The Maximal Digit Property of numeration systems

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1. A particular class of Rényi numeration systems

2. Moving to alternate bases

Rényi numeration system

Given a *base* $\beta > 1$, represent numbers using a greedy algorithm.
If $x \in [0, 1]$, define $r_0 = x$ then $d_{i-1} = \lfloor \beta r_i \rfloor$ and $r_i = \beta r_i - d_{i-1}$.
Obtain a word $d_{-1} d_{-2} \cdots$ such that $\sum_{i=1}^{\infty} \frac{d_{-i}}{\beta^i} = x$. This word is called $d_\beta(x)$. It is on the alphabet $\{0, \ldots, \lfloor \beta \rfloor\}$.
Then, define the representation $\langle x \rangle_\beta$ as follows. If $x \in [0, 1)$, $\langle x \rangle_\beta = {}^\omega 0 \cdot d_\beta(x)$. If $x \geq 1$, let N be the least integer such that $\frac{x}{\beta^N} < 1$. Then, $\langle x \rangle_\beta = \sigma^N({}^\omega 0 \cdot d_B(\frac{x}{\beta^N}))$.

Example

If β is the golden ratio, we have, for instance, $\langle 1/2 \rangle_\beta = \cdot(010)^\omega$ and $\langle 2 \rangle_\beta = 10 \cdot 01$.
If we let γ be the root of $x^3 - 3x^2 - 2x - 1$, then $\langle \gamma - 3 \rangle_\gamma = \cdot 210^\omega$.

Note that while $d_\beta(1)$ is often nontrivial, $\langle 1 \rangle_\beta$ is always $1\cdot$.

Normalization

The *value* of the word $a_N a_{N-1} \cdots a_0 \cdot a_{-1} \cdots$ is

$$\sum_{i=0}^{N} a_i \beta^i + \sum_{i=1}^{\infty} \frac{a_{-i}}{\beta^i}.$$

Several words may have the same value. Finding the particular word which is the representation of this value is called *normalization*.
In this work, we would like to normalize words using a system of rewriting rules.

Example

Let β be the golden ratio. The word $2\cdot$ also has value 2. We could normalize it with the sequence $2\cdot \to 1 \cdot 11 \to 10 \cdot 01$.

From a numeration system to a rewriting system

A *rewriting system* is a pair $(E, \to)$ where $\to$ is a binary relation on a set E.

In our work, the set E is $A^{\mathbb{Z}}$, and we assume that $\to$ rewrites a word to a word of the same length and is context-free: if $x \to y$, then there exist α, γ, s, t such that $x = \alpha s \gamma$, $y = \alpha t \gamma$, $|s| = |t|$, and furthermore $\alpha' s \gamma' \to \alpha' t \gamma'$ for all α', γ'.

Additionally, if $\oplus$ is the operation of digitwise addition, we would like to have $x \to y \Leftrightarrow x \oplus z \to y \oplus z$ for all $z \in A^{\mathbb{Z}}$ such that all involved words are still in $A^{\mathbb{Z}}$.

We call $s \to t$ a *core rule* in this construction.

Given β a simple Parry number, we define a rewriting system R_β associated with β from the core rule

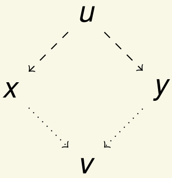
$$0 t_1 \cdots t_\ell \to 10^\ell$$

where $d_\beta(1) = t_1 \cdots t_\ell$.

Confluence

Let $\to^*$ be the reflexive transitive closure of $\to$. Then $\to$ is *confluent* if

$$\forall u, x, y \in E : u \to^* x, u \to^* y, \exists v : x \to^* v, y \to^* v.$$



Proposition (Frougny, 1992)

The system R_β is confluent over the alphabet $\{0, \ldots, c\}$ if and only if $d_\beta(1) = t_1 \ldots t_\ell$ with $t_1 = \ldots = t_{\ell-1} = c$ and $t_\ell \leq c$.

The numbers β that have this property are sometimes called *generalized multinacci numbers*. For these numbers, we can normalize finite words by applying rewriting rules.

Spectrum and Beta-integers

Resulting from normalization is the equality between two sets of numbers associated with β.

The *spectrum* of β (with respect to the alphabet A, which is implicit) is $X_\beta = \{\sum_{i=0}^{\ell} a_i \beta^i \,:\, i \in \mathbb{N},\, a_i \in A \forall i\}$.

The β-integers are the set $\mathbb{N}_\beta = \{x : \langle x \rangle_\beta \in A^* \cdot 0^\omega\}$

If normalization can be done without adding more digits to the right, it follows that all elements of the spectrum are β-integers.

Conversely, if an element cannot be normalized without adding digits to the right, one can find an associated element in $X_\beta \setminus \mathbb{N}_\beta$.

Optimality

Optimality was first considered by Dajani, de Vries, Komornik and Loreti. Optimal bases are such that the "local" conditions imposed by the greedy algorithm generalize to a more global condition, considering multiple digits at a time.

Example

Let γ be the positive root of $x^3 - 3x^2 - 2x - 1$ and $x = \frac{3}{\gamma^2} + \frac{3}{\gamma^3} \simeq 0.290$. Then, we have $d_\gamma(x) = \cdot 1002210^\omega$. However, the length-3 prefix of $d_{\gamma(x)}$ is not the best under-approximation of x by a word of length 3, as we have $\mathsf{val}_\gamma(\cdot 100) < \mathsf{val}_\gamma(\cdot 033) \leq x$. Thus, base γ is not optimal.

Proposition (Dajani et. al., 2011)

When considering the alphabet $\{0, \ldots, c\}$, a base β is optimal if and only if $d_\beta(1) = t_1 \ldots t_\ell$ with $t_1 = \ldots = t_{\ell-1} = c$ and $t_\ell \leq c$.

Alternate base numeration system

The aim of this work is to find a generalization of this class of numeration systems to *alternate base numeration systems*. These generalize Rényi numeration systems by allowing an alternation between multiple bases.

We consider a sequence $\mathcal{B} = (\beta_n)_{n\in\mathbb{Z}}$ periodic of period p.

The representation map is given by a greedy algorithm. For x in $[0,1]$, we set $r_0 = x$ and for $n \geq 1$ we set $x_{-n} = \lfloor \beta_{-n} r_{n-1} \rfloor$ and $r_n = \beta_{-n} r_{n-1} - x_{-n}$. We denote the infinite word $\cdot(x_{-n})_{n\geq 1}$ by $d_{\mathcal{B}}(x)$.

Now for $x \geq 1$, let $N \geq 0$ be the unique N such that $\prod_{i=0}^{N-1} \beta_i \leq x < \prod_{i=0}^{N} \beta_i$. Then $\langle x \rangle_{\mathcal{B}} = \sigma^N({}^{\omega}0 \cdot d_{\sigma^{-N}(\mathcal{B})}(\frac{x}{\prod_{i=0}^{N} \beta_i}))$ for $x \geq 1$, and $\langle x \rangle_{\mathcal{B}} = d_{\mathcal{B}}(x)$ for $x \in [0,1)$.

The evaluation map is given by

$$\mathrm{val}_{\mathcal{B}}(a) = \sum_{n=0}^{N} a_n \prod_{i=0}^{n-1} \beta_i + \sum_{n=1}^{+\infty} \frac{a_{-n}}{\prod_{i=1}^{n} \beta_{-i}}.$$

Maximal digit property

Definition

An alternate base $\mathcal{B}$ of period p has the *maximal digit property* (MDP) if for every $k \in \mathbb{Z}_p$ the expansion $d_{\sigma^{-k}\mathcal{B}}(1)$ satisfies

$$(d_{\sigma^{-k}(\mathcal{B})}(1))_j = \lceil \beta_{k-j} \rceil - 1 \qquad (1)$$

for every $j \in \mathbb{N},\ 1 < j < \left| d_{\sigma^{-k}(\mathcal{B})}(1) \right|$.

Example

If $d_{\mathcal{B}}(1) = \cdot 3231$ and $d_{\sigma(\mathcal{B})}(1) = \cdot 2322$, then $\mathcal{B}$ has the maximal digit property.
If $d_{\mathcal{B}}(1) = \cdot 32$ and $d_{\sigma(\mathcal{B})}(1) = \cdot 2222$, then $\mathcal{B}$ does not have the maximal digit property.

Notice that this implies that all $d_{\sigma^{-k}(\mathcal{B})}(1)$ are finite, and that when $p = 1$ this is the condition we have seen before.

Associated rewriting system

With an alternate base $\mathcal{B}$, we can associate a rewriting system $\rho_{\mathcal{B}}$. We assume that all $d_{\sigma^{-k}(\mathcal{B})}(1)$ are finite, for simplicity.
If $d_{\sigma^{-k}(\mathcal{B})}(1) = t_1^{(k)} \cdots t_\ell^{(k)}$, we allow the core rule

$$0t_1^{(k)} \cdots t_\ell^{(k)} \to 10^\ell,$$

but *only* if it is "correctly aligned", i.e. if the leftmost digit is at a position congruent to $k \bmod p$.
We then build a rewriting system $\rho_{\mathcal{B}}$ from the p core rules associated with $d_{\mathcal{B}}(1), \ldots, d_{\sigma^{-p+1}(\mathcal{B})}(1)$.

Main result

Theorem

Let $\mathcal{B}$ be an alternate base of period p. Then the following statements are equivalent:

(a) $X_{\sigma^i(\mathcal{B})} = \mathbb{N}_{\sigma^i(\mathcal{B})}$ *for all integers i.*

(b) *$\mathcal{B}$ is optimal.*

(c) *$\mathcal{B}$ has the maximal digit property (MDP).*

Theorem

Let $\mathcal{B}$ be an alternate base of period p such that the $d_{\sigma^{-i}(\mathcal{B})}(1)$ are all finite. Then the following statements are equivalent.

(c) *The base $\mathcal{B}$ has the MDP.*

(d) *The rewriting system $\rho_{\mathcal{B}}$ is confluent over finite words on the alphabet A.*

(e) *The rewriting system $\rho_{\mathcal{B}}$ allows normalization of finite words in base $\mathcal{B}$.*

Sketch of proof (1)

When $\mathcal{B}$ does not have the maximal digit property, one can build counterexamples to (a)-(e) from a similar scheme.

Example

Consider a base of period 2 such that $d_{\mathcal{B}}(1) = \cdot 32$ and $d_{\sigma^{-1}(\mathcal{B})}(1) = \cdot 412$ (no MDP). The representations in the last three rows have the same value.

0	4	1	2		
		0	4	1	2
0	4	1	4	1	2
1	0	0	2	1	2
0	4	2	0	0	0

These three representations provide a counterexample for confluence and normalization, and when correctly scaled the last two provide a counterexample for optimality and equality of spectrum.

Sketch of proof (2)

When $\mathcal{B}$ has the MDP, normalization is proven by showing that all non-admissible representations can be reduced, confluence follows from normalization, and optimality and equality of spectrum are proven similarly.

Example

Consider a base of period 2 such that $d_{\mathcal{B}}(1) = \cdot 341$ and $d_{\sigma^{-1}(\mathcal{B})}(1) = \cdot 431$.
Then a non-admissible representation must contain one of the factors $|34|2$, $|34|3$, $4|32|$, $4|33|$ or $4|34|$, all of which can be rewritten by $\rho_{\mathcal{B}}$.

Conclusion

The assumption that the $d_{\sigma^{-i}(\boldsymbol{\mathcal{B}})}(1)$ are all finite can be dropped in the second theorem.

Take-home message: there is a family of numeration systems with additional nice properties (normalization, confluence, optimality, equality of spectrum) that generalizes to the alternate base case.

Thank you for your attention!