement: a different threshold level is determined for each marking. When a marking is received, production of a new item is triggered if there is no item being produced at that moment and the inventory level is less than or equal to the threshold associated with that marking. The completion of a production triggers a production if the inventory level is less than the threshold associated with the last observed marking. If the inventory level reaches the threshold, the production stops until a new demand arrives, in which case it will stay idle if the newly observed marking corresponds to a threshold level lower than the inventory level.

We propose an analytical method based on the matrix geometric method for evaluating the performance of a production/inventory system controlled with the marking-dependent threshold policy where the demand, production, and information arrival processes are modeled as Marked Markovian Arrival Processes. We introduce an MIP formulation for determining the optimal thresholds based on this analytical method.

For the data-driven JSO method, we give the mathematical programming representation of the problem as a mixed integer program. Given the considerable increase in the power of solvers in solving mathematical programming problems, the MIP formulation is given as the first step in devising efficient solution methods for the problem with shorter trace lengths. In addition, we give the discrete event simulation procedure for the system if longer traces are to be used to determine the parameters of the marking-dependent control policy.

We use an experimental setup for a feeder line for an assembly system to evaluate the performance of the marking-dependent threshold policy that uses different markings to control the release of material into the feeder line. We use this setup to determine the effects of using different markings and also to evaluate the JSO approach with simulated shop-floor data with different trace lengths.

Our numerical results show that the JSO-based approach is an effective method for controlling the system. With sufficiently long traces, the proposed method is able to use the information effectively and yields results that are close to the performance obtained with the analytical solution for the same information level. The value of information about the status of a machine increases as the variability of the repair times decreases and the repair time increases. The information about the machine status is more valuable than the information about its downstream buffer. The JSO-based approach is an efficient way for determining the optimal threshold levels given limited data. Furthermore, this method outperforms parametric methods that first fit distributions ignoring the autocorrelation of the demand process and then determine the control parameters based on the analytical model with the fitted parameters.

We are working on implementing the marking-dependent control policy as a release control policy for a wafer fab in order to manage the flows into a die bank in a semiconductor manufacturing plant. In this implementation, markings such as the availability status of the stations that are identified as the bottleneck resources based on historical data and the level of work in progress inventory are used in the marking-dependent threshold policy. Given the complexity of semiconductor manufacturing, there may be many other sources of information that can be used to control the system. However, considering all sources of information will result in optimization problems that are computationally intractable. Furthermore, there is no simple way of deciding which sources of information to use.

This work can be extended in different ways. There are two fundamental questions regarding the marking-dependent threshold policy. First, the optimality of the marking-dependent threshold policies for the system considered in this study can be proven formally by formulating the problem as a partially observable Markov decision process. Furthermore, using a large number of markings to control a system is computationally prohibitive due to the difficulty of determining the optimal thresholds. Therefore, this work can be extended to develop a method to select a small number of markings from a set of possible markings. Given a large number of markings, controlling the system with a given smaller number of thresholds can be posed as a clustering problem. However, attacking this problem as a pure optimization problem can be computationally intractable. Therefore, there is a need to develop efficient clustering methods based on different data analytics approaches. In addition, the pattern of the arrivals can help in inferring the marking information. These are left for future research.

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Appendix A. Specifying the intermediate matrices for MIP

Let $\bar{X}$ denote the maximum threshold level admissible. Let $\psi_{P,i,j}$: $1 \leq i \leq C_D, 1 \leq j \leq C_W, 0 \leq X < \bar{X}$ denote the binary variable for the decision to continue production when the inventory position is X and marking pair (i,j) has been observed last. Hence, $\psi_{X,i,j} = \delta_{\{S_{i,j}>X\}}$ holds. Given this definition, for a policy to be a threshold policy, the following set of constraints must hold. Note that the constraints given by Eq. (67) correspond to Eq. (34) in the general formulation:

$$\psi_{P,i,j} \leq \psi_{P-1,i,j} : 1 \leq i \leq C_D, 1 \leq j \leq C_W, 0 < X < \bar{X}. \tag{67}$$

For determining $\mathbf{E}$ and $\mathbf{g}$, we first choose an order for matching $\psi_{X,i,j}$ values and the elements of the vector $\mathbf{y}$. This order is given in Eqs. (68), where $\ddot{\psi}_X$ and $\dot{\psi}_{X,i}$ are intermediary sub-vectors of $\mathbf{y}$. Let $f(X,i_1,i_2) = XC_DC_W + (i_1-1)C_W + i_2$ denote a function from $\{0,\dots,\bar{X}-1\} \times \{1,\dots,C_D\} \times \{1,\dots,C_W\}$ to $\{1,\dots,\bar{X}C_DC_W\}$, used for pointing to ψ_{X,i_1,i_2} in $\mathbf{y}$. Using this order and the function f, the constraints given in Eq. (67) can be specified by $\mathbf{E}$ and $\mathbf{g}$ given in Eqs. (69)–(70):

$$\begin{aligned} &\mathbf{y} = \left[\ddot{\psi}_0 \ \cdots \ \ddot{\psi}_{\bar{X}-1}\right], \ddot{\psi}_X = \left[\dot{\psi}_{X,1} \ \cdots \ \dot{\psi}_{X,C_D}\right], \\ &\dot{\psi}_{X,i} = \left[\psi_{X,i,1} \ \cdots \ \psi_{X,i,C_W}\right], \end{aligned} \tag{68}$$

$$\begin{aligned} \mathbf{E} = &\sum_{X=1}^{\bar{X}-1}\sum_{i_1=1}^{C_D}\sum_{i_2=1}^{C_W}\left(\mathbf{e}_{(X-1)C_DC_W+(i_1-1)C_W+i_2,(\bar{X}-1)C_DC_W}\right)^T \\ &\otimes\left(\mathbf{e}_{f(X,i_1,i_2),\bar{X}C_DC_W} - \mathbf{e}_{f(X-1,i_1,i_2),\bar{X}C_DC_W}\right), \end{aligned} \tag{69}$$

$$\mathbf{g} = \mathbf{0}_{(\bar{X}-1)C_DC_W,1}. \tag{70}$$

Moreover, given a vector $\mathbf{y}$, the threshold level for marking pair (i_1,i_2) can be calculated as

$$S_{i_1,i_2} = \sum_{X=0}^{\bar{X}-1}\mathbf{y}_{f(X,i_1,i_2)}. \tag{71}$$

In the following we describe determining the alternative forms of the boundary state transitions. Let $\tilde{\mathbf{Q}}_{X_1,x_2}(\Psi)$ denote the sub-block of $\mathbf{Q}$ corresponding to transitions from inventory position X_1 to inventory position X_2, as a function of the binary variables $\Psi \in \{0,1\}^{\bar{X}C_DC_W}$. Given this notation, the sub-blocks of $\mathbf{Q}$ as a function of Ψ can be specified by the following equations:

$$\begin{aligned} \tilde{\mathbf{Q}}_{X,X-1}(\Psi) = &\begin{bmatrix}1&0\\0&0\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\left(1-\psi_{X-1,i_2,j_2}\right)\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{j_1=j_2\}}\mathbf{D1}_{i_2}\otimes\mathbf{I}_w\right) \\ &+\begin{bmatrix}0&1\\0&0\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\psi_{X-1,i_2,j_2}\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{j_1=j_2\}}\mathbf{D1}_{i_2}\otimes\mathbf{I}_w\right) \\ &+\begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes\mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ &+\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\left(\sum_{i=1}^{C_D}\left(\left(\mathbf{e}_{i,C_D}\otimes\mathbf{1}_{C_D}\right)\otimes\mathbf{I}_{C_W}\otimes\mathbf{D1}_i\otimes\mathbf{I}_w\right)\right), \\ &X \leq \bar{X}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{Q}}_{X,X}(\Psi) = &\ \delta_{\{X<\bar{X}\}}\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\mathbf{I}_{C_D}\otimes\mathbf{I}_{C_W}\otimes\mathbf{I}_d\otimes\mathbf{W0}\right) \\ &+\left(\mathbf{I}_2\otimes\mathbf{I}_{C_D}\otimes\mathbf{I}_{C_W}\otimes\mathbf{D0}\otimes\mathbf{I}_w\right) \\ &+\begin{bmatrix}1&0\\0&0\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\left(1-\psi_{X,i_2,j_2}\right)\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{j_1=j_2\}}\mathbf{D2}_{i_2}\otimes\mathbf{I}_w\right) \\ &+\begin{bmatrix}0&1\\0&0\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\psi_{X,i_2,j_2}\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{j_1=j_2\}}\mathbf{D2}_{i_2}\otimes\mathbf{I}_w\right) \\ &+\begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes\mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ &+\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\left(\sum_{i=1}^{C_D}\left(\left(\mathbf{e}_{i,C_D}\otimes\mathbf{1}_{C_D}\right)\otimes\mathbf{I}_{C_W}\otimes\mathbf{D2}_i\otimes\mathbf{I}_w\right)\right), \\ &X \leq \bar{X}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{Q}}_{X,X+1}(\Psi) = &\begin{bmatrix}1&0\\0&0\end{bmatrix}\otimes\mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ &+\begin{bmatrix}0&1\\0&0\end{bmatrix}\otimes\mathbf{0}_{C_DC_Wdw,C_DC_Wdw} \\ &+\begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\left(1-\psi_{X+1,i_2,j_2}\right)\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{i_1=i_2\}}\mathbf{I}_d\otimes\mathbf{W1}_{j_2}\right) \\ &+\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\sum_{i_1=1}^{C_D}\sum_{j_1=1}^{C_W}\sum_{i_2=1}^{C_D}\sum_{j_2=1}^{C_W}\psi_{X+1,i_2,j_2}\mathbf{J}_{i_1j_1,i_2j_2,C_DC_W} \\ &\otimes\left(\delta_{\{i_1=i_2\}}\mathbf{I}_d\otimes\mathbf{W1}_{j_2}\right), \qquad X < \bar{X}. \end{aligned}$$

Then, the boundary transitions denoted by $\tilde{\mathbf{B}}_{0,0}(\Psi)$ can be derived by Eqs. (72). In addition, $\mathbf{B}^0_{0,0}, \mathbf{B}^k_{0,0}$ can be calculated accordingly using Eqs. (73)–(74).

$$\tilde{\mathbf{B}}_{0,0}(\Psi) = \begin{bmatrix} \tilde{\mathbf{Q}}_{\bar{X},\bar{X}}(\Psi) & \tilde{\mathbf{Q}}_{\bar{X},\bar{X}-1}(\Psi) & & & & & \\ \tilde{\mathbf{Q}}_{\bar{X}-1,\bar{X}}(\Psi) & \tilde{\mathbf{Q}}_{\bar{X}-1,\bar{X}-1}(\Psi) & \tilde{\mathbf{Q}}_{\bar{X}-1,\bar{X}-2}(\Psi) & & & & \\ & \tilde{\mathbf{Q}}_{\bar{X}-2,\bar{X}-1}(\Psi) & \tilde{\mathbf{Q}}_{\bar{X}-2,\bar{X}-2}(\Psi) & \tilde{\mathbf{Q}}_{\bar{X}-2,\bar{X}-3}(\Psi) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tilde{\mathbf{Q}}_{0,1}(\Psi) & \tilde{\mathbf{Q}}_{0,0}(\Psi) & \tilde{\mathbf{Q}}_{0,-1}(\Psi) \\ & & & & \tilde{\mathbf{Q}}_{-1,0}(\Psi) & \tilde{\mathbf{Q}}_{-1,-1}(\Psi) \end{bmatrix} \tag{72}$$

$$\mathbf{B}^0_{0,0} = \tilde{\mathbf{B}}_{0,0}(\Psi), \psi_{X,i,j} = 0 : 1 \leq i \leq C_D, 1 \leq j \leq C_W, 0 \leq X < \bar{X} \tag{73}$$

$$\begin{aligned} &\mathbf{B}^k_{0,0} = \tilde{\mathbf{B}}_{0,0}(\Psi) - \mathbf{B}^0_{0,0}, \\ &\psi_{X,i,j} = \delta_{\{f(X,i,j)=k\}} : 1 \leq i \leq C_D, 1 \leq j \leq C_W, 0 \leq X < \bar{X} \end{aligned} \tag{74}$$

Given this representation, in terms of the general formulation, $n = 2C_DC_Wdw$ and $m = n(\bar{X}+2)$ and $K = C_DC_W\bar{X}$ and $\mathbf{E}$ has $C_DC_W\bar{X}$ rows. Hence, the final formulation has $2C_DC_W(1+(2+\bar{X})(1+2C_DC_W\bar{X}))dw$ continuous and $C_DC_W\bar{X}$ binary variables and $1+C_DC_W(\bar{X}+2(3+\bar{X}(1+4C_DC_W(2+\bar{X})))dw)$ constraints. However, in practice, as $\bar{X}$ increases, $\mathbf{B}^k_{0,0}$ matrices become more sparse, and as a result, a considerable number of the continuous variables and constraints can be eliminated prior to solving the problem using the respective big-M values (The big-M values for the redundant variables will be zero).

Appendix B. Discrete event simulation algorithm for evaluating the system performance for given traces

Algorithm 1 generates the inventory process of a system controlled with marking-dependent threshold policy. In this pseudo-code, $\hat{x}$ represents variable x's value at the current time $\hat{T}$, and x_e represents the value of x immediately after event e. Once the inventory process for a threshold matrix S has been calculated, it can be used for evaluating $S + x\mathbf{1}_{C_D,C_W}$ by adding x to P_e.

The algorithm starts with initializing the variables in line 1. Then one arrival event is placed in the future event list (FEL) in line 2. Then, while FEL is not empty, the earliest event, denoted by E, is identified in line 5. Then based on the type of this event (d for arrivals and s for productions), the last observed marking ($\hat{\alpha}$ or $\hat{\beta}$) is updated in lines 7–8. Then, the current threshold level $\hat{S}$ is set based on the last observed markings $\hat{\alpha}$ and $\hat{\beta}$ in line 9. After the current threshold level is set, the effect of event E on the other system variables must be assessed.

If event E is an arrival, the inventory level will decrease if the arrival is an arrival of information and demand. This is expressed in line 11. After each arrival, the next arrival is scheduled and added to FEL. This is expressed in line 12. If the inventory level is less than the current threshold level and the machine is idle, production must start, meaning a production completion event should be added to FEL and the machine's current status WS must become *working*. This is expressed in lines 14–15.

Alternatively, if the earliest event E is a production completion event, the inventory level should increase by one, as expressed in line 19. Then, if the inventory level is less than the current threshold level, the current machine status WS should remain *working* and a new production completion must be scheduled accordingly. Otherwise, the machine must become *idle*. Lines 21–24 express these tasks. Finally, the event that has took place must be erased from FEL. This is expressed in line 27.

Algorithm 1 DES algorithm

1: $i \leftarrow 1,\ k \leftarrow 1,\ e \leftarrow 1,\ T_e \leftarrow 0,\ X_e \leftarrow 0,\ e \leftarrow e+1$, WS $\leftarrow$ idle, $\hat{T} \leftarrow 0$
2: $FEL \leftarrow \left\{ \left(\mathrm{d}, \hat{T} + \tau_i, \alpha_i, \xi_i, \phi \right) \right\},\ i \leftarrow i+1$
3: $\hat{\beta} \leftarrow 1$
4: **while** $|FEL| > 0$ **do**
5: $\hat{E} \leftarrow E : E_2 < E_2^{'} \forall E, E^{'} \in FEL$
6: $\hat{T} \leftarrow \hat{E}_2,\ T_e \leftarrow \hat{T}$
7: **if** $E_1 = \mathrm{d}$ **then** $\hat{\alpha} \leftarrow E_3$ **end if**
8: **if** $E_1 = \mathrm{s}$ **then** $\hat{\beta} \leftarrow E_5$ **end if**
9: $\hat{S} \leftarrow S_{\hat{\alpha},\hat{\beta}}$
10: **if** $E_1 = \mathrm{d}$ **then**
11: $X_e \leftarrow X_{e-1} - \hat{E}_4$
12: **if** $i \leq |\mathcal{T}|$ **then** $FEL \leftarrow FEL \cup \left\{ \left(\mathrm{d}, \hat{T} + \tau_i, \alpha_i, \xi_i, \phi \right) \right\},\ i \leftarrow i+1$ **end if**
13: **if** WS = idle, $X_e < \hat{S}$ **then**
14: WS $\leftarrow$ working
15: **if** $k \leq |\mathcal{T}|$ **then** $FEL \leftarrow FEL \cup \left\{ \left(\mathrm{s}, \hat{T} + g_k, \phi, \phi, \beta_k \right) \right\},\ k \leftarrow k+1$ **end if**
16: **end if**
17: **end if**
18: **if** $E_1 = \mathrm{s}$ **then**
19: $X_e \leftarrow X_{e-1} + 1$
20: **if** $X_e < \hat{S}$ **then**
21: WS $\leftarrow$ working
22: **if** $k \leq |\mathcal{T}|$ **then** $FEL \leftarrow FEL \cup \left\{ \left(\mathrm{s}, \hat{T} + g_k, \phi, \phi, \beta_k \right) \right\},\ k \leftarrow k+1$ **end if**
23: **else**
24: WS $\leftarrow$ idle
25: **end if**
26: **end if**
27: $FEL \leftarrow FEL \backslash \hat{E}$
28: $e \leftarrow e+1$
29: **end while**

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