12th ECCOMAS Thematic Conference on Multibody Dynamics
July 13–18, 2025, Innsbruck, Austria

A Tutorial on the Invariant Filtering Framework

Sven Goffin[1], Axel Barrau[2], Silvère Bonnabel[3], Olivier Brüls[4], Pierre Sacré[1]

[1]Montefiore Institute
University of Liège, Belgium
[sven.goffin, p.sacre]@uliege.be

[2]OFFROAD
France
axel@offroad.works

[3]Center for Robotics
Mines Paris – PSL, France
silvere.bonnabel@mines-paristech.fr

[4]Aerospace and Mech. Engineering
University of Liège, Belgium
o.bruls@uliege.be

EXTENDED ABSTRACT

1 Introduction

Filtering is the process of estimating the state of a dynamic system from noisy measurements by combining observed data with a mathematical model of the system. Developed during the Apollo program in the 1960s, the Extended Kalman Filter (EKF) remains the most widely used filter for industrial nonlinear filtering problems today [1]. It extends the linear Kalman filter methodology to linearized system equations. While being straightforward, this approach does not leverage the geometric structure or symmetries inherent in many systems. This limitation has led to the development of geometric filtering methods. Among these methods, the Invariant Extended Kalman Filter (IEKF) has gained significant popularity in the past two decades, with notable industrial successes [2]. The IEKF offers improved convergence properties for systems that are group affine and/or involve observations expressed in invariant form—characteristics commonly found in attitude and pose estimation tasks [3, 4].

In mechanical systems, the state is often constrained within a specific subspace of the state space. For example, a pendulum consisting of a rigid bar of length L is constrained to move along a circular arc of radius L centered on its pivot point. Incorporating such constraints into a probabilistic filtering framework remains a challenging task. For nonlinear constraints, a common approach involves (re)projecting the estimated state onto the constrained subspace [5]. However, this method introduces some arbitrariness in how the projection is performed and offers no guarantees of consistency with the underlying estimation problem.

This work provides a comprehensive introduction to the invariant filtering framework. To make the concepts more accessible, we focus on the concrete problem of estimating the extended pose—including orientation, velocity, and position—of an inertial measurement unit (IMU) mounted on a pendulum. Alongside presenting the general theory of the IEKF, we propose a systematic method to effectively integrate kinematic constraints within the framework.

2 About the Invariant Kalman Framework

Let G be a matrix Lie group $G \subset M_N(\mathbb{R})$ of dimension n. The invariant framework is designed to estimate the state $\boldsymbol{\chi}_k \in G$ of nonlinear dynamical systems of the following form:

$$\boldsymbol{\chi}_{k+1} = \mathbf{f}(\boldsymbol{\chi}_k, \mathbf{u}_k)\exp_G(\mathbf{w}_k), \tag{1a}$$

$$\mathbf{y}_k = \boldsymbol{\chi}_k \mathbf{d}_k + \mathbf{n}_k, \tag{1b}$$

where $\mathbf{u}_k \in \mathbb{R}^b$ is the system input, $\mathbf{d}_k \in \mathbb{R}^N$, $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$ and $\mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{N}_k)$ are respectively the process and measurement noise, $\mathbf{f}$ is a nonlinear function, and $\exp_G(\cdot)$ is the exponential map of group G. The IEKF proceeds recursively through two main steps: the propagation, that propagates the state estimate through the system dynamics (1a), and the update, that refines it using incoming measurements (1b). Depending on whether group composition involves left or right multiplication, the IEKF can be formulated using either the left or right formalism. For conciseness, we focus on the left formalism.

In the left-invariant formalism, the state uncertainty is modeled as a concentrated Gaussian distribution on G [6]:

$$\boldsymbol{\chi}_k = \hat{\boldsymbol{\chi}}_k \exp_G(\boldsymbol{\xi}_k), \quad \text{with } \boldsymbol{\xi}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_k), \tag{2}$$

where $\hat{\boldsymbol{\chi}}_k$ is the state estimate and $\mathbf{P}_k$ is the covariance of the linearized error $\boldsymbol{\xi} \in \mathbb{R}^n$. The pair $(\hat{\boldsymbol{\chi}}, \mathbf{P})$ represents the filter estimate.

If the function $\mathbf{f}$ in (1a) is group affine, as defined in Equation (11) of [2], its Jacobian $\mathbf{F}_k$ with respect to the linearized error is entirely independent of the current state estimate and wholly encodes the nonlinear nature of $\mathbf{f}$ in the absence of process noise, as if the dynamics were linear. This property is called log-linearity and holds for a wide range of problems in attitude and pose estimation [3]. A similar property holds for measurements of the form (1b). Indeed, when a new measurement is received, the IEKF updates its estimate using the innovation:

$$\mathbf{z}_k = \hat{\boldsymbol{\chi}}_k^{-1}\mathbf{y}_k - \mathbf{d}_k = \exp_G(\boldsymbol{\xi}_k)\mathbf{d}_k - \mathbf{d}_k = \mathbf{H}_k \boldsymbol{\xi}_k + \mathcal{O}(\|\boldsymbol{\xi}_k\|^2), \tag{3}$$

where the output Jacobian $\mathbf{H}_k$ is entirely independent of $\hat{\boldsymbol{\chi}}_k$. This stands in sharp contrast to the EKF framework, where $\mathbf{H}_k$ depends on the current state estimate. These key properties grant the IEKF its strong convergence characteristics [2].

3 Handling Kinematic Constraints within the Invariant Filtering Framework

Consider the illustrative task of estimating the extended pose of a pendulum equipped with an IMU. The system consists of a mass m attached to a rigid bar of length L, which swings around a spherical joint. An inertial frame, denoted as F_I, is placed at the pendulum's joint, while the frame attached to the IMU is referred to as F_s. The extended pose of the IMU is defined as

$$\boldsymbol{\chi}_k = \begin{bmatrix} \mathbf{R}_k & \mathbf{v}_k & \mathbf{p}_k \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \in SE_2(3), \tag{4}$$

where $\mathbf{R}_k$ is the rotation matrix from the IMU frame F_s to the inertial frame F_I, and $\mathbf{v}_k, \mathbf{p}_k \in \mathbb{R}^3$ represent the IMU's velocity and position vectors in F_I, respectively. The notation $SE_2(3)$ refers to the matrix Lie group of extended poses [3]. Assuming no prior knowledge about the motion and neglecting biases, the general dynamics of $\boldsymbol{\chi}_k$ follows a group affine structure and writes

$$\mathbf{R}_{k+1} = \mathbf{R}_k \exp_G((\boldsymbol{\omega}_k + \mathbf{w}_k^{\boldsymbol{\omega}})\, dt), \qquad \mathbf{v}_{k+1} = \mathbf{v}_k + (\mathbf{R}_k(\mathbf{a}_k + \mathbf{w}_k^{\mathbf{a}}) + \mathbf{g})\, dt, \qquad \mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{v}_k\, dt, \tag{5}$$

where the angular velocity $\boldsymbol{\omega}_k$ and linear acceleration $\mathbf{a}_k$ measured by the IMU are treated as inputs to the dynamical system.

The spherical joint imposes translational constraints on the motion of the pendulum's suspension point. Mathematically, this constraint is represented as

$$\mathbf{p}_k + \mathbf{R}_k \mathbf{a} = \mathbf{0}, \tag{6}$$

where $\mathbf{a}$ is the known position of the spherical joint in frame F_s. Expression (6) provides exact information about the IMU's pose. We advocate this constraint can be treated as a noise-free pseudo-measurement in the invariant framework. Specifically, it can be reformulated as

$$\boldsymbol{\chi}_k \begin{bmatrix} \mathbf{a} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \tag{7}$$

which has the same invariant form as measurement (1b) when the noise is turned off. This formulation ensures the independence of the output Jacobian $\mathbf{H}_k$, allowing the kinematic constraint to be incorporated during the update step while preserving the filter's convergence properties.

Incorporating exact information as noise-free pseudo-measurements introduces several challenges in the Kalman methodology, such as instabilities in the Kalman gain computation and inconsistencies in the update, as discussed in [7]. To address these issues, we propose an enhancement of the IEKF update inspired by the Gauss-Newton algorithm, resulting in the Iterated IEKF. This iterative algorithm ensures that the estimated state's probability distribution remains entirely within the constrained state space, thereby excluding all states that violate the imposed constraints.

4 Conclusion and Perspective

We demonstrated how to effectively incorporate kinematic constraints from a spherical joint into the invariant framework. By using a pertinent model for the extended pose of a multibody system, we believe that the invariant framework can be generalized to estimate the poses of such systems. In this context, the kinematic constraints that connect the individual body parts can be treated as pseudo-measurements. This promising direction is currently the focus of ongoing research in our lab.

Acknowledgments

Sven Goffin is a FRIA grantee of the Fonds de la Recherche Scientifique - FNRS.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Transactions of the ASME–Journal of Basic Engineering*, vol. 82, no. Series D, pp. 35–45, 1960.

[2] A. Barrau and S. Bonnabel, "Invariant Kalman filtering," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 1, no. 1, pp. 237–257, 2018.

[3] ——, "The geometry of navigation problems," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, pp. 689–704, 2023.

[4] R. Hartley, M. Ghaffari, R. M. Eustice, and J. W. Grizzle, "Contact-aided invariant extended Kalman filtering for robot state estimation," *The International Journal of Robotics Research*, vol. 39, no. 4, pp. 402–430, 2020.

[5] D. Simon and T. L. Chia, "Kalman filtering with state equality constraints," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 38, no. 1, pp. 128–136, 2002.

[6] G. S. Chirikjian, *Stochastic Models, Information Theory, and Lie Groups, Volume 1: Classical Results and Geometric Methods*. Springer Science & Business Media, 2009.

[7] S. Goffin, S. Bonnabel, O. Brüls, and P. Sacré, "Invariant Kalman filtering with noise-free pseudo-measurements," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 8665–8671.