Automatic proofs in combinatorial game theory

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Joint work with Bastien Mignoty, Michel Rigo & Markus A. Whiteland

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Overview

1. Wanna play?
2. Wythoff's game
3. Using `Walnut` for Wythoff's game
4. Some generalisations

Wanna play?

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Game of Nim

- ***Rules:***
  - → 2 players, taking turns, they may not pass;
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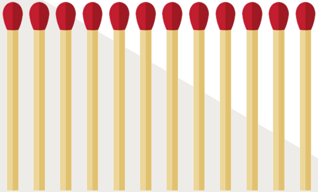

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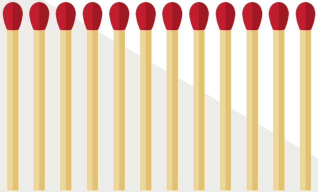

"Never be so kind, you forget to be clever"

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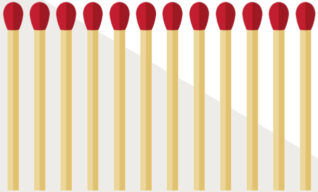

12 matches

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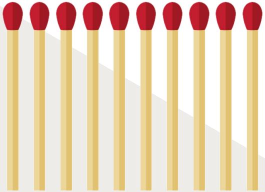

10 matches

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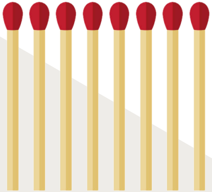

8 matches

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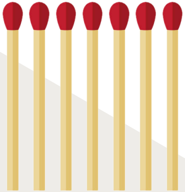

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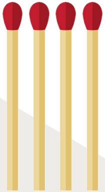

4 matches

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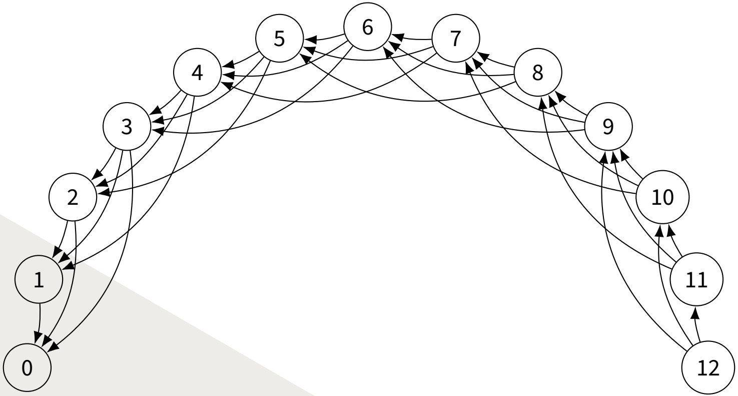
1 match

Wanna play?

Switching to graphs



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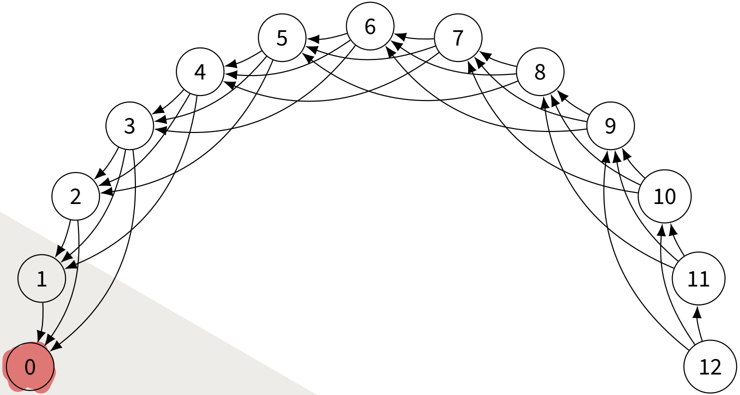

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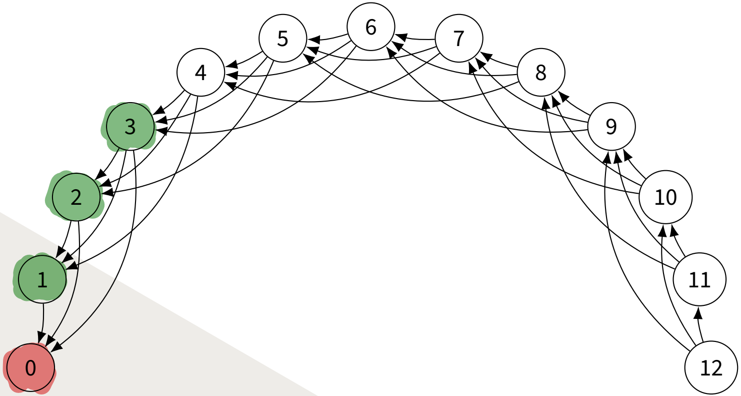

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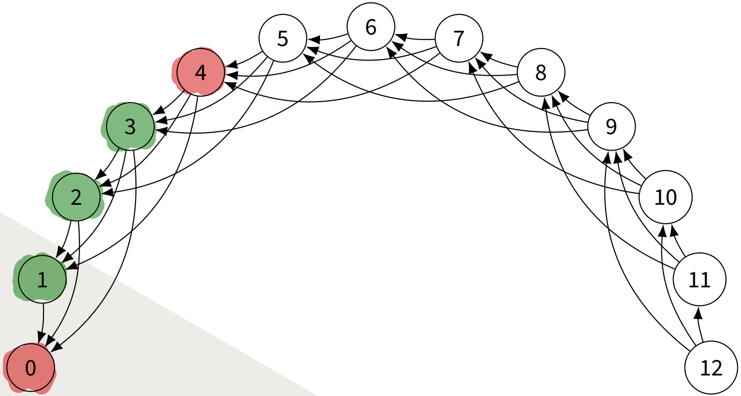

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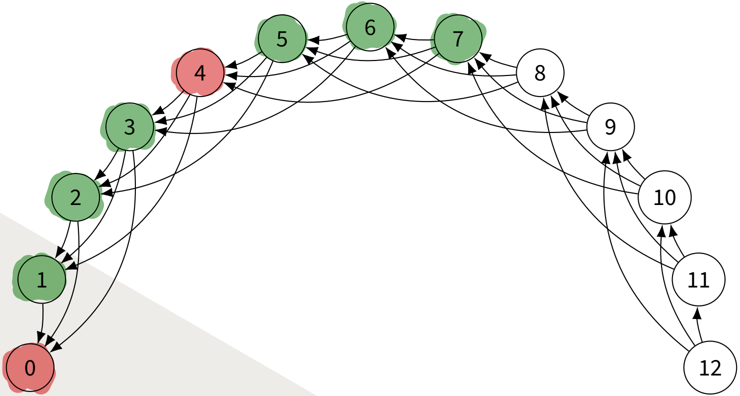

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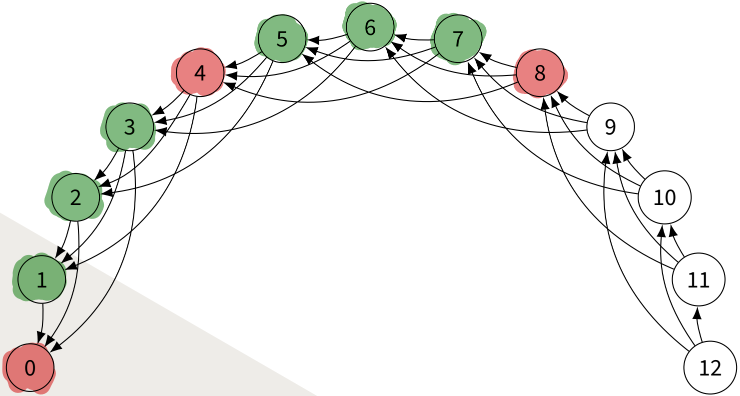

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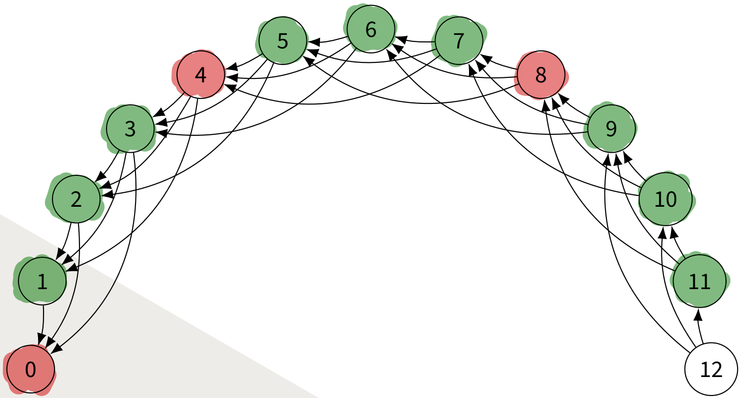

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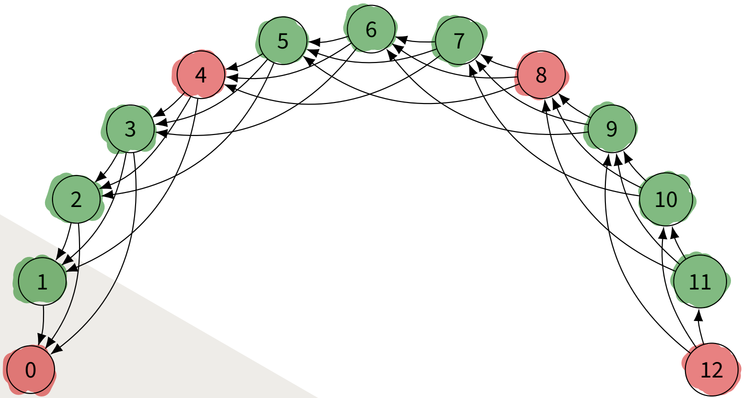

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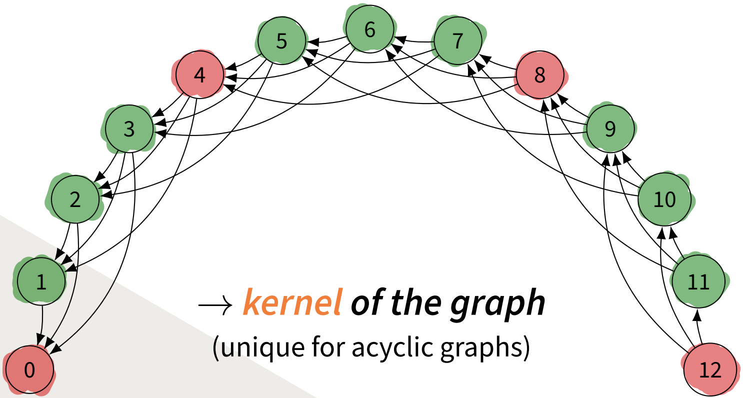

→ ***kernel of the graph***

(unique for acyclic graphs)

Wanna play?

$\mathcal{P}$- and $\mathcal{N}$-positions

A ***position*** of a game describes the actual configuration of the game before one of the two players makes a ***move***.

Ex: In the previous game of Nim, the *position* i depicts the situation with i remaining sticks on the table.

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Two types of positions:

- ***$\mathcal{P}$-position*** $=$ position of the game where the ***previous*** player is ensured to win; it is thus a *losing* position for the person about to play.
- ***$\mathcal{N}$-position*** $=$ position of the game where the ***next*** player is ensured to win; it is thus a ***winning*** position for the person about to play.

Wanna play?

$\mathcal{P}$-positions and kernel

Proposition (folklore)

The sets of $\mathcal{P}$- and $\mathcal{N}$-positions of an impartial acyclic game are uniquely determined by the following two properties:

- Every move from a $\mathcal{P}$-position leads to an $\mathcal{N}$-position; equivalently there is no move between two $\mathcal{P}$-positions (***stability property*** of $\mathcal{P}(G)$).
- From every $\mathcal{N}$-position, there exists a move leading to a $\mathcal{P}$-position (***absorbing property*** of $\mathcal{P}(G)$).

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$\rightarrow$ ***For an impartial acyclic game,***

set of $\mathcal{P}$-position $=$ kernel of the associated graph

Wythoff’s game

Wythoff's game

The rules

Wythoff's game is a 2-player impartial game which is a variation of the game of Nim.

- 2 piles of tokens,
- the players play one after another, they may not pass,
- the first player unable to play loses.

Allowed moves:

- removing a positive number of tokens from one pile,
- removing the same number of tokens from both piles.

Wythoff's game

Characterising the $\mathcal{P}$-positions

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- the pairs (a_n, b_n), defined by $(a_0, b_0) = (0, 0)$ and

$$a_n = \operatorname{mex}\{a_i, b_i : 0 \leq i < n\} \quad \text{and} \quad b_n = a_n + n,$$

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- the positions of 0's and 1's in the ***Fibonacci word* f**:

$$\text{fixed point of} \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases} \rightsquigarrow \quad \mathbf{f} =$$

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First $\mathcal{P}$-positions: $(0, 0), (1, 2), (3, 5), (4, 7), (6, 10)$, etc.

Wythoff's game

Characterising the $\mathcal{P}$-positions – A famous result

Recall that any positive integer can be written as a sum of Fibonacci numbers in a "greedy" way $\rightsquigarrow$ Zeckendorf representation.

Ex: Considering $F_0 = 1, F_1 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, we have

$$11 = 8 + 3 \quad \to \quad \mathrm{rep}_F(11) = 10100.$$

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Theorem (Fraenkel, 1982)

A pair (a, b) of integers such that $a \leq b$ is a $\mathcal{P}$-position of Wythoff's game if and only if

- $\text{rep}_F(a)$ *ends with an even number of zeroes,*
- $\text{rep}_F(b)$ *is a left-shift of* $\text{rep}_F(a)$*, i.e.* $\text{rep}_F(b) = \text{rep}_F(a)0$.

Using `Walnut` for Wythoff's game

Using `Walnut` for Wythoff's game

What is `Walnut`?

`Walnut` is a free software system originally created by Hamoon Mousavi.

- extensively used for proving results in combinatorics on words and additive number theory;
- relies on *Büchi's theorem*: transform first-order logical formulas into finite automata for which decision procedures can be applied.

Hence, if a problem of interest can be expressed in a convenient extension of Presburger arithmetic $\langle \mathbb{N}, + \rangle$, it can then receive an automatic treatment.

Using `Walnut` for Wythoff's game

Fraenkel's theorem using `Walnut`

Step 1: Defining the candidate set for $\mathcal{P}$-positions in `Walnut`

```
reg end_even_zeros msd_fib "0*(00|0*1)*":
reg left_shift {0,1} {0,1} "([0,0]|([0,1][1,1]*[1,0]))*":
def ppos_asym "?msd_fib $end_even_zeros(a) & $left_shift(a,b)":
def ppos "?msd_fib $ppos_asym(a,b) | $ppos_asym(b,a)":
```

Ex: we can evaluate `$ppos(6,10)`:

images/ppos_paper.pdf

$$\mathrm{rep}_F(6) = 01001$$

$$\mathrm{rep}_F(10) = 10010$$

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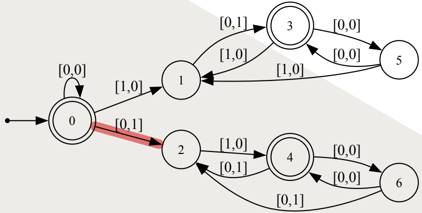

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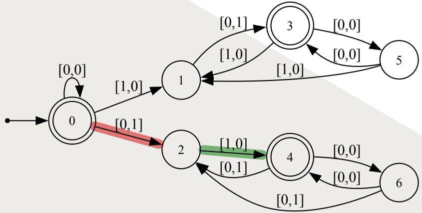

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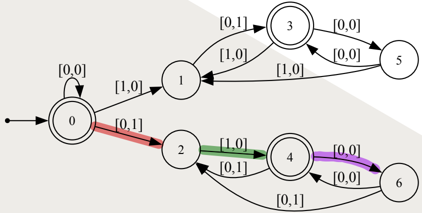

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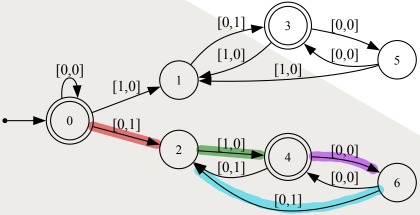
reg end_even_zeros msd_fib "0*(00|0*1)*":
reg left_shift {0,1} {0,1} "([0,0]|([0,1][1,1]*[1,0]))*":
def ppos_asym "?msd_fib $end_even_zeros(a) & $left_shift(a,b)":
def ppos "?msd_fib $ppos_asym(a,b) | $ppos_asym(b,a)":
```



Ex: we can evaluate `$ppos(6,10)`:

$$\mathrm{rep}_F(6) = 01001$$
$$\mathrm{rep}_F(10) = 10010$$

Using Walnut for Wythoff's game

Fraenkel's theorem using Walnut

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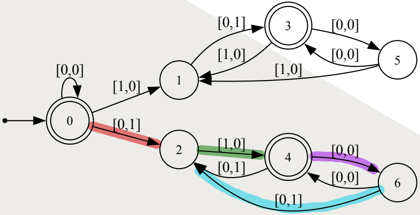

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Step 2: Verifying that this set is exactly the set of $\mathcal{P}$-positions

Recall that the sets of $\mathcal{P}$- and $\mathcal{N}$-positions of an impartial acyclic game are uniquely determined by the two following properties:

Stability, i.e. there is no move between two $\mathcal{P}$-positions

```
eval w_stable "?msd_fib Ap,q,r,s
(($ppos(p,q) & $ppos(r,s) & p >= r & q >= s)
=> ((p=r & q=s) | (p>r & q>s & p+s!=q+r)) )":
```

Absorbing, i.e. for each $\mathcal{N}$-position, there exists a move leading to a $\mathcal{P}$-position

```
eval w_absorbing "?msd_fib Ap,q (~$ppos(p,q) => Ex,y
( x<=p & y<=q & $ppos(x,y) & (p+y=q+x | p=x | q=y) )) ":
```

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Fraenkel's theorem using `Walnut`

Both commands evaluate to `TRUE`, which proves Fraenkel's theorem!

We were able to use `Walnut` because:

- the rules of the game can be expressed using first-order logic,
- we have a "regular" candidate for the set of $\mathcal{P}$-positions,
- the Fibonacci numeration system is addable.

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Note also that we did have a candidate for the set of $\mathcal{P}$-positions.

Using `Walnut` for Wythoff's game

Other results about Wythoff's game

Thanks to `Walnut`, we aslo managed to:

- prove a 15-year-old conjecture describing the set of *forbidden moves* in Wythoff's game (i.e., the set of moves which would allow to play between two $\mathcal{P}$-positions),
- show that all allowed moves in Wythoff's game are *non-redundant* (a move is said to be *redundant* if removing it from the rule-set does not affect the set of $\mathcal{P}$-positions),
- study some extensions/restrictions of Wythoff's game.

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What more can we do with this software?

Some generalisations

Some generalisations

Slightly changing the rules

Same rules as Wythoff's game, but a player may now remove $k > 0$ tokens from one heap and $\ell > 0$ from the other, provided that $|k - \ell| < m$ for a fixed integer $m \geq 1$.

Note that $m = 1 \rightsquigarrow$ Wythoff's game.

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Here, we make use of ***Ostrowski numeration systems***: consider the quadratic irrational

$$[1, \overline{m}] = \frac{2 - m + \sqrt{m^2 + 4}}{2}$$

$\rightarrow$ the convergents give two numeration systems: the p-system (using numerators) and the q-system (using denominators).

Rem.: for $m = 1$, $[1, \overline{m}] = \varphi$ the golden ratio $\rightsquigarrow$ Fibonacci numeration system.

Some generalisations

Slightly changing the rules

We get the same result as the one for Wythoff's game, but with the associated Ostrowski numeration system:

Theorem (Fraenkel, 1982)

A pair (a, b) of integers such that $a \leq b$ is a $\mathcal{P}$-position of m-Wythoff's game if and only if

- $\mathrm{rep}_p(a)$ *ends with an even number of zeroes,*
- $\mathrm{rep}_p(b)$ *is a left-shift of* $\mathrm{rep}_p(a)$*, i.e.* $\mathrm{rep}_p(b) = \mathrm{rep}_p(a)0$*,*

where $\mathrm{rep}_p(x)$ *is the representation of x in the p-system associated to* $[1, \overline{m}]$.

$\rightarrow$ we can handle such systems with `Walnut` (it builds the required automata), and thus prove results automatically!

Some generalisations

Using `Walnut` to find a new conjecture

Again here, we can ask what are the redundant moves of m-Wythoff's game. Using `Walnut`, we get the following conjecture:

Conjecture (Mignoty, R., Rigo, Whiteland, 2025+)

Let $m \geq 2$. The set of redundant moves of the variation of Wythoff's game where one is allowed to remove $k > 0$ and $\ell > 0$ provided that $|k - \ell| < m$ is

$$\bigcup_{1 \leq i < m} \{(n, n+i), (n+i, n) \mid n \geq m - i + 2\}.$$

→ proved for $m = 2$ ($\sim$ 7Gb), $m = 3, 4$ ($\sim$ 45Gb, 20 minutes);

→ for $m = 5$, we quickly run out of memory $\rightsquigarrow$ use `Walnut` differently.

Some generalisations

Beyond Ostrowski systems

Same rules as Wythoff's game, but a player may now remove $k > 0$ tokens from one heap and $\ell > 0$ from the other, provided that $0 < k \leq \ell < sk + m$ for two positive integers m, s.
For $s = 1 \rightsquigarrow m$-Wythoff's game; for $m = s = 1 \rightsquigarrow$ Wythoff's game.

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We do get the same result (Fraenkel, 1998) using the following numeration system: we define the linear recurrence sequence $(U_i)_{i\geq 0}$ by

$$U_0 = 1, \quad U_1 = s + m \quad \text{and} \quad U_i = (s + m - 1)U_{i-1} + sU_{i-2} \; \forall i \geq 2.$$

We then have that each integer $n > 0$ has a unique representation $d_\ell \cdots d_0$ such that

$$n = \sum_{i=0}^{\ell} d_i U_i, \quad \text{with } d_\ell \neq 0.$$

Some generalisations

Beyond Ostrowski systems

We are lucky with this numeration system, for we are in the *Pisot* case, which means:

- we have a regular candidate,
- we have an addable system.

So, in principle, we can use `Walnut`!

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Beyond Ostrowski systems

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Some more work here all the same: no "built-in" automata for this system, so we have to provide `Walnut` with

- an automaton recognising U-representations,
- another one computing the addition $\rightsquigarrow$ using the work of C. Frougny and J. Sakarovitch to build the ***zero-automaton***.

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Fraenkel's combinatorial games and `Walnut`:
It's a ***match***!

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However...

- automatic proofs are obtained for ***fixed*** parameters,
- ***state complexity*** could be problematic,
- difficult to cope with ***Tribonacci adder***.

Thank you for your attention!