# $p$-Large Deviation Wavelet Formalism


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Joint work with

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# Multifractal analysis

Given a signal $X$, we want to study

- the regularity in each of its points : $H_X(x) \quad \forall x \in \mathbb{R}$,
- the significance of the different singularities by computing the Hausdorff dimension of the sets of points sharing a common regularity :

$$\dim_{\mathcal{H}} \left( \{ x \in \mathbb{R} : H_X(x) = h \} \right) \quad \forall h \in \mathbb{R} .$$

$\rightarrow$ Not reachable numerically, hence the need for computable formulas providing good estimates, called *formalisms*.

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# Hölder regularity

Let $\alpha \geq 0$, $x_0 \in \mathbb{R}$ and $f$ a function. Then $f \in C^\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$|f(x) - P(x)| \leq C\,|x - x_0|^\alpha \quad \forall x \in B(x_0, R),$$

i.e.

$$\|f - P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha \quad \forall r \leq R.$$

The Hölder exponent of $f$ at $x_0$ is

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}.$$

Drawback : $f \in C^\alpha(x_0) \Rightarrow f$ bounded on a neighbourhood of $x_0$.

$\rightarrow$ Limited to locally bounded functions.

$\rightarrow$ Definition of the $p$-regularity by replacing $L^\infty$ with $L^p$ (Calderòn and Zygmud, 1961).

# Hölder regularity

Let $\alpha \geq 0$, $x_0 \in \mathbb{R}$ and $f \in L^\infty_{\text{loc}}(\mathbb{R})$ . Then $f \in C^\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

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The Hölder exponent of $f$ at $x_0$ is

$$h_f^{(+\infty)}(x_0) = h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}.$$

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# $p$-Regularity

Let $p \geq 1, \alpha \geq \frac{-1}{p}$, $x_0 \in \mathbb{R}$ and $f \in L^p_{\text{loc}}(\mathbb{R})$. Then $f \in T^p_\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$\left(\frac{1}{r}\int_{B(x_0,r)} |f(x) - P(x)|^p \ dx\right)^{\frac{1}{p}} \leq Cr^\alpha \quad \forall r \leq R,$$

i.e.

$$\|f - P\|_{L^p(B(x_0,r))} \leq Cr^{\alpha+\frac{1}{p}} \quad \forall r \leq R.$$

The $p$-exponent of $f$ at $x_0$ is

$$h_f^{(p)}(x_0) = \sup\left\{\alpha \geq \frac{-1}{p} : f \in T^p_\alpha(x_0)\right\}.$$

# Examples

- Cusp (canonical singularity) : if

$$f(x) = |x - x_0|^\alpha \quad (\alpha < 1)\,,$$

then

$$h_f^{(p)}(x_0) = \alpha \quad \forall p \leq \frac{-1}{\alpha} \text{ if } \alpha < 0.$$

- Chirp (oscillating balanced singularity) : if

$$f(x) = |x - x_0|^\alpha \sin\left(|x - x_0|^{-\beta}\right) \quad (\alpha < 1 \text{ and } \beta > 0)\,,$$

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- Lacunary comb (oscillating lacunary singularity) : if

$$f(x) = \begin{cases} 2^{-\alpha j} & \text{if } x \in \left[2^{-j}, 2^{-j} + 2^{-\gamma j}\right] \\ 0 & \text{if } x \notin \bigcup_{j \in \mathbb{N}} \left[2^{-j}, 2^{-j} + 2^{-\gamma j}\right] \end{cases} \quad (\alpha \in \mathbb{R},\ \gamma > 1)\,,$$

then

$$h_f^{(p)}(0) = \alpha + \frac{\gamma - 1}{p} \quad \forall p \leq \frac{-1}{\alpha} \text{ if } \alpha < 0$$

since

$$\left(\frac{1}{r}\int_{-r}^{r} |f(x)|^p \, dx\right)^{\frac{1}{p}} \sim \left(2^{J(r)} \sum_{j \geq J(r)} 2^{-\alpha p j} 2^{-\gamma j}\right)^{\frac{1}{p}} \sim \left(2^{-(\alpha p + \gamma - 1)J(r)}\right)^{\frac{1}{p}}.$$

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# Multifractal spectrum and formalism

$p$-Spectrum (called Hölder spectrum when $p = +\infty$) :

$$\mathscr{D}_f^{(p)} : h \in \left[\frac{-1}{p}, +\infty\right] \mapsto \dim_{\mathcal{H}} \left\{ x \in \mathbb{R} : h_f^{(p)}(x) = h \right\}.$$

Multifractal formalism : numerically computable formula which

- always provides an upper bound for the spectrum,
- gives the exact spectrum for a large class of functions,
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- Gauge function :

$$\xi : [0,+\infty] \to [0,+\infty] \text{ such that } \xi(0)=0 \text{ and } \xi \text{ is increasing near } 0.$$

- Hausdorff outer measure associated with $\xi$ at scale $t$ :

$$\mathcal{H}^{\xi}_t(A) = \inf\left\{ \sum_n \xi(\mathrm{diam}(E_n)) : \mathrm{diam}(E_n) \le t, A \subseteq \bigcup_n E_n \right\}.$$

- Hausdorff measure associated with $\xi$ :

$$\mathcal{H}^{\xi}(A) = \lim_{t\to 0^+} \mathcal{H}^{\xi}_t(A).$$

- $s$-Dimensional Hausdorff measure :

$$\xi(x) = x^s \ (s \ge 0) \ \Rightarrow \ \mathcal{H}^{\xi} = \mathcal{H}^s$$




- Hausdorff dimension :

$$\dim_{\mathcal{H}}(A) = \begin{cases} \inf\{s \ge 0 : \mathcal{H}^s(A) = 0\} & \text{if } A \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

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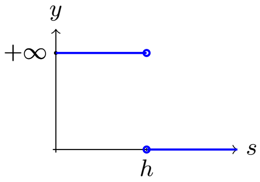


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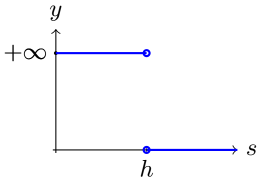


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Orthonormal basis of $L^2(\mathbb{R})$ of the form

$$\left\{2^{\frac{j}{2}}\psi_{j,k} : j,k \in \mathbb{Z}\right\},$$

where

$$\psi_{j,k}(x) = \psi(2^j x - k).$$

To any function $f$, we can associate a sequence $(c_{j,k})_{j\in\mathbb{N},\, k\in\{0,\ldots,2^j-1\}}$ such that

$$f = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} \psi_{j,k},$$

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$$c_{j,k} = 2^j \int_0^1 f(x)\psi_{j,k}\,dx.$$

($L^\infty$ normalization)

# Dyadic intervals

- $$\lambda_{j,k} = \left[k2^{-j}, (k+1)2^{-j}\right), \; j \text{ is the scale and } k \text{ is the position}$$
- $$\psi_{\lambda_{j,k}} = \psi_{j,k}, \quad c_{\lambda_{j,k}} = c_{j,k}$$
- $$\Lambda_j = \left\{\lambda_{j,k} : k \in \{0, \ldots, 2^j - 1\}\right\} \simeq \{0, \ldots, 2^j - 1\}$$
- $$\Lambda = \bigcup_{j \in \mathbb{N}} \Lambda_j \simeq \{(j,k) : j \in \mathbb{N}, k \in \Lambda_j\}$$
- $$\lambda_j(x_0) = \text{ only dyadic interval of scale } j \text{ containing } x_0$$
- $$f \text{ function } \leftrightarrow \vec{c} = (c_{j,k})_{(j,k)\in\Lambda} = (c_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda$$

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# Wavelets and functional spaces

Aim : condition over wavelet coefficients ensuring $L^p_{\text{loc}}$.

Uniform Hölder exponent :

$$h^{\min} = \liminf_{j\to+\infty} \frac{\log_2\left(\sup_{\lambda\in\Lambda_j} |c_\lambda|\right)}{-j}.$$

Scaling function :

$$\eta_f : p > 0 \mapsto \liminf_{j\to+\infty} \frac{\log_2\left(2^{-j}\sum_{\lambda\in\Lambda_j} |c_\lambda|^p\right)}{-j}.$$

- For $p = +\infty$,

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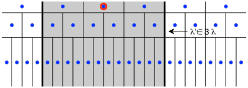
# Wavelets and pointwise regularity (through leaders)

Fix $j \in \mathbb{N}$ and $\lambda \in \Lambda_j$.

Leaders :

$$l_\lambda^{(+\infty)} = l_\lambda = \sup_{j' \geq j} \sup_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|$$




$p$-Leaders :

$$l_\lambda^{(p)} = \left( \sum_{\lambda' \in \Lambda_{j'},\, \lambda' \subseteq 3\lambda} |c_{\lambda'}|^p \, 2^{-(j'-j)} \right)^{\frac{1}{p}}$$

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$$\rho_{\vec{c}}(\alpha) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log_2 \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq |c_\lambda| \leq 2^{-(\alpha-\varepsilon)j}\}}{j}.$$

**Intuitively :**

$$\#\left\{\lambda \in \Lambda_j : |c_\lambda| \sim 2^{-\alpha j}\right\} \sim 2^{\rho_{\vec{c}}(\alpha) j} \quad \forall j >> .$$

**Wavelet profile :**

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$p$-Leaders density / $p$-Large deviation spectrum :

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# Known results

- Upper bound for the Hölder spectrum.

**Theorem (J.M. Aubry and S. Jaffard, 2002)**

*If $f \in \bigcup_{\varepsilon>0} C^{\varepsilon}$, then for every $h \geq 0$,*

$$\mathscr{D}_f^{(+\infty)}(h) \leq h \sup_{\alpha \in (0,h]} \frac{\rho_{\vec{c}}(\alpha)}{\alpha}.$$

- Exact $p$-spectrum for Lacunary Wavelet Series, i.e. wavelet series of the form

$$f_{\alpha,\eta} = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} 2^{-\alpha j} \xi_{j,k} \psi_{j,k},$$

with $\xi_{j,k} \overset{\text{i.i.d.}}{\sim}$ Bern$(2^{(\eta-1)j})$, $\eta \in (0,1)$, $\alpha \in \mathbb{R}$.

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*Almost surely, for every $p < \begin{cases} \frac{\eta-1}{\alpha} & \text{if } \alpha < 0 \\ +\infty & \text{otherwise} \end{cases}$ and every $h \geq \frac{-1}{p}$,*

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$$\mathscr{D}_{f_{\alpha,\eta}}^{(p)}(h) = \begin{cases} \left(h + \frac{1}{p}\right) \sup\limits_{\alpha' \in \left(\frac{-1}{p}, h\right]} \frac{\rho_{\vec{c}}(\alpha')}{\alpha' + \frac{1}{p}} & \text{if } h \in \left[\alpha, \frac{\alpha}{\eta} + \left(\frac{1}{\eta} - 1\right)\frac{1}{p}\right] \\ -\infty & \text{otherwise} \end{cases}.$$

# Upper bound for the $p$-spectrum

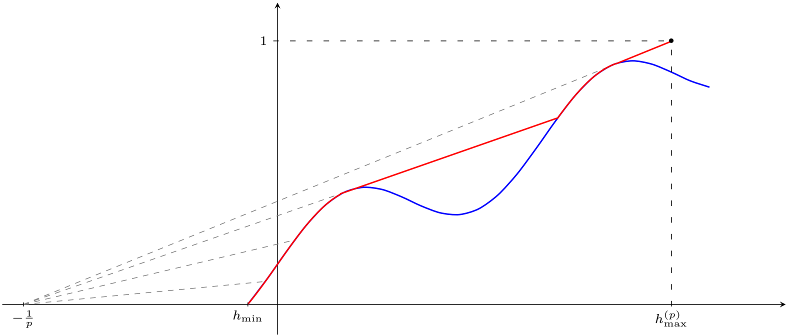
**Theorem**

*If $f$ is a function for which*

$$p_0(f) = \sup\{p > 0 : \eta_f(p) > 0\},$$

*then for every $0 < p < p_0(f)$ and every $h \geq \frac{-1}{p}$,*

$$\mathscr{D}_f^{(p)}(h) \leq \rho_{\vec{c}}^{(p),*}(h) \leq \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left(\frac{-1}{p}, h\right]} \frac{\rho_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}} = \left(h + \frac{1}{p}\right) \sup_{\alpha \in \left(\frac{-1}{p}, h\right]} \frac{\nu_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}}.$$



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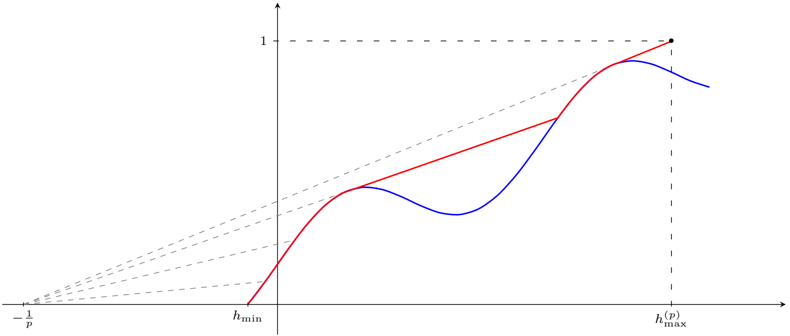

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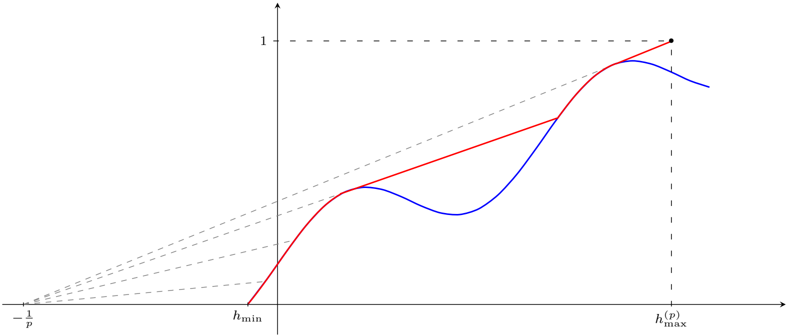

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$$f = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k},$$

where the $2^j$ random variables $\frac{-\log_2|c_{j,k}|}{j}$ are drawn independently according to a given probability law $\boldsymbol{\rho_j}$, hence

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# Sketch of the proof : case $h \in \left[h_{\min}, h_{\max}^{(p)}\right]$

**Theorem (E. Daviaud, 2025)**

*Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of balls of $[0,1]$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of $(0,1]$. If*

$$s = \sup\left\{\gamma : \mathcal{L}\left(\limsup_{n\,:\,\gamma_n\leq\gamma} B_n^{\gamma_n}\right) = 1\right\},$$

*then there exists a gauge function $\xi$ such that*

$$\mathcal{H}^{\xi}\left(\limsup_{n\to+\infty} B_n\right) > 0 \quad \textit{and} \quad \lim_{r\to 0^+} \frac{\log \xi(r)}{\log r} = s.$$

*In particular,*

$$\dim_{\mathcal{H}}\left(\limsup_{n\to+\infty} B_n\right) \geq s.$$

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$$E(\alpha, \delta) = \limsup_{j \to +\infty} \bigcup_{k \,:\, \left|c_{j,k}\right| \geq 2^{-\alpha j}} B\left(k2^{-j}, 2^{-\delta j}\right) \quad \forall \delta \leq 1,$$

then, almost surely,

$$E\left(\alpha, \frac{\alpha + \frac{1}{p}}{h + \frac{1}{p}}\right) \subseteq \left\{x \in [0,1] : h_f^{(p)}(x) \leq h\right\} \quad \forall \alpha \leq h$$

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0 1

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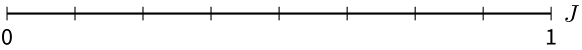

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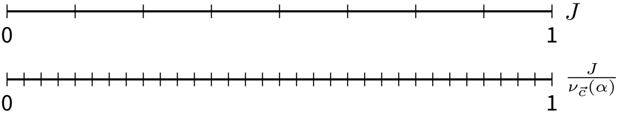

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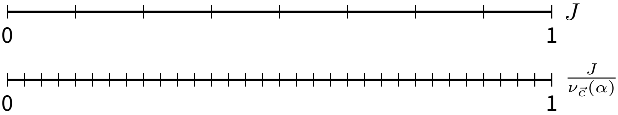


about $2^{\nu_{\vec{c}}(\alpha)\cdot\frac{J}{\nu_{\vec{c}}(\alpha)}}$ coeff $\gtrsim 2^{-\alpha j}$

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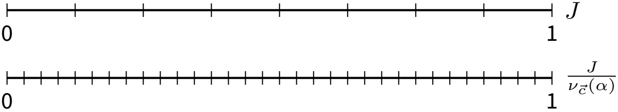


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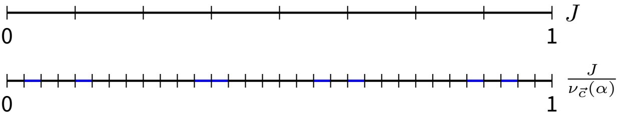


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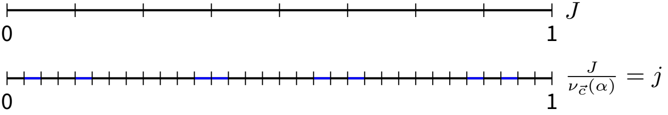


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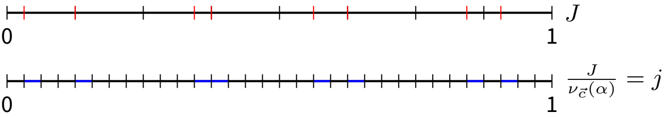


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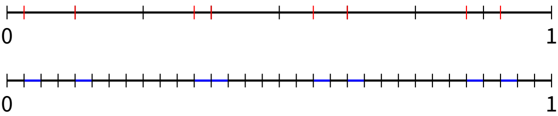


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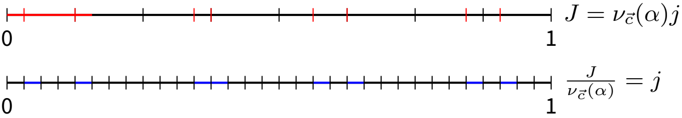


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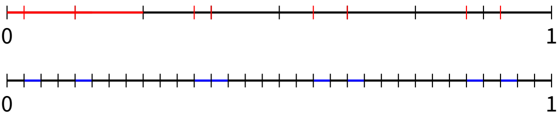


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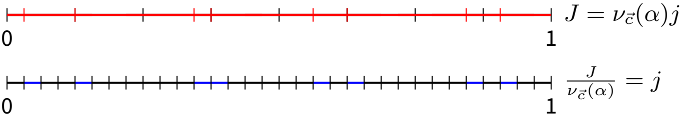


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# $S^\nu$ spaces and prevalence

Hölder case proved by J.M. Aubry, F. Bastin and S. Dispa (2007).

If $\nu$ is an admissible profile, then

$$S^\nu = \{\vec{c} \in \mathbb{R}^\Lambda : \nu_{\vec{c}} \leq \nu\}.$$

(S. Jaffard, 1996)

A subset $A$ of a Polish space $X$ is prevalent if there exists a Borel subset $B$ of $X$ and a non-trivial Borel probability measure $\mu$ on $X$ such that $\mu(B+x)=0$ for every $x \in X$ and $X \setminus A \subseteq B$.
(B.R. Hunt, T. Sauer and J.A. Yorke, 1992)

**Theorem**

*For a prevalent set of functions $f$ in $S^\nu$,*

- $\nu_{\vec{c}} = \nu$,
- *for every $0 < p < p_\nu$ and every $h \geq \frac{-1}{p}$,*

$$\mathscr{D}_f^{(p)}(h) = \begin{cases} \left(h + \frac{1}{p}\right) \sup\limits_{\alpha \in \left(\frac{-1}{p}, h\right]} \dfrac{\nu_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}} & \text{if } h \in \left[h_{\min}^{(\nu)}, h_{\max}^{(p),(\nu)}\right], \\ -\infty & \text{otherwise.} \end{cases}$$

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$$\mathscr{D}_f^{(p)}(h) = \begin{cases} \left(h + \frac{1}{p}\right) \sup\limits_{\alpha \in \left(\frac{-1}{p}, h\right]} \dfrac{\nu_{\vec{c}}(\alpha)}{\alpha + \frac{1}{p}} & \text{if } h \in \left[h_{\min}^{(\nu)}, h_{\max}^{(p),(\nu)}\right], \\ -\infty & \text{otherwise.} \end{cases}$$

Hölder case proved by J.M. Aubry, F. Bastin and S. Dispa (2007).

If $\nu$ is an admissible profile, then

$$S^\nu = \{\vec{c} \in \mathbb{R}^\Lambda : \nu_{\vec{c}} \leq \nu\}.$$

(S. Jaffard, 1996)

A subset $A$ of a Polish space $X$ is prevalent if there exists a Borel subset $B$ of $X$ and a non-trivial Borel probability measure $\mu$ on $X$ such that $\mu(B + x) = 0$ for every $x \in X$ and $X \setminus A \subseteq B$.
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# Sketch of the proof

- We define $p_\nu > 0$ such that for every $p < p_\nu$ and every $f \in S^\nu$, $\eta_f(p) > 0$.
- Let $X$ be a Random Wavelet Series for which $\boldsymbol{\nu} = \nu$, where

$$\boldsymbol{\nu}(\alpha) = \lim_{\varepsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log_2\left(2^j \boldsymbol{\rho_j}((-\infty, \alpha + \varepsilon])\right)}{j}$$

  satisfies

$$\boldsymbol{\nu} = \nu_{\vec{x}} \text{ on } [h_{\min}, +\infty),$$

  hence $X \in S^\nu$ almost surely. It is enough to prove that for every $f \in S^\nu$, almost surely, $X + f$ has the expected spectrum.
- Almost surely,

$$h_{\max}^{(p),(\nu)} = h_{\max}^{(p)}, \quad h_{\min}^{(\nu)} = \inf\{\alpha \geq 0 : \nu(\alpha) \geq 0\} = h_{\min} \quad \text{and} \quad \nu_{\vec{x}} = \nu.$$

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