Multiple Timescale Spectral Analysis in Wind Engineering

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Multiple Timescale Spectral Analysis

$$u''(t) + \varepsilon f\left(u, u'(t)\right) + u(t) = 0 \qquad \varepsilon \ll 1 \qquad \varepsilon = 0 \longrightarrow u(t) = R\cos\left(t + \varphi_0\right)$$

Perturbation methods

Slow timescale : $T = \varepsilon t$ Fast timescale : t

$$d_t \to \partial_t + \varepsilon \partial_T$$

$$u(t) \to u_0\left(t, T\right) + \varepsilon u_1\left(t, T\right) + \cdots$$

Leading order solution :

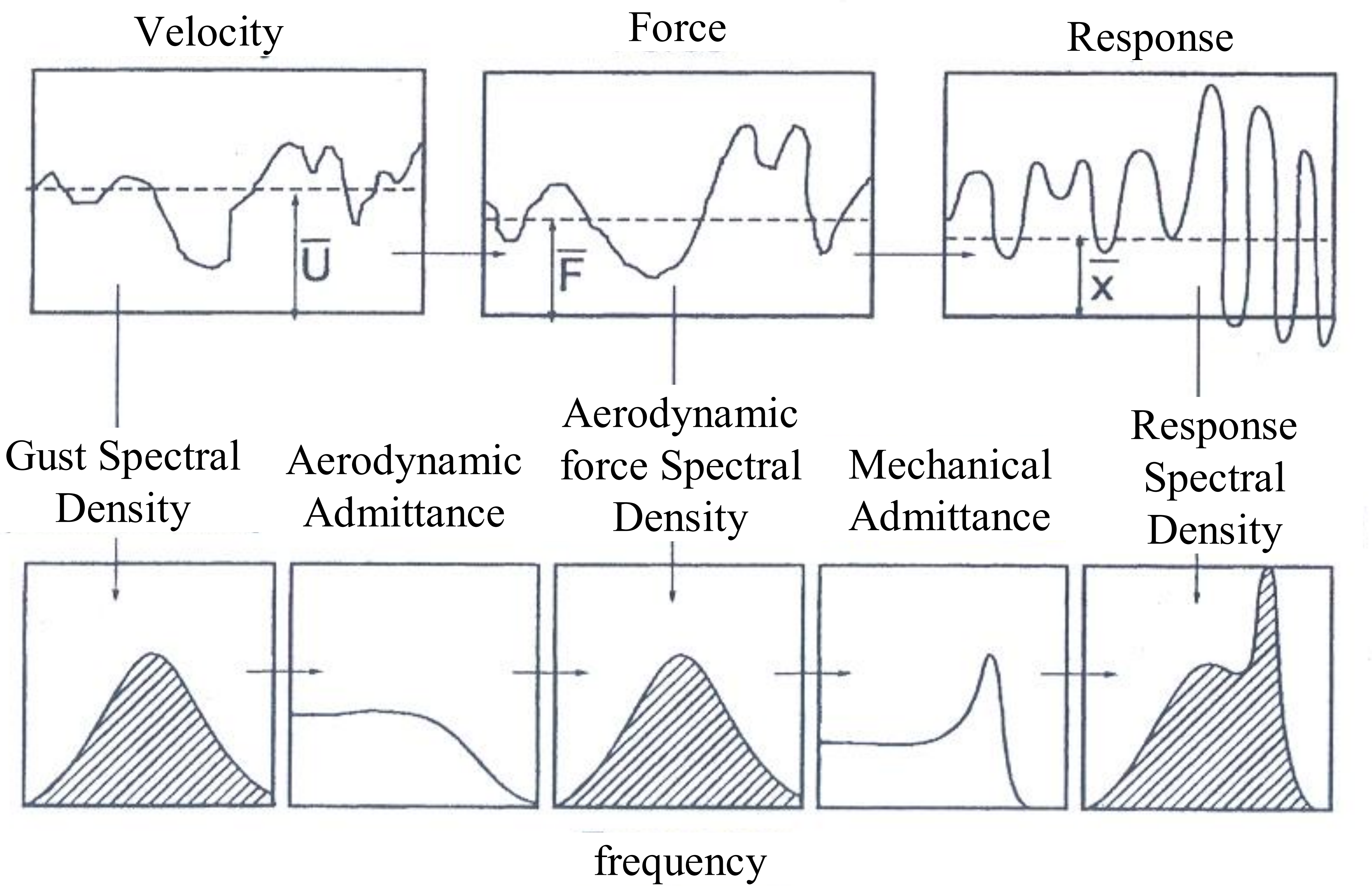
$$u_0\left(t, T\right) = R\left(T\right)\cos\left(t + \varphi\left(T\right)\right)$$

Secularity conditions : $R'(T) = \cdots$
$\varphi'(T) = \cdots$

- Multiple timescale, in time domain
- Specific features :
  - Drop derivatives by one order
  - Slow is momentarily considered as constant for fast dynamics

Multiple Timescale Spectral Analysis

- Spectral analysis = integration of a spectrum over frequency

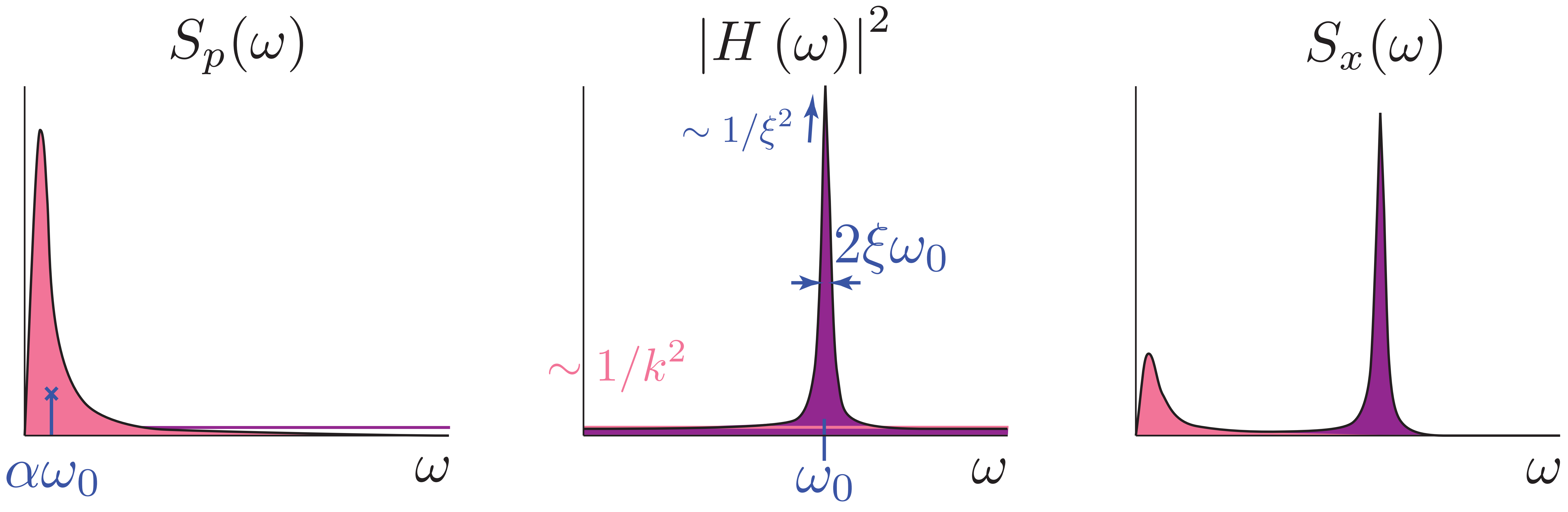


Multiple Timescale Spectral Analysis

- Spectral analysis = integration of a spectrum over frequency
- Spectrum (of response) = product of 2 functions
- Drop by one order of derivatives integration
- Notice existence of **two timescales** : fast t, and slow ξt, αt
- Slow is momentarily considered as constant for fast dynamics
- Notice two small parameters in integrand : α and ξ

$$S_x(\omega) = |H(\omega)|^2 S_p(\omega)$$

$$\sigma_x^2 = \int_{-\infty}^{+\infty} S_x(\omega)\mathrm{d}\omega$$



$$B = \left(\frac{\sigma_p}{k}\right)^2 \quad \text{Background}$$

$$R = \int_{-\infty}^{+\infty} |H(\omega)|^2 S_p(\omega_0)\,\mathrm{d}\omega$$

$$R = \frac{\pi\omega_0}{2\xi}\frac{S_p(\omega_0)}{k^2} \quad \text{Resonant}$$

Another example of spectral analysis: covariance

- **covariances** of modal responses obtained from cross-PSDs

$$\sigma_{ij} = \int_{-\infty}^{+\infty} H_i(\omega)\, H_j^*(\omega)\, S_{p_{ij}}(\omega)\, \mathrm{d}\omega$$

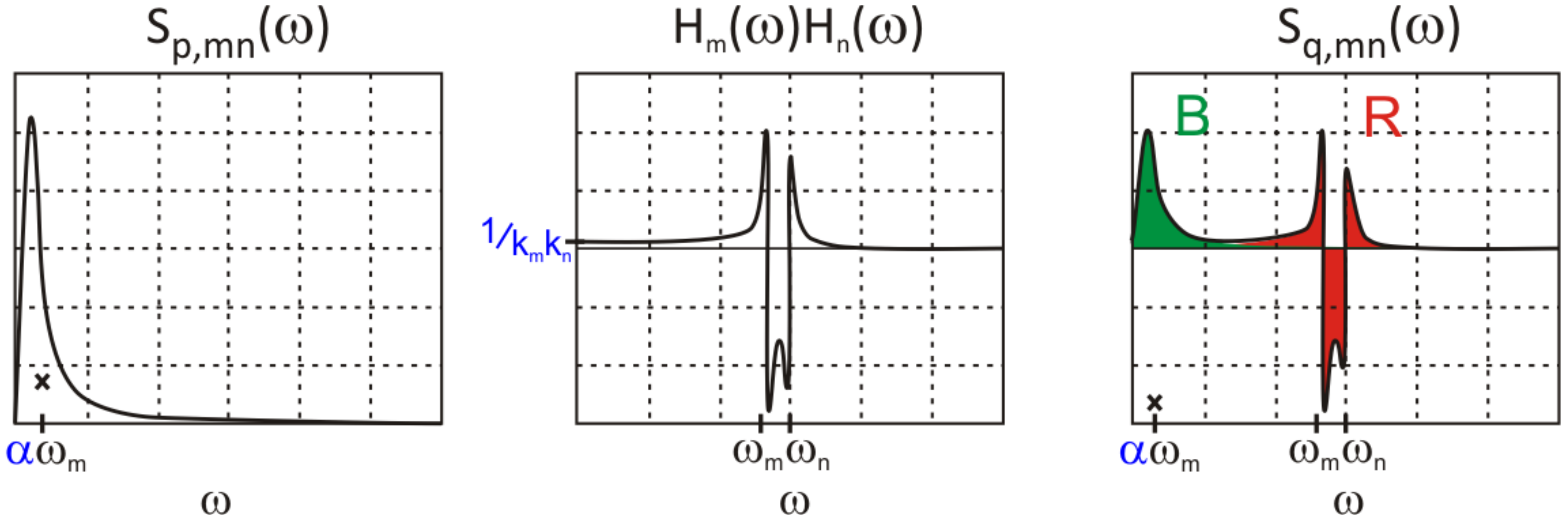
Background : replace $H_i(\omega)\, H_j^*(\omega)$ by $1/k_i k_j$

Resonant : replace $S_{p_{ij}}(\omega)$ by $S_{p_{ij}}(\bar{\omega})$

Variance

Background : replace $\left|H(\omega)\right|^2$ by $1/k^2$

Resonant : replace $S_p(\omega)$ by $S_p(\omega_0)$



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Variance

$$R = \int_{-\infty}^{+\infty} |H(\omega)|^2 S_p(\omega_0) \, \mathrm{d}\omega$$

$$R = \frac{\pi \omega_0}{2\xi} \frac{S_p(\omega_0)}{k^2}$$ Resonant

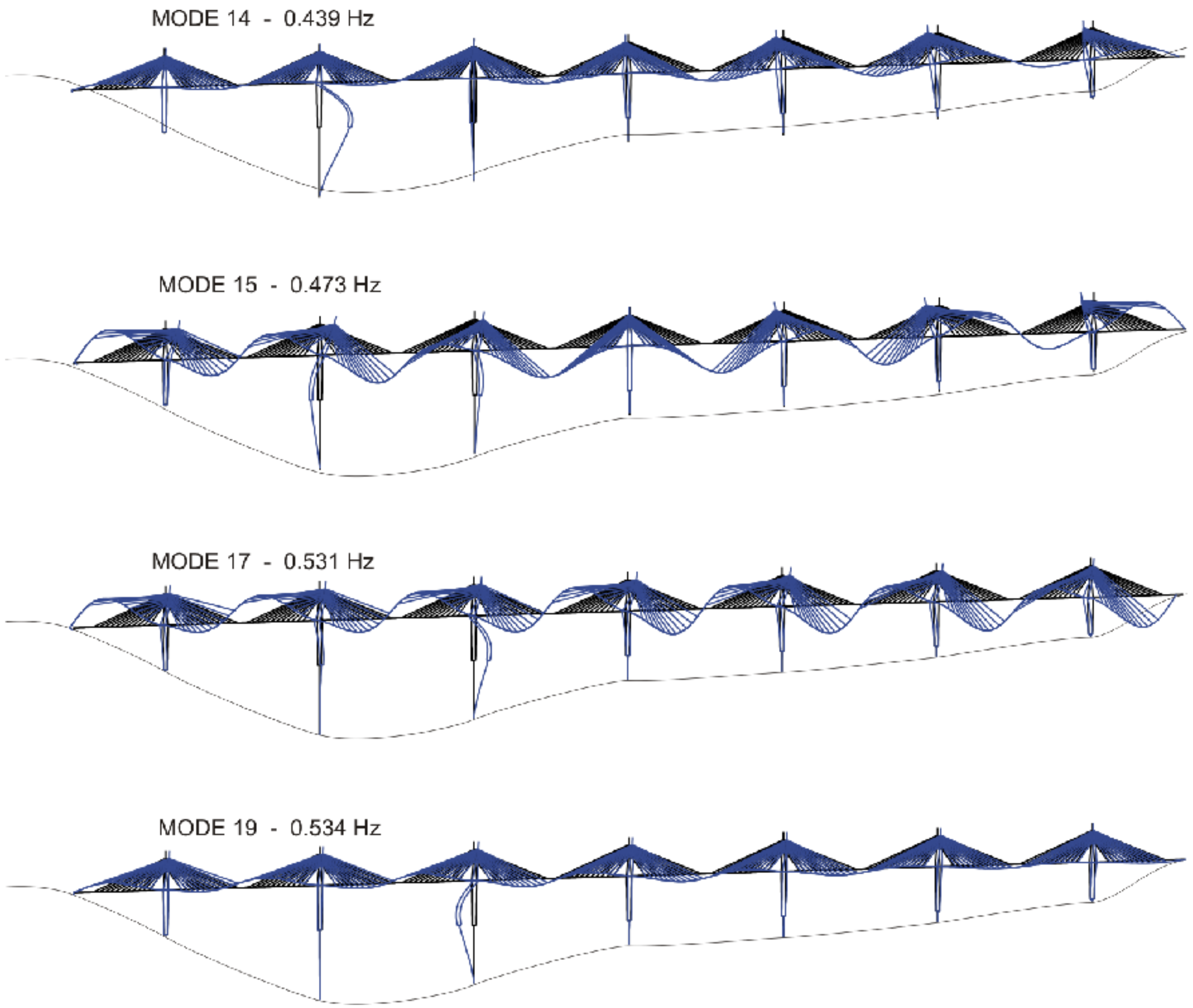
Resonant component requires :

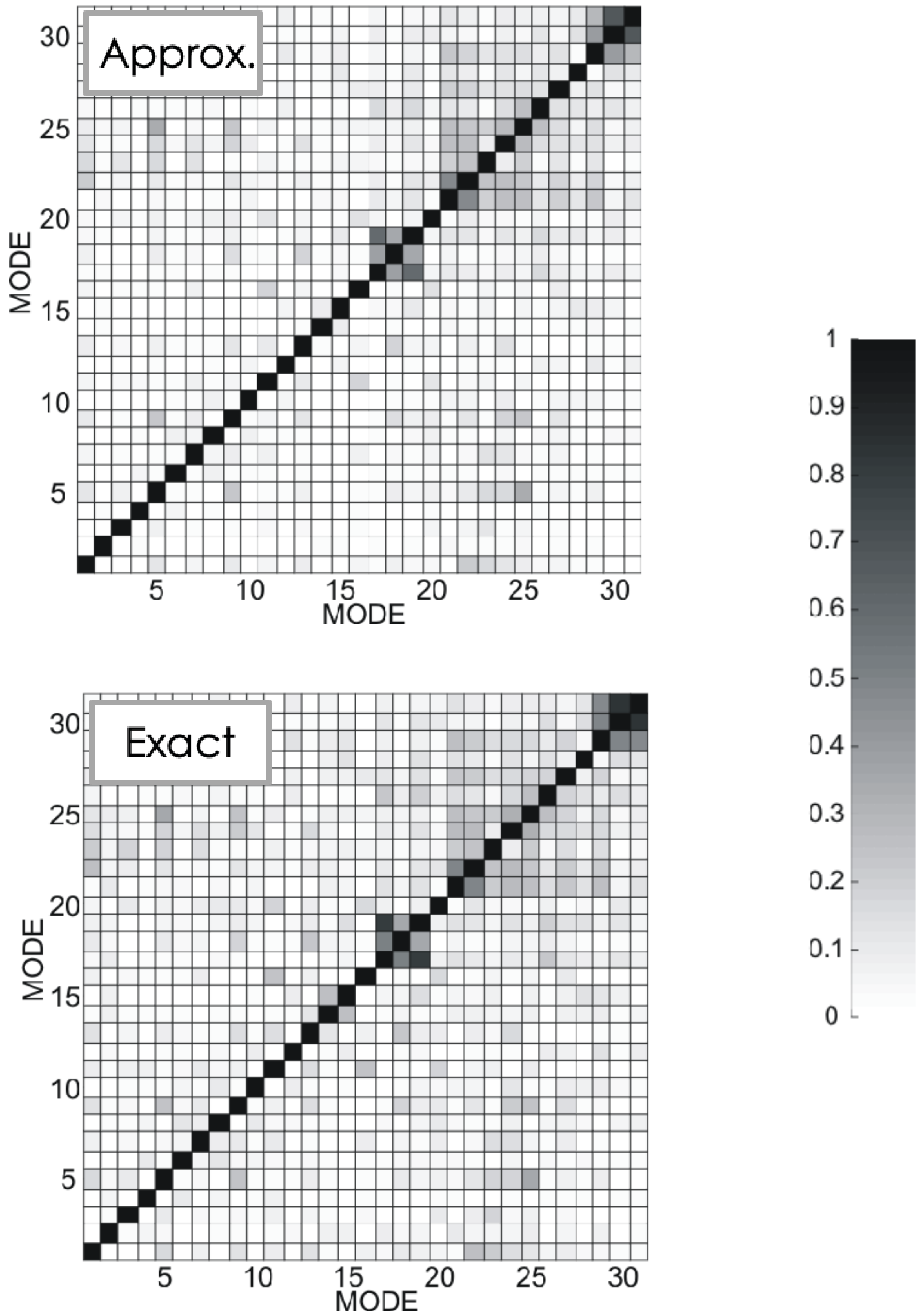
$$\int_{-\infty}^{+\infty} H_i(\omega) H_j^*(\omega) \, d\omega = \frac{1}{k_i k_j} \frac{4\pi \omega_i^2 \omega_j^2 (\xi_i \omega_i + \xi_j \omega_j)}{2\omega_i^2 \omega_j^2 \left(2\xi_i^2 + 2\xi_j^2 - 1\right) + 4\xi_i \xi_j \omega_i^3 \omega_j + 4\xi_i \xi_j \omega_i \omega_j^3 + \omega_i^4 + \omega_j^4}$$

(Not bad but quite long — Same as der Kiureghian's formula for correlation in earthquake engineering, 1993)

Another example of spectral analysis: covariance

- **covariances** of modal responses obtained from cross-PSDs





A counter-example : bispectral analysis

- Non-Gaussian stochastic analysis : **skewness** obtained from integration of bispectrum

$$B_x(\omega_1, \omega_2) = H(\omega_1)\, H(\omega_2)\, H^*(\omega_1 + \omega_2)\, B_p(\omega_1, \omega_2)$$

$$m_3 = \iint_{-\infty}^{+\infty} B_x(\omega_1, \omega_2)\, \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

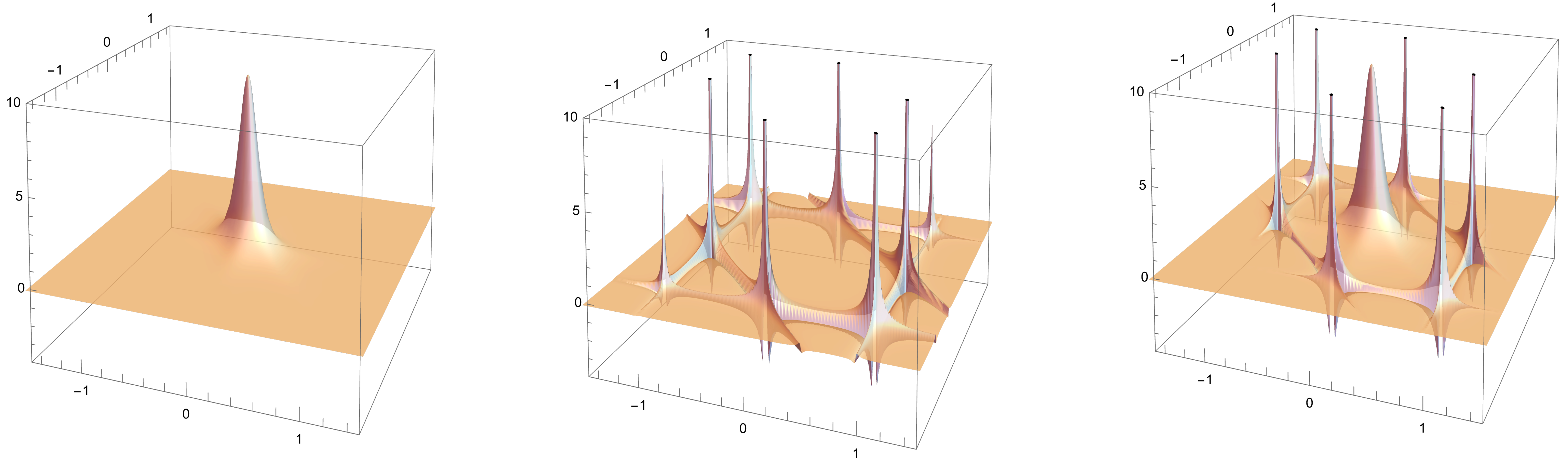
Background : replace $H(\omega_1)\, H(\omega_2)\, H^*(\omega_1 + \omega_2)$ by $1/k^3$

Resonant : replace $B_p(\omega_1, \omega_2)$ by $B_p(\omega_0, \omega_0)$

Variance

Background : replace $\left|H(\omega)\right|^2$ by $1/k^2$

Resonant : replace $S_p(\omega)$ by $S_p(\omega_0)$



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Variance

Background : replace $\left|H(\omega)\right|^2$ by $1/k^2$

Resonant : replace $S_p(\omega)$ by $S_p(\omega_0)$

Resonant component requires :

$$\iint_{-\infty}^{+\infty} H_i(\omega_1) H_j(\omega_2) H_k^*(\omega_1 + \omega_2) \, \mathrm{d}\omega_1 \mathrm{d}\omega_2 = \ldots$$

(Much longer than covariance, but still ok)

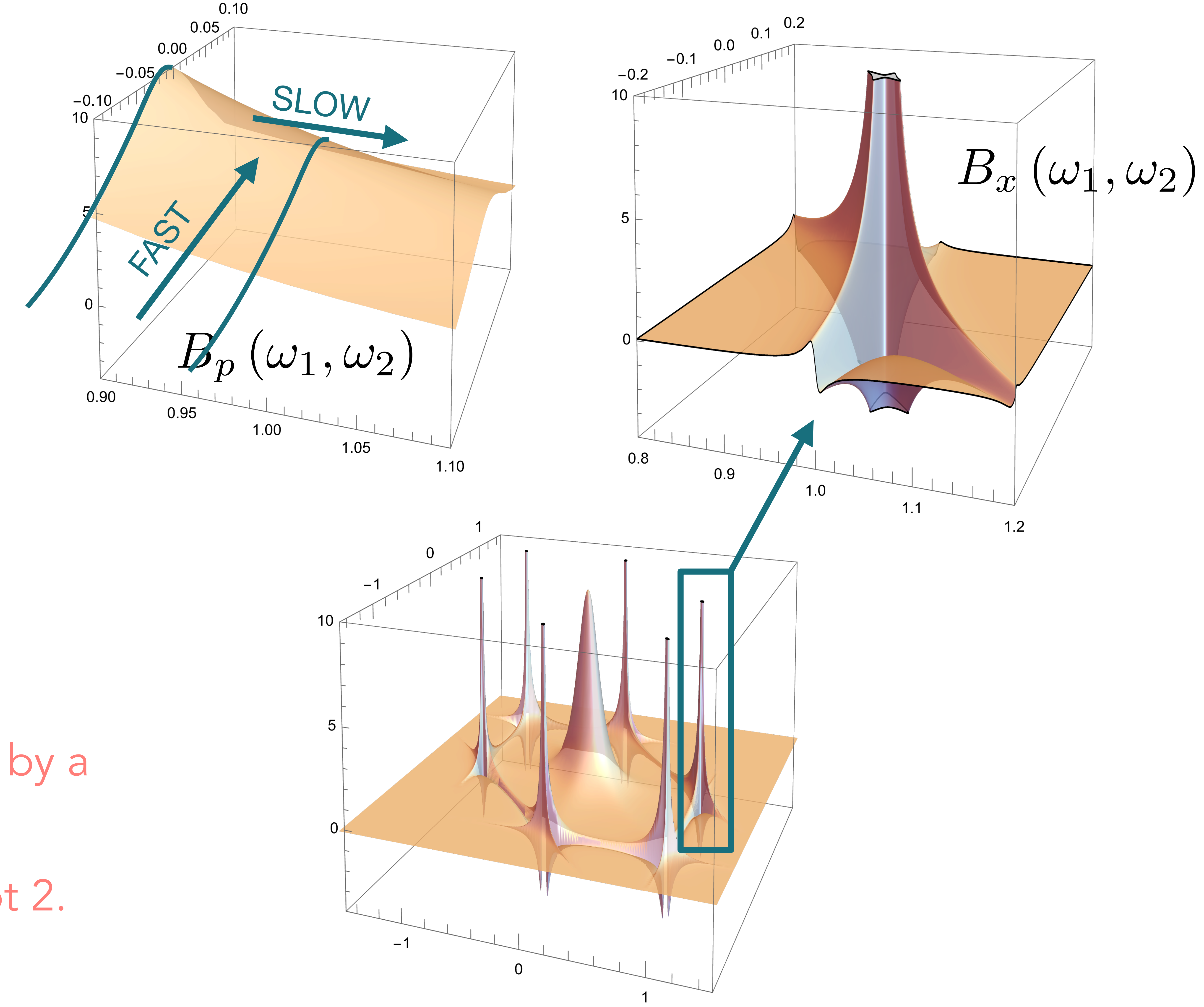
This approximation unfortunately fails. One order of magnitude error.

A counter-example : bispectral analysis

- **zoom** on a peak

- The bispectrum of the loading changes:
  - slowly along an axis
  - fast in another direction

- The replacement by a constant in one direction is fine, but not both



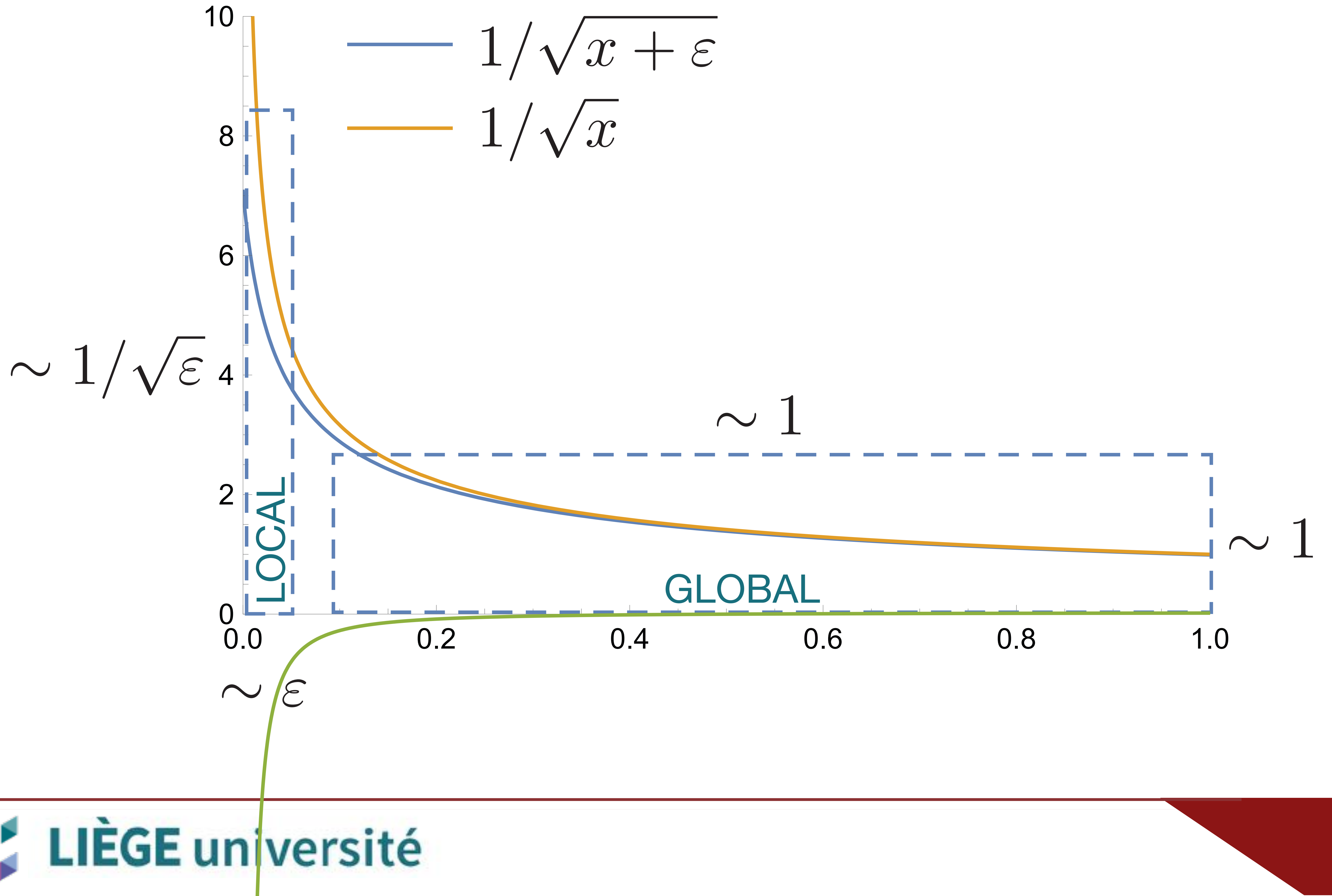
1. White noise approximation : the replacement by a constant is not universally correct.
2. We can drop the order of integration by 1, not 2.

Integral with small parameters $\varepsilon \ll 1$

Method of ranked contributions

Evaluate :

$$\int_0^1 \frac{1}{\sqrt{x+\varepsilon}}\mathrm{dx} = 2\sqrt{1+\varepsilon} - 2\sqrt{\varepsilon}$$
$$\sim \underline{2} - 2\sqrt{\varepsilon} + \varepsilon + \cdots$$



Two contributions to the integral :

- **Global** (background) : ~ 1
- **Local** (resonant) : $\sim \sqrt{\varepsilon}$

Global contribution : $f \sim \frac{1}{\sqrt{x}}$

$$\int_0^1 \frac{1}{\sqrt{x+\varepsilon}}\mathrm{d}x \sim \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x = \underline{2}$$

Create **residual**, …

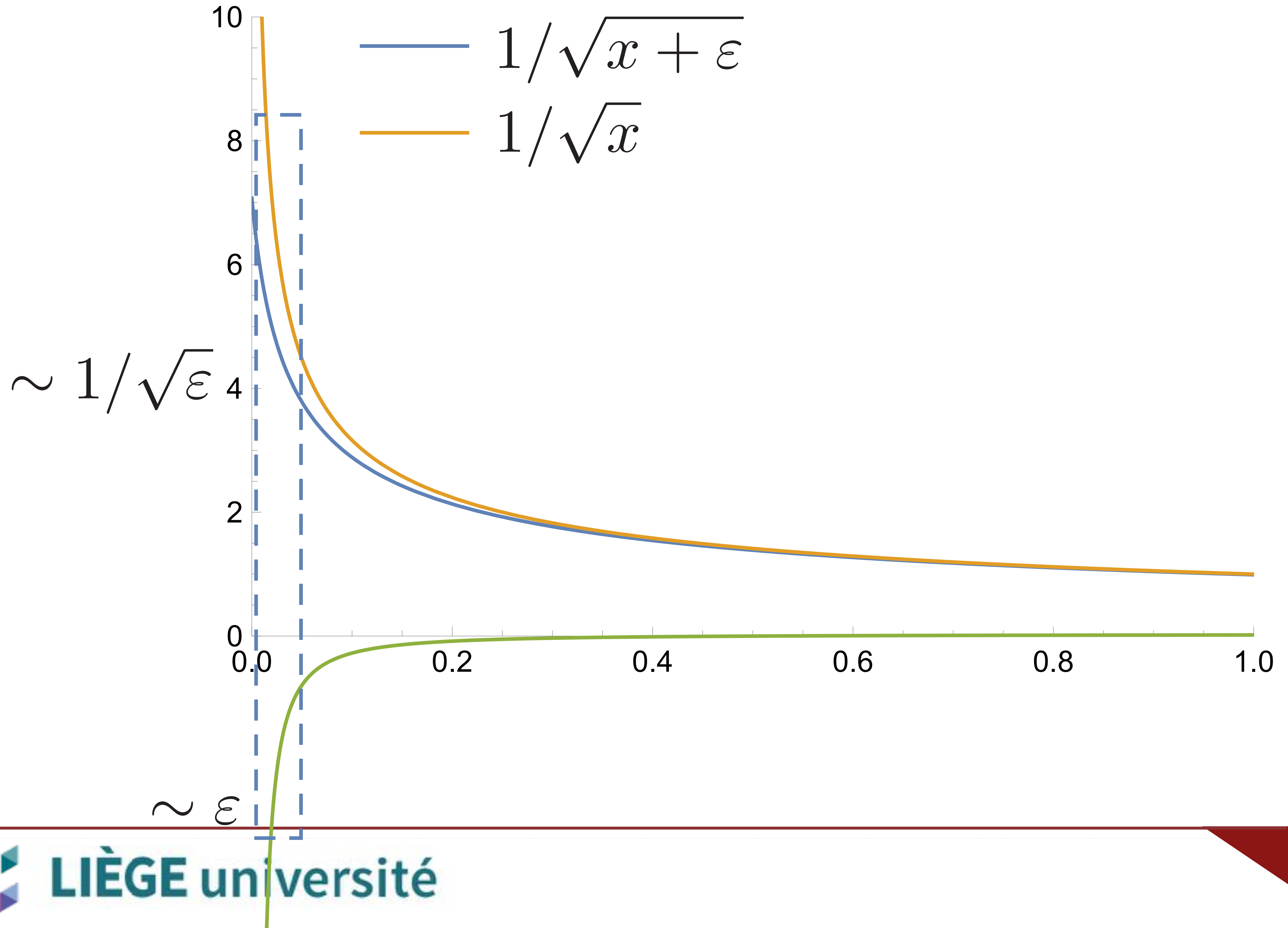
… focus on following contribution.

The Multiple Timescale Spectral Analysis

Method of ranked contributions

Evaluate :

$$\int_0^1 \frac{1}{\sqrt{x+\varepsilon}}\mathrm{dx} = 2\sqrt{1+\varepsilon} - 2\sqrt{\varepsilon}$$
$$\sim \underline{2} - \underline{2\sqrt{\varepsilon}} + \varepsilon + \cdots$$



Two contributions to the integral :

- **Global** (background) : ~ 1
- **Local** (resonant) : $\sim \sqrt{\varepsilon}$

Local contribution : ZOOM with stretched coord.

$x = \varepsilon u$

$$f - f_0 \sim \frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{\sqrt{1+u}} - \frac{1}{\sqrt{u}}\right)$$

$$\int_0^{1/\varepsilon \to +\infty} \frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{\sqrt{1+u}} - \frac{1}{\sqrt{u}}\right)\mathrm{d}u = -\frac{2}{\sqrt{\varepsilon}}$$

Second contribution : $-\frac{2}{\sqrt{\varepsilon}} \cdot \varepsilon = \underline{-2\sqrt{\varepsilon}}$

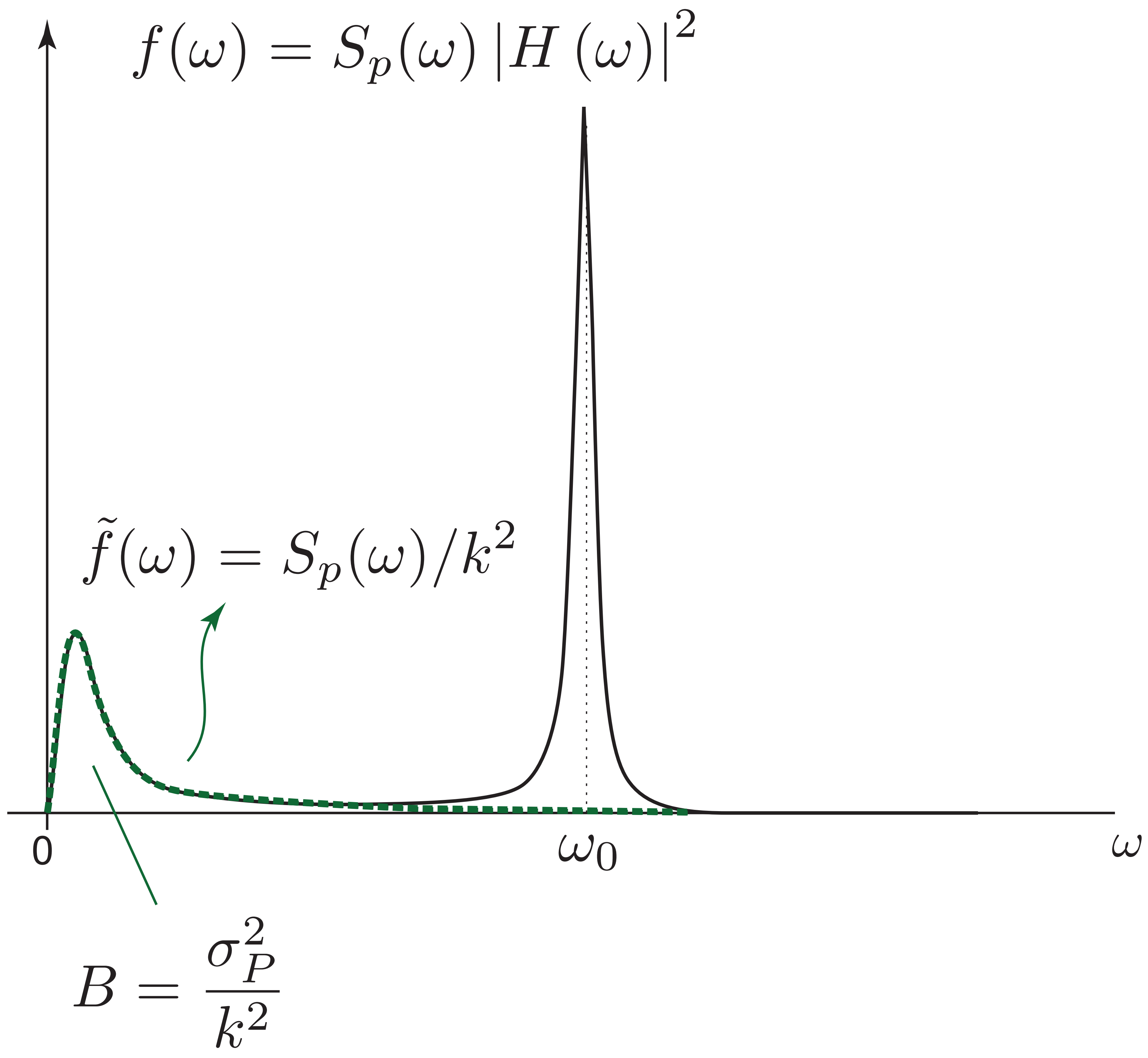
The Multiple Timescale Spectral Analysis

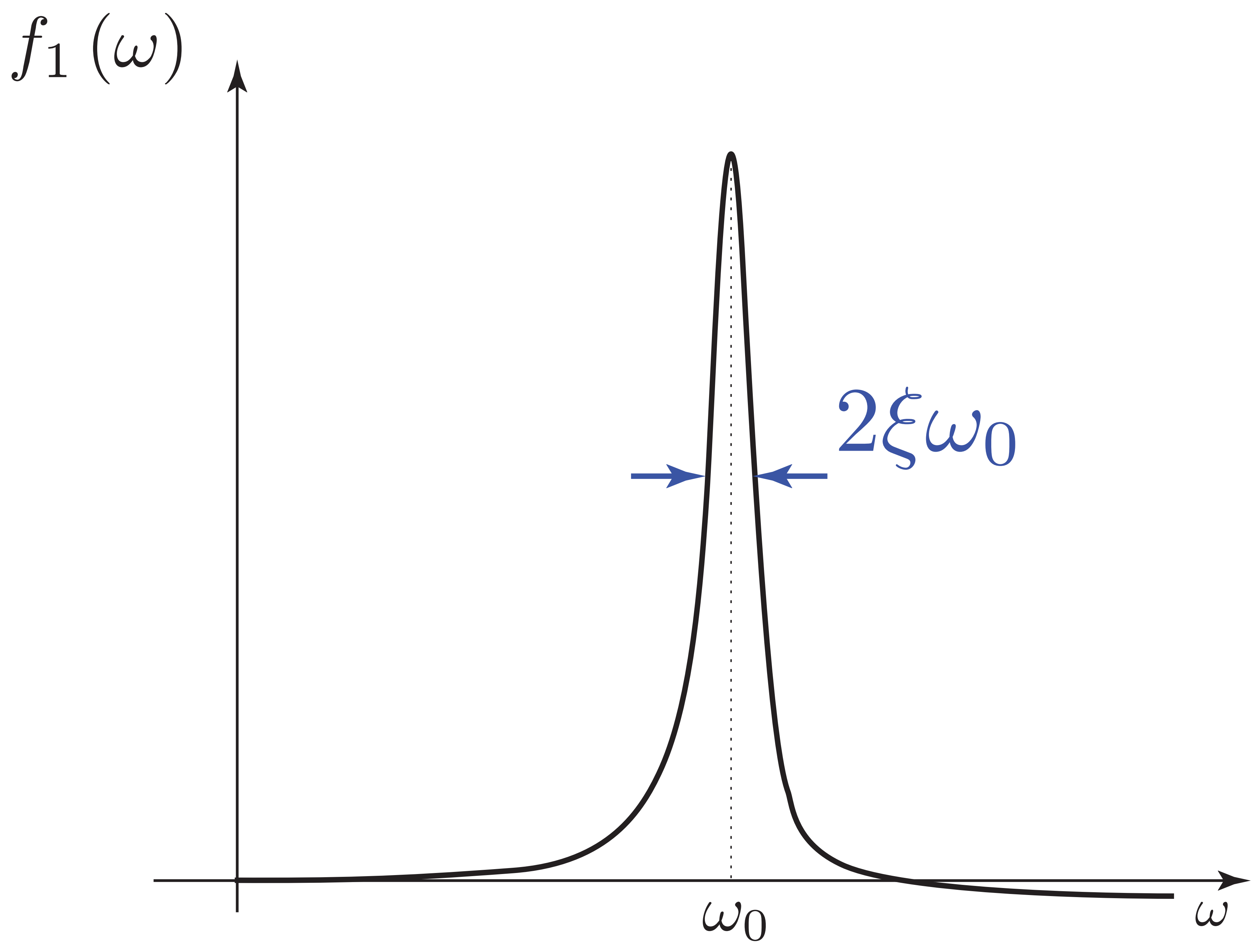
- By means of local/global approximations we can provide the asymptotic expansion of integral
- Background = global — Resonant = local

- Important to **<u>zoom</u>** in the neighborhood of local contributions, then only, approximate locally

Revisiting the estimation of variance

1. Identify the two components to the integral
2. Choose the **Background** to start, determine first contribution
3. Subtract off approximation and construct first residual

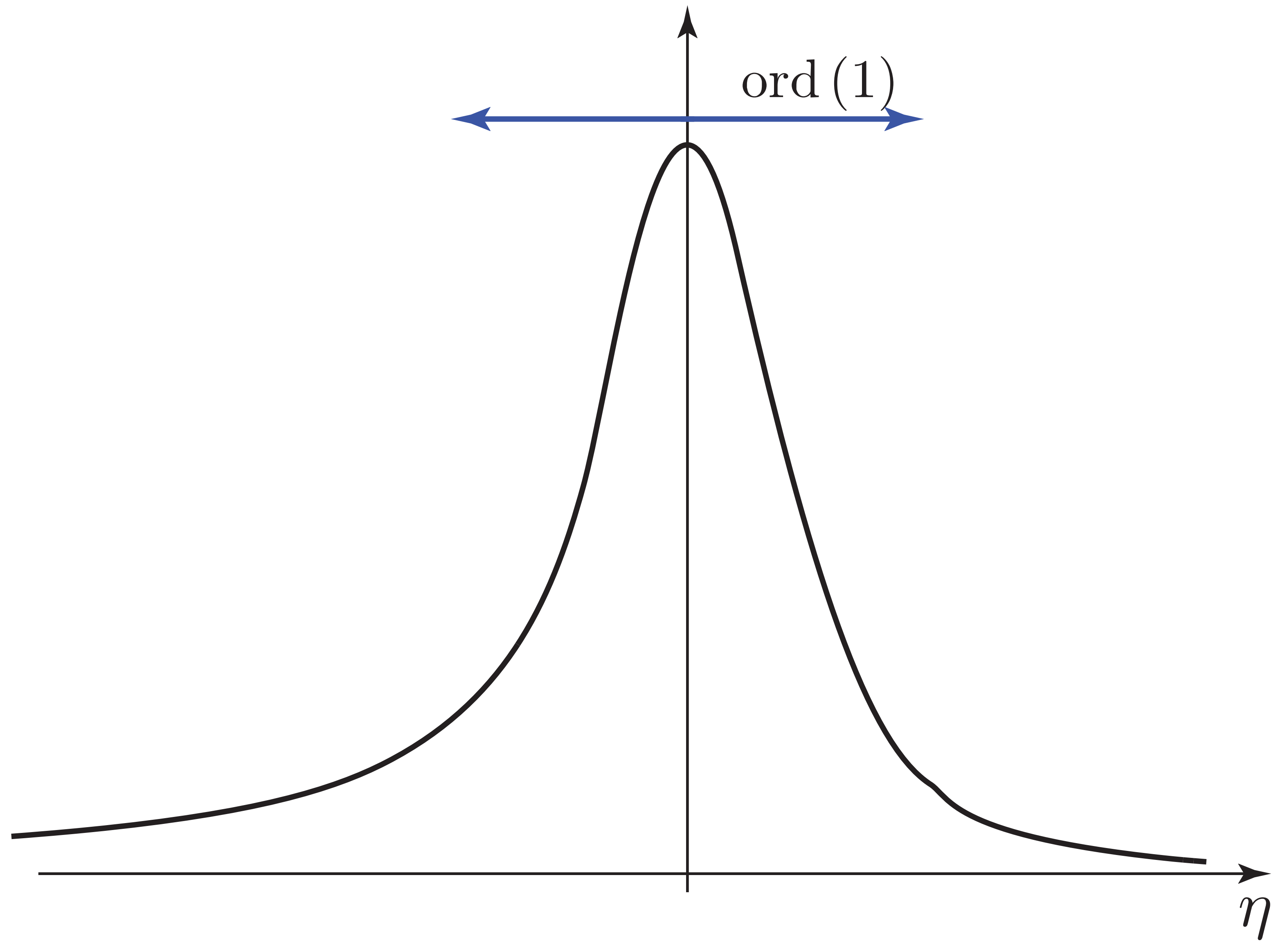




$$f_1(\omega) = f(\omega) - \tilde{f}(\omega) = S_p(\omega)\left(|H(\omega)|^2 - \frac{1}{k^2}\right)$$

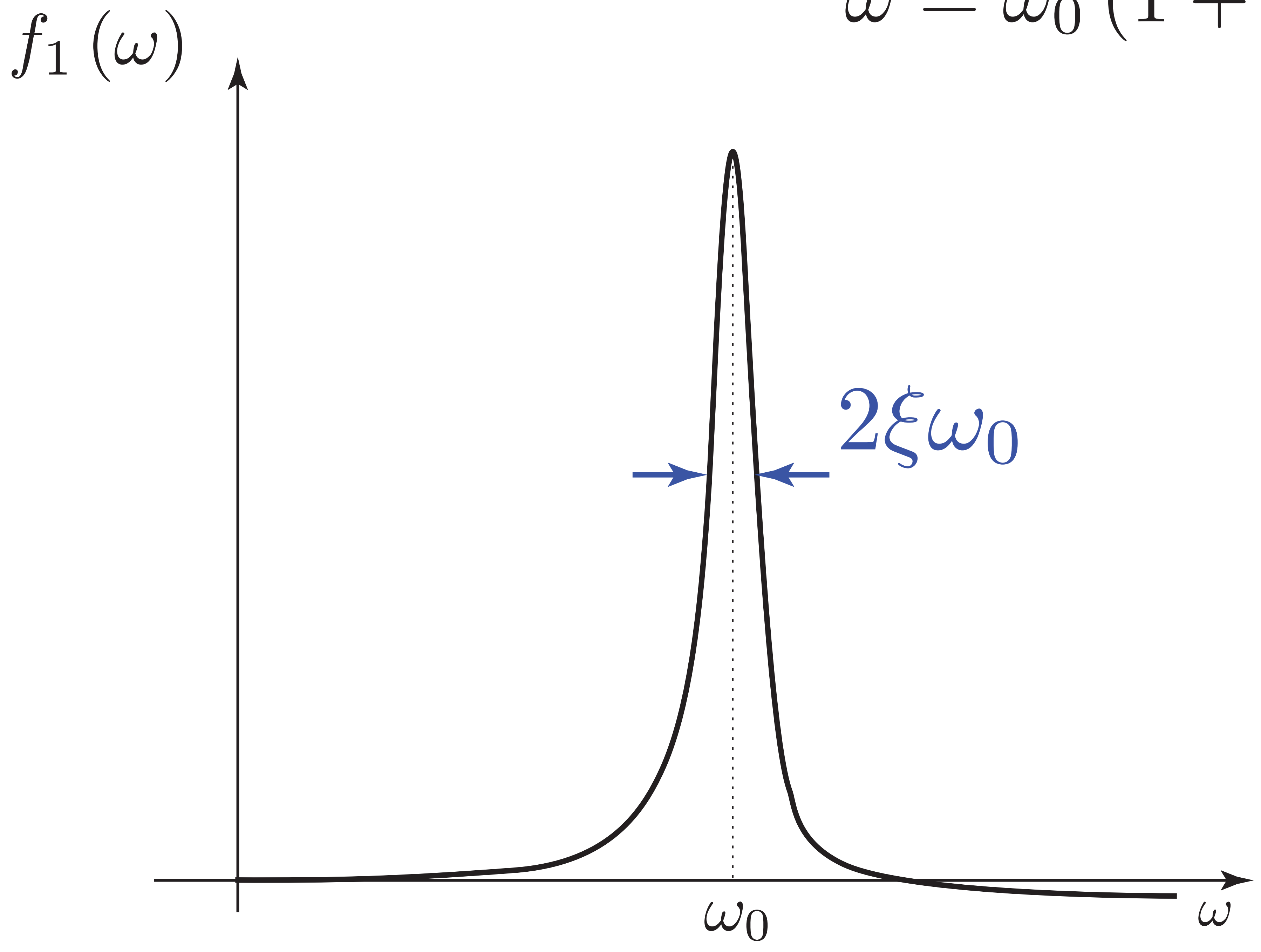
Revisiting the estimation of variance

4. Now focus on the Resonant component
5. Zoom-in with the appropriate rescaling > create a domain of order 1



Use stretched coordinate

$$\omega = \omega_0 (1 + \xi\eta)$$



$$f_1(\omega) = f(\omega) - \tilde{f}(\omega) = S_p(\omega)\left(|H(\omega)|^2 - \frac{1}{k^2}\right)$$

Revisiting the estimation of variance

6. Find a local approximation
7. Integrate and determine Resonant contribution

We recovered Davenport B/R's approximation

Major difference :
the use of a stretched coordinate makes it neat and avoids « replace the PSD by constant value (equivalent white noise) »

$$f_1(\omega) = S_p(\omega)\left(\underbrace{\left|H\left(\omega\right)\right|^2 - \frac{1}{k^2}}\right)$$

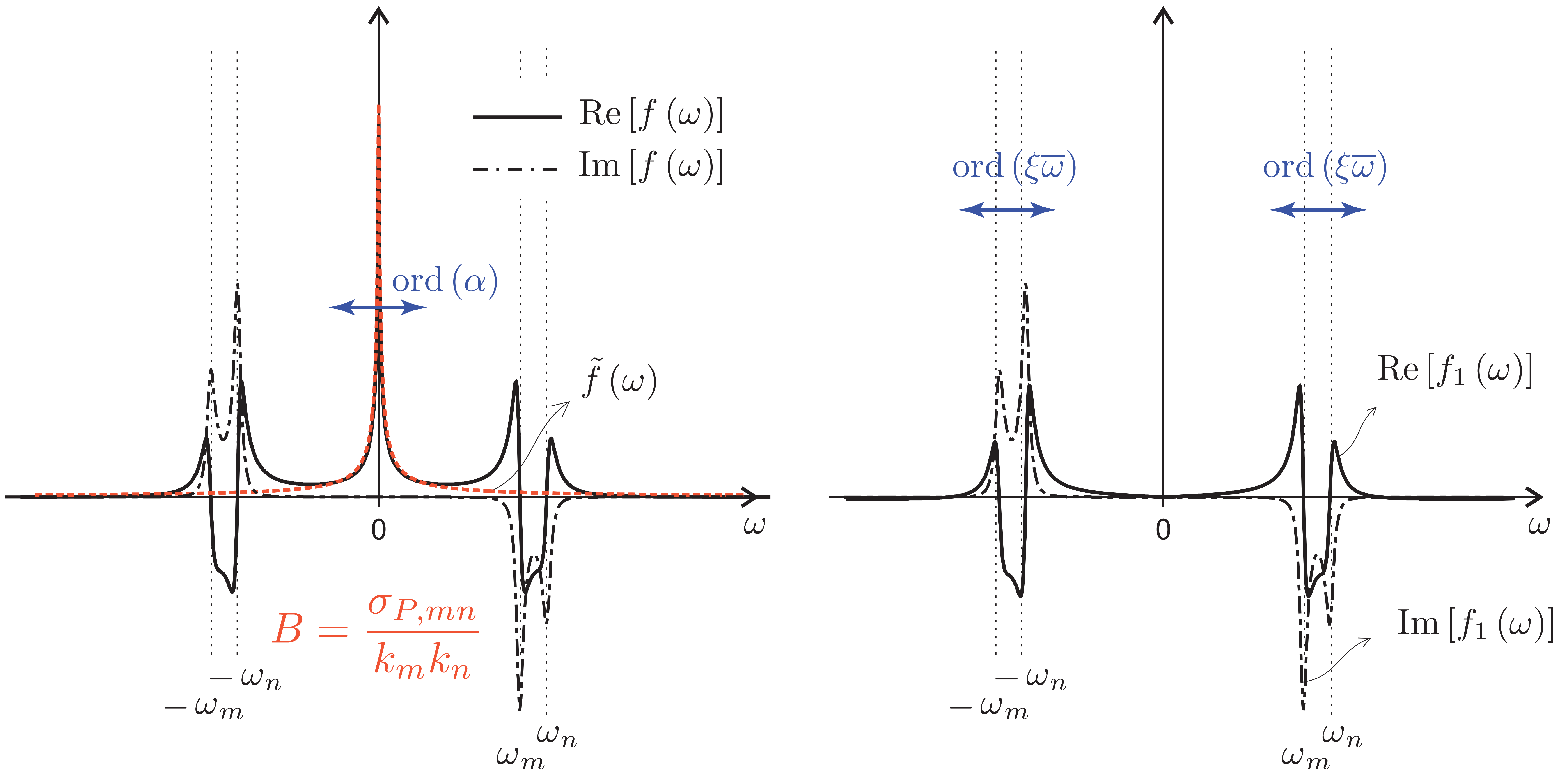
$$\frac{1}{4\xi^2 k^2}\frac{1}{1+\eta^2} + \mathrm{ord}\left(\xi^{-1}\right)$$

$$\tilde{f}_1(\omega) = S_p(\omega_0)\frac{1}{4k^2}\frac{1}{\xi^2 + \left(\frac{\omega}{\omega_0} - 1\right)^2}$$

$$\longrightarrow \int \tilde{f}_1(\omega) d\omega \quad \rightarrow \quad R = \frac{\pi\omega_0}{2\xi}\frac{S_p(\omega_0)}{k^2}$$

Revisiting the estimation of covariance

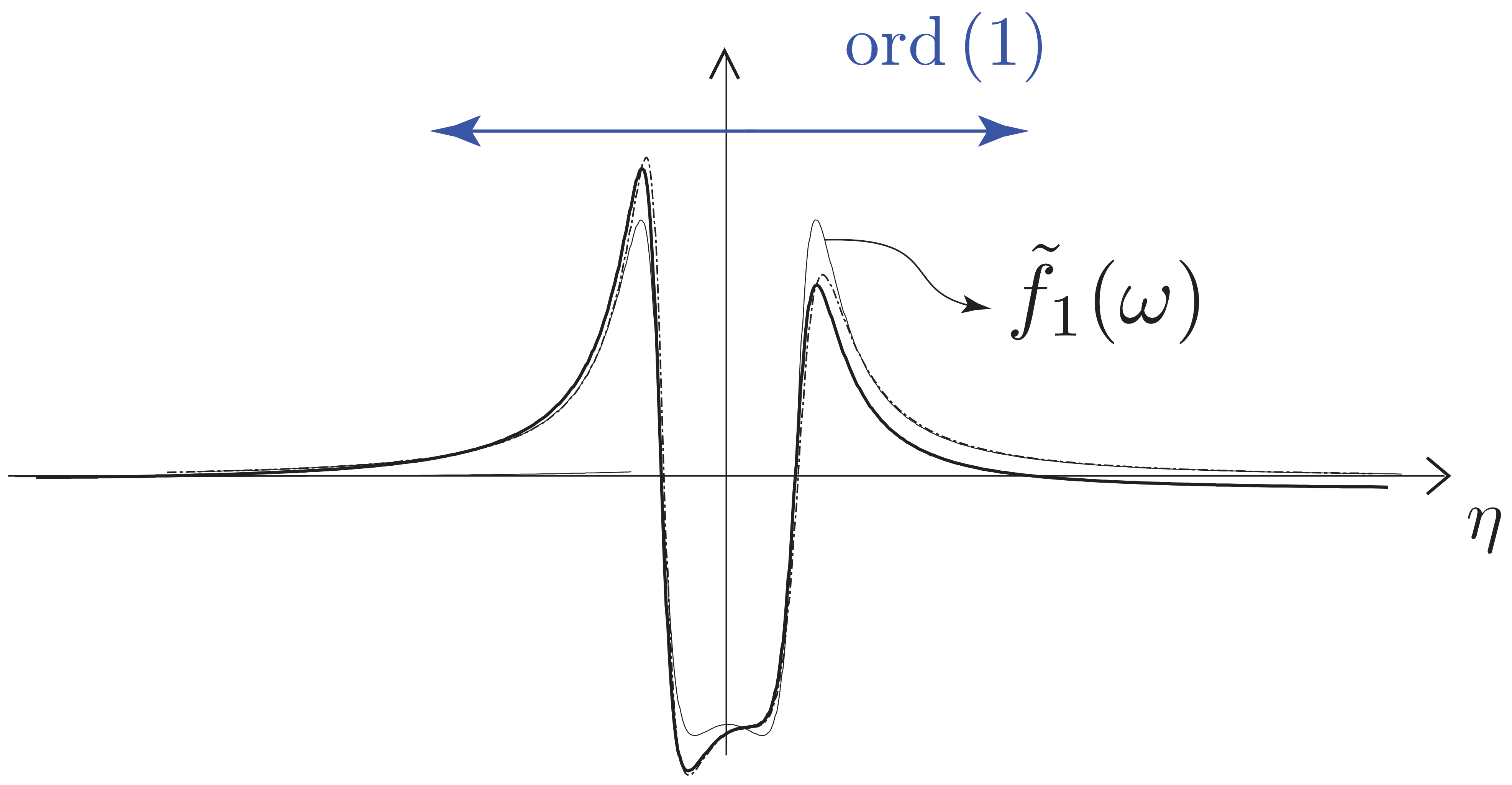
Develop same approach for the covariance



Revisiting the estimation of covariance

$$\omega = \frac{\omega_m + \omega_n}{2} + \eta(\omega_n - \omega_m)$$

Relative separation of natural frequencies : $\varepsilon = \frac{\omega_n - \omega_m}{\omega_n + \omega_m}$



$$\int \tilde{f}_1(\omega) d\omega$$

$$\downarrow$$

$$R = \frac{1}{k_m k_n} \mathrm{Re} \left[\frac{S_p(\omega_m) + S_p(\omega_n)}{2} \frac{\omega_m + \omega_n}{2} \frac{\pi(\xi - i\varepsilon)}{2(\varepsilon^2 + \xi^2)} \right]$$

"White noise approximation"

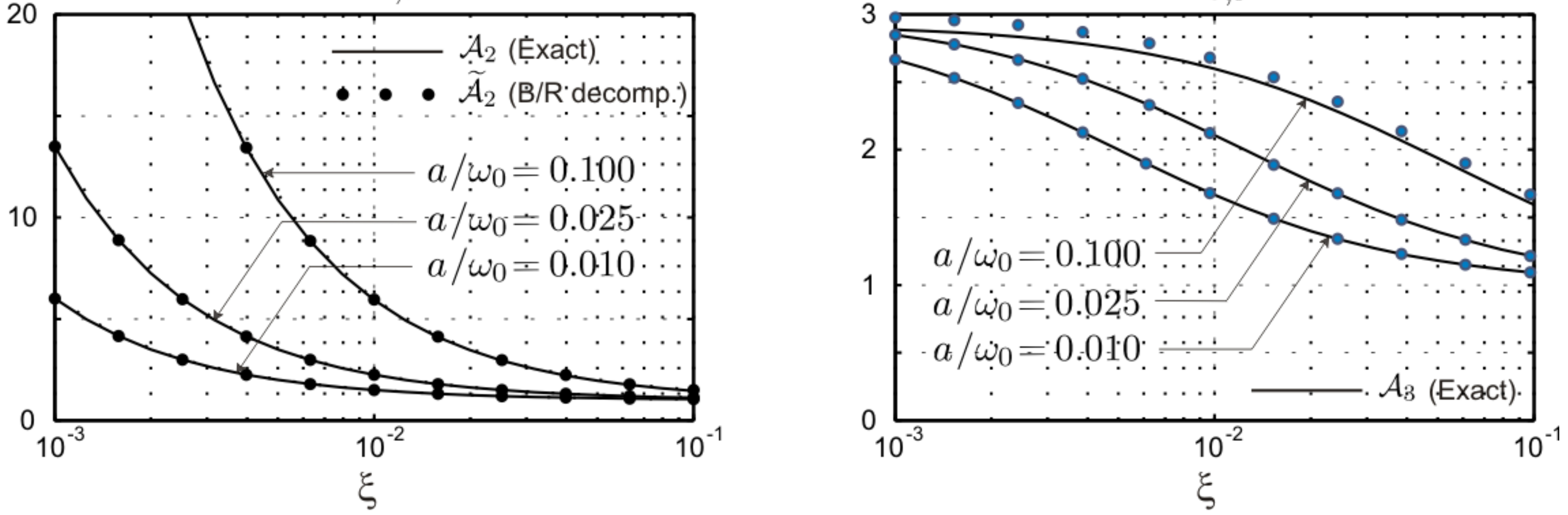
$$\int_{-\infty}^{+\infty} H_i(\omega) H_j^*(\omega) d\omega = \frac{1}{k_i k_j} \frac{4\pi \omega_i^2 \omega_j^2 (\xi_i \omega_i + \xi_j \omega_j)}{2\omega_i^2 \omega_j^2 \left(2\xi_i^2 + 2\xi_j^2 - 1\right) + 4\xi_i \xi_j \omega_i^3 \omega_j + 4\xi_i \xi_j \omega_i \omega_j^3 + \omega_i^4 + \omega_j^4}$$

Revisiting the estimation of bispectrum

Develop same approach for bispectrum

$$m_3 = \iint_{-\infty}^{+\infty} B_x(\omega_1, \omega_2)\, \mathrm{d}\omega_1 \mathrm{d}\omega_2 \quad \longrightarrow \quad m_3 = B + bR = \frac{m_{3,f}}{k^3} + 6\pi \frac{\xi\, \omega_0^3}{k^3} \int_{-\infty}^{+\infty} \frac{B_f(\omega_0, \omega_2)}{(2\xi\, \omega_0)^2 + \omega_2^2} d\omega_2$$

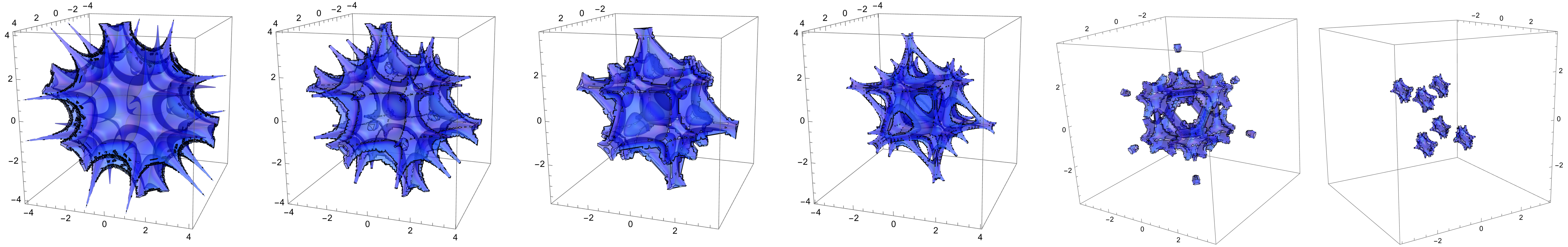
Variance $\frac{m_{2,x}}{m_{2,b}}$ Skewness $\frac{m_{3,x}}{m_{3,b}}$



MTSA integration of trispectrum

Develop same approach for trispectrum (4th order non-Gaussian analysis, kurtosis)

$$m_4 = \iiint_{-\infty}^{+\infty} T_x\left(\omega_1, \omega_2, \omega_3\right) \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3 \longrightarrow m_4 = B + tR = \frac{m_{4,f}}{k^4} + \iint_{-\infty}^{+\infty} \cdots d\omega_1 d\omega_2$$



Contours of fourth kernel

The other applications of the MTSA

- 1963 : Variance 1-DOF linear system [Davenport]
- 2005 : Covariance, modal responses
- 2011 : Skewness 1-DOF linear system (NG loading)
- 2012 : Kurtosis 1-DOF linear system (NG loading)
- 2015 : **Multiple Timescale Spectral Analysis**
- 2015 : Nonlinear 1-DOF system, Volterra series
- 2018 : Variance 1-DOF with fractional derivatives
- 2019-20 : Wind and waves : B-R-I decomposition
- 2022-23 : Floating structures
- 2022 : SDOF Flutter (frequency dependent systems)
- 2023 : Non-Gaussian M-DOF linear system
- 2025 : MDOF Flutter (long-span bridges)

Takehome messages

1. The (naive) replacement of PSD by a constant does not always work
2. MTSA = very generic can be applied to many contexts (integral with small numbers)
3. Drop by one order of integration > comes with significant speedup, therefore nice for parametric studies, wrapping up in probabilistic framework or sensitivity analysis
4. Still room for applications in many other contexts
   - 3rd order Non-Gaussian MDOF to real structures (real finite element code)
   - 4th order Non-Gaussian MDOF (to be completely developed)
   - Transient loads (e.g. evolutionary spectral analysis)
   - Time-varying systems
5. Implementation in your own finite element code

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