Spot-fare inspection in urban bus transportation systems: strategy and unpredictability under a Stackelberg game approach

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Abstract

This study addresses the operational implementation of a spot-fare inspection strategy on a proof-of-payment urban bus transportation system, where opportunistic passengers can evade fare payment by the most convenient path. The spot-fare inspection strategy defines the frequency at which the transit authority should control sites of the transportation network to inhibit the action of opportunistic passengers. The operational implementation is done using an unpredictable allocation schedule, where the transit authority selects an allocation schedule of $\boldsymbol{n}$ sites to be controlled (one for each inspection team) each day with some probability. The challenge is to determine the set of allocation schedules and their respective probabilities of being selected whose systematic day-to-day application matches the inspection frequencies defined by the spot strategy. The interaction between transit authority and opportunistic passengers is modeled as a Leader-Follower Stackelberg game, where the decision of opportunistic passengers to evade the fare payment and the path to take depends on the passengers' observations on the inspection frequencies set by the transit authority. We consider that the transit authority implements a vehicle selective inspection policy and an on board passengers mass inspection policy, with and without interruption of the bus schedule, representing two real approaches to fare inspection.

Keywords: Urban bus system; fare evasion; spot strategy; unpredictable allocation schedule; Stackelberg game.

1 Introduction

Fare evasion has a significant financial impact on proof-of-payment (POP) transportation systems without barriers (Cantillo et al, 2022), denoted as POP_S systems (Barabino et al, 2020). In these public transportation systems, passengers purchase a ticket before boarding the vehicles; however, they are not automatically controlled and may opportunistically decide to evade the fare (Barabino et al, 2023). When passengers evade fare payment, operating companies experience economic losses that affect their revenues. This fare evasion can accumulate over time, resulting in a negative impact on the financial sustainability of the transportation system (Barabino and Salis, 2023) and representing an obstacle to its adequate expansion, operation, and service provision (Guzman et al, 2021). Furthermore, fare evasion causes other externalities, such as a negative impact on the perception of safety and service quality (Reddy et al, 2011).

To counter fare evasion in POP_S systems, the transit authority implements several methods, with passenger inspections and fine enforcement being the most relevant ones (Barabino et al, 2024, 2023; Board et al, 2022; Multisystems et al, 2002). However, not all evaders in a public transportation system react in the same way to inspections and fine enforcement. For instance, *chronic fare evaders* (Barabino et al, 2024) or *recidivist evaders* (Currie and Delbosc, 2021) do not pay the ticket or fines for economic or ideological reasons and are insensitive to the inspection level or fines defined by the transit authority. This group of evaders is not numerous (Barabino and Salis, 2023; Currie and Delbosc, 2021; Salis et al, 2017). On the other hand, *opportunistic evaders* or *calculated risk-related evaders* choose to evade based on the risk of being inspected and caught. Thus, this group of evaders is the most sensitive to the inspection system defined by the transit authority and largely explains the negative correlations between fare evasion and fare inspection (the larger the inspection, the lesser fare evasion) found by Dauby and Kovacs (2007) and Currie and Delbosc (2021). However, other research, which is mostly practitioner-oriented, finds contrasting results (Board et al, 2022; Multisystems et al, 2002; Clarke et al, 2010).

Passenger inspection in a POP_S system is performed by inspection teams and is defined by the *deployment policy* (spot or patrolling), the *location policy* (on board or in-station), and the *inspection policy* (mass or selective) whose combination results in different *fare inspection systems* (Escalona et al, 2023). However, given a fare inspection system, the key characteristic of passenger inspections should be their unpredictability from the passengers' perspective because predictable inspections can be exploited by potential evaders for their benefit.

A spot deployment policy considers that each inspection team is allocated to a specific site of the transportation network where they remain verifying fare payment during a given time interval, e.g., peak hours. This approach contrasts with the patrolling deployment policy where each inspection team follows a time-space path verifying fare payment at different sites of the transportation network during their workday. The location policy defines whether fare payment verification is performed on board the vehicles or in-station when passengers alight or board the vehicles. The mass inspection policy considers that fare payment verification is applied to all passengers and/or vehicles at the site controlled by the transit authority. On the contrary, the selective inspection policy considers that the inspectors randomly select passengers and/or vehicles to be controlled. It should be noted that the design of a selective passenger inspection policy is a complex issue in terms of equity when inspections are socially biased towards certain groups (Barabino et al, 2024).

The strategy followed by opportunistic passengers to evade fare payment depends on the fare inspection system and the risk of being inspected, where the risk is commonly measured by the inspection probability. For instance, opportunistic passengers choose the path with the lowest inspection probability in response to a fare inspection system based on an on board location policy, or choose to alight or board the vehicle at stations with low or null inspection probability in response to a fare inspection system based on an in-station location policy. Thus, determining the distribution of inspection probabilities over the transportation network that inhibits the action of the largest number of opportunistic passengers under a given fare inspection system is a relevant issue for the transit authority seeking to reduce the evasion rate. However, the distribution of inspection probabilities is only useful to the transit authority if there is a mechanism for its practical implementation.

The problem of determining the probability distribution of fare inspections over a transportation network under a fare inspection system based on a spot or patrolling deployment policy, denoted as *fare inspection strategy* according to Correa et al (2017), is addressed in the literature mainly as a Leader-Follower Stackelberg game (Barabino et al, 2020). In this game, the transit authority (*the leader*) sets the inspection probabilities over the transportation network, and opportunistic passengers (*the followers*) decide their evasion strategy according to their knowledge of the fare inspection system, the risk of being inspected, and travel times. Thus, opportunistic passengers act strategically when they observe and learn the inspection pattern of the transit authority. Two variants can be distinguished for the evasion strategy followed by opportunistic passengers (Correa et al, 2017; Bahamondes et al, 2017). In the first one, denoted as *non-adaptive strategy*, opportunistic passengers choose a priori the most convenient path (station to alight or board the vehicle) from a finite set of alternatives under a fare inspection system based on an on board (in-station) location policy. In the second one, denoted as *adaptive strategy*, opportunistic passengers obtain information along the way and use it to update their path (station to alight or board the vehicle) under a fare inspection system based on an on board (in-station) location policy.

Two types of fare inspection strategies can be distinguished. The *spot-fare inspection strategy* considers the spatial inspection probability distribution over the transportation network under a spot deployment policy. The *fare inspection patrolling strategy* aims to determine the temporal-spatial inspection probability distribution over the transportation network under a patrolling deployment policy. The first one can be interpreted as the frequency with which inspection teams must be allocated to network sites during a given time interval, while the second one can be interpreted as the frequency with which the network sites should be patrolled.

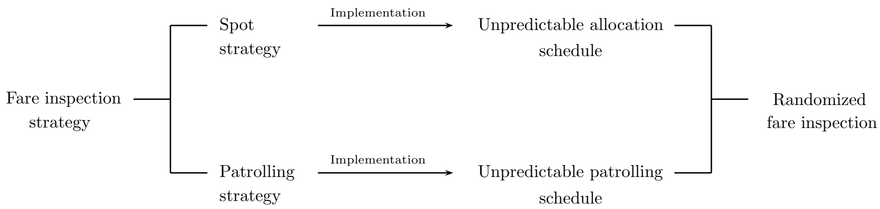


Fig. 1: Fare inspection strategy implementation using randomized fare inspection

The operational implementation of a fare inspection strategy, summarized in Figure 1, can be done through a randomized fare inspection. More precisely, a spot-fare inspection strategy can be operationally implemented through a set of allocation schedules, where each allocation schedule is a collection of sites to be controlled (one for each fare inspection team) with an associated probability of being selected, denoted as *unpredictable allocation schedule* of inspection teams. Similarly, a fare inspection patrolling strategy can be operationally implemented through a set of patrolling schedules, where each schedule is a collection of temporal-spatial patrol paths (one for each fare inspection team) with an associated probability of being selected, denoted as *unpredictable patrolling schedule*. Thus, the transit authority can select each day an allocation schedule (patrolling schedule) with a probability, avoiding any regularity that could be exploited by opportunistic passengers. The challenge is to determine the unpredictable allocation schedule (unpredictable patrolling schedule) whose systematic day-to-day application matches in the medium term the inspection probabilities defined by the spot strategy (patrolling strategy).

Large-scale urban bus POP_S systems are particularly vulnerable to fare evasion due to the number of bus lines, the number of bus stops, the diversity of service routes, the extension of the bus transportation networks, and the limited fare inspection resources. For instance, in the Transmilenio bus system in Bogotá, Colombia, evasion reached 15% in 2023 (DTS, 2024), and in the bus system of Santiago, Chile, evasion reaches 42.8% in 2023 (DTPM, 2024). On the other hand, even if the evasion rate in developed countries is usually less than 5%, this rate may be underestimated, as mentioned by Ramos and Silva (2023). For instance, the Metropolitan Transportation Authority of New York, USA, reported for the first quarter of 2023 a 41% evasion rate in the bus system (MTA, 2023), and Egu and Bonnel (2020) studied evasion in Lyon, France, and found that the evasion rate reached 33% in the city's bus system.

This paper studies the design of a spot-fare inspection strategy and their practical implementation in public bus transportation networks using an unpredictable allocation schedule. We formulate the spot strategy as a Leader-Follower Stackelberg game. The transit authority determines the spatial distribution of inspection probabilities over the transit network under a spot-fare inspection system, with selective inspection of buses and a mass inspection of passengers on board the bus. Opportunistic passengers respond by evaluating a finite set of alternative paths and choosing the most convenient for them. This path-based approach to defining the spot strategy leads to a nonlinear optimization problem (NLP) for which a relaxation-based heuristic is proposed. Then, we present an equivalent formulation for the unpredictable allocation schedule, which ensures that for each spot strategy there is a corresponding unpredictable allocation schedule. The unpredictable allocation schedule is determined using column generation, where the pricing problem generates allocation schedules and the master problem determines the probability of being selected. Two variants of the spot-fare inspection system, with selective inspection of buses and mass inspection of passengers on board are explored. The first one considers that passengers are inspected while the bus is stopped, representing the system used by the transit authority in Santiago, Chile (Delgado et al, 2018). The second one considers the inspection of passengers with the bus in motion, which does not alter the bus schedule and is the most widely used.

The main contributions of this study are summarized as follows. (1) To the best of our knowledge, it is the first time that a spot-fare inspection strategy and their operational implementation in public transportation networks are addressed jointly under a Stackelberg game approach. (2) A mechanism to operationally implement a spot-fare inspection strategy is proposed. (3) An efficient approach to generate an unpredictable allocation schedule that takes advantage of the spot inspection strategy is defined. (4) A mechanism for evaluating the time it takes for a spot-fare inspection strategy to be operationally implemented is provided.

The remainder of this paper is structured as follows. A review of related work is discussed in Section 2. In Section 3, we formulate the spot-fare inspection strategy as a Leader-Follower Stackelberg game. In Section 4, we derive bounds for the spot-fare inspection strategy. Section 5 presents the unpredictable allocation schedule formulation. In Section 6, a brief discussion of the practical implementation of the spot-fare inspection strategy is presented. Computational results are reported in Section 7. Finally, conclusions and suggested future extensions are described in Section 8.

2 Related works

The spot-fare inspection strategy and the unpredictable allocation schedule in public transportation networks under an adaptive or non-adaptive path-finding strategy are particular topics of network security games. In a network security game, a defender can check a limited number of edges in a graph and the attacker computes from a given set of source and target nodes on a path connecting a source and a target (Correa et al, 2017). Network security games (Casorrán et al, 2019) are used to great effect in applications such as airport security (Pita et al, 2008), maritime security (Shieh et al, 2012), border security (Lessin et al, 2019), wildlife poaching (Fang et al, 2016), mitigating pyro-terrorism attacks and natural wildfires (Rashidi et al, 2018), fare inspection patrolling (Escalona et al, 2023; Brotcorne et al, 2021; Jiang et al, 2013; Yin et al, 2012; Jiang et al, 2012), minimizing the misinformation spread in social networks (Tanınmış et al, 2020), drunk drivers' interdiction (Jie et al, 2020), and drug traffic interdiction (Washburn and Wood, 1995).

Different from other network security games, the spot-fare inspection strategy and the unpredictable allocation schedule consider multiple attackers, each with multiple strategies to evade fare payment. Indeed, in a public transportation network there are as many attackers as origin-destination pairs, and the number of strategies an attacker has is equal to the number of alternative paths between its origin and destination. In Table 1, we classify the papers related to spot strategy and unpredictable allocation schedule according to: (i) the problem addressed (spot strategy or unpredictable schedule), (ii) path-finding strategy (adaptive or non-adaptive), (iii) inspection policy (mass or selective), and (iv) application (Toll on Motorways, Metro System, Bus System, or General).

Borndörfer et al (2012) studied the optimal spot inspection strategy and the unpredictable inspectors allocation schedule to enforce the toll payment on transportation networks (highways) under a Stackelberg game approach. They adopt a directed graph as a spatial network type where inspection activity occurs on the edges. The path that a driver follows is predefined a priori. Using an upper bound for the inspection probabilities, they formulated two spot inspection strategies as a single-level linear problem (LP) and mixed integer linear problem (MIP) when the leader maximizes the revenue generated by the toll and minimizes the number of evaders, respectively. On the other hand, the unpredictable inspector's allocation is formulated as an LP and solved using a column generation approach. The pricing problem, which generates spatial inspector allocations, is formulated as a NLP and solved by

Table 1: Studies on spot strategy and unpredictable allocation schedule

	Problem		Path-finding		Inspection		Application			
Author	Strategy	Schedule	Adaptive	Non-adaptive	Mass	Selective	TM	MS	BS	G
Borndörfer et al (2012)	X	X		X		X	X			
Borndörfer et al (2013)	X			X		X	X			
Borndörfer et al (2015)	X			X		X	X			
Correa et al (2017)	X		X	X	X			X		
Bahamondes et al (2017)	X		X	X	X					X
Holzmann and Smith (2021)	X		X	X	X					X
This work	X	X		X	X	X			X	

TM: Toll on Motorways, MS: Metro system, BS: Bus system, G: General a greedy heuristic. The formulation of the spot inspection strategy and the unpredictable inspectors allocation are independent. Borndörfer et al (2012) assume a selective inspection policy because the probability of inspecting a driver on an edge is equal to the fraction of controlled drivers on that edge. In contrast, in our paper the followers select the most convenient path from a finite set of alternatives, we address jointly the spot strategy and the unpredictable allocation schedule, and we implement a selective inspection policy of buses and an on board passengers mass inspection policy.

Borndörfer et al (2013) extend the spot inspection strategy presented in Borndörfer et al (2012) by considering that each driver is free to choose his path in the network to reach his destination. They simplify the followers' problem to a standard shortest path problem, where drivers choose the path that minimizes the sum of the expected travel cost and fines. Based on this simplification, they formulate the spot inspection strategy as a single-level MIP. Borndörfer et al (2015) generalize the spot inspection strategy of Borndörfer et al (2013) to transportation networks with fixed and distance-based tolls, respectively.

Correa et al (2017) address the spot-fare inspection strategy in public transportation networks under adaptive and non-adaptive passengers using a Stackelberg game approach. They adopt a directed graph to represent the spatial transit network, where nodes are stations and edges represent trips between stations. The spot-fare inspection strategy considering non-adaptive passengers is formulated as a single-level mixed integer nonlinear (path-based) problem (MINLP). The spot-fare inspection strategy considering adaptive opportunistic passengers is formulated as single-level MINLP taking advantage of the primal-dual relation of the follower's problem. They present a general local search framework, computing near-optimal solutions for both path-finding strategies. Correa et al (2017) assume a massive inspection policy because the probability of inspecting an opportunistic passenger at an edge of the network is equal to the probability that the transit authority controls that edge. We study a problem similar to the non-adaptive passengers variant of Correa et al (2017). However, our spot-fare inspection strategy is an NLP, and we move towards an unpredictable allocation schedule of inspection teams. Bahamondes et al (2017) study the same problem as Correa et al (2017) as a network interdiction problem under Stackelberg game approach. They consider that the checkpoint can only be located on a single edge, leading to a different optimization problem for opportunistic passengers. The spot strategy is formulated as a single-level path-based LP and solved using metaheuristics. In contrast, we consider that fare inspections can be performed simultaneously at several sites in the transit network.

Holzmann and Smith (2021) consider the work of Borndörfer et al (2013), Correa et al (2017), and Bahamondes et al (2017) as stochastic interdiction actions that focus on the enforcement of fare evasion deterrence in public transportation networks. Holzmann and Smith (2021) study the *stochastic* *shortest-path interdiction problem* where the leader selects a policy of random interdiction actions, and the follower only knows the probability of where interdictions are deployed in the network and identifies a path that has the minimum expected cost. They generalize the work of Borndörfer et al (2013), Correa et al (2017), and Bahamondes et al (2017) by considering that the leader's objective function could be either concave or convex.

In summary, no previous work has jointly addressed the design of a spot-fare inspection strategy and an unpredictable allocation schedule that ensures that for any spot strategy there is a corresponding unpredictable allocation schedule in an interconnected bus network. Furthermore, no previous work considers that opportunistic passengers choose the most convenient path in response to a policy of selective bus inspection and mass on board passenger inspection.

In this paper we address the design of a fare inspection strategy and its operational implementation in a public transportation network. However, it should be noted that there are other research lines to address passenger inspections and fine enforcement. For example, one research line is to determine the optimal level of inspection in a proof-of-payment public transportation system, i.e., the optimal number of inspectors, in order to maximize the network operator's profit. This research line is initiated by Boyd et al (1989) and exploited by Kooreman (1993), Barabino et al (2013), Barabino et al (2014), and Barabino and Salis (2019). A comprehensive review of fare inspection in proof-of payment transit networks can be found in Barabino et al (2024).

3 Spot-fare inspection strategy formulation

Let us consider an interconnected urban public bus transportation system with L $(l = 1, ..., |L|)$ bus lines. The streets where buses travel and the stations where passengers board and alight are used by one or more bus lines. In this bus network, passengers behave according to two profiles: honest (always purchasing a ticket) and opportunist (assessing the specific situation before buying a ticket). To prevent fare evasion during a given time interval Π (e.g., 4-6 pm), the transit authority is committed to a spot-fare inspection system with selective inspection of buses and a mass inspection of passengers on board. Opportunistic passengers react by evading fare payment by using the most convenient path for them, which may involve the use of different combinations of bus lines. The notations used in this section are summarized in Appendix A.

The bus network is represented by an undirected graph $G = (V, E)$, where V $(v = 1, ..., |V|)$ is the set of nodes representing bus stops and E $(e = 1, ..., |E|)$ is the set of edges representing the connection between bus stops. Nodes and edges can be used by one or more bus lines. Let $\mathcal{E}_l$ be the set of edges followed by bus line l in G. An illustrative example of G is shown in Figure 2 where six bus lines operate in the network.

For instance, in Figure 2 bus line 1 $(l = 1)$ travels from $v = 1$ to $v = 6$ using edges e_1, e_5, and e_6, with stations $v = 1, 2, 7$, and 6. Furthermore, e_1 is

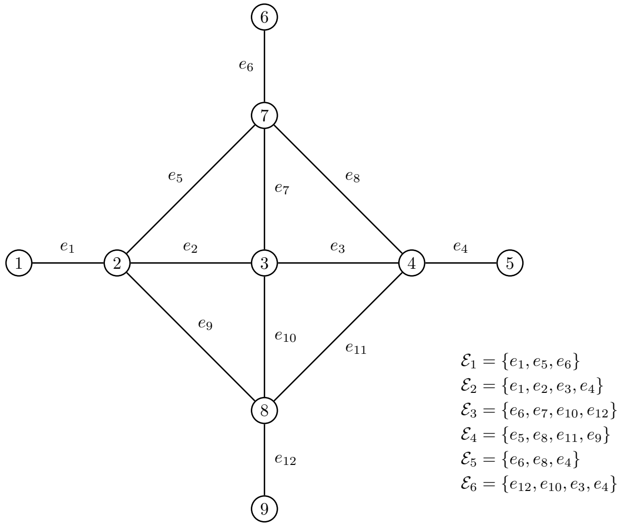


Fig. 2: Illustrative example of G

used by bus lines 1 and 2, e_5 is used by bus lines 1 and 4, and e_6 is used by bus lines 1, 3, and 5. We assume that the route of each bus line is a round trip by the same nodes and edges.

Each day the transit authority allocates inspection resources to simultaneously control sites during Π. Let X $(x = 1, ..., |X|)$ be the set of sites in G that can be inspected by the transit authority. Two mutually exclusive spot-fare inspection systems with selective inspection of buses and a mass inspection of passengers on board are considered by the transit authority:

- The first one considers inspections at the edges of G, where buses are randomly stopped by the transit authority between stations and passengers subject to mass inspection on board of the vehicle. The bus controlled by the authority continues on its route after the inspection. Thus, $X = E$ $(x = e)$ under edge inspection.
- The second one considers that inspection resources are allocated to control *districts*, where inspection teams randomly board buses and check all passengers on board with the bus in motion without disrupting the bus schedule. Let D $(d = 1, ..., |D|)$ be the set of districts. Thus, $X = D$ $(x = d)$ under district inspections.

We define a district for each $v \in V$ as a subgraph of G. More precisely, let $\mathcal{G}(v) \subseteq G$ be the subgraph defined by v that contains all nodes and edges of G such that the travel time of an inspection team boarding a bus at v and alighting at v' is less than or equal to τ_d time units for any $l \in \mathcal{N}_v$, where $\mathcal{N}_v$ is the set of bus lines that use v. Thus, we define as many districts as the number of nodes in the transportation network, i.e., $d = v$ and $|D| = |V|$. For instance, consider the district defined by $v = 1$ in Figure 2, which is used by the $l = 1$ and $l = 2$ bus lines. Let τ_e^l be the bus travel time of bus line l in e. If it is satisfied that $\tau_{e_1}^{l_1} + \tau_{e_5}^{l_1} \leq \tau_d$ and $\tau_{e_1}^{l_1} + \tau_{e_5}^{l_1} + \tau_{e_6}^{l_1} > \tau_d$ for bus line $l = 1$, and $\tau_{e_1}^{l_2} + \tau_{e_2}^{l_2} \leq \tau_d$ and $\tau_{e_1}^{l_2} + \tau_{e_2}^{l_2} + \tau_{e_3}^{l_2} > \tau_d$ for bus line $l = 2$, then the district associated with $v = 1$ is defined by the subgraph $\mathcal{G}(v = 1)$ in Figure 3.

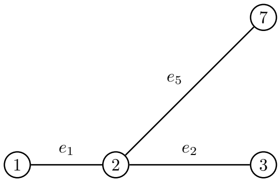


Fig. 3: Illustrative example of $\mathcal{G}(v = 1)$

Passengers are defined by their origin-destination. Thus, a type $p \in \mathcal{P}$ of passengers is defined as a set of $d_p \geq 0$ passengers with origin $v_p^+ \in V$ and destination $v_p^- \in V$ during Π. Let the discrete random variable d_p^o be the number of opportunistic passengers of type p during Π. We consider that d_p^o follows a discrete uniform distribution, i.e., $d_p^o \sim \mathcal{U}\{0, ..., \hat{d}_p^o\}$ with $\hat{d}_p^o \leq d_p$ for any $p \in \mathcal{P}$.

3.1 The follower's problem

Opportunistic passengers are assumed to be rational and non-adaptive, meaning that they make their decision about buying a ticket or not, and the path to take, based solely on the expected cost of their trip which depends on the probability of being inspected. More precisely, opportunistic passengers of type p can either buy a ticket at price B and take the shortest path (with respect to travel time) or choose a path without paying for the ticket but with the risk of incurring a fine F ($F >> B$) if they are caught. For each opportunistic passenger of type $p \in \mathcal{P}$, let $\mathcal{N}_p$ be the set of k-shortest paths from v_p^+ to v_p^- (with respect to travel time). In particular, let the first element of $\mathcal{N}_p$ be the shortest path through p.

Let W_p^o be equal to 1 if an opportunistic passenger of type p purchases a ticket, and 0 otherwise. Let H_p^i be equal to 1 if an opportunistic passenger of type p takes the path $i \in \mathcal{N}_p$, and 0 otherwise. Furthermore, let the random variable μ_p^o be the fee paid by the opportunistic passenger of type p. Thus, $\mu_p^o = BW_p^o + F(1 - W_p^o)\sum_{i\in\mathcal{N}_p} \mathbb{1}_p^i H_p^i$ with $\sum_{i\in\mathcal{N}_p} H_p^i = 1$, where $\mathbb{1}_p^i$ is an indicator function equal to 1 if an opportunistic passenger of type p is inspected on path $i \in \mathcal{N}_p$, and 0 otherwise. Let U_p^o be the expected value of the amount paid by an opportunistic passenger of type p, i.e., $U_p^o = \mathbb{E}(\mu_p^o) = B\, W_p^o + F(1 - W_p^o)\sum_{i\in\mathcal{N}_p} \mathbb{P}_p^i H_k^i$, where $\mathbb{P}_p^i = \mathbb{E}(\mathbb{1}_p^i)$ is the probability that an opportunistic passenger of type p is inspected in the path $i \in \mathcal{N}_p$. Thus, the optimization problem for an opportunistic passenger of type p is formulated as follows:

$$\text{FMP}(p): \quad \min_{W_p^o, \mathbf{H}_p} \quad B\ W_p^o + F\ (1 - W_p^o) \sum_{i \in \mathcal{N}_p} \mathbb{P}_p^i H_p^i \tag{1}$$

$$\text{s.t:} \quad \sum_{i \in \mathcal{N}_p} H_p^i = 1 \tag{2}$$

$$W_p^o \in \{0, 1\} \tag{3}$$

$$H_p^i \in \{0, 1\} \qquad \forall i \in \mathcal{N}_p, \tag{4}$$

where constraint (2) ensures that an opportunistic passenger of type p can only choose one path between v_p^+ and v_p^-. The FMP(p) model is a separable integer nonlinear problem (INLP) because there are no linking constraints (no capacity constraint). Thus, the expected value of the amount paid by an opportunistic passenger of type p is:

$$U_p^o = \min \left\{ B, F \min_{i \in \mathcal{N}_p} \{\mathbb{P}_p^i\} \right\} \qquad \forall p \in \mathcal{P}. \tag{5}$$

From (5) it is easy to infer that the opportunistic passenger's decision depends on the risk of being inspected, i.e., $\min_{i \in \mathcal{N}_p}\{\mathbb{P}_p^i\}$, and the threshold $c = B/F$. Indeed, if $\min_{i \in \mathcal{N}_p}\{\mathbb{P}_p^i\} < c$, then an opportunistic passenger of type p does not buy the ticket ($W_p^o = 0$) and takes the path $i = \arg\min_{i \in \mathcal{N}_p}\{\mathbb{P}_p^i\}$. In contrast, if $\min_{i \in \mathcal{N}_p}\{\mathbb{P}_p^i\} \geq c$, then an opportunistic passenger of type p buys the ticket ($W_p^o = 1$) and takes the shortest path in terms of travel time ($i = 0$).

3.2 The leader's problem

The spot-fare inspection strategy consists of determining the inspection probability distribution on G such that the action of opportunistic passengers is inhibited. Let n be the number of fare inspection teams and let $\mathbb{1}_x$ be an indicator function equal to 1 if site x is controlled during Π, and 0 otherwise. Under the spot-fare inspection, the transit authority simultaneously deploys all inspection resources on G during Π. Then, $\sum_{x \in X} \mathbb{1}_x \leq n$, as the transit authority cannot simultaneously control more than n sites during Π. Taking expected values, we obtain the following constraint:

$$\sum_{x \in X} \mathbb{P}_x \leq n, \tag{6}$$

where $\mathbb{P}_x = \mathbb{E}(\mathbb{1}_x)$ is the probability that the transit authority controls x during Π.

Let $\mathcal{X}_p^i$ be the set of sites and bus line pairs that an opportunistic passenger of type p follows if he takes the path $i \in \mathcal{N}_p$, i.e., $\mathcal{X}_p^i = \{(x, l) :$ p goes through x on bus line l when it takes path $i\}$ for any $i \in \mathcal{N}_p$ and $p \in \mathcal{P}$. Let $\mathbb{P}_{xl}^p$ be the probability that an opportunistic passenger of type p is inspected at $(x,l) \in \mathcal{X}_p^i$. Thus, the probability that an opportunistic passenger of type p is inspected on path $i \in \mathcal{N}_p$ is defined as follows:

$$\mathbb{P}_p^i = 1 - \prod_{(x,l)\in\mathcal{X}_p^i} (1 - \mathbb{P}_{xl}^p) \qquad \forall p \in \mathcal{P}, i \in \mathcal{N}_p, \tag{7}$$

since an opportunistic passenger of type p, on path $i \in \mathcal{N}_p$, is not inspected if the transit authority does not control any $(x,l) \in \mathcal{X}_p^i$ during Π.

Let the random variable ω_{xl} be the number of buses of line l inspected on x during Π, with $(x,l) \in \mathcal{X}_p^i$ for any $i \in \mathcal{N}_p$ and $p \in \mathcal{P}$. We consider that ω_{xl} follows a hypergeometric distribution since during Π an inspection team controls r_x vehicles (without replacement) out of a total of $\sum_{l\in\mathcal{L}_x} q_{xl}$ vehicles of which q_{xl} are on line l. That is, $\omega_{xl} \sim Hypergeometric\left(\sum_{l\in\mathcal{L}_x} q_{xl}, q_{xl}, r_x\right)$, where q_{xl} is the number of buses of line l running on x, $\mathcal{L}_x$ is the set of bus lines passing through x, and r_x is the number of buses that an inspection team controls in x, all during Π. A passenger of type p is inspected at $(x,l) \in \mathcal{X}_p^i$ if the transit authority controls x and the passenger is traveling on a bus of line l at x. Assuming independence, we have:

$$\mathbb{P}_{xl}^p = \mathbb{P}_x \sum_{j=1}^{\min\{q_{xl},r_x\}} \mathbb{P}(\omega_{xl} = j)\frac{j}{q_{xl}} \qquad \forall (x,l) \in \mathcal{X}_p^i, i \in \mathcal{N}_p, p \in \mathcal{P}. \tag{8}$$

The transit authority's objective function is to maximize the expected revenue from ticket sales and fines collected from passengers. Thus, the leader's problem under a Stackelberg game approach is defined as:

$$
\begin{aligned}
x\text{-SSP}: \quad \max_{U,\mathbb{P}} \quad & \sum_{p\in\mathcal{P}} \mathbb{E}(d_p^o)\ U_p^o && && (9)\\
\text{s.t:} \quad & U_p^o \leq B && \forall p \in \mathcal{P} && (10)\\
& U_p^o \leq F\left(1 - \prod_{(x,l)\in\mathcal{X}_p^i} (1 - f_{xl}\mathbb{P}_x)\right) && \forall p \in \mathcal{P}, i \in \mathcal{N}_p && (11)\\
& \sum_{x\in X} \mathbb{P}_x \leq n && && (6)\\
& U_p^o \geq 0 && \forall p \in \mathcal{P} && (12)\\
& \mathbb{P}_x \in [0,1] && \forall x \in X, && (13)
\end{aligned}
$$

where $\mathbb{E}(d_p^o) = 0.5\hat{d}_p^o$, because d_p^o follows a discrete uniform distribution, and

$$f_{xl} = \sum_{j=1}^{\min\{q_{xl},r_x\}} \mathbb{P}(\omega_{xl} = j)\frac{j}{q_{xl}}.$$

Constraints (10) and (11) represent the optimal reaction of the followers to the transit authority decision because $U_p^o = \min\{B, F \min_{i\in\mathcal{N}_p}\{\mathbb{P}_p^i\}\}$ for any $p \in \mathcal{P}$, and the x-SSP model is a maximization optimization problem. The expected revenue of honest passengers is omitted from the objective function since it is a constant.

The difficulty in solving the x-SSP model is related to the inequality constraint (11) since $U_p^o - F\left(1 - \prod_{(x,l)\in\mathcal{X}_p^i}(1 - f_{xl}\mathbb{P}_x)\right)$ is neither convex nor concave.

It should be noted that $U_p^o = F \min_{i\in\mathcal{N}_p}\{\mathbb{P}_p^i\} < B$ when an opportunistic passenger of type p evades fare payment, and $U_p^o = B$ otherwise. Thus, the transit authority has no incentive to encourage evasion because the upper bound on the amount that an opportunistic passenger of type p is willing to pay is equal to B. The real objective of the transit authority is to inhibit the action of as many opportunistic passengers as possible by making efficient and effective use of its resources.

The optimal spot-fare inspection strategy for the transit authority, which refers to the inspection probabilities in the transportation network, is defined by $\{\mathbb{P}_x\}_{x\in X}$, where $\mathbb{P}_x$ is the optimal variable of x-SSP for any $x \in X$.

4 A relaxation-based heuristic for x-SSP model

In this section, we present a relaxation-based heuristic that provides a lower bound for the x-SSP objective function, a feasible spot-fare inspection strategy, a quality measure of the feasible spot-fare inspection strategy in terms of the optimality gap, and evasion and inspection measures induced by the feasible spot-fare inspection strategy.

The relaxation-based heuristic exploits the geometric structure of the non-linear constraint (11). Let $\mathcal{C}_p^i$ be the set of $\mathbb{P}_x$ and U_p^o that satisfy (10), (11), (12), and (13) for any $i \in \mathcal{N}_p$ and $p \in \mathcal{P}$. The set $\mathcal{C}_p^i$ is non-convex because of (11). However, the set of $\mathbb{P}_x$ and U_p^o that satisfy (10), (12), (13), and $F\left(1 - \prod_{(x,l)\in\mathcal{X}_p^i}(1 - f_{xl}\mathbb{P}_x)\right) - U_p^o \leq 0$ is convex, which defines a particular geometry for $\mathcal{C}_p^i$.

The geometry of $\mathcal{C}_p^i$ is based on the unitary hypercube defined by $0 \leq \mathbb{P}_x \leq 1$ for all $(x,l) \in \mathcal{X}_p^i$. The dimension of the hypercube is equal to $|\mathcal{X}_p^i|$. Constraint (11) cuts the hypercube by defining $\mathbf{a}_i$ vertices with $i = 1, ..., |\mathcal{X}_p^i|$, where $\mathbf{a}_i \in \mathbb{R}^{|\mathcal{X}_p^i|}$ with all its components equal to zero except the ith component equal to $\frac{U_p^o}{f_{xl}F}$. To illustrate the $\mathcal{C}_p^i$ geometry, we consider a simple example with $\mathcal{X}_p^i = \{(x',l'),(x'',l'')\}$, as shown in Figure 4.

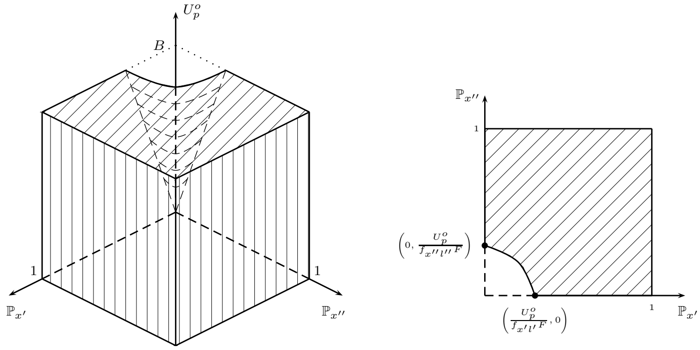


Fig. 4: Illustrative example of set $\mathcal{C}_p^i$

The left-hand side of Figure 4 shows the set of $\mathbb{P}_{x'}$, $\mathbb{P}_{x''}$, and U_p^o that satisfy (10), (11), (12), and (13), and the right-hand side shows the set of $\mathbb{P}_{x'}$ and $\mathbb{P}_{x''}$ that satisfy (11) and (13) for some $U_p^o \in [0, B]$.

We generate the convex hull of $\mathcal{C}_p^i$ by using the hyperplane defined by the vertices $a_1, ..., a_{|\mathcal{X}_p^i|}$. The hyperplane is defined as $F \sum_{(x,l)\in\mathcal{X}_p^i} f_{xl}\mathbb{P}_x = U_p^o$, and constraint (11) can be replaced by the linear constraint:

$$U_p^o \leq F \sum_{(x,l)\in\mathcal{X}_p^i} f_{xl}\mathbb{P}_x \qquad \forall p \in \mathcal{P}, i \in \mathcal{N}_p, \tag{14}$$

thereby providing an upper bound for x-SSP. The left-hand side of Figure 5 shows the convex hull for Figure 4 and the right-hand side shows the set of all $\mathbb{P}_{x'}$ and $\mathbb{P}_{x''}$, that satisfy (13) and (14) for some $U_p^o \in [0, B]$.

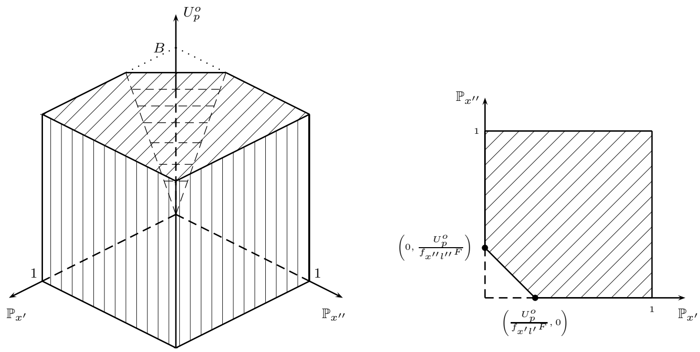


Fig. 5: Illustrative example of the convex hull of $\mathcal{C}_p^i$

Using (14), we define the following LP relaxation of x-SSP:

$$
\begin{aligned}
x\text{-}\mathcal{R}(\text{SSP}): \quad & \max_{U,\mathbb{P}} \quad \sum_{p\in\mathcal{P}} \mathbb{E}(d_p^o)\, U_p^o && && (9)\\
& \text{s.t:} \quad U_p^o \le B && \forall p \in \mathcal{P} && (10)\\
& \qquad U_p^o \le F \sum_{(x,l)\in\mathcal{X}_p^i} f_{xl}\mathbb{P}_x && \forall p\in\mathcal{P}, i\in\mathcal{N}_p && (14)\\
& \qquad \sum_{x\in X} \mathbb{P}_x \le n && && (6)\\
& \qquad U_p^o \ge 0 && \forall p\in\mathcal{P} && (12)\\
& \qquad \mathbb{P}_x \in [0,1] && \forall x \in X. && (13)
\end{aligned}
$$

Problem x-$\mathcal{R}$(SSP) is a relaxation of x-SSP since the feasible region of x-SSP is included in that of x-$\mathcal{R}$(SSP). Thus, the optimal objective function of x-$\mathcal{R}$(SSP) is an upper bound of the optimal objective function of x-SSP, i.e., $Z^*_{x\text{-SSP}} \le Z^*_{x\text{-}\mathcal{R}(\text{SSP})}$, where $Z^*_{x\text{-SSP}}$ and $Z^*_{x\text{-}\mathcal{R}(\text{SSP})}$ are the optimal objective function of x-SSP and x-$\mathcal{R}$(SSP) models, respectively.

The spot-fare inspection strategy resulting from x-$\mathcal{R}$(SSP) satisfies constraint (6) and is therefore feasible in x-SSP, i.e., $\{\mathbb{P}_x\}_{x\in X}$ is feasible in x-SSP where $\mathbb{P}_x$, for any $x \in X$, is the optimal variable of x-$\mathcal{R}$(SSP).

To define a lower bound for the optimal objective function value of the x-SSP model, we use the spot-fare inspection strategy resulting from x-$\mathcal{R}$(SSP) to determine the response of each opportunistic passenger. For each opportunistic passenger of type p, we compute the expected value they pay according to:

$$\widehat{U}_p^o = \min_{i\in\mathcal{N}_p} \left\{ B, F\left(1 - \prod_{(x,l)\in\mathcal{X}_p^i} (1 - f_{xl}\mathbb{P}_x)\right)\right\}, \tag{15}$$

where $\mathbb{P}_x$, for any $x \in X$, is the optimal variable of x-$\mathcal{R}$(SSP). We then compute the lower bound for the optimal objective function of the x-SSP model, defined as $\hat{Z}_x = \sum_{p\in\mathcal{P}} \mathbb{E}(d_p^o)\widehat{U}_p^o$.

The quality of the feasible spot-fare inspection strategy resulting from the x-$\mathcal{R}$(SSP) model is measured using the optimality gap between the upper bound given by $Z^*_{x\text{-}\mathcal{R}(\text{SSP})}$ and the lower bound given by $\hat{Z}_x$, i.e.,

$$Gap_x(\%) = 100 \times \frac{(Z^*_{x\text{-}\mathcal{R}(\text{SSP})} - \hat{Z}_x)}{\hat{Z}_x}. \tag{16}$$

The procedure to obtain a feasible spot-fare inspection strategy is summarized in Algorithm 1.

Using the feasible spot-fare inspection strategy resulting from Algorithm 1, the evasion rate and inspection rate can be computed. Following Brotcorne et al (2021) and Escalona et al (2023), the evasion rate is defined as the ratio

Algorithm 1 Feasible spot-fare inspection strategy

1: $\{\mathbb{P}_x\}_{x\in X}, Z^*_{x\text{-}\mathcal{R}(\mathrm{SSP})} \leftarrow$ solve x-$\mathcal{R}$(SSP)
2: $p = 0$
3: $\hat{Z}_x^{(0)} = 0$
4: **while** $p \leq |\mathcal{P}|$ **do**
5: $\quad \widehat{U}^o_p \leftarrow$ solve (15)
6: $\quad \hat{Z}_x^{(p)} = \hat{Z}_x^{(p-1)} + \mathbb{E}(d^o_p)\widehat{U}^o_p$
7: $\quad p = p + 1$
8: **end while**
9: $\hat{Z}_x = \hat{Z}_x^{(|\mathcal{P}|)}$
10: Return feasible spot strategy and Gap: $\{\mathbb{P}_x\}_{x\in X}, Gap_x$

between the expected number of evaders and the total number of passengers. Thus, the evasion rate during Π is:

$$ER = \frac{\sum_{p\in\mathcal{P}:\hat{U}^o_p<B}\mathbb{E}(d^o_p)}{\sum_{p\in\mathcal{P}} d_p}, \tag{17}$$

where $\hat{U}^o_p$ is computed according to (15). It should be noted that (17) is a population measure because the total number of opportunistic passengers who choose to evade in the transportation network during Π is determined using (15). This population measure diverges from the sample measures determined empirically in real transportation systems where the evasion rate is the ratio of evaders to inspected passengers (Barabino et al, 2020).

The inspection rate is defined as the ratio between the expected number of checked passengers and all passengers. Thus, the inspection rate during Π is:

$$IR = \frac{\sum_{p\in\mathcal{P}}\left\{(d_p - \mathbb{E}(d^o_p))g_0 + \mathbb{E}(d^o_p)\min_{i\in\mathcal{N}_p}\{g_i\}\right\}}{\sum_{p\in\mathcal{P}} d_p}, \tag{18}$$

where $g_0 = 1 - \prod_{(x,l)\in\mathcal{X}^0_p}(1 - f_{xl}\mathbb{P}_x)$, $g_i = 1 - \prod_{(x,l)\in\mathcal{X}^i_p}(1 - f_{xl}\mathbb{P}_x)$, and $\mathbb{P}_x$, for any $x \in X$, is the optimal variable of x-$\mathcal{R}$(SSP).

5 An equivalent unpredictable allocation schedule formulation

In this section, we first formulate the unpredictable allocation schedule as an optimization problem equivalent to the x-SSP model by assuming that the set of all allocation schedules of n inspection teams are known. Then, a relaxation of the unpredictable allocation schedule is presented as an optimization problem equivalent to x-$\mathcal{R}$(SSP).

Let $\mathcal{S}$ be the set of allocation schedules. Each allocation schedule defines the n sites of the transportation network that must be controlled simultaneously during Π. Under a Stackelberg game approach, the unpredictable allocation schedule involves determining the set of allocation schedules and their respective probabilities of being selected. Let $\pi_s \in [0,1]$ be the probability of selecting the allocation schedule $s \in \mathcal{S}$, which can be interpreted as the frequency at which the transit authority implements that allocation schedule.

Assuming that the transit authority knows all the allocation schedules of n inspection teams, the probability that site x is controlled during Π can be expressed as a convex combination of the inspection teams allocations. That is, $\mathbb{P}_x = \sum_{s\in\mathcal{S}} \pi_s \, Y_{x|s}$ with $\sum_{s\in\mathcal{S}} \pi_s = 1$, where $Y_{x|s}$ is equal to 1 if the transit authority controls x during Π given the allocation schedule s, and 0 otherwise. By implementing the change of variables in the x-SSP model, the resulting optimization problem can be reformulated as the following NLP:

$$
\begin{aligned}
x\text{-UAS}: \quad \max_{\mathbf{U},\boldsymbol{\pi}} \quad & \sum_{p\in\mathcal{P}} \mathbb{E}(d_p^o)\; U_p^o && (19)\\
\text{s.t:} \quad & U_p^o \le B \qquad \forall p\in\mathcal{P} && (10)\\
& U_p^o \le F\left(1 - \prod_{(x,l)\in\mathcal{X}_p^i}\left(1 - f_{xl}\sum_{s\in\mathcal{S}}\pi_s Y_{x|s}\right)\right) \qquad \forall p\in\mathcal{P}, i\in\mathcal{N}_p && (20)\\
& \sum_{s\in\mathcal{S}} \pi_s = 1 && (21)\\
& \pi_s \ge 0 \qquad \forall s\in\mathcal{S}, && (22)
\end{aligned}
$$

where each allocation schedule satisfies $\sum_{x\in E} Y_{x|s} \le n$, i.e., the number of sites that the transit authority can control simultaneously during Π is limited to the number of inspection resources available. The x-UAS model is difficult to solve since the function of the inequality constraint (20) written in standard form is neither convex nor concave.

In a similar way as x-SSP, we propose the following LP relaxation of x-UAS:

$$
\begin{aligned}
x\text{-}\mathcal{R}(\text{UAS}): \quad \max_{\mathbf{U},\boldsymbol{\pi}} \quad & \sum_{p\in\mathcal{P}} \mathbb{E}(d_p^o)\; U_p^o && (19)\\
\text{s.t:} \quad & U_p^o \le B \qquad \forall p\in\mathcal{P} && (10)\\
& U_p^o \le F \sum_{(x,l)\in\mathcal{X}_p^i}\sum_{s\in\mathcal{S}} f_{xl}\pi_s Y_{x|s} \qquad \forall p\in\mathcal{P}, i\in\mathcal{N}_p && (23)\\
& \sum_{s\in\mathcal{S}} \pi_s = 1 && (21)
\end{aligned}
$$

$$\pi_s \geq 0 \qquad \forall s \in \mathcal{S}, \tag{22}$$

because $1 - \prod_{(x,l)\in\mathcal{X}_p^i} \left(1 - f_{xl} \sum_{s\in\mathcal{S}} \pi_s \; Y_{x|s}\right) \leq \sum_{(x,l)\in\mathcal{X}_p^i} \sum_{s\in\mathcal{S}} f_{xl}\pi_s \; Y_{x|s}$. It should be noted that x-$\mathcal{R}$(UAS) is a reformulation of x-$\mathcal{R}$(SSP), and consequently an equivalent optimization problem, since it results from the change of variable in x-$\mathcal{R}$(SSP).

Clearly, for a realistic public bus transportation network, it is impossible to enumerate all the allocation schedules since $\mathcal{S}$ grows exponentially with the size of G. Thus, a Column Generation (CG) approach must be applied to solve x-$\mathcal{R}$(UAS).

The CG process is defined such that an arbitrarily given subset of allocation schedules $\mathcal{S}_r(\subset \mathcal{S})$ is first considered, and further allocation schedules are then added to reach optimality. The CG solution procedure starts by defining a restricted master problem denoted as RMP by replacing $\mathcal{S}$ in x-$\mathcal{R}$(UAS) with $\mathcal{S}_r$. Let Z^*_{RMP} be the optimal objective function of RMP, $\beta^*_{pi} \geq 0$ and θ^* the values of the dual variables corresponding to constraints (23) and (21), respectively. Then, Z^*_{RMP} is an optimal solution to x-$\mathcal{R}$(UAS) if all reduced costs are non-positive, i.e., $\bar{c}_s = F \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{N}_p} \sum_{(x,l)\in\mathcal{X}_p^i} \beta^*_{pi} f_{xl} Y_{x|s} - \theta^* \leq 0$.

To generate further inspection team allocations or verify the optimality of the current solution Z^*_{RMP}, we need to solve the following pricing problem:

$$\text{SP}: \quad \max_{\mathbf{Y}} \quad \bar{c}_s = F \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{N}_p} \sum_{(x,l)\in\mathcal{X}_p^i} \beta^*_{pi} f_{xl} Y_{x|s} - \theta^* \tag{24}$$

$$\text{s.t:} \quad \sum_{x\in X} Y_{x|s} \leq n \tag{25}$$

$$Y_{x|s} \in \{0,1\} \qquad \forall x \in X. \tag{26}$$

If the optimal objective value of SP is positive, then the allocation schedule with the maximum reduced cost $\bar{c}_s$ is added to RMP as a new entering allocation schedule, and the updated RMP is solved again. Otherwise, Z^*_{RMP} is an optimal solution to x-$\mathcal{R}$(UAS), i.e., $Z^*_{x\text{-}\mathcal{R}(\text{UAS})} = Z^*_{\text{RMP}}$, where $Z^*_{x\text{-}\mathcal{R}(\text{UAS})}$ is the optimal objective function of the x-$\mathcal{R}$(UAS) model. Furthermore, since x-$\mathcal{R}$(UAS) and x-UAS are reformulations of x-$\mathcal{R}$(SSP) and x-SSP, respectively, we conclude that $\hat{Z}_x \leq Z^*_{x\text{-SSP}} = Z^*_{x\text{-UAS}} \leq Z^*_{x\text{-}\mathcal{R}(\text{SSP})} = Z^*_{x\text{-}\mathcal{R}(\text{UAS})} = Z^*_{\text{RMP}}$, where $Z^*_{x\text{-UAS}}$ is the optimal objective function of x-UAS. It should be noted that the SP model is an 0-1 Knapsack problem. Although the 0–1 knapsack problem is NP-hard, it is one of the easiest problems within this complexity class (Jooken et al, 2022). Following Bellman (1966), the SP model can be solved in pseudopolynomial time with a time complexity of $O(|X|n)$.

We take advantage of the optimal variables resulting from x-$\mathcal{R}$(SSP) to improve the CG algorithm. More precisely, let $\mathcal{X}_0 = \{x \in X : \mathbb{P}_x = 0\}$ be the set of not controlled sites where $\mathbb{P}_x$ is the optimal variable of x-$\mathcal{R}$(SSP). Consequently, for any $x \in \mathcal{X}_0$ we have $\pi_s = 0$ or $Y_{x|s} = 0$ for any $s \in S$ because

$\mathbb{P}_x = \sum_{s \in \mathcal{S}} \pi_s Y_{x|s}$. Thus, an equivalent pricing problem can be defined by including $Y_{x|s} = 0$ for any $x \in \mathcal{X}_0$ in the SP model, denoted RSP.

Solving x-$\mathcal{R}$(UAS) by the CG procedure using RMP-RSP leads to a feasible unpredictable allocation schedule of inspection teams. Let $\{(s, \pi_s) : \pi_s > 0\}_{s \in \mathcal{S}_r}$ be the feasible unpredictable allocation schedule resulting from solving x-$\mathcal{R}$(UAS), and $\hat{\mathcal{S}}_r = \{s \in \mathcal{S}_r : \pi_s > 0\}$ be the set of *useful allocation schedules*. The procedure to obtain a feasible unpredictable allocation schedule of inspection teams is summarized in Algorithm 2.

Algorithm 2 Feasible unpredictable allocation schedule

```
Feasible Spot Strategy: {P_x}_{x∈X} ← solve x-R(SSP)
Set X_0 = {x ∈ X : P_x = 0}
s = 0
Set c̄_s^(0) = ∞, Y^(0) = 0
β^(0), θ^(0) ← solve RMP
while c̄_s^(s) > 0 do
    s = s + 1
    c̄_s^(s), Y^(s) ← solve RSP
    β^(s), θ^(s) ← solve RMP
end while
Return feasible Unpredictable Allocation Schedule : {(s, π_s) : π_s > 0)}_{s∈S_r}
```

The systematic day-to-day application of the unpredictable allocation schedule resulting from Algorithm 2 converges at steady-state to the evasion rate defined in (17) since x-$\mathcal{R}$(SSP) and x-$\mathcal{R}$(UAS) are equivalent optimization problems. However, a steady-state solution is not operationally useful to the transit authority. Surely the transit authority is interested in knowing after how much time the evasion rate defined by (17) is reached. We propose to use Monte Carlo simulation to reproduce the transit authority's daily choice of an inspection team allocation $s \in \hat{\mathcal{S}}_r$ with probability π_s and compute the evasion rate as a function of time. Let $g_{s,\tau}$ be the frequency with which the inspection teams allocation $s \in \hat{\mathcal{S}}_r$ is selected after τ days. Thus, the evasion rate after τ days is:

$$ER_\tau = \frac{\sum_{p \in \mathcal{P}: U^o_{p,\tau} < B} \mathbb{E}(d^o_p)}{\sum_{p \in \mathcal{P}} d_p}, \tag{27}$$

where $U^o_{p,\tau} = \min\left\{B, F \min_{i \in \mathcal{N}_p}\left\{1 - \prod_{(x,l) \in \mathcal{X}^i_p}\left(1 - f_{xl} \sum_{s \in \hat{\mathcal{S}}_r} g_{s,\tau} Y_{x|s}\right)\right\}\right\}$ is the amount paid by an opportunistic passenger of type p after τ days and $Y_{x|s}$ is the optimal variable of x-$\mathcal{R}$(UAS).

6 On the practical implementation of the spot-fare inspection strategy

This paper has a practical orientation. The objective is to provide the transit authority with an unpredictable allocation schedule that inhibits the action of as many opportunistic evaders as possible considering the number of inspection teams available to the transit authority. The practical implementation of the spot-fare inspection strategy using an unpredictable allocation schedule is as follows.

- The first step consists of generating a graph G representing the urban bus network during Π and to characterize the bus lines and passengers in G. More precisely, the path of each bus line in G and the number of buses of each line per time unit in G must be determined. Furthermore, the k-shortest paths that can be used by each type of passenger in G to avoid fare payment must be determined.
- The second step consists of solving x-$\mathcal{R}$(SSP) to obtain a feasible spot-fare inspection strategy (SSP), i.e., the frequency with which the transit authority should control each site in G during Π. Using the spot-fare inspection strategy, the evasion and inspection rates during Π (ER, IR) are determined given the number of inspection teams available to the transit authority. Thus, the transit authority can estimate whether or not the number of inspection teams available is sufficient to reduce evasion to acceptable levels. The transit authority can also determine the smallest number of fare inspection teams that induce an evasion rate below an acceptable level. This acceptable level may be the evasion rate induced by chronic or recidivist evaders, since the fare inspections have no effect on them.
- The third step consists in solving x-$\mathcal{R}$(UAS) from which an unpredictable allocation schedule (UAS) is obtained, i.e., a set of allocation schedules each with its respective probability of being selected. More precisely, each allocation schedule defines the sites in G to be controlled (one for each inspection team) and has an associated probability of being selected. Using Monte Carlo simulation to reproduce the daily choice of an allocation schedule, the convergence time to the evasion rate induced by the spot-fare inspection strategy is determined. Thus, the transit authority has clarity on the implementation period of the spot-fare inspection strategy. It should be noted that convergence is guaranteed since x-$\mathcal{R}$(SSP) and x-$\mathcal{R}$(UAS) are two equivalent optimization problems.
- The fourth step considers the real implementation of the unpredictable allocation schedule. Each day the transit authority chooses an allocation schedule s with probability π_s and allocates its inspection resources to the sites in G as defined by the allocation schedule. This is repeated systematically day by day during the implementation period defined in step three. At the end of the implementation period, the transit authority must implement a sampling schedule to verify whether the evasion rate defined in step two was achieved.

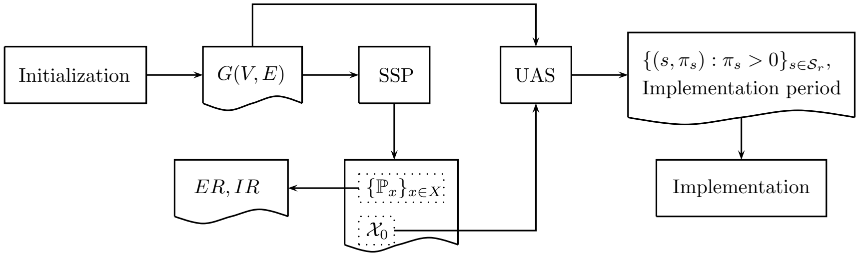


Fig. 6: Practical implementation of the spot-fare inspection strategy

The four steps, which are summarized in Figure 6, are denoted as i) initialization, ii) spot-fare inspection strategy, iii) unpredictable allocation schedule, and iv) implementation, respectively. It should be noted that in this paper we cover up to the third step.

7 Numerical results

In this section, we first present numerical results to evaluate the performance of the feasible spot-fare inspection strategy and the feasible unpredictable allocation schedule of inspection teams resulting from algorithms 1 and 2, respectively, when the sites to be controlled are edges or districts, i.e., $x = e$ or $x = d$. Then, management insights for the transit authority are discussed, and an illustrative example is presented.

The computational experiments are based on the Berlin urban bus system operating in the city center (Zone AB), where 43 bus lines serve 417 bus stops. Berlin's bus network was selected for the numerical tests due to the availability of official data and its suitability as a case study. The network's complexity and size make it ideal for showcasing both the computational challenges and the performance of our approach. Relying on the geographical reference system of the Berlin urban bus system and the route followed by each bus line, we generate G, with $|V| = 417$ and $|E| = 487$. Figure 7 shows G.

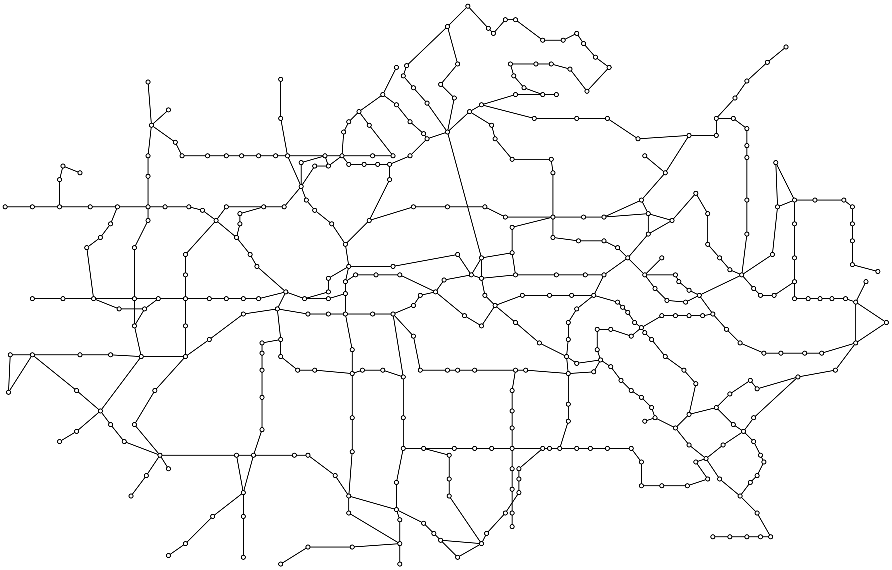


Fig. 7: Zone AB Berlin urban bus system

We consider all origin-destination pairs combinations in G, resulting in a total of $|\mathcal{P}| = 86293$ passenger types. Furthermore, we consider spot-fare inspections at 4-6 pm, i.e., $\Pi = [16:00, 18:00]$, with 400 000 passengers on average.

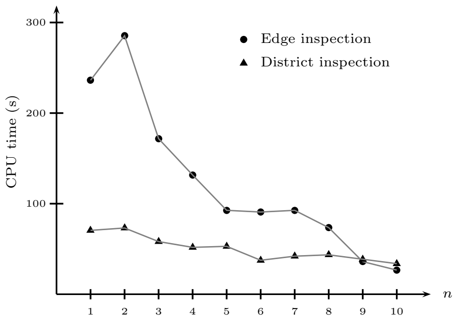
To determine the set of k-shortest paths that an opportunistic passenger of type p can take in G, we consider: (1) the line l bus travel time at e, τ_e^l, for any $l \in \mathcal{N}_e$ and $e \in E$, where $\mathcal{N}_e$ is the set of bus lines using e, (2) the line l bus stopping time at v, τ_v^l, for any $l \in \mathcal{N}_v$ and $v \in V$, where $\mathcal{N}_v$ is the set of bus lines using v, and (3) the time to change from a bus of line l to a bus of line l', $\tau_v^{l,l'}$, for any $l \neq l' \in \mathcal{N}_v$ and $v \in V$. In particular, $\tau_e^l = \frac{l_e}{v_e^l}$, where l_e is the length of the edge e and v_e^l is the average speed of a bus of line l at e during Π.

Twenty randomized test problems (test set) were generated, where each of them considers $n \in \{0, ..., 10\}$, leading to a total of 200 instances with the next common parameters. The set of k-shortest paths for an opportunistic passenger of type p has at most ten alternative paths, i.e., $|\mathcal{N}_p| \leq 10$ for any $p \in \mathcal{P}$. The number of passengers of type p is uniformly distributed according to $d_p \sim \mathcal{U}\{2, ..., 7\}$, ensuring that the expected total demand $\mathbb{E}(d_p)|\mathcal{P}|$ aligns with the approximate real demand of 400,000 passengers during $\Pi = [16:00, 18:00]$ in the Berlin bus network. Opportunistic passengers are uniformly distributed according to $d_p^o \sim \mathcal{U}\{0, ..., \hat{d}_p^o\}$ with $\hat{d}_p^o = \lfloor 0.4 d_p \rfloor + 1$ for any $p \in \mathcal{P}$. The ticket price is $B = 1.5$, and the fine is $F = 75$, such that the ratio between the ticket price and fine is $c = 0.02$. $v_e^l = U[20, 30]$ (km/hr) for any $l \in \mathcal{N}_e$ and $e \in E$. $\tau_v^l = U[1, 3]$ (min) for any $l \in \mathcal{N}_v$ and $v \in V$. $\tau_v^{l,l'} = U[3, 6]$ (min) for any $l \neq l' \in \mathcal{N}_v$ and $v \in V$. The length of edge e, l_e, is determined using geo-referenced data on the Berlin street network. The number of buses on line l running on site x, q_x^l, is determined using the frequency of buses during $\Pi = [16:00, 18:00]$ defined by the Berlin Transport Company (BVG) and $r_x \sim \mathcal{U}\{3, ..., 5\}$ for $x = e$ and $x = d$. In order to determine the set of districts, we consider $\tau_d = 20$ minutes.

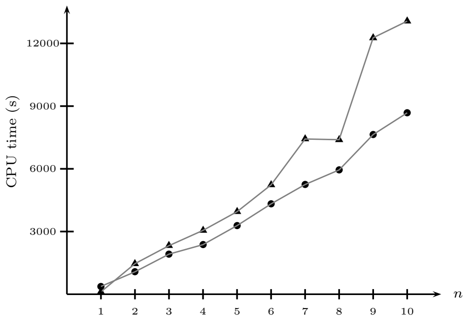
Graph G and the districts were constructed using Python 3.7. Figures of the network were generated in TikZ using available information on the coordinates of the bus stops. The set of k-shortest paths that an opportunistic passenger of type p can use in G is computed using Python's Networkx library. Models x-$\mathcal{R}$(SSP), RMP, and RSP were solved using CPLEX 20.1. The stopping criterion was based on an optimality gap lower than or equal to 10^{-4}. All tests are performed on a PC with an Intel Core i7 2.3 GHz processor and 16 GB RAM.

7.1 Performance of feasible spot strategy and feasible unpredictable inspection teams allocation

We compute the CPU time to obtain a feasible spot strategy and a feasible unpredictable allocation schedule using algorithms 1 and 2, respectively, when $x = e$ and $x = d$. The maximum CPU times to obtain the feasible spot strategy and the feasible unpredictable allocation schedule under edge inspection are 426 s and 11937 s, respectively, and under district inspection are 96 s and 18881 s, respectively. Figure 8 shows the average CPU times to obtain the feasible spot strategy and the feasible unpredictable allocation schedule for all instances and fare inspection teams when $x = e$ and $x = d$.



(a) Feasible spot strategy



(b) Feasible unpredictable allocation schedule

Fig. 8: Average CPU times

The average CPU time to obtain the feasible spot strategy and the feasible unpredictable allocation schedule are decreasing and increasing, respectively, with the number of inspection teams. Furthermore, the average CPU time to obtain the feasible spot strategy is higher when the fare inspection is done on network edges ($x = e$). On the contrary, the average CPU time to obtain the feasible unpredictable allocation schedule is higher when the fare inspection is done in districts ($x = d$).

The x-$\mathcal{R}$(SSP) and x-$\mathcal{R}$(UAS) models are two equivalent optimization problems because x-$\mathcal{R}$(UAS) is a reformulation of x-$\mathcal{R}$(SSP). Thus, $Z^*_{x\text{-}\mathcal{R}(\text{SSP})} = Z^*_{x\text{-}\mathcal{R}(\text{UAS})} = Z^*_{\text{RMP}}$. To measure the quality of the feasible spot strategy and the feasible unpredictable allocation schedule resulting from algorithms 1 and 2, respectively, we compute the relative optimality gap between the upper bound given by $Z^*_{x\text{-}\mathcal{R}(\text{SSP})}$ and the lower bound given by $\hat{Z}_x$ according to (16) when $x = e$ and $x = d$. The relative optimality gap of the feasible spot strategy and the feasible unpredictable allocation schedule resulting from algorithms 1 and 2, respectively, are shown in Figure 9 for all instances and fare inspection teams.

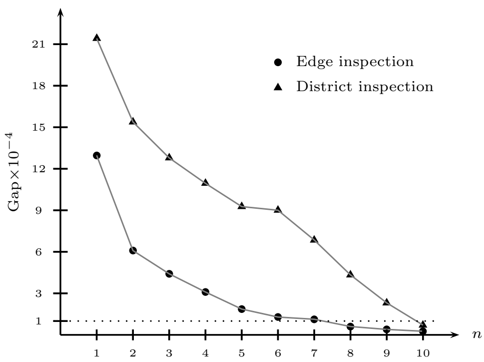


Fig. 9: Average optimality gap

Algorithms 1 and 2 provide good-quality spot strategies and unpredictable allocation schedules, respectively, because the maximum optimality gaps are 0.14% and 0.23% when $x = e$ and $x = d$, respectively. Furthermore, the average optimality gap is monotonically decreasing with the number of fare inspection teams. Indeed, the average optimality gap is less than or equal to 1.0×10^{-4} for $x = e$ and $x = d$ when the number of fare inspection teams is greater than or equal to eight and ten, respectively.

The unpredictable allocation schedule resulting from algorithm 2 takes advantage of the set of non-controlled sites, i.e., $\mathcal{X}_0 = \{x \in X : \mathbb{P}_x = 0\}$ where $\mathbb{P}_x$ is the optimal variable of x-$\mathcal{R}$(SSP). Figure 10 shows the average percentage of sites that do not need to be controlled under inspection in edges and districts, respectively, for all instances and fare inspection teams.

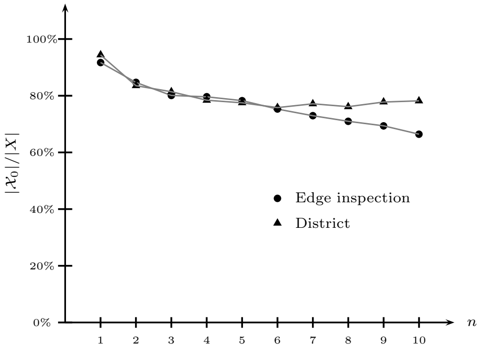


Fig. 10: Percentage of sites with zero inspection probability

As expected, only a few edges and districts need to be inspected to reach a good unpredictable allocation schedule because on average only 23% and 20% of edges and districts need to be inspected, respectively.

For all instances, the algorithm 2 efficiency is defined as the ratio between the number of useful allocation schedules and the number of allocation schedules generated by the column generation procedure using RMP-RSP, i.e., $\eta(\%) = 100 \times |\hat{\mathcal{S}}_r|/|\mathcal{S}_r|$, where $|\mathcal{S}_r|$ is the number of team allocation schedules, and $|\hat{\mathcal{S}}_r|$ is the number of useful team allocation schedules.

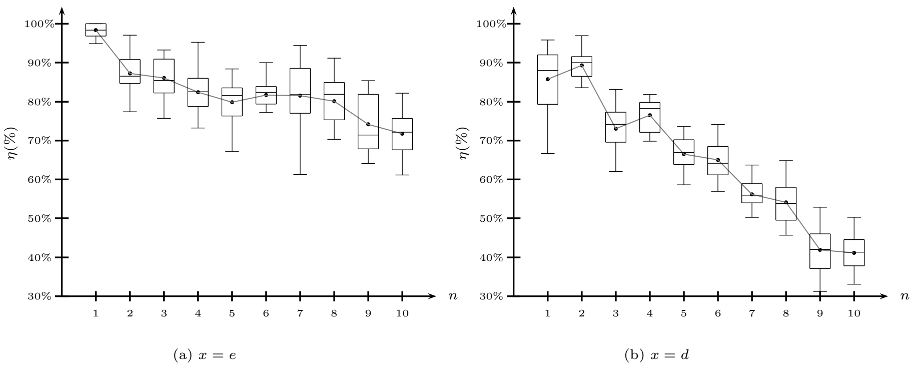


Fig. 11: Efficiency of Algorithm 2

As shown in Figure 11, the algorithm 2 efficiency is decreasing with the number of inspection teams. For instance, under edge inspection ($x = e$) and the deployment of one inspection team ($n = 1$), an average of 40 allocation schedules were generated ($|\mathcal{S}_r| = 40$), from which 39 were useful ($|\hat{\mathcal{S}}_r| = 39$), defining an efficiency value of 98% ($\eta = 0.98$). On the contrary, under edge inspection ($x = e$) and the deployment of ten inspection team ($n = 10$) an average of 218 allocation schedules were generated ($|\mathcal{S}_r| = 218$) but only 155 were useful ($|\hat{\mathcal{S}}_r| = 155$), an efficiency value of 71% ($\eta = 0.71$). The efficiency of algorithm 2 when the fare inspections are performed in districts ($x = d$) is lower than when it is performed in the network edges ($x = e$). Table 2 shows the average, maximum, and minimum generated and useful allocation schedules resulting from the Algorithm 2 when $x = e$ and $x = d$.

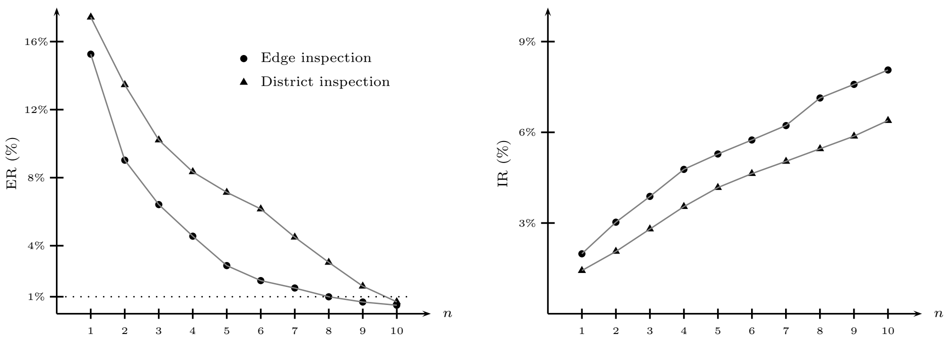
Table 2: Generated and useful allocation schedules

	$x = e$						$x = d$					
	$\|\mathcal{S}_r\|$			$\|\hat{\mathcal{S}}_r\|$			$\|\mathcal{S}_r\|$			$\|\hat{\mathcal{S}}_r\|$		
n	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.
1	40	42	38	39	40	37	27	39	24	23	26	22
2	75	85	67	66	73	58	70	74	65	62	65	59
3	97	112	88	83	90	79	91	108	82	66	69	63
4	104	115	84	85	91	80	105	116	99	80	85	75
5	118	137	107	94	99	88	124	145	106	82	89	78
6	136	162	120	111	115	107	137	153	116	88	93	84
7	151	204	125	121	126	114	163	202	131	90	95	86
8	166	192	147	132	140	127	172	197	145	92	94	90
9	194	223	161	142	146	136	222	288	174	91	98	84
10	218	260	185	155	160	151	231	288	171	93	99	86

7.2 Managerial insights for the transit authority

In this section, we derive managerial insights for the transit authority related to the evasion rate, inspection rate, and the systematic day-to-day application of the unpredictable allocation schedule of inspection teams during Π.

For each instance, we compute the evasion rate and inspection rate according to (17) and (18), respectively. Figure 12 shows the evasion and inspection rates induced by the spot-fare inspection strategy under inspection at edges ($x = e$) and districts ($x = d$) for all fare inspection teams.



(a) Evasion rate (b) Inspection rate

Fig. 12: Evasion and inspection rates

Figure 12a shows how the opportunistic passenger's decision to evade fare payment, and the path they take to reach their destination, changes as the number of inspection teams increases. As expected, the evasion rate is decreasing with the number of fare inspection teams. On the other hand, the inspection rate (Figure 12b) is monotonically increasing with the number of inspection teams. Under edge inspection ($x = e$) and one inspection team deployed ($n = 1$), the average inspection rate is equal to 2.0% and the average evasion rate is equal to 15.3%. On the contrary, when ten inspection teams are deployed ($n = 10$), the average inspection rate equals 8.6% and the average evasion rate equals 0.5%. Under district inspection ($x = d$) and one inspection team deployed ($n = 1$), the average inspection rate is equal to 1.4% and the average evasion rate is equal to 17.4%. On the contrary, when ten inspection teams are deployed ($n = 10$), the average inspection rate equals 6.4% and the average evasion rate equals 0.7%.

As a practical result for the transit authority, we observe the smallest number of fare inspection teams inducing an evasion rate less than or equal to 1%, which represents passengers who do not pay the fine or who do not react to the fare inspection as a mechanism to inhibit evasion, i.e., chronic fare evaders or recidivist evaders. Under edge inspection and district inspection (Figure 12a), the smallest number of inspection teams that induce an evasion rate less than or equal to 1% is achieved with the deployment of eight and ten inspection teams, respectively.

Under edge inspection, the systematic day-to-day application of the unpredictable allocation schedule when eight inspection teams are deployed induce at steady state an average evasion rate equal to 0.99% and an average inspection rate equal to 7.1%, all with average optimality gap equal to 0.006%. Similarly, under district inspection, the systematic day-to-day application of the unpredictable allocation schedule when ten inspection teams are deployed induce at steady state an average evasion rate equal to 0.7% and an average inspection rate equal to 6.4%, all with average optimality gap equal to 0.007%. Using Monte Carlo simulation to reproduce the transit authority's daily choice of an inspection team allocation $s \in \hat{\mathcal{S}}_r$ with probability π_s, we compute the evasion rate after τ days according to (27). Figure 13 shows the daily simulation of the unpredictable allocation schedule over a 150-day horizon when $n = 8$ and $n = 10$ under edge and district inspection, respectively.

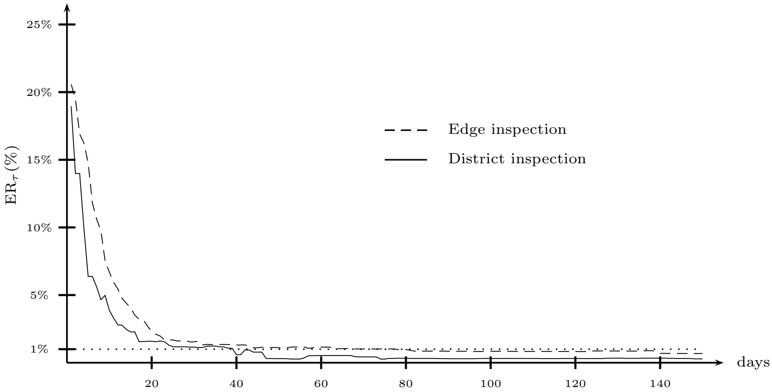


Fig. 13: Unpredictable allocation schedule simulation under edge and district inspection

The daily evasion rate decreases as the unpredictable allocation schedule is systematically applied day after day. From Figure 13, we observe that after 80 and 40 days an average evasion rate of 1% is reached under edge and district inspection, respectively. Consequently, we conclude that the unpredictable allocation schedule can be operationally implemented in the medium term under edge and district inspection.

The Monte Carlo simulation chooses a daily allocation schedule based on the selection probability π_s with $s \in \hat{\mathcal{S}}_r$. We compute the empirical probability distribution function of the allocation schedule selection, i.e., $\sum_{s\in\hat{\mathcal{S}}_r} \pi_s$, resulting from x-$\mathcal{R}$(UAS) when $x = e$ and $x = d$. In particular, the average number of useful allocation schedules are $|\hat{\mathcal{S}}_r| = 132$ and $|\hat{\mathcal{S}}_r| = 93$ when eight and ten inspection teams are deployed under edge and district inspection, respectively (see Table 2).

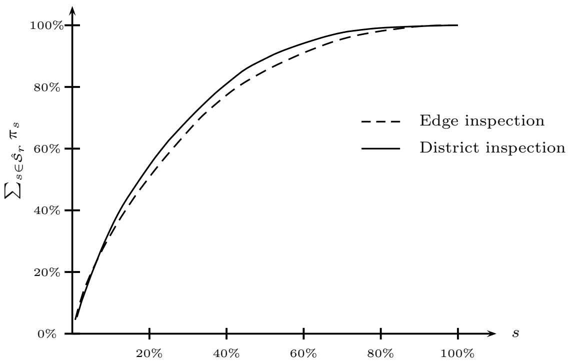


Fig. 14: Probability distribution function of the allocation schedule selection

From Figure 14, we observe that 43% and 39% of the useful allocation schedules accumulate 80% of the selection under edge and district inspection, respectively. Thus, we infer that few allocation schedules accumulate a high probability of selection.

7.3 Illustrative example

We use the first instance of the test set as an illustrative example to show, on G, the spot strategy and the unpredictable allocation schedule resulting from x-$\mathcal{R}$(SSP) and x-$\mathcal{R}$(UAS) under inspection in edges and districts, respectively. Both with eight and ten inspection teams deployed during Π, the evasion rate is maintained at less than or equal to 1%.

Under edge inspection ($x = e$) and eight inspection team deployed ($n = 8$), the evasion rate is equal to 0.84% and the inspection rate is equal to 7.1%, all with an optimality gap equal to 0.004%. Figure 15 shows the spot strategy on the Berlin bus system when inspections are performed on the edges and eight inspection teams are deployed, i.e., $\{\mathbb{P}_e\}_{e \in E}$ where $\mathbb{P}_e$ is the optimal variable of e-$\mathcal{R}$(SSP).

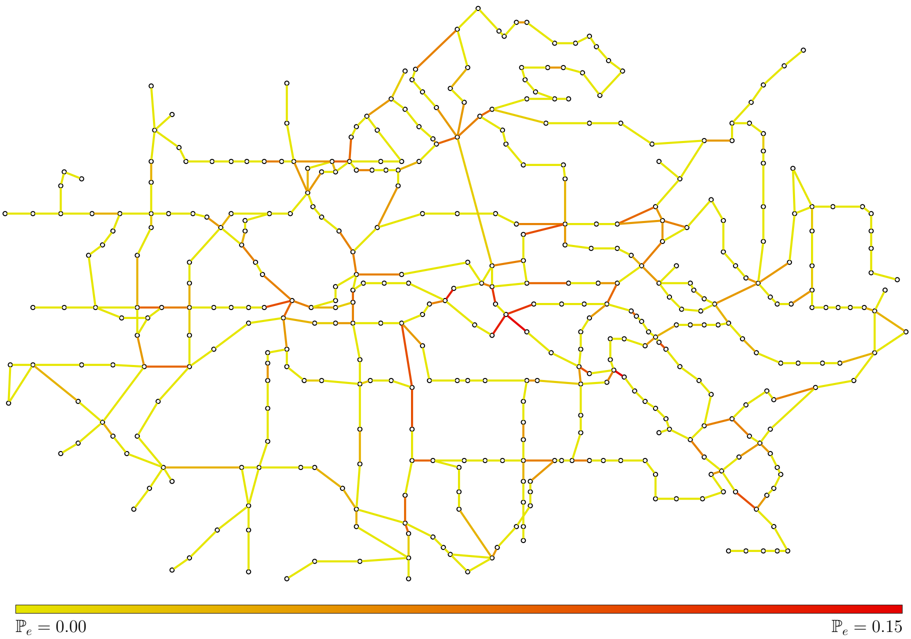
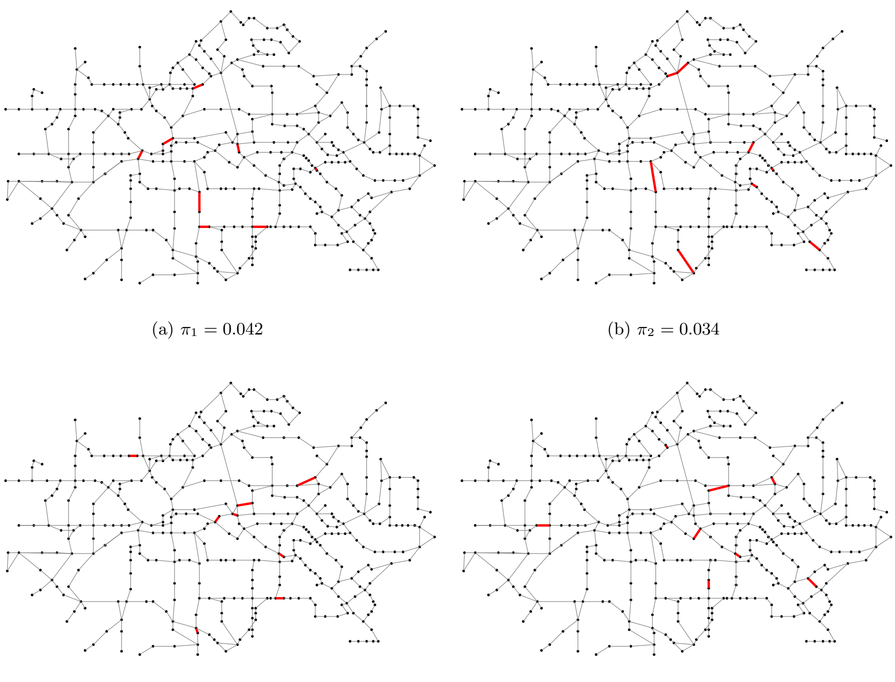


Fig. 15: Spot-fare inspection strategy on Berlin urban bus system under edge inspection with $n = 8$

Figure 15 shows that 71% of the edges of G are not inspected, i.e., $|\mathcal{X}_0| = 348$. The spot-fare inspection strategy is concentrated on red edges around bus lines interconnecting stations. Thus, the spot strategy, illustrated in Figure 15, shows the edges where the transit authority should frequently perform fare inspections during Π. However, the practical implementation of this spot strategy under edge inspection is done using the allocation schedules resulting from e-$\mathcal{R}$(UAS) solved using algorithm 2. The algorithm generated 164 allocation schedules ($|\mathcal{S}_r| = 159$) of which 128 are useful ($|\hat{\mathcal{S}}_r| = 130$). Figure 16 shows the four allocation schedules most likely to be selected under edge inspection and deployment of eight fare inspection teams.



(a) $\pi_1 = 0.042$ (b) $\pi_2 = 0.034$

(c) $\pi_3 = 0.029$ (d) $\pi_4 = 0.027$

Fig. 16: Allocation schedules under edge inspection with $n = 8$

Figures 16a, 16b, 16c, and 16d are the four allocation schedules most likely to be selected in the day-by-day implementation of the unpredictable allocation schedule under edge inspection. The probabilities of selecting allocation schedules $s = 1, 2, 3, 4$ are $\pi_1 = 4.2\%$, $\pi_2 = 3.4\%$, $\pi_3 = 2.9\%$, and $\pi_4 = 2.7\%$, respectively, i.e., out of 1000 days of implementation of the unpredictable allocation schedule, schedules $s = 1, 2, 3, 4$ are expected to be implemented 42, 34, 29, and 27 days, respectively. It should be noted that the allocation schedules are consistent with the spot strategy because the edges inspected in 16a, 16b, 16c, and 16d have high probability of being inspected in the spot strategy (Figure 15). Furthermore, the four allocation schedules are different in terms of the inspected edges which ensures unpredictability from the passengers' perspective.

Under district inspection ($x = d$) and ten inspection teams deployed ($n = 10$), the evasion rate is equal to 0.31% and the inspection rate is equal to 6.3%, all with optimality gap equal to 0.002%. Figure 17 shows the spot strategy on the Berlin bus system when inspections are performed on districts and ten inspection teams are deployed.

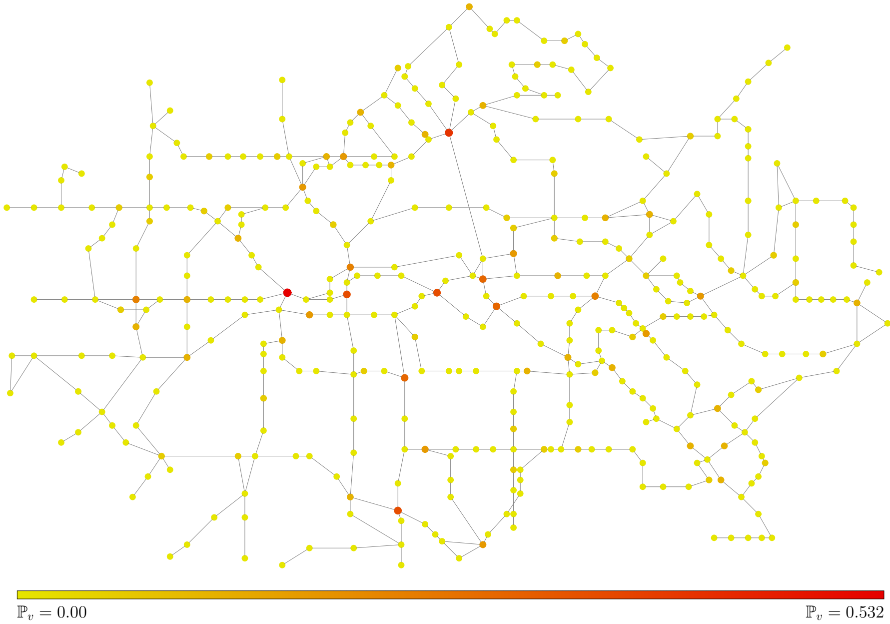
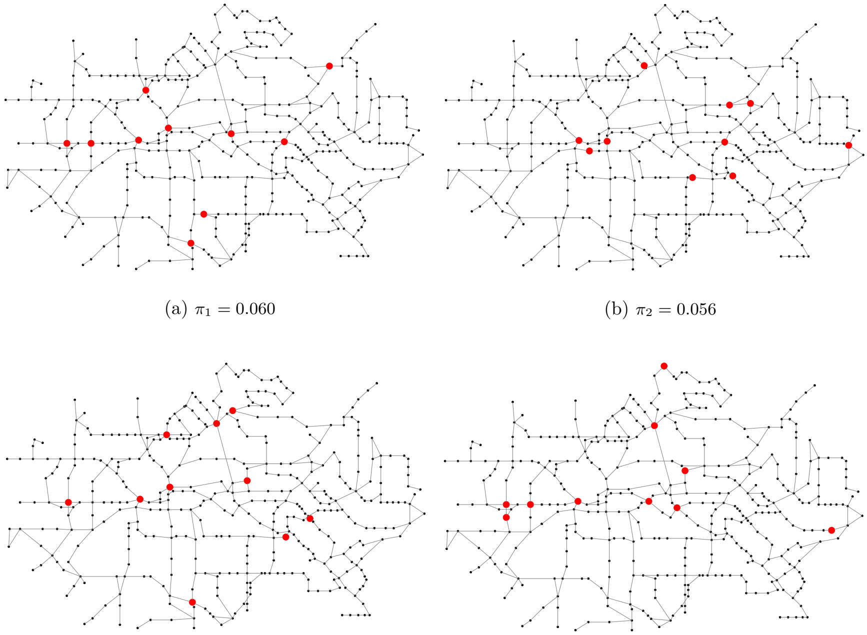


Fig. 17: Spot-fare inspection strategy on the Berlin urban bus system under district inspection with $n = 10$

Figure 17 shows that 79% of the districts of G are not inspected, i.e., $|\mathcal{X}_0| = 329$. The spot-fare inspection strategy concentrates on districts $\mathcal{G}(v)$ where v are interconnection nodes through which several bus lines pass. Thus, the spot strategy, illustrated in Figure 17, shows the districts where the transit authority should frequently perform fare inspections during Π. However, the practical implementation of this spot strategy under district inspection is done using the allocation schedules resulting from d-$\mathcal{R}$(UAS) solved using algorithm 2. The algorithm generated 257 allocation schedules ($|\mathcal{S}_r| = 257$) of which 91 are useful ($|\hat{\mathcal{S}}_r| = 91$). Figure 18 shows the four allocation schedules most likely to be selected under district inspection and deployment of ten fare inspection teams.



(a) $\pi_1 = 0.060$

(b) $\pi_2 = 0.056$

(c) $\pi_3 = 0.046$

(d) $\pi_4 = 0.045$

Fig. 18: Allocation schedules under district inspection with $n = 10$

Figures 18a, 18b, 18c, and 18d are the four allocation schedules most likely to be selected in the day-by-day implementation of the unpredictable allocation schedule under district inspection. The probabilities of selecting allocation schedules $s = 1, 2, 3, 4$ are $\pi_1 = 6.0\%$, $\pi_2 = 5.6\%$, $\pi_3 = 4.6\%$, and $\pi_4 = 4.5\%$, respectively, i.e., out of 1000 days of implementation of the unpredictable allocation schedule, schedules $s = 1, 2, 3, 4$ are expected to be implemented 60, 56, 46, and 45 days, respectively. It should be noted that the solution structure of the allocation schedules are consistent with the spot strategy because the inspected districts in 18a, 18b, 18c, and 18d have high probability of being inspected in the spot strategy (Figure 17).

8 Conclusion

This study addresses the design of a spot-fare inspection strategy and its operational implementation in a proof-of-payment public bus transportation network using an unpredictable allocation schedule of inspection teams. Opportunistic passengers are free to choose the most convenient path within a finite set of alternatives. The transit authority is committed to a selective inspection policy of buses and mass inspection of passengers on board the bus. Two realistic approaches to a spot inspection system are explored. The first one, denoted as edge inspection, considers the inspection of passengers on board while the bus is stopped. The second one, denoted as district inspection, considers inspecting passengers on board while the bus is in motion.

We model the spot-fare inspection strategy as a Leader-Follower Stackelberg game, where the transit authority determines the spatial inspection probability distribution on the transportation network, and opportunistic passengers respond by optimizing their decision whether or not to purchase a ticket and the path to follow by evaluating a finite set of paths. This path-based approach to define the spot-fare inspection strategy leads to a NLP for which a relaxation-based heuristic is proposed to compute a feasible spot strategy. On the other hand, defining the inspection probabilities as a convex combination over the set of all allocation schedules, we formulate the unpredictable allocation schedule as a NLP for which an LP relaxation is derived. This LP relaxation is solved using a column generation approach. The pricing problem generates allocation schedules, and the master problem defines the probability with which these allocation schedules should be selected. Thus, the resulting set of allocation schedules, where each of them has an associated probability of being selected, defines a feasible unpredictable allocation schedule for the transit authority.

Numerical experiments were conducted on a large-scale interconnected bus transportation network. The relaxation-based heuristic to compute a feasible spot inspection strategy, and the column generation approach to compute the feasible unpredictable allocation schedule, provide good-quality solutions. Indeed, the smaller number of inspection teams that induces an evasion rate less than or equal to 1% leads to a maximum optimality gap of less than 0.01%. In terms of managerial insights, numerical experiments show that a spot strategy based on edge inspection requires less inspection teams than one based on district inspection. We simulate the daily application of the unpredictable allocation schedule under inspection on edges and districts using Monte Carlo simulation and observe that the evasion rate of 1% is achieved after 80 and 40 days, respectively. Thus, we conclude that the unpredictable allocation schedule under edge or district inspection leads to solutions that can be applied in the medium term.

We next suggest future research prospects. The first one is to address the spot-fare inspection strategy and the unpredictable inspection teams allocation schedule under adaptive passengers, i.e., when opportunistic passengers obtain information to adapt their path as they travel in the transportation network. The second one is to address the spot strategy and the unpredictable allocation schedule under a different policy (e.g., selective in-station inspection) and considering heterogeneous opportunistic passengers in terms of their risk-prone to be captured. The third is to address the optimal design of balanced, contiguous, and compact fare inspection districts. The fourth one is to explore the relationship between inspection costs and expected revenues or reductions in subsidies, particularly in contexts where public entities manage inspections and private operators benefit indirectly. The fifth one is to validate the models in a real bus transportation network.

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Compliance with Ethical Standards

- **Conflicts of interest**. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.
- **Research involving human participants and/or animals**. This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix A Glossary of terms

Table A1: Sets, parameters, and variables

Sets	
V	Set of nodes representing bus stops, indexed by $v = 1, ..., \|V\|$
E	Set of edges representing the connection between bus stops, indexed by $e = 1, ..., \|E\|$
L	Set of bus lines, indexed by $l = 1, ..., \|L\|$
X	Set of sites in the bus network, indexed by $x = 1, ..., \|X\|$
$\mathcal{P}$	Set of type of passengers, indexed by $p = 1, ..., \|\mathcal{P}\|$
$\mathcal{S}$	Set of all allocation schedules
$\mathcal{E}_l$	Indexed set of edges followed by bus line l
$\mathcal{N}_p$	Indexed set of k-shortest paths from v_p^+ to v_p^-
$\mathcal{X}_p^i$	Indexed set of sites and bus line pairs that an opportunistic passenger of type p follows if he takes the path $i \in \mathcal{N}_p$
$\mathcal{L}_x$	Indexed set of bus lines passing through x during Π
$\mathcal{X}_0$	Set of not controlled sites
Parameters	
Π	Time interval
B	Ticket price
F	Fine with $F >> B$
d_p	Expected number of passengers of type p during Π
d_p^o	Number of opportunistic passengers of type p during Π
n	Number of fare inspection teams
q_{xl}	Number of buses of line l running on x during Π
r_x	Number of buses that an inspection team controls in x during Π
Variables	
W_p^o	1 if an opportunistic passenger of type p purchases a ticket, and 0 otherwise
H_p^i	1 if an opportunistic passenger of type p takes the path $i \in \mathcal{N}_p$, and 0 otherwise
U_p^o	Expected value of the amount paid by an opportunistic passenger of type p
$\mathbb{P}_x$	Probability that the transit authority controls $x \in X$ during Π
π_s	Probability of selecting the allocation schedule $s \in \mathcal{S}$
$Y_{x\|s}$	1 if the transit authority controls x during Π given the allocation schedule s, and 0 otherwise

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