Absolute continuity of finite-dimensional distributions of Hermite processes via Malliavin calculus

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Abstract

We investigate the existence of densities for finite-dimensional distributions of Hermite processes of order $q \geq 1$ and self-similarity parameter $H \in (\frac{1}{2}, 1)$. Whereas the Gaussian case $q = 1$ (fractional Brownian motion) is well understood, the non-Gaussian situation has not yet been settled. In this work, we extend the classical three-step approach used in the Gaussian case: factorization of the determinant into conditional terms, strong local nondeterminism, and non-degeneracy. We transport this strategy to the Hermite setting using Malliavin calculus. Specifically, we establish a determinant identity for the Malliavin matrix, prove strong local nondeterminism at the level of Malliavin derivatives, and apply the Bouleau-Hirsch criterion. Consequently, for any distinct times $t_1, \ldots, t_n$, the vector $(Z^{H,q}_{t_1}, \ldots, Z^{H,q}_{t_n})$ of a Hermite process admits a density with respect to the Lebesgue measure. Beyond the result itself, the main contribution is the methodology, which could extend to other non-Gaussian models.

Keywords: Hermite processes; Rosenblatt process; Malliavin calculus; Bouleau–Hirsch criterion; strong local nondeterminism; density of finite-dimensional distributions

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1 Introduction

A cornerstone of Malliavin calculus is the *Bouleau–Hirsch criterion* [2], which asserts that a random vector $(Z_1, \ldots, Z_n)$ with Malliavin differentiable components has an absolutely continuous distribution with respect to the Lebesgue measure on $\mathbb{R}^n$ as soon as its *Malliavin matrix*

$$\Big(\langle DZ_i, DZ_j \rangle \Big)_{1 \leq i,j \leq n}$$

is almost surely non-singular. This extends the Gaussian situation, where absolute continuity is equivalent to the non-degeneracy of the covariance matrix, which in that case coincides with the Malliavin matrix.

The case of vectors of multiple Wiener–Itô integrals has been recently investigated. For $n = 1$, the situation is fully understood: a classical theorem of Shigekawa [15] ensures that any nontrivial

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multiple integral has a density with respect to Lebesgue measure. For higher-dimensional vectors, the problem becomes subtler. Nourdin, Nualart and Poly [7] established a general dichotomy: if the components of a vector belong to a finite sum of Wiener chaoses, then either its law is absolutely continuous, or there exists a nontrivial polynomial relation annihilating the vector; in particular, the existence of a density can often be decided from structural properties of the chaos expansions. In a complementary direction, Nualart and Tudor [10] analyzed the case of a *pair* $(I_q(f), I_q(g))$ of multiple integrals of the same order. They proved that such a pair fails to have a density if and only if the two integrals are proportional. They also noticed that this criterion does not hold in dimension bigger or equal than three. Their result provides a sharp criterion in dimension two, while the present paper focuses on vectors of *arbitrary* size with a specific stochastic process structure.

Our object of study is the class of *Hermite processes* of order $q \geq 1$ and self-similarity parameter $H \in (\frac{1}{2}, 1)$. They are non-Gaussian (except for $q = 1$), self-similar, have stationary increments, live in the q-th Wiener chaos, and arise naturally as limits in the non-central limit theorems of Dobrushin–Major [4] and Taqqu [17]. For $q = 1$, one recovers *fractional Brownian motion*, the unique Gaussian self-similar process with stationary increments. The case $q = 2$ corresponds to the *Rosenblatt process*, first identified by Taqqu [16] in the study of quadratic functionals of long-range dependent Gaussian sequences, and also appearing in Rosenblatt's work [13] on non-Gaussian limit laws. The Rosenblatt process has since become the canonical example of a non-Gaussian, long-memory, self-similar process, often regarded as the natural analogue of fractional Brownian motion in the non-Gaussian world (see also Tudor [19]). For higher orders $q \geq 3$, Hermite processes provide further canonical models of long-range dependence beyond Gaussianity.

From a probabilistic perspective, Hermite processes occupy a central place in the limit theory of dependent time series. Whenever one considers normalized partial sums of a Gaussian sequence with long memory, transformed by a non-linear function of Hermite rank q, the limit in distribution is a Hermite process of order q. This universality explains their fundamental role in non-central limit theorems, time series analysis, and statistical inference for long-memory models. We refer to the surveys of Taqqu [18], the monograph of Tudor [20], and the book by Pipiras and Taqqu [11] for a detailed account of their properties and applications.

By the Bouleau–Hirsch criterion [2], to prove that the finite-dimensional vector $(Z_1, \ldots, Z_n)$ has a density it is enough to show that its Malliavin matrix is almost surely non-singular. In the case of Hermite processes, our strategy is to control the determinant of this matrix by combining two ingredients: a factorization identity for the Malliavin matrix, and a strong local nondeterminism estimate formulated at the level of Malliavin derivatives. This leads to the following theorem.

Theorem 1.1. *Let $n \geq 2$ and let $t_1, \cdots, t_n$ be distinct times in $(0, \infty)$. Let $Z^{H,q}$ be a Hermite process of order $q \geq 1$ and self-similarity index $H \in (\frac{1}{2}, 1)$. Then, the random vector $(Z^{H,q}_{t_1}, \ldots, Z^{H,q}_{t_n})$, admits a density with respect to Lebesgue measure on $\mathbb{R}^n$.*

In order to motivate our approach for proving Theorem 1.1, let us first recall the classical Gaussian case $q = 1$ (fractional Brownian motion). In that case, a simple yet elegant three-step argument establishes that the finite-dimensional distributions possess a density; in other words, Theorem 1.1

holds when $q = 1$. This classical scheme will serve as a blueprint for our generalization to the Hermite setting.

Sketch of the proof of Theorem 1.1 when $q = 1$. Let $B^H = (B_t^H)_{t\geq 0}$ denote a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Fix $0 < t_1 < \cdots < t_n$ and set $Z_k := B_{t_k}^H$.

Step 1 (determinant and conditional variances). For any Gaussian vector, the determinant of its covariance matrix is equal to the product of conditional variances (see, e.g., [1, Eq. (2.8) p. 71]):

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) \;=\; \mathrm{Var}(Z_1) \prod_{k=2}^{n} \mathrm{Var}(Z_k \mid Z_1, \ldots, Z_{k-1}). \tag{1}$$

Step 2 (SLND). By using the self-similarity and the stationarity of the increments, Pitt [12, Lemma 7.1] showed that the fractional Brownian motion enjoys the *strong local nondeterminism* property: there exists $c_H > 0$ such that

$$\mathrm{Var}\Big(Z_k \,\Big|\, Z_1, \ldots, Z_{k-1}\Big) \;\geq\; c_H\,(t_k - t_{k-1})^{2H}, \qquad k = 2, \ldots, n. \tag{2}$$

Step 3 (conclusion). Since $(Z_1, \ldots, Z_n)$ is Gaussian, non-degeneracy of its covariance matrix implies absolute continuity. Combining Steps 1 and 2, we obtain

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) \;\geq\; c_H^n \prod_{k=1}^{n} (t_k - t_{k-1})^{2H} \;>\; 0,$$

so the law of $(Z_1, \ldots, Z_n)$ has a density.

In the non-Gaussian case $q \geq 2$, we adopt the same guiding philosophy, but the passage from the Gaussian to the non-Gaussian setting requires substantial innovations: the covariance matrix is replaced by the Malliavin matrix, and the determinant identity (1) for the covariance matrix by the corresponding identity for the Malliavin matrix, see Lemma 4.1 below. Likewise, SLND is replaced by a new analogue at the level of Malliavin derivatives. To establish this extension of SLND, we prove a self-similarity and stationarity of increments property for the Malliavin derivative.

This new framework not only yields absolute continuity of Hermite finite-dimensional distributions but, more importantly, it provides a robust methodological advance: we design a systematic way to transport the classical Gaussian scheme into the non-Gaussian Malliavin setting. Beyond the scope of Hermite processes, we believe this methodology opens the door to treating absolute continuity questions for a wide range of non-Gaussian processes representable as multiple Wiener–Itô integrals.

The rest of the paper is organized as follows. Section 2 recalls the necessary background on Malliavin calculus. In Section 3 we present the definition and main properties of Hermite processes. Section 4 contains a determinant identity for the Malliavin matrix. In Section 5 we establish self-similarity and stationarity of increments for the Malliavin derivative. Section 6 develops the analogue of the strong local nondeterminism property in this framework. Finally, Section 7 combines these ingredients with the Bouleau–Hirsch criterion to prove Theorem 1.1.

2 Preliminaries on Malliavin calculus

This section recalls the basic tools of Malliavin calculus that will be used throughout the paper. We refer to the standard references [8, 9, 14] for a complete account.

Isonormal Gaussian process. Let $B = \{B(t), t \in \mathbb{R}\}$ be a two-sided Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $B(0) = 0$. We denote by $\mathfrak{H} = L^2(\mathbb{R})$ the canonical Hilbert space associated with B. For each $h \in \mathfrak{H}$, the Wiener integral

$$B(h) = \int_{\mathbb{R}} h(t)\, dB(t)$$

is defined first for step functions and then extended by density. The family $\{B(h) : h \in \mathfrak{H}\}$ forms an *isonormal Gaussian process* over $\mathfrak{H}$, that is, a centred Gaussian family satisfying

$$\mathbb{E}[B(h)B(g)] = \langle h, g\rangle_{\mathfrak{H}}, \qquad h, g \in \mathfrak{H}.$$

Hermite polynomials and Wiener chaoses. Let $(H_q)_{q\geq 0}$ be the Hermite polynomials, defined by the recursion

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_{q+1}(x) = xH_q(x) - qH_{q-1}(x).$$

For $q \geq 1$, the *qth Wiener chaos* $\mathcal{H}_q$ is defined as the closed linear span in $L^2(\Omega)$ of the random variables

$$\{H_q(B(h)) : h \in \mathfrak{H}, \ \|h\|_{\mathfrak{H}} = 1\}.$$

In particular, $\mathcal{H}_1$ is the Gaussian chaos generated by $B(h)$, while $\mathcal{H}_q$ is non-Gaussian as soon as $q \geq 2$. The family $(\mathcal{H}_q)_{q\geq 0}$ forms an orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

known as the *Wiener–Itô chaos expansion.*

Multiple Wiener–Itô integrals. For $q \geq 1$ and $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$, we define

$$I_q(h^{\otimes q}) := H_q(B(h)).$$

If $\|h\|_{\mathfrak{H}} \neq 1$, one sets $I_q(h^{\otimes q}) := \|h\|_{\mathfrak{H}}^q H_q(B(h/\|h\|_{\mathfrak{H}}))$. For a general elementary tensor $h_1 \otimes \cdots \otimes h_q \in \mathfrak{H}^{\otimes q}$ (the symmetric tensor product), one defines $I_q(h_1 \otimes \cdots \otimes h_q)$ by polarization, and for a general symmetric function $f \in \mathfrak{H}^{\odot q}$, one extends linearly and by density. This yields a well-defined linear operator

$$I_q : \mathfrak{H}^{\odot q} \to L^2(\Omega).$$

satisfying the isometry

$$\mathbb{E}[I_p(f)I_q(g)] = \begin{cases} p!\, \langle f, g\rangle_{\mathfrak{H}^{\otimes p}}, & p = q, \\ 0, & p \neq q, \end{cases}$$

for all $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$.

One has the fundamental equivalence

$$\mathcal{H}_q = \{I_q(f) : f \in \mathfrak{H}^{\odot q}\}, \qquad q \geq 1,$$

which shows that the two constructions of Wiener chaos (via Hermite polynomials or via multiple integrals) are identical.

Product formula for multiple integrals. A key tool is the product formula: for $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$,

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}\big(f\widetilde{\otimes}_r g\big), \tag{3}$$

where $f \otimes_r g$ denotes the r-contraction of f and g, defined by

$$\begin{aligned}(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{\mathbb{R}^r} & f(u_1, \ldots, u_r, t_1, \ldots, t_{p-r}) \\ & \times g(u_1, \ldots, u_r, t_{p-r+1}, \ldots, t_{p+q-2r})\, du_1 \cdots du_r, \end{aligned} \tag{4}$$

for $1 \leq r \leq p \wedge q$, and $f \otimes_0 g = f \otimes g$ is the tensor product. In general, $f \otimes_r g \in \mathfrak{H}^{\otimes(p+q-2r)}$ is not symmetric, and we denote by $f\widetilde{\otimes}_r g$ its symmetrization. For more details, see e.g. Section 1.1.2 in [9].

Malliavin derivative. Let $\mathcal{S}$ be the class of *smooth cylindrical random variables* of the form

$$F = f(B(g_1), \ldots, B(g_n)),$$

with $n \geq 1$, $g_1, \ldots, g_n \in \mathfrak{H}$, and $f \in C_P^\infty(\mathbb{R}^n)$ (i.e., smooth with all derivatives of polynomial growth). The *Malliavin derivative* of F is the stochastic process

$$D_r F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(g_1), \ldots, B(g_n))\, g_j(r), \qquad r \in \mathbb{R}.$$

By iteration, one defines higher-order derivatives $D^k F$. For $k, p \geq 1$, the Sobolev-type space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}\big[\|D^j F\|_{\mathfrak{H}^{\otimes j}}^p\big],$$

and we set $\mathbb{D}^\infty = \bigcap_{k,p\geq 1} \mathbb{D}^{k,p}$. When $F, G \in \mathbb{D}^{1,2}$, the inner product of their derivatives reads

$$\langle DF, DG\rangle_{\mathfrak{H}} = \int_{\mathbb{R}} D_r F\, D_r G\, dr.$$

Bouleau–Hirsch criterion. A central application of Malliavin calculus is the analysis of regularity of distributions. As already mentioned, the following classical result will play a key role.

Theorem 2.1 (Bouleau–Hirsch [2])**.** *Let $\mathbf{Z} = (Z_1, \ldots, Z_n)$ be a random vector such that each $Z_i \in \mathbb{D}^{1,p}$ for some $p > 1$. If its Malliavin matrix*

$$\Gamma_{\mathbf{Z}} = \big(\langle DZ_i, DZ_j\rangle_{\mathfrak{H}}\big)_{1\leq i,j\leq n}$$

is a.s. non-degenerate, i.e. $\det(\Gamma_{\mathbf{Z}}) > 0$ *a.s., then the law of $\mathbf{Z}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.*

3 Background on Hermite processes

We now introduce Hermite processes and recall some of their fundamental properties. For a detailed account, we refer to Tudor's monograph [20].

Definition 3.1. Let $q \geq 1$ and $H \in (\frac{1}{2}, 1)$. The *Hermite process* $(Z_t^{H,q})_{t\in\mathbb{R}}$ of order q and self-similarity parameter H is defined by

$$Z_t^{H,q} \; = \; I_q(L_t^{H,q}), \qquad L_t^{H,q}(\xi) = c(H,q) \int_0^t \prod_{j=1}^q (s-\xi_j)_+^{H_0 - \frac{3}{2}}\, ds, \;\; t \in \mathbb{R}, \tag{5}$$

where $H_0 = 1 + \frac{H-1}{q} \in (1 - \frac{1}{2q}, 1)$ and

$$c(H,q) = \sqrt{\frac{H(2H-1)}{q!\, \beta^q(H_0 - \frac{1}{2},\, 2 - 2H_0)}}.$$

Here $\beta(\cdot,\cdot)$ denotes the Beta function, and I_q is the multiple Wiener–Itô integral of order q.

Remark 3.2.

- The constant $c(H,q)$ has been chosen so that $\mathbb{E}[(Z_1^{H,q})^2] = 1$.
- We use the convention $\theta_+^\alpha = \theta^\alpha$ if $\theta > 0$, and 0 otherwise. Integrals on negative intervals are defined by $\int_0^t = -\int_t^0$ when $t < 0$. More generally, for any $t < s$, we take $\int_s^t = -\int_t^s$.
- In the literature, Hermite processes are usually defined only for $t \geq 0$. In analogy with the moving-average representation of fractional Brownian motion (see, e.g., [5], pp. 9–10), we have extended their definition here to the whole real line.
- For each fixed $t \in \mathbb{R}$, the random variable $Z_t^{H,q}$ is a multiple Wiener–Itô integral.
- The class of Hermite processes contains important examples:
  - for $q = 1$, one recovers fractional Brownian motion,
  - for $q = 2$, one obtains the Rosenblatt process,
  - for $q \geq 3$, one obtains higher-order non-Gaussian processes.

  Fractional Brownian motion is the only Gaussian Hermite process; all others are non-Gaussian.

We state the key properties of Hermite processes (see [20] for proofs in the case $t \geq 0$; the extension to $t \in \mathbb{R}$ being immediate). To this aim, all along this paper, $\overset{\text{law}}{=}$ stands for the equality in finite-dimensional distributions for stochastic processes.

- *Self-similarity.* For every $a > 0$,

$$(Z_{at}^{H,q})_{t\in\mathbb{R}} \;\overset{\text{law}}{=}\; (a^H Z_t^{H,q})_{t\in\mathbb{R}}.$$

- *Stationary increments.* For every $h \in \mathbb{R}$,

$$(Z_{t+h}^{H,q} - Z_h^{H,q})_{t\in\mathbb{R}} \;\overset{\text{law}}{=}\; (Z_t^{H,q})_{t\in\mathbb{R}}.$$

- *Covariance function.* For all $s, t \in \mathbb{R}$,

$$\mathbb{E}[Z_t^{H,q} Z_s^{H,q}] \; = \; \tfrac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big),$$

which coincides with the covariance of fractional Brownian motion with Hurst index H.

- *Adaptedness.* The process $(Z_t^{H,q})_{t\in\mathbb{R}}$ is adapted to the following filtration: for $t \in \mathbb{R}$,

$$\mathcal{F}_t = \sigma\{B(\mathbb{1}_A) \, : A \text{ is a Borel } \lambda\text{-finite subset of } (-\infty, t]\} \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the $\mathbb{P}$-null subsets of Ω and λ is the Lebesgue measure on $\mathbb{R}$. Hence, according to [9, Corollary 1.2.1], we have, for every $t \in \mathbb{R}$,

$$D_r Z_t^{H,q} = 0 \qquad \text{for } \mathbb{P}\otimes\lambda\text{-a.e. } (\omega, r) \in \Omega \times (t, \infty). \tag{6}$$

Remark 3.3. *From stationarity of increments, one readily checks that* $Z_t^{H,q} \overset{law}{=} -Z_{-t}^{H,q}$ *for every* $t \in \mathbb{R}$.

Finally, the following lemma will be useful.

Lemma 3.4. *Let* $Z^{H,q} = \left(Z_t^{H,q}\right)_{t\in\mathbb{R}}$ *be the Hermite process of order* $q \geqslant 1$ *and self-similarity parameter* $H \in \left(\frac{1}{2}, 1\right)$. *We have*

$$\|DZ_1^{H,q}\|^2_{\mathfrak{H}_1} > 0 \qquad \mathbb{P}\text{-a.s.} \tag{7}$$

where $\mathfrak{H}_1 = L^2([0,1])$ *and* $\|f\|^2_{\mathfrak{H}_1} = \int_0^1 |f(x)|^2 dx$.

In order to prove this lemma, we will need the following result.

Lemma 3.5. *Let F be a nonnegative random variable belonging to a finite sum of Wiener chaoses. Then, the following are equivalent:*

(i) $\mathbb{E}[F] > 0$;

(ii) $F > 0$ $\mathbb{P}$-*a.s.*

Proof of Lemma 3.5. We just need to prove $(i) \Rightarrow (ii)$. Let N be a positive integer such that $F \in \bigoplus_{k=0}^N \mathcal{H}_k$. In the proof of [7, Theorem 3.1] (implication $(a) \Rightarrow (c)$), it is shown that, for all such nonnegative F, there exists an universal constant $c > 0$ such that, for any $\lambda > 0$, one has the small ball probability estimate

$$\mathbb{P}(F \leq \lambda) \leq cN\lambda^{1/N}(\mathbb{E}[F])^{-1/N}.$$

Thus, letting $\lambda \to 0$, we obtain $\mathbb{P}(F = 0) = 0$. This completes the proof. □

Now, we are able to give the proof of Lemma 3.4.

Proof of Lemma 3.4. Let us notice that $\|DZ_1^{H,q}\|^2_{\mathfrak{H}_1} \in \bigoplus_{k=0}^{2(q-1)} \mathcal{H}_k$ and

$$\begin{aligned} &\mathbb{E}\left[\|DZ_1^{H,q}\|^2_{\mathfrak{H}_1}\right] \\ &= c(H,q)^2\, q^2 (q-1)! \int_0^1 dx_1 \int_{\mathbb{R}^{q-1}} \prod_{l=2}^q dx_l \left\{ \int_0^1 ds \prod_{k=1}^q (s - x_k)_+^{H_0 - \frac{3}{2}} \right\}^2 > 0. \end{aligned}$$

Therefore, by Lemma 3.5, we conclude that $\|DZ_1^{H,q}\|^2_{\mathfrak{H}_1} > 0$ $\mathbb{P}$-a.s. □

4 A determinant identity for the Malliavin matrix

We now state our determinant identity for the Malliavin matrix, which should be seen as the analogue of identity (1) in the Gaussian case for the covariance matrix. Our proof is nothing but an adaptation of the classical argument used to establish the determinant of a Gram matrix.

Lemma 4.1. *Let $\mathbf{Z} = (Z_1, \dots, Z_n) \in (\mathbb{D}^{1,2})^n$, and let $\Gamma_{\mathbf{Z}}$ denote its Malliavin matrix. Then, almost surely,*

$$\det(\Gamma_{\mathbf{Z}}) = \|DZ_1\|_{\mathfrak{H}}^2 \prod_{j=2}^{n} \left\| DZ_j - \operatorname{proj}_{E_{j-1}}(DZ_j) \right\|_{\mathfrak{H}}^2, \tag{8}$$

where E_{j-1} is the linear span in $\mathfrak{H}$ of $\{DZ_1, \dots, DZ_{j-1}\}$, and $\operatorname{proj}_{E_{j-1}}$ denotes the orthogonal projection onto E_{j-1}.

Remark 4.2. *Before digging into the proof of Lemma 4.1, let us mention that for almost every $\omega \in \Omega$, $DZ_1(\omega), \dots, DZ_{j-1}(\omega) \in \mathfrak{H}$ and thus the subspace $E_{j-1}(\omega)$ is spanned by finitely many vectors and is therefore finite-dimensional. Hence, for any such ω, $E_{j-1}(\omega)$ is automatically closed in $\mathfrak{H}$. It means that, the orthogonal projection $\operatorname{proj}_{E_{j-1}(\omega)}$ is almost surely well-defined.*

Proof of Lemma 4.1. Recall that the Malliavin matrix of $\mathbf{Z}$ has entries

$$\Gamma_{i,j} = \langle DZ_i, DZ_j \rangle_{\mathfrak{H}}, \qquad 1 \le i, j \le n.$$

Fix $i = n$. We decompose

$$\begin{aligned} \langle DZ_j, DZ_n \rangle_{\mathfrak{H}} &= \langle DZ_j, \operatorname{proj}_{E_{n-1}}(DZ_n) \rangle_{\mathfrak{H}} + \langle DZ_j, DZ_n - \operatorname{proj}_{E_{n-1}}(DZ_n) \rangle_{\mathfrak{H}} \\ &=: A_{j,n} + B_{j,n}. \end{aligned}$$

Since $DZ_n - \operatorname{proj}_{E_{n-1}}(DZ_n)$ is orthogonal to E_{n-1}, we immediately obtain

$$B_{j,n} = 0, \qquad j = 1, \dots, n-1.$$

By multilinearity of the determinant, we may split

$$\det(\Gamma_{\mathbf{Z}}) = \det(A_{\mathbf{Z}}) + \det(B_{\mathbf{Z}}),$$

where $\det(A_{\mathbf{Z}})$ is obtained by replacing the last column of $\Gamma_{\mathbf{Z}}$ with $(A_{1,n}, \dots, A_{n,n})^\top$, and $\det(B_{\mathbf{Z}})$ with $(B_{1,n}, \dots, B_{n,n})^\top$. But by definition of the projection, the last column of $A_{\mathbf{Z}}$ is a linear combination of the first $n-1$, hence $\det(A_{\mathbf{Z}}) = 0$.

On the other hand, $\det(B_{\mathbf{Z}})$ has the block form

$$\begin{aligned} \det(B_{\mathbf{Z}}) &= \det \begin{pmatrix} \left(\langle DZ_i, DZ_j \rangle_{\mathfrak{H}} \right)_{1 \le i,j \le n-1} & 0 \\ * & B_{n,n} \end{pmatrix} \\ &= B_{n,n} \det \left(\langle DZ_i, DZ_j \rangle_{\mathfrak{H}} \right)_{1 \le i,j \le n-1}. \end{aligned}$$

Since $B_{n,n} = \|DZ_n - \operatorname{proj}_{E_{n-1}}(DZ_n)\|_{\mathfrak{H}}^2$, this yields

$$\det(\Gamma_{\mathbf{Z}}) = \|DZ_n - \operatorname{proj}_{E_{n-1}}(DZ_n)\|_{\mathfrak{H}}^2 \det \left(\langle DZ_i, DZ_j \rangle_{\mathfrak{H}} \right)_{1 \le i,j \le n-1}.$$

The desired factorization (8) then follows by induction on n. □

5 Malliavin self-similarity and Malliavin stationarity of increments of Hermite processes

To establish the analogue of the strong local nondeterminism estimate (2) for Hermite processes, we first need to strengthen the classical properties of self-similarity and stationarity of increments, now at the level of their Malliavin derivative. This is the content of the following proposition.

Proposition 5.1. *Let $Z^{H,q} = \{Z_t^{H,q},\, t \in \mathbb{R}\}$ be the Hermite process of order $q \geq 1$ and self-similarity parameter $H \in (\frac{1}{2}, 1)$ given by (5). Then the Malliavin derivative of $Z^{H,q}$ inherits both stationarity of increments and self-similarity: for all $a \in \mathbb{R}$ and $c > 0$,*

$$\Big(\langle DZ_{s+a}^{H,q} - DZ_a^{H,q},\, DZ_{t+a}^{H,q} - DZ_a^{H,q} \rangle_{\mathfrak{H}} \Big)_{s,t\in\mathbb{R}} \overset{\text{law}}{=} \Big(\langle DZ_s^{H,q},\, DZ_t^{H,q} \rangle_{\mathfrak{H}} \Big)_{s,t\in\mathbb{R}},$$

and

$$\Big(\langle DZ_{cs}^{H,q},\, DZ_{ct}^{H,q} \rangle_{\mathfrak{H}} \Big)_{s,t\in\mathbb{R}} \overset{\text{law}}{=} c^{2H} \Big(\langle DZ_s^{H,q},\, DZ_t^{H,q} \rangle_{\mathfrak{H}} \Big)_{s,t\in\mathbb{R}}.$$

In order to prove this proposition, we will need the following lemma.

Lemma 5.2. *Let $L_t^{H,q}$ be given by (5). Hence, we have the following identities:*

(a) *For any $c > 0$, $t_1, t_2 \in \mathbb{R}$, $r = 0, \ldots, q-1$, and $y^1, y^2 \in \mathbb{R}^{q-1-r}$,*

$$(L_{ct_1}^{H,q} \otimes_{r+1} L_{ct_2}^{H,q})(y^1, y^2) = c^{2H-(q-1-r)} a(H,q,r) \int_0^{t_1} dv_1 \int_0^{t_2} dv_2 |v_1 - v_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2} \prod_{\ell=1}^{q-(r+1)} (v_j - \frac{y_\ell^j}{c})_+^{\frac{H-1}{q} - \frac{1}{2}}, \tag{9}$$

where $a(H,q,r) := c(H,q)^2 \beta\left(\frac{2-2H}{q}, \frac{1}{2} - \frac{1-H}{q}\right)^{r+1}$ and β is the beta function.

(b) *For any $a \in \mathbb{R}$, $t_1, t_2 \in \mathbb{R}$, $r = 0, \ldots, q-1$, and $y^1, y^2 \in \mathbb{R}^{q-1-r}$,*

$$(L_{t_1+a}^{H,q} - L_a^{H,q}) \otimes_{r+1} (L_{t_2+a}^{H,q} - L_a^{H,q})(y^1, y^2) = a(H,q,r) \int_a^{t_1+a} du_1 \int_a^{t_2+a} du_2 |u_1 - u_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2} \prod_{\ell=1}^{q-1-r} (u_j - y_\ell^j)_+^{\frac{H-1}{q} - \frac{1}{2}}.$$

Proof. Let us first consider the point (a). Let $c > 0$, $t_1, t_2 \in \mathbb{R}$, $r = 0, \ldots, q-1$, and $y^1, y^2 \in \mathbb{R}^{q-1-r}$. Hence, we have

$$\begin{aligned}
&(L_{ct_1}^{H,q} \otimes_{r+1} L_{ct_2}^{H,q})(y^1, y^2) \\
&= c(H,q)^2 \int_{\mathbb{R}^{r+1}} \prod_{i=1}^{r+1} dx_i \prod_{j=1}^{2} \int_0^{ct_j} du_j \prod_{k=1}^{r+1} (u_j - x_k)_+^{\frac{H-1}{q} - \frac{1}{2}} \prod_{\ell=1}^{q-1-r} (u_j - y_\ell^j)_+^{\frac{H-1}{q} - \frac{1}{2}} \\
&= a(H,q,r) \int_0^{ct_1} du_1 \int_0^{ct_2} du_2 |u_1 - u_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2} \prod_{\ell=1}^{q-1-r} (u_j - y_\ell^j)_+^{\frac{H-1}{q} - \frac{1}{2}},
\end{aligned} \tag{10}$$

where $a(H,q,r) := c(H,q)^2\beta\left(\frac{2-2H}{q}, \frac{1}{2} - \frac{1-H}{q}\right)^{r+1}$, β is the beta function and, in the last equality, we used the following formula: for $-1 < a < -\frac{1}{2}$,

$$\int_{\mathbb{R}} (u-y)_+^a (v-y)_+^a dy = \beta(-1-2a, a+1)|u-v|^{2a+1}.$$

Now, By using the changes of variables $v_j = \frac{u_j}{c}$ of Jacobian c, for $j = 1,2$, we obtain that the right-hand side of (10) is equal to

$$c^2 a(H,q,r)\int_0^{t_1} dv_1 \int_0^{t_2} dv_2 |cv_1 - cv_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2}\prod_{\ell=1}^{q-(r+1)} (cv_j - y_\ell^j)_+^{\frac{H-1}{q}-\frac{1}{2}}.$$

We reach the equality (9) by factoring out c in the term $|cv_1 - cv_2|^{\frac{2(H-1)(r+1)}{q}}$ and the $2\times(q-1-r)$ terms of the forms $(cv_j - y_\ell^j)_+^{\frac{H-1}{q}-\frac{1}{2}}$ in this last expression.

Concerning the point (b), recalling the convention in the second point of Remark 3.2, we check that, for any $a, t \in \mathbb{R}$ and $y \in \mathbb{R}^q$, one has

$$(L_{t+a}^{H,q} - L_a^{H,q})(y) = c(H,q)\int_a^{t+a} du \prod_{j=1}^{q}(u-y_j)_+^{\frac{H-1}{q}-\frac{1}{2}}.$$

The proof of point (b) follows then the same spirit as the one of point (a). □

Proof of Proposition 5.1. Let us start by establishing that the Malliavin derivative of $Z^{H,q}$ is self-similar. Let $c > 0$ and $s, t \in \mathbb{R}$. According to the product formula (3) and the definition of the contraction (4), we have

$$\begin{aligned}\langle DZ_{cs}^{H,q}, DZ_{ct}^{H,q}\rangle_{\mathfrak{H}} &= q^2 \int_{\mathbb{R}} dx_q\, I_{q-1}\big(L_{cs}^{H,q}(\star, x_q)\big) I_{q-1}\big(L_{ct}^{H,q}(\star, x_q)\big)\\ &= q^2\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2(q-1-r)}\left(L_{cs}^{H,q}\widetilde{\otimes}_{r+1}L_{ct}^{H,q}\right), \qquad (11)\end{aligned}$$

where $L^{H,q}$ is given by (5) and, in the notation $L_{cs}^{H,q}(\star, x_q)$, the symbol $\star$ denotes the remaining $(q-1)$ free variables (i.e. the unfixed arguments of the kernel). The point (a) in Lemma 5.2 yields

$$\begin{aligned}&\langle DZ_{cs}^{H,q}, DZ_{ct}^{H,q}\rangle_{\mathfrak{H}}\\ &= c^{2H-(q-1-r)} q^2 \sum_{r=0}^{q-1} a(H,q,r) r!\binom{q-1}{r}^2 \int_{\mathbb{R}^{2(q-1-r)}} \prod_{m=1}^{2}\prod_{k=1}^{q-1-r} dB(y_k^m)\\ &\qquad \times \int_0^s dv_1 \int_0^t dv_2 |v_1 - v_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2}\prod_{\ell=1}^{q-(r+1)} (v_j - \frac{y_\ell^j}{c})_+^{\frac{H-1}{q}-\frac{1}{2}}.\end{aligned}$$

Now, by the change of variables $z_\ell^j = \frac{y_\ell^j}{c}$, for $j = 1,2$, and $\ell = 1, ..., q-1-r$, we obtain that the right-hand side of the previous equality is equal to

$$\begin{aligned}&c^{2H-(q-1-r)} q^2 \sum_{r=0}^{q-1} a(H,q,r) r!\binom{q-1}{r}^2 \int_{\mathbb{R}^{2(q-1-r)}} \prod_{m=1}^{2}\prod_{k=1}^{q-1-r} dB(cz_k^m)\\ &\times \int_0^s dv_1 \int_0^t dv_2 |v_1 - v_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2}\prod_{\ell=1}^{q-(r+1)} (v_j - z_\ell^j)_+^{H_0-\frac{3}{2}},\end{aligned}$$

Using the scaling property of the Wiener process, let us observe that the finite-dimensional distributions of the two-parameter process on the right-hand side of the previous equality coincide in law with those of the two-parameter process given by:

$$
\begin{aligned}
& c^{2H-(q-1-r)} c^{q-r-1} q^2 \sum_{r=0}^{q-1} a(H,q,r) r! \binom{q-1}{r}^2 \int_{\mathbb{R}^{2(q-1-r)}} \prod_{m=1}^{2} \prod_{k=1}^{q-1-r} dB(z_k^m) \\
& \times \int_0^s dv_1 \int_0^t dv_2 |v_1 - v_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2} \prod_{\ell=1}^{q-(r+1)} (v_j - z_\ell^j)_+^{H_0 - \frac{3}{2}} \\
& = c^{2H} \langle DZ_s^{H,q},\, DZ_t^{H,q} \rangle_{\mathfrak{H}}.
\end{aligned}
$$

We now prove the stationarity of increments property for the Malliavin derivative of $Z^{H,q}$. By using the point (b) in Lemma 5.2, we have

$$
\begin{aligned}
& \langle D(Z_{s+a}^{H,q} - Z_a^{H,q}), D(Z_{t+a}^{H,q} - Z_a^{H,q}) \rangle_{\mathfrak{H}} \\
& = q^2 \sum_{r=0}^{q-1} r! \binom{q-1}{r}^2 I_{2(q-1)-2r} \left((L_{s+a}^{H,q} - L_a^{H,q}) \widetilde{\otimes}_{r+1} (L_{t+a}^{H,q} - L_a^{H,q}) \right) \\
& = q^2 \sum_{r=0}^{q-1} r! a(H,q,r) \binom{q-1}{r}^2 \int_{\mathbb{R}^{2(q-1)-2r}} \prod_{m=1}^{2} \prod_{k=1}^{q-1-r} dB(y_k^m) \\
& \times \int_a^{s+a} du_1 \int_a^{t+a} du_2 |u_1 - u_2|^{\frac{2(H-1)(r+1)}{q}} \prod_{j=1}^{2} \prod_{\ell=1}^{q-1-r} (u_j - y_\ell^j)_+^{H_0 - \frac{3}{2}}.
\end{aligned}
$$

Using the change of variable $v_j = u_j - a$, for $j = 1, 2$, and the stationarity of increments for the Wiener process B, we deduce:

$$
\left(\langle DZ_{s+a}^{H,q} - DZ_a^{H,q},\, DZ_{t+a}^{H,q} - DZ_a^{H,q} \rangle_{\mathfrak{H}} \right)_{s,t \in \mathbb{R}} \overset{\text{law}}{=} \left(\langle DZ_s^{H,q},\, DZ_t^{H,q} \rangle_{\mathfrak{H}} \right)_{s,t \in \mathbb{R}},
$$

which completes the proof of Proposition 5.1. □

6 Strong local nondeterminism for Hermite processes

In this section we establish the analogue of the strong local nondeterminism property for Hermite processes. The following theorem provides its precise formulation in the Malliavin framework.

Theorem 6.1. *Let $Z^{H,q} = \{Z_t^{H,q},\, t \in \mathbb{R}\}$ be the Hermite process of order $q \geq 1$ with self-similarity parameter $H \in (\frac{1}{2}, 1)$. For any $j \in \mathbb{N}^*$ and any time grid $0 = t_0 < t_1 < \cdots < t_{j-1} < t_j$, one has*

$$
\left\| DZ_{t_j}^{H,q} - \mathrm{proj}_{E_{j-1}} \left(DZ_{t_j}^{H,q} \right) \right\|_{\mathfrak{H}}^2 \overset{law}{\geq} (t_j - t_{j-1})^{2H} \, \| DZ_1^{H,q} \|_{\mathfrak{H}_1}^2, \tag{12}
$$

where $\mathfrak{H}_1 = L^2([0,1])$, $E_{j-1} = \mathrm{span}\{DZ_{t_1}^{H,q}, \ldots, DZ_{t_{j-1}}^{H,q}\}$, and the notation $X \overset{law}{\geq} Y$ means that

$$
\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)] \tag{13}
$$

for all non-decreasing bounded functions $f : \mathbb{R}_+ \to \mathbb{R}$.

Proof. Fix $j \in \mathbb{N}^*$ and a time grid $0 = t_0 < t_1 < \cdots < t_{j-1} < t_j$. Consider the random vector space

$$\widetilde{E}_{j-1} = \operatorname{span}\left\{D(Z^{H,q}_{t_k} - Z^{H,q}_{t_\ell}) : k, \ell = 0, \ldots, j-1\right\}.$$

Let $(a_1, \ldots, a_{j-1})$ be the random variables such that

$$\left\|DZ^{H,q}_{t_j} - \operatorname{proj}_{E_{j-1}}(DZ^{H,q}_{t_j})\right\|^2_{\mathfrak{H}} = \left\|DZ^{H,q}_{t_j} - \sum_{k=1}^{j-1} a_k DZ^{H,q}_{t_k}\right\|^2_{\mathfrak{H}}.$$

We claim that, almost surely

$$\begin{aligned} &\left\|DZ^{H,q}_{t_j} - \operatorname{proj}_{E_{j-1}}(DZ^{H,q}_{t_j})\right\|^2_{\mathfrak{H}} \\ \geq\ &\left\|D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}) - \operatorname{proj}_{\widetilde{E}_{j-1}}\big(D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}})\big)\right\|^2_{\mathfrak{H}}. \end{aligned} \tag{14}$$

Indeed, one can rewrite

$$\left\|DZ^{H,q}_{t_j} - \sum_{k=1}^{j-1} a_k DZ^{H,q}_{t_k}\right\|^2_{\mathfrak{H}} = \left\|D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}) - \sum_{k,\ell=0}^{j-1} b_{k,\ell} D(Z^{H,q}_{t_k} - Z^{H,q}_{t_\ell})\right\|^2_{\mathfrak{H}},$$

with coefficients $b_{k,0} = a_k$ for $k = 1, \ldots, j-2$, $b_{j-1,0} = a_{j-1} - 1$, and $b_{k,\ell} = 0$ otherwise. Taking the infimum over $(b_{k,\ell})$ yields (14).

Now, for any $(b_{k,\ell})$,

$$\begin{aligned} &\left\|D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}) - \sum_{k,\ell=0}^{j-1} b_{k,\ell} D(Z^{H,q}_{t_k} - Z^{H,q}_{t_\ell})\right\|^2_{\mathfrak{H}} \\ &= \Bigg\|D\big(Z^{H,q}_{t_{j-1}+(t_j-t_{j-1})} - Z^{H,q}_{t_{j-1}}\big) - \sum_{k,\ell=0}^{j-1} b_{k,\ell}\, D\big(Z^{H,q}_{t_{j-1}+(t_k-t_{j-1})} - Z^{H,q}_{t_{j-1}}\big) \\ &\qquad\qquad + \sum_{k,\ell=0}^{j-1} b_{k,\ell}\, D\big(Z^{H,q}_{t_{j-1}+(t_\ell-t_{j-1})} - Z^{H,q}_{t_{j-1}}\big)\Bigg\|^2_{\mathfrak{H}} \\ &=: X(b_{k,\ell}). \end{aligned}$$

By Malliavin stationarity of increments (Proposition 5.1), X has the same law as

$$\widetilde{X}(b_{k,\ell}) = \left\|DZ^{H,q}_{t_j-t_{j-1}} - \sum_{k,\ell=0}^{j-1} b_{k,\ell}\, D(Z^{H,q}_{t_k-t_{j-1}} - Z^{H,q}_{t_\ell-t_{j-1}})\right\|^2_{\mathfrak{H}}.$$

By Malliavin self-similarity (Proposition 5.1), the law of $\widetilde{X}$ is the same as that of

$$\hat{X}(b_{k,\ell}) = (t_j - t_{j-1})^{2H} \left\|DZ^{H,q}_1 - \sum_{k,\ell=0}^{j-1} b_{k,\ell}\, D\Big(Z^{H,q}_{\frac{t_k-t_{j-1}}{t_j-t_{j-1}}} - Z^{H,q}_{\frac{t_\ell-t_{j-1}}{t_j-t_{j-1}}}\Big)\right\|^2_{\mathfrak{H}}.$$

Therefore,

$$\begin{aligned} &\left\|D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}) - \operatorname{proj}_{\widetilde{E}_{j-1}}(D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}))\right\|^2_{\mathfrak{H}} \\ \overset{\text{law}}{=}\ &(t_j - t_{j-1})^{2H} \left\|DZ^{H,q}_1 - \operatorname{proj}_{\hat{E}_{j-1}}(DZ^{H,q}_1)\right\|^2_{\mathfrak{H}}, \end{aligned}$$

where

$$\hat{E}_{j-1} = \operatorname{span}\Big\{D\Big(Z^{H,q}_{\frac{t_k - t_{j-1}}{t_j - t_{j-1}}} - Z^{H,q}_{\frac{t_\ell - t_{j-1}}{t_j - t_{j-1}}}\Big) : k, \ell = 0, \dots, j-1\Big\}.$$

Finally, almost surely, for any $(b_{k,\ell})$,

$$\Big\|DZ^{H,q}_1 - \sum_{k,\ell=0}^{j-1} b_{k,\ell}\, D\Big(Z^{H,q}_{\frac{t_k - t_{j-1}}{t_j - t_{j-1}}} - Z^{H,q}_{\frac{t_\ell - t_{j-1}}{t_j - t_{j-1}}}\Big)\Big\|^2_{\mathfrak{H}} \;\geq\; \|DZ^{H,q}_1\|^2_{\mathfrak{H}_1},$$

since, according to (6), $D_r Z^{H,q}_{\frac{t_k - t_{j-1}}{t_j - t_{j-1}}}(\omega) = 0$ for all $k = 1, \dots, j-1$ and almost every $(\omega, r) \in \Omega \times [0,1]$. Thus, almost surely,

$$\big\|D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}) - \operatorname{proj}_{\hat{E}_{j-1}}(D(Z^{H,q}_{t_j} - Z^{H,q}_{t_{j-1}}))\big\|^2_{\mathfrak{H}} \;\geq\; \|DZ^{H,q}_1\|^2_{\mathfrak{H}_1},$$

which completes the proof. □

7 Proof of Theorem 1.1

We are now in position to prove our main result. Combining the determinant identity for the Malliavin matrix (Lemma 4.1) with the strong local nondeterminism property for Hermite processes (Theorem 6.1), we can apply the Bouleau–Hirsch criterion to conclude.

Proof of Theorem 1.1. Up to a permutation of the indices, one may assume $t_1 < \cdots < t_n$. Let $\mathbf{Z} = (Z^{H,q}_{t_1}, \dots, Z^{H,q}_{t_n})$, and denote by $\Gamma_{\mathbf{Z}}$ its Malliavin matrix. By the Bouleau–Hirsch criterion (Theorem 2.1), it is enough to prove that

$$\mathbb{P}(\det(\Gamma_{\mathbf{Z}}) > 0) = 1.$$

By Lemma 4.1, we can factorize the determinant as

$$\det(\Gamma_{\mathbf{Z}}) = \|DZ^{H,q}_{t_1}\|^2_{\mathfrak{H}} \prod_{j=2}^{n} \big\|DZ^{H,q}_{t_j} - \mathrm{proj}_{\mathrm{E}_{\mathrm{j}-1}}(\mathrm{DZ}^{\mathrm{H,q}}_{\mathrm{t_j}})\big\|^2_{\mathfrak{H}}, \tag{15}$$

where $E_{j-1} = \operatorname{span}\{DZ^{H,q}_{t_1}, \dots, DZ^{H,q}_{t_{j-1}}\}$.

Now, by the self-similarity of the Malliavin derivative of $Z^{H,q}$ (Proposition 5.1) and Lemma 3.4, the first factor is almost surely positive. For each $j = 2, \dots, n$, Theorem 6.1 yields

$$\big\|DZ^{H,q}_{t_j} - \operatorname{proj}_{E_{j-1}}(DZ^{H,q}_{t_j})\big\|^2_{\mathfrak{H}} \overset{\text{law}}{\geq} (t_j - t_{j-1})^{2H}\, \|DZ^{H,q}_1\|^2_{\mathfrak{H}_1},$$

with $\mathfrak{H}_1 = L^2([0,1])$. Since $\|DZ^{H,q}_1\|^2_{\mathfrak{H}_1} > 0$ almost surely (again by Lemma 3.4), and by taking $f = \mathbb{1}_{(0,\infty)}$ in (13), it follows that each factor in (15) is almost surely positive.

Therefore $\det(\Gamma_{\mathbf{Z}}) > 0$ almost surely, and by Theorem 2.1 the law of

$$(Z^{H,q}_{t_1}, \dots, Z^{H,q}_{t_n})$$

admits a density with respect to Lebesgue measure. This completes the proof of Theorem 1.1. □

Remark 7.1. *Let us keep the notation of the previous proof. In this work, we established the almost sure positivity of* $\det(\Gamma_Z)$*, which ensures the existence of a density for* Z*. However, this property alone is not sufficient to guarantee the smoothness of this density. It is well known that smoothness requires the finiteness of negative moments of* $\det(\Gamma_Z)$*, the so-called* non-degeneracy condition *(see, for instance, [9, Section 2.1.4]). A natural question arising from the present work is therefore whether our strategy can be extended to address this stronger requirement. In the case of the second Wiener chaos, partial answers in this direction are given in [6].*

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