Numerical investigation of a multi-step one-shot method for frequency domain acoustic full waveform inversion

A. Sior, B. Martin and C. Geuzaine University of Liège, Belgium

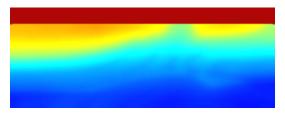
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Introduction

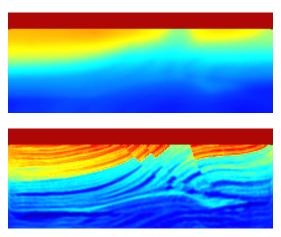
Full waveform inversion (FWI) (I)

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Full waveform inversion (FWI) (II)

In the frequency domain, FWI of the squared wave slowness field m amounts to the inversion of the forward Helmholtz problem

$$\begin{cases} \Delta_x u + \omega^2 m u &= b \\ \partial_n u + i\omega \sqrt{m} u &= 0 \end{cases} \Longrightarrow A(m)u = b$$

by minimizing an error functional J between observed data at receivers d and the waveform u induced by the squared wave slowness m

$$\underset{m}{\arg\min} J(u(m))$$
 with $u(m)=u$ such that $A(m)u=b$

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Minimization with first- and second-order methods requires the gradient of J(u(m)), obtained by solving the **forward** and **adjoint** problems.

Linearized inverse problem

We focus on the one-shot paradigm applied to solve of the **linearized inverse problem**. Linearizing around the background slowness \tilde{m} and waveform field \tilde{u} such that

$$m = \tilde{m} + \delta m$$
 and $u = \tilde{u} + \delta u$

Solving the linearized inverse problem therefore becomes finding δm such that

$$\mathop{\arg\min}_{\delta m} \tilde{J}(\delta u(\delta m)) \text{ with } \delta u(\delta m) = \delta u \text{ with } A(\tilde{m}) \delta u = -\omega^2 \tilde{U} \delta m$$

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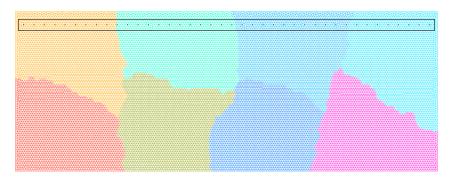
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The linearized problem has a **constant system matrix** and its misfit functional is **convex**

Domain decomposition method

Computing the gradient requires solving the **forward** and **adjoint** problems. We focus on iteratively solving these linear systems using an **ORAS** preconditioner for the partitioned domain



When the forward and adjoint problems are solved iteratively, the inversion algorithm has the form

```
1: \delta u^0 = 0: \lambda^0 = 0
 2: for n = 1, 2, ... do
             \delta u_0^n = \delta u^{n-1}; \lambda_0^n = \lambda^{n-1}
 3:
         for \ell = 1, 2, ..., a do
 4:
                    \delta u_{\ell}^{n} = \langle \ell \text{-th iterate of solve for } \tilde{A} \delta u = -\omega^{2} \tilde{U} \delta m^{n} \rangle
 5:
              end for
 6:
 7:
             for \ell = 1, 2, ..., b do
                    \lambda_{\ell}^{n} = \langle \ell \text{-th iterate of solve for } \tilde{A}^{*} \lambda = P_{\Gamma}^{*} (P_{\Gamma} \delta u_{\sigma}^{n} - \delta d) \rangle
             end for
 9:
            \delta u^n = \delta u_a^n; \lambda^n = \lambda_a^n
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11: q^n = \operatorname{grad}(\delta u^n, \lambda^n)
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One-step one-shot

The one-shot paradigm consists in iterating on the forward & adjoint solvers concurrently to the minimization

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One-step one-shot FWI

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- 2: for n = 1, 2, ... do

5:
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$$\lambda^n = \langle \text{Next iterate of solve for } \tilde{A}^*\lambda = P_\Gamma^*(P_\Gamma \delta u^{n-1} - \delta d) \rangle$$

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When the system solver is the **stationary iteration** and the minimization scheme is **fixed-step gradient descent**, this converges under some conditions (Bonazzoli, Haddar and Vu, 2022)

Multi-step one-shot

Multi-step one-shot increases the amount of forward & adjoint solver iterations done per minimization step and limits them to \boldsymbol{k}

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Fixed-step gradient descent

The convergence behavior of the multi-step one-shot algorithm is mapped as a function of k and the step size τ for the Marmousi reference model at $1\,\mathrm{Hz}$

• 100 multi-step one-shot gradient descent iterations with a stationary forward & adjoint solver are run with a given k and τ

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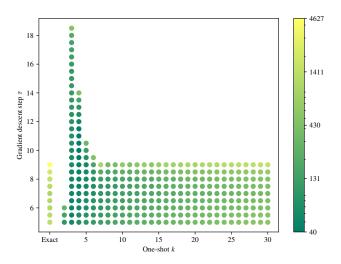
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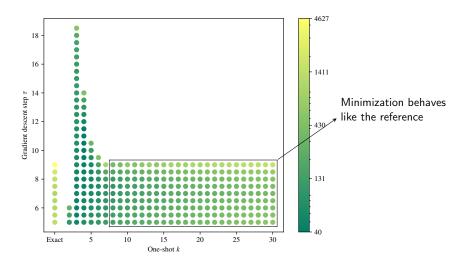
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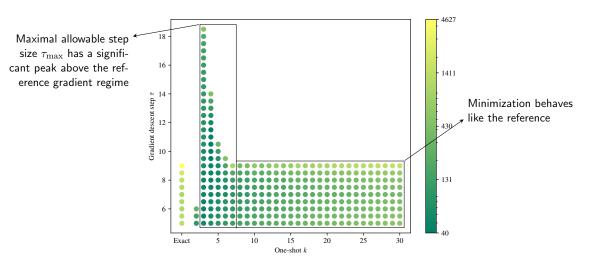
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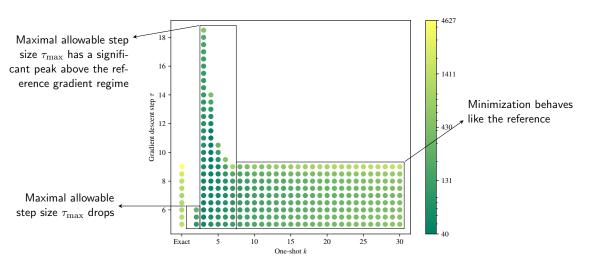
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- The cost is the number of ORAS preconditioner applications for 8 subdomains
- This cost is compared with the **reference iterative FWI** with a relative residual criterion of 10^{-6} for solves



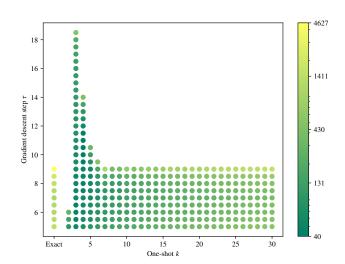






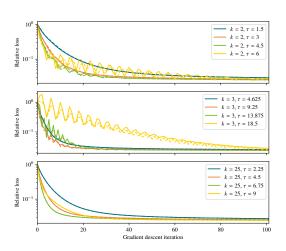
The convergence behavior of the multi-step one-shot algorithm is mapped as a function of k and the step size τ for the Marmousi reference model at $1\,\mathrm{Hz}$

For a given step size τ , choosing an adequate value of k is **effective** at reducing the cost to reach convergence, even if this value is within the intermediate range



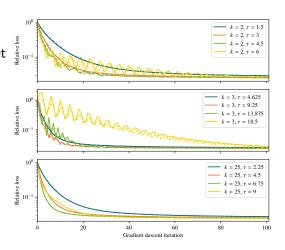
The behavior relative loss $J(\delta u(\delta m))$ is monitored as τ approaches its maximal allowable value for a given k

• For k=25, the convergence remains monotonous as τ approaches $\tau_{\rm max}$



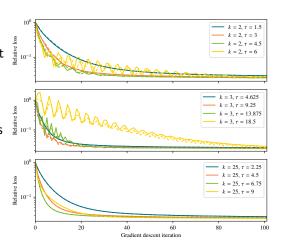
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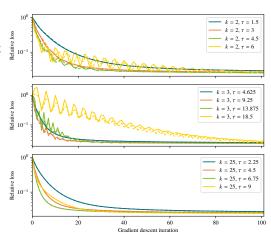
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Overall, one-shot "loosens" the limit between monotonous convergence and diverging oscillations by facilitating oscillating convergent behaviors



Stationary iteration: gradient evolution with k (I)

To obtain an intuitive understanding, we compare the k-step one-shot estimated gradient g is compared to the LU-computed reference gradient $g_{\rm ref}$

• 100 k-step one-shot gradient descent iterations with a direct forward & adjoint solver for a given τ

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- At each step, a **one-shot estimation** of the gradient is computed for a given k, using a **stationary forward & adjoint solver**

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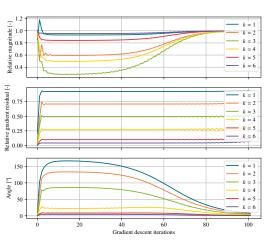
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- 100 k-step one-shot gradient descent iterations with a direct forward & adjoint solver for a given τ
- At each step, a one-shot estimation of the gradient is computed for a given k, using a stationary forward & adjoint solver
- This one-shot gradient is compared in relative magnitude, angle, and relative difference with the reference LU gradient

Stationary iteration: gradient evolution with k (II)

This experiment is performed on the linearized inverse problem at 1 Hz using the Marmousi case (8 subdomains, METIS) for $\tau=9$

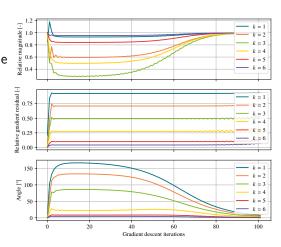
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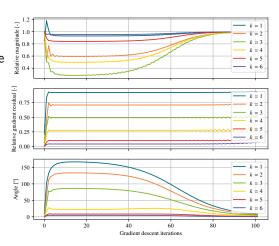
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- As k increases, the angle between the estimated and reference gradients monotonously decreases
- The relative magnitude first decreases then increases as it approaches the reference gradient
- The relative gradient difference is roughly constant for a given k



Adaptive step one-shot

Barzilai-Borwein method

Step size adaptivity for the descent can be implemented with the **Barzilai-Borwein** method

• The method introduces a **short** and a **long** step size

$$p_{\mathrm{short}}^n = -\frac{\langle \Delta \delta m, \Delta g \rangle}{\langle \Delta g, \Delta g \rangle} g^n \quad \text{and} \quad p_{\mathrm{long}}^n = -\frac{\langle \Delta \delta m, \Delta \delta m \rangle}{\langle \Delta \delta m, \Delta g \rangle} g^n$$

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 The geometric mean of the long and short step size is also useful in non-convex scenarios

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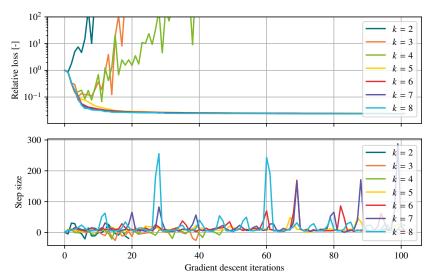
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 For minimization of convex functionals, the short and long step sizes converge, albeit non-monotonously (Raydan, 1991)

Barzilai-Borwein: short step relative losses

The short and mean step sizes were found to be robust to the one-shot estimated gradients



$\label{eq:Adaptive } \textbf{Adaptive } k \text{ one-shot}$

Negative step ramp-up

Requirement that the Barzilai-Borwein step size be positive. A strategy to upgrade \boldsymbol{k} is therefore to

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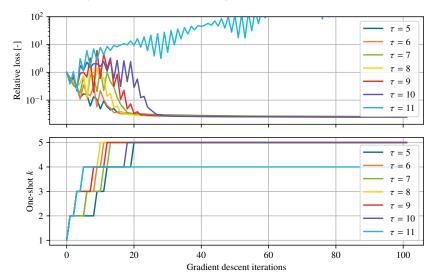
- Set k to an initial value of k_0
- Increase it every time the value of

$$\langle \Delta \delta m, \Delta g \rangle$$

is negative, as sampled by the one-shot estimated gradients

Negative step ramp-up: relative losses and k

This strategy was tested for the inversion of the linearized inverse problem at 1 Hz on the Marmousi model (8 subdomains, METIS) for various step sizes τ , with $k_0 = 1$



Non-linearized inverse problem

Gauss-Newton one-shot

The non-linearized inverse problem can be solved as a succession of linearized inverse problems using the Gauss-Newton method. A globalized scheme for the complete problem optimization becomes

1. Find the step direction p, the solution of the locally linearized problem

$$p = \operatorname*{arg\,min}_{\delta m} \tilde{J}(u(\delta m))$$

2. Find a scaling factor α that enforces the Armijo-Goldstein condition

$$J(u(m_k + \alpha p)) \le J(u(m_k)) + c_1 D_m J(u(m_k))(\alpha p)$$

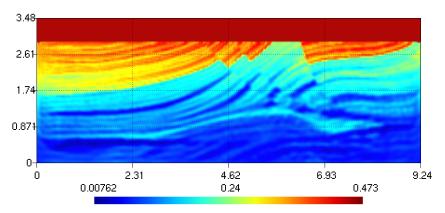
3. Update m such that

$$m_{k+1} = m_k + \alpha p$$

Step 1 may be done using one-shot gradient descent with adaptive step sizes and \boldsymbol{k} schemes

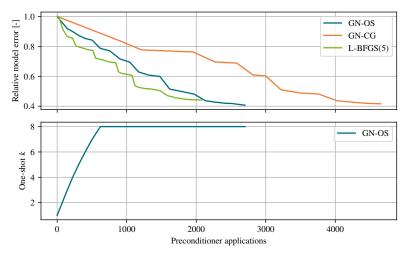
Test case 1: Marmousi (I)

This scheme was tested for the sequential inversion of the Marmousi case (Versteeg, 1994) at frequencies $\{1, 1.6, 2.5, 4, 6.5, 10.4\}$ Hz



Test case 1: Marmousi (II)

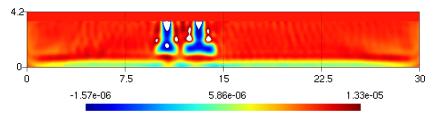
The inversion is compared with L-BFGS(5) and the Gauss-Newton-CG



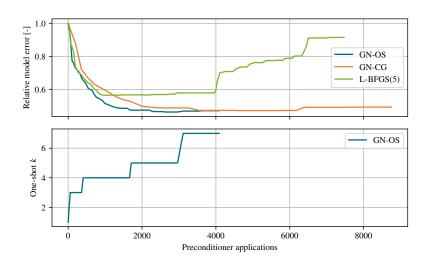
Using one-shot optimization to solve the Gauss-Newton equation leads to performances comparable to that of state-of-the-art algorithms

Test case 2: T-shaped reflectors (I)

This scheme was also tested for the inversion of near-surface imaging application consisting of T-shaped reflectors in a soil (Métivier et al., 2017) (with frequencies $\{100, 125, 150, 175, 200, 225, 250, 275, 300\}$ Hz)



Test case 2: T-shaped reflectors (II)



Coupling forward & adjoint iterations with the optimization is an interesting avenue for potential performance gains when solving inverse problems:

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- Applying one-shot on the non-linearized problem remains to be explored
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- Gauss-Newton one-shot seems competitive with state-of-the-art methods!
- There is a trade-off between the estimated gradient accuracy and the complexity of descent methods that can be employed
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- Applying one-shot on the non-linearized problem remains to be explored
 - → No convexity guarantee
 - → Need a one-shot compatible globalization strategy
- The choice of an adequate k remains elusive

Contact

 \boxtimes alejandro.sior@student.uliege.be

Access my masters' thesis:

