Binomial Coefficients of Multidimensional Arrays

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Abstract. Motivated by Parikh matrices of picture arrays introduced in combinatorial image analysis, we propose a generalization of binomial coefficients of words to multidimensional arrays. These coefficients recursively count prescribed patterns occurring in an array. The base case is the one of binomial coefficients of words.
With our definition we extend Pascal's rule, the Chu–Vandermonde identity and therefore, the concept of Parikh matrices, in a natural way. We further present some more binomial-related identities and introduce (q,t)-deformations, i.e., multivariate polynomials whose evaluation at $(q,t) = (1,1)$ recovers the value of the classical coefficients. We explain the additional combinatorial information encoded in the coefficients of these (q,t)-polynomials compared to their integer-valued counterparts.

Keywords: Binomial coefficient of words ; Parikh matrix ; Combinatorial image analysis ; Multidimensional array ; Gaussian coefficients.

1 Introduction

Combinatorial properties of multidimensional words have been thoroughly investigated by many authors: periodicity, primitiveness, Lyndon words, automaticity, etc. [6,10,14]. Much attention has been given to two-dimensional arrays in image processing. For instance, one of the main motivations for research in two-dimensional pattern matching is related to searching aerial photographs where array elements encode the color of pixels. See, for instance, [1,2].

Let A be a finite alphabet, and let $A^{m\times p}$ denote the set of $m \times p$ *arrays* (or *pictures*), i.e., functions from $\{1,\dots,m\}\times\{1,\dots,p\}$ to A. Although the concepts we develop apply to both the rows and columns, we focus on columns for concise notation; analogous definitions for rows are straightforward.

Recall that the *binomial coefficient* of two words $u = u_1 \cdots u_k$ and v (with each u_i a letter) is defined by

$$\binom{u}{v} = \#\{1 \leq i_1 < \cdots < i_{|v|} \leq k \mid u_{i_1} \cdots u_{i_{|v|}} = v\}. \tag{1}$$

For further details, see, for instance, [8, Chap. 6].

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Let $A = \{\mathtt{a}, \mathtt{b}\}$ and $M = \begin{pmatrix} M_1 \cdots M_p \end{pmatrix}$ be an array whose columns are $M_i \in A^{m\times 1}$ for each $i \in \{1, \dots, p\}$. Recent work (see, e.g., [3,7,16]) has focused on Parikh matrices of binary picture arrays, with particular emphasis on combinatorial image analysis. These matrices capture information about the occurrence of letters and of the scattered subword ab in the array M. Precisely, the *column-Parikh matrix* associated with M is an upper-triangular matrix of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{N}^{3\times 3} \quad \text{where} \quad z = \sum_{i=1}^{p} \binom{M_i^\intercal}{\mathtt{ab}} \quad \text{and} \quad x + y = m \cdot p.$$

Here, x (resp. y) denotes the total number of occurrences of the letter a (resp. b) in M; hence, z counts the total number of subwords ab occurring locally in the columns.

In this article, arrays of the form $m \times 1$ or $1 \times p$ are treated as conventional unidimensional words. In fact, the transpose of an $m \times 1$ array yields a word of length m. Consequently, horizontal and vertical words can be compared directly, making it legitimate to state that ab may appear as a subword of the column M_i without further notice.

Example 1. The column-Parikh matrix associated with the three arrays

$$M = \begin{pmatrix} \mathtt{a\,b\,a\,a} \\ \mathtt{b\,a\,a\,a} \\ \mathtt{a\,a\,b\,a} \\ \mathtt{a\,b\,b\,a} \end{pmatrix}, \; M' = \begin{pmatrix} \mathtt{a\,a\,a\,a} \\ \mathtt{a\,b\,b\,a} \\ \mathtt{a\,a\,b\,b} \\ \mathtt{a\,a\,a\,b} \end{pmatrix} \; \text{and} \; M'' = \begin{pmatrix} \mathtt{a\,a\,a\,a} \\ \mathtt{b\,b\,a\,a} \\ \mathtt{a\,b\,b\,a} \\ \mathtt{a\,a\,b\,a} \end{pmatrix} \; \text{is} \; \begin{pmatrix} 1 & 11 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

because, considering columns of M from left to right, gives

$$\binom{\mathtt{abaa}}{\mathtt{ab}} + \binom{\mathtt{baab}}{\mathtt{ab}} + \binom{\mathtt{aabb}}{\mathtt{ab}} + \binom{\mathtt{aaaa}}{\mathtt{ab}} = 1 + 2 + 4 + 0 = 7.$$

Therefore, these matrices (which could easily be extended to larger alphabets) contain information about the occurrence of specific patterns within the columns of an array. This example shows that several arrays have the same column-Parikh matrix: any permutation of columns does not affect the Parikh matrix. Replacing a column by an equivalent one also leads to the same matrix. For instance, if the fourth column of M is placed in the first position and if the column containing baab is replaced by abba which, also contains the subword ab twice and the same number of a's and b's, then the array M' has the same column-Parikh matrix as above. This is a reason why many authors are interested in M-ambiguity, i.e., an equivalence relation defined by arrays having the same Parikh matrix. This is a difficult problem of genuine interest around Parikh matrices (of unidimensional words). Thus, if one considers the problem of reconstructing an image made up of pixels, the Parikh matrix alone is insufficient to guarantee a unique reconstruction.

Classical Parikh matrices associated with unidimensional words have been generalized in [15]. With a word v over an alphabet A, for each $a \in A$, one associates a square unitriangular matrix M_a of dimension $|v|+1$ encoding the positions of a within v. For any word $u = a_1 \cdots a_{|u|} \in A^*$, compute the corresponding product of $|u|$ matrices $M_{a_1} \cdots M_{a_{|u|}}$. The upper-right element of the resulting matrix is exactly $\binom{u}{v}$. Therefore, Parikh matrices and binomial coefficients of words are intimately linked [4].

By convention, the binomial of integers $\binom{m}{p} = 0$ if $m < p$ and similarly, the binomial of words $\binom{u}{v} = 0$ whenever $|u| < |v|$. These conventions are extended to a multidimensional setting. Henceforth, it will not be necessary to specify whether one array is smaller than another.

1.1 Main Object of Study

Our aim is to introduce generalized binomial coefficients that allow counting the occurrence of patterns beyond just subwords in columns. In particular, we want to count the number of occurrences of certain (scattered) subarrays. Several definitions could be considered. To stay in line with [3,7,16], we have adopted an approach that focuses on subwords appearing in the different columns. Our choice also relies on a definition that can be recursively extended to higher dimensions. For the sake of readability, we will focus primarily on two-dimensional arrays.

Reconsider (1), which defines binomial coefficients of words. Let $u = a_1 \cdots a_k$ and $v = b_1 \cdots b_\ell$ be words where a_i and b_j are letters. We can write

$$\binom{u}{v} = \sum_{1 \le i_1 < \cdots < i_\ell \le k} \delta_{a_{i_1}, b_1} \cdots \delta_{a_{i_\ell}, b_\ell}.$$

The Kronecker symbol can be thought of as a zero-dimensional binomial coefficient. Roughly speaking, letters are 0-dimensional objects and words are 1-dimensional. So for any two letters $\mathtt{a}, \mathtt{b}$, $\binom{\mathtt{a}}{\mathtt{b}} = \delta_{\mathtt{a},\mathtt{b}} = 1$ if and only if $\mathtt{a} = \mathtt{b}$. Such a trivial observation permits us to introduce our object of study in a natural way. As binomial coefficients of words are built from 0-dimensional coefficients, our 2-dimensional coefficients are built from 1-dimensional ones.

Definition 1. *Let $M = \begin{pmatrix} M_1 \cdots M_m \end{pmatrix}$ and $P = \begin{pmatrix} P_1 \cdots P_p \end{pmatrix}$ be two-dimensional arrays. We define the* column-binomial coefficient *of the arrays M and P by*

$$\binom{M}{P} = \sum_{1 \le i_1 < \cdots < i_p \le m} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_p}}{P_p}$$

where on the r.h.s. we have classical binomial coefficients of (unidimensional) words, with columns M_i and P_j interpreted as words. In particular, if $m < p$ or if the number of rows of M is less than the number of rows of P, then the column-binomial coefficient is zero.

Note that in the above definition, if $m = p = 1$, then we get back the usual binomial of words.

Example 2. Consider the array M from Example 1 and the sixteen 2×2 arrays

$$\begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{a} \end{pmatrix} \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{a} & \mathtt{b} \end{pmatrix} \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} \end{pmatrix} \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{b} \end{pmatrix} \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{a} \end{pmatrix} \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{a} & \mathtt{b} \end{pmatrix} \cdots \begin{pmatrix} \mathtt{b} & \mathtt{b} \\ \mathtt{b} & \mathtt{b} \end{pmatrix}$$

As P ranges over these sixteen arrays, the column-binomial coefficient $\binom{M}{P}$ takes the values

$$37, 22, 46, 14, 6, 7, 2, 4, 30, 20, 13, 4, 4, 6, 0, 1. \tag{2}$$

Computations show that there is exactly one other 4×4 array with the same column-binomial coefficient, the array M'' from Example 1. Unlike column-Parikh matrices, when P has two columns, permuting the columns of M alters the corresponding binomial coefficient.

Let us briefly discuss a notion of ambiguity. Out of the 2^{16} binary matrices of size 4×4, there are $49\,497$ pairwise distinct vectors of binomial values such as (2). There are at most 16 matrices having the same vector (and this is the single class with 16 elements). One representative is composed of four identical columns, namely $\begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{b} & \mathtt{a} \end{pmatrix}^{\mathsf{T}}$. There are four equivalence classes, and each consists of 12 matrices.

Remark 1. A natural candidate for extracting subarrays is to consider submatrices A *submatrix* of a matrix is obtained by deleting any set of rows and/or columns. In this case, counting the number of occurrences of a submatrix P within a matrix M is bounded from above by our column-binomial coefficient $\binom{M}{P}$. However, choosing to count submatrices does not lead to a Pascal's rule as in Proposition 1. Indeed, the constraint of considering the same set of rows in every column is too strong. Whenever a product such as $\binom{C}{D}\binom{M}{P}$ occurs, as in (3), there is no easy way to ensure that the rows selected in C and M coincide.

Let $k \geq 1$ and M be a k-dimensional array with dimension $(d_1, \ldots, d_k)$. For each $j \in \{1, \ldots, d_k\}$, we let M_j denote the $(k-1)$-dimensional *section* (or slice) of M where the last component is set to j. Hence, M_j is a $(k-1)$-dimensional word of dimension $(d_1, \ldots, d_{k-1})$ where

$$M_j[i_1, \ldots, i_{k-1}] = M[i_1, \ldots, i_{k-1}, j].$$

In particular, if M is a 1-dimensional word, M_j denotes its j^{th} letter and if M is a 2-dimensional array, M_j is its j^{th} column.

Definition 2. *Let $M \in A^{d_1 \times \cdots \times d_k}$ and $P \in A^{e_1 \times \cdots \times e_k}$ be k-dimensional arrays. The* binomial coefficient *of M and P is given by*

$$\binom{M}{P} = \sum_{1 \leq i_1 < \cdots < i_{e_k} \leq d_k} \binom{M_{i_1}}{P_1} \cdots \binom{M_{i_{e_k}}}{P_{e_k}}$$

where, on the r.h.s., we have binomial coefficients of $(k-1)$-dimensional section arrays.

1.2 Our Results

One could debate our choice of Definition 1, as other counting functions could be considered. However, besides Remark 1, another argument in favor of our choice is that the classical formulas for binomials naturally extend. We list those below.

We define the *concatenation* of two arrays. Let $A \oplus B$ denote the matrix obtained by juxtaposing, side by side, the two arrays A and B that have the same number of rows. As a fundamental property, we have Pascal's rule:

Proposition 1. *Let $M \in A^{r\times m}$, $P \in A^{s\times p}$, $C \in A^{r\times 1}$ and $D \in A^{s\times 1}$.*

$$\binom{M \oplus C}{P \oplus D} = \binom{M}{P \oplus D} + \binom{C}{D}\binom{M}{P}. \tag{3}$$

This generalizes the relation for words u, v and letters $\mathtt{c}, \mathtt{d}$ [8, Prop. 6.3.2]

$$\binom{u\mathtt{c}}{v\mathtt{d}} = \binom{u}{v\mathtt{d}} + \delta_{\mathtt{c},\mathtt{d}}\binom{u}{v}.$$

It is worth noting that in (3), $\binom{C}{D}$ serves as a 1-dimensional coefficient replacing the 0-dimensional coefficient $\delta_{\mathtt{c},\mathtt{d}}$ (whereas the other coefficients are column-binomial coefficients). We also have a Chu–Vandermonde identity, [8, Cor. 6.3.7].

Proposition 2. *Let $N \in A^{r\times n}$ so that $M \oplus N$ belongs to $A^{r\times(m+n)}$*

$$\binom{M \oplus N}{P} = \sum_{P_1 \oplus P_2 = P} \binom{M}{P_1}\binom{N}{P_2}.$$

Since we have established Pascal's rule, it is not surprising that we can obtain an analogue of (generalized) Parikh matrices. The strategy is the same as in Definition 1: we replace the 0-dimensional Kronecker symbols in the classical definition [4] with 1-dimensional coefficients.

Definition 3. *Let $M = \begin{pmatrix} M_1 \cdots M_m \end{pmatrix}$ and $P = \begin{pmatrix} P_1 \cdots P_p \end{pmatrix}$. Let $\Psi : A^{r\times 1} \to (\mathbb{N}^{(p+1)\times(p+1)}, \cdot)$ be a morphism of monoids such that, for all $k \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, p+1\}$,*

$$[\Psi(M_k)]_{i,i} = 1, \quad [\Psi(M_k)]_{i,i+1} = \binom{M_k}{P_i} \text{ for } i \le p$$

and $[\Psi(M_k)]_{i,j} = 0$ otherwise.

Hence $\Psi(M) := \Psi(M_1)\cdots\Psi(M_m)$ is a *unitriangular* matrix, i.e., an upper-triangular matrix with ones on the diagonal.

Theorem 1. *Let $M = \begin{pmatrix} M_1 \cdots M_m \end{pmatrix}$ and $P = \begin{pmatrix} P_1 \cdots P_p \end{pmatrix}$ be two-dimensional arrays. With the above notation, the product $\Psi(M_1)\cdots\Psi(M_m)$ is unitriangular matrix.*

For each entry above the diagonal, i.e., at position (i, j) with $i < j$, the value is given by the column-binomial coefficient

$$\binom{M}{(P_i \cdots P_{j-1})}.$$

In particular, the upper-right element of the matrix $\Psi(M)$ is $\binom{M}{P}$.

Example 3. With the array M from Example 1 and $P = \begin{pmatrix} \mathtt{a} & \mathtt{a} \\ \mathtt{b} & \mathtt{a} \end{pmatrix}$, we obtain

$$\Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{b} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{b} \\ \mathtt{b} \end{pmatrix} \Psi\begin{pmatrix} \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \\ \mathtt{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 46 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, the entries of the matrix correspond to

$$\binom{M}{(\mathtt{a}\ \mathtt{b})^\intercal} = 7, \quad \binom{M}{(\mathtt{a}\ \mathtt{a})^\intercal} = 11 \quad \text{and} \quad \binom{M}{P} = 46.$$

In Section 2, we prove these results and discuss more properties. We sum the total values of the coefficients $\binom{M}{P}$ when P ranges over all arrays of a given size. We obtain a generalization of Manvel–Meyerowitz–Schwenk–Smith–Stockmeyer formula [9]. We also discuss the product $\binom{M}{P}\binom{M}{Q}$. Then in Section 3, we introduce multivariate deformations of these coefficients. The number of variables is equal to the dimension of the arrays. The so-called *q-analogue* of a counting function reduces to the original function when q tends to 1. For a deformation to be useful, it should preserve at least some if not all of the algebraic properties of the original function.

2 Proofs and Extra Results

The proofs of Pascal's rule, Chu–Vandermonde identity, and the structure theorem of Parikh matrices are rather straightforward. Readers familiar with this type of combinatorial identities will not be surprised. After presenting these proofs, we consider some more relations.

Proof (of Proposition 1). Let $M \in A^{r\times m}$ and $P \in A^{s\times p}$. Let $C \in A^{r\times 1}$ and $D \in A^{s\times 1}$. By definition of the column-binomial coefficient, we have

$$\binom{M \oplus C}{P \oplus D} = \sum_{1 \le i_1 < \cdots < i_{p+1} \le m+1} \prod_{j=1}^{p} \binom{(M \oplus C)_{i_j}}{P_j} \binom{(M \oplus C)_{i_{p+1}}}{D},$$

where $(M \oplus C)_k = M_k$ for $1 \le k \le m$ and $(M \oplus C)_{m+1} = C$. Similarly, $(P \oplus D)_j = P_j$ for $1 \le j \le p$ and $(P \oplus D)_{p+1} = D$. Partitioning the sum based on whether the last column index i_{p+1} is $m + 1$ yields the desired result. □

Due to space constraints, we provide only the proof of its generalization in Proposition 6, since the arguments are similar.

Proof (of Theorem 1). Define $\Phi_j = \Psi(M_1)\cdots\Psi(M_j)$. We use the same notation as in the previous proof. We show, by induction on $j \geq 1$, that for $1 \leq i < r \leq p+1$,

$$[\Phi_j]_{i,r} = \binom{M_{[1..j]}}{P_{[i..(r-1)]}}.$$

In particular, $[\Phi_j]_{i,r} = 0$ whenever $j < r - i$ because $M_{[1..j]}$ has fewer columns than $P_{[i..r-1]}$. For $j = 1$, $\Phi_1 = \Psi(M_1)$ and the conclusion follows from the definition of Ψ. For the inductive step, observe that $[\Psi(M_{j+1})]_{\ell,r}$ is nonzero only when $\ell = r$ or $\ell = r-1$. Hence the off-diagonal entry $[\Phi_{j+1}]_{i,r}$ with $i < r$, is

$$[\Phi_j]_{i,r} + [\Phi_j]_{i,r-1}\binom{M_{j+1}}{P_{r-1}} = \binom{M_{[1..j]}}{P_{[i..(r-1)]}} + \binom{M_{[1..j]}}{P_{[i..(r-2)]}}\cdot\binom{M_{j+1}}{P_{r-1}}$$

by induction hypothesis. The conclusion then follows from Proposition 1. □

Summing over all possible patterns P is equivalent to consider arrays over a unary alphabet $\{\mathtt{a}\}$. This generalizes the fact that for words, $\sum_{v\in A^k}\binom{u}{v} = \binom{|u|}{k}$.

Proposition 3. *Let $M \in A^{r\times m}$*

$$\sum_{P\in A^{s\times p}}\binom{M}{P} = \binom{\mathtt{a}^{r\times m}}{\mathtt{a}^{s\times p}} = \binom{m}{p}\cdot\binom{r}{s}^p.$$

Proof. The sum in the l.h.s. counts all possible ways to choose p columns from M and, within each chosen column, select s rows. The number of ways to select p columns is $\binom{m}{p}$. For each selected column, there are $\binom{r}{s}$ ways to choose s rows. Since the choices between columns are independent, the total number of configurations is $\binom{m}{p}\cdot\binom{r}{s}^p$. This count is independent of the entries in M because all possible subarrays P are considered. □

Applying the above proposition to Example 2, we get $\binom{4}{2}\binom{4}{2}^2 = 6^3 = 216$. This is indeed the total sum of (2). The relation can easily be extended to an arbitrary dimension. Let $k \geq 2$ and $M \in A^{d_1\times\cdots\times d_k}$. Proceeding by induction on k, we easily get

$$\sum_{P\in A^{e_1\times\cdots\times e_k}}\binom{M}{P} = \binom{d_k}{e_k}\binom{d_{k-1}}{e_{k-1}}^{e_k}\binom{d_{k-2}}{e_{k-2}}^{e_k e_{k-1}}\cdots\binom{d_1}{e_1}^{e_k e_{k-1}\cdots e_2}.$$

Similarly to the case of words, we consider the (column)-shuffle ⧢ of two arrays $P \in A^{s\times p}$, $Q \in A^{s\times q}$ having the same number s of rows. It is a formal polynomial in $\mathbb{N}\langle\langle A^{s\times 1}\rangle\rangle$. As an example, we have

$$\begin{pmatrix}\mathtt{a}&\mathtt{b}\\\mathtt{a}&\mathtt{a}\end{pmatrix} ⧢ \begin{pmatrix}\mathtt{a}&\mathtt{b}\\\mathtt{a}&\mathtt{b}\end{pmatrix} = \begin{pmatrix}\mathtt{a}&\mathtt{b}&\mathtt{a}&\mathtt{b}\\\mathtt{a}&\mathtt{a}&\mathtt{a}&\mathtt{b}\end{pmatrix} + 2\begin{pmatrix}\mathtt{a}&\mathtt{a}&\mathtt{b}&\mathtt{b}\\\mathtt{a}&\mathtt{a}&\mathtt{a}&\mathtt{b}\end{pmatrix} + 2\begin{pmatrix}\mathtt{a}&\mathtt{a}&\mathtt{b}&\mathtt{b}\\\mathtt{a}&\mathtt{a}&\mathtt{b}&\mathtt{a}\end{pmatrix} + \begin{pmatrix}\mathtt{a}&\mathtt{b}&\mathtt{a}&\mathtt{b}\\\mathtt{a}&\mathtt{b}&\mathtt{a}&\mathtt{a}\end{pmatrix}.$$

Note that the concatenation of the two arrays appears in the shuffle with a coefficient of at least one. If f is a polynomial or a formal series, the coefficient of an element R is denoted by $\langle f, R\rangle$. For the definition of the shuffle product on the module $\mathbb{Z}\langle\langle A\rangle\rangle$, see [8, p. 126].

Proposition 4. *Let $M \in A^{r\times m}$, $P \in A^{s\times p}$, and $Q \in A^{s\times q}$. We have:*

$$\binom{M}{P}\binom{M}{Q} = \sum_{R\in(A^{s\times 1})^*} \langle P \sqcup\!\sqcup Q, R\rangle \binom{M}{R} + \sum_{\substack{1\le i_1<\cdots<i_p\le m\\ 1\le j_1<\cdots<j_q\le m\\ \{i_1,\ldots,i_p\}\cap\{j_1,\ldots,j_q\}\neq\emptyset}} \binom{(M_{i_1}\cdots M_{i_p})}{P}\cdot\binom{(M_{j_1}\cdots M_{j_q})}{Q}.$$

In particular, the above result permits us to express $\binom{M}{P\varobar Q}$ occurring in the first sum of the r.h.s. in the above formula as an expression involving $\binom{M}{P}\binom{M}{Q}$ divided by $\langle P \sqcup\!\sqcup Q, P \varobar Q\rangle$. This is similar to [13, Thm. 6.4], which is used to express binomial coefficients using only Lyndon words.

Proof. By definition of the column-binomial coefficient, we have

$$\binom{M}{P}\binom{M}{Q} = \sum_{1\le i_1<\cdots<i_p\le m} \binom{M_{i_1}}{P_1}\cdots\binom{M_{i_p}}{P_p} \sum_{1\le j_1<\cdots<j_q\le m} \binom{M_{j_1}}{Q_1}\cdots\binom{M_{j_q}}{Q_q}.$$

We expand this product and focus on the $\binom{m}{p}\binom{m-p}{q}$ terms where $\{i_1,\ldots,i_p\}\cap\{j_1,\ldots,j_q\}=\emptyset$, i.e., $p+q$ pairwise distinct columns from M are selected. Since all these indices are distinct, they are linearly ordered. Each such ordering is in one-to-one correspondence with a shuffle of columns. For example, $i_1 < i_2 < j_1 < i_3 < j_2$ ($p = 3$ and $q = 2$) corresponds to $(P_1P_2Q_1P_3Q_2)$. Since the indices belong to $\{1,\ldots,m\}$, several choices of the $p+q$ indices lead to the same ordering. Continuing the example, if we fix one ordering, e.g., $i_1 < i_2 < j_1 < i_3 < j_2$, and consider all choices leading to this ordering, we get

$$\sum_{1\le i_1<i_2<j_1<i_3<j_2\le m} \binom{M_{i_1}}{P_1}\binom{M_{i_2}}{P_2}\binom{M_{j_1}}{Q_1}\binom{M_{i_3}}{P_3}\binom{M_{j_2}}{Q_2} = \binom{M}{(P_1P_2Q_1P_3Q_2)}.$$

Now we consider all the orderings of the $p + q$ indices and we thus see all the elements of the shuffle $P \sqcup\!\sqcup Q$. Different orderings may lead to the same resulting shuffle of columns. As an example, if $P_2 = Q_1$, then $i_1 < i_2 < j_1 < i_3 < j_2$ and $i_1 < j_1 < i_2 < i_3 < j_2$ lead to the same array $(P_1P_2Q_1P_3Q_2) = (P_1Q_1P_2P_3Q_2)$. If we express the sum as a sum over these elements occurring in the shuffle, this explains the coefficient $\langle P \sqcup\!\sqcup Q, R\rangle$ in the first part of the formula. □

Note that we could refine the above formula by summing over the size of the intersection of the two sets of indices. We now extend a result from Manvel et al. [9]. See also [5].

Theorem 2. *Let $M \in A^{r\times m}$, $P \in A^{s\times p}$ and $r \geq q \geq s$, $m \geq t \geq p$. We have*

$$\sum_{T\in A^{q\times t}} \binom{M}{T}\binom{T}{P} = \binom{m-p}{t-p}\binom{r}{q}^{t-p}\binom{r-s}{q-s}^{p}\binom{M}{P}.$$

Proof. In the r.h.s., as in Proposition 3, we review all possible choices of $q \cdot t$ elements that give rise to a valid pattern T and within each such T count the occurrences of P. However, a fixed occurrence of P appears many times in this counting, as we now explain.

Fix a particular *occurrence* of $P \sqsubseteq M$. Concretely, $P \sqsubseteq M$ means selecting p columns of M (out of m) and within each column, independently select s rows (out of r), matching P. Next, to build a $q \times t$ subarray $T \sqsubseteq M$ containing that chosen copy of P:

- In each of the p columns used by P, we independently pick $q-s$ more elements (out of $r-s$ available). Hence we have $\binom{r-s}{q-s}^p$ ways to do so.
- Since P already occupies p columns, we must choose $t-p$ columns from the remaining $m-p$. There are $\binom{m-p}{t-p}$ such choices. In each of these $t-p$ columns, we independently pick q elements. Hence we have $\binom{m-p}{t-p}\binom{r}{q}^{t-p}$ ways in total. □

Corollary 1. *Let $M, M' \in A^{r\times m}$. Let q, t, s, p be such that $r \geq q \geq s$ and $m \geq t \geq p$. If $\binom{M}{T} = \binom{M'}{T}$ for all $T \in A^{q\times t}$, then $\binom{M}{P} = \binom{M'}{P}$ for all $P \in A^{s\times p}$.*

3 Deformations of these Coefficients

The q-deformations of binomial coefficients of words have been introduced in [11] as a generalization of Gaussian coefficients. As discussed in that paper, two definitions may coexist: letters can be processed either from the left or from the right. The two resulting univariate polynomials are different, but their properties are similar. Equivalently, one may consider reversals of the words. Since we index rows and columns from the upper left corner starting with index 1, we choose the recursive definition of q-binomials processed from the left, i.e., for $u, v \in A^*$ and $\mathtt{c}, \mathtt{d} \in A$

$$\binom{\mathtt{c}u}{\mathtt{d}v}_q = \binom{u}{\mathtt{d}v}_q \cdot q^{|\mathtt{d}v|} + \delta_{\mathtt{c},\mathtt{d}}\binom{u}{v}_q. \tag{4}$$

We recall a useful result in that setting, see again [11] for details.

Theorem 3. *Let u be a word over A, $k \geq 0$, and $a_1, \ldots, a_k \in A$. Then*

$$\binom{u}{a_1\cdots a_k}_q = \sum_{\substack{u_1,u_2,\ldots,u_{k+1}\in A^* \\ u=u_1a_1\cdots u_k a_k u_{k+1}}} q^{\sum_{i=1}^{k}(k+1-i)|u_i|}.$$

As an example of classical q-binomial coefficients of words, we have

$$\binom{\mathtt{aab}}{\mathtt{ab}}_q = q^2 + q \quad \text{and} \quad \binom{\mathtt{aba}}{\mathtt{ab}}_q = 1. \tag{5}$$

For instance, the word aab can be factorized as $u_1\mathtt{a}u_2\mathtt{b}u_3$ with $u_1 = u_3 = \varepsilon$ and $u_2 = \mathtt{a}$, which yields the term q in the first q-binomial above (resp. $u_2 = u_3 = \varepsilon$ and $u_1 = \mathtt{a}$ giving q^2). This combinatorial interpretation is important for the next definition, because a similar idea applies to the exponent of the second variable t.

For bidimensional arrays, we consider the following definition.

Definition 4. *Let $M = \begin{pmatrix} M_1 \cdots M_m \end{pmatrix}$ and $P = \begin{pmatrix} P_1 \cdots P_p \end{pmatrix}$ be two-dimensional arrays. We define the (q,t)-*column-binomial coefficient *of M and P by*

$$\binom{M}{P}_{q,t} = \sum_{1 \le i_1 < \cdots < i_p \le m} t^{\sum_{j=1}^{p}(i_j - j)} \binom{M_{i_1}}{P_1}_q \cdots \binom{M_{i_p}}{P_p}_q.$$

Here, the l.h.s. we have q-binomial coefficients of words (obtained by considering the transpose of the columns).

As expected with such deformations, when q and t are equal to 1, we recover the column-binomial coefficient discussed in the first part of this paper.

Example 4. We make use of the q-binomials of the columns computed in (5). Let

$$M = \begin{pmatrix} \mathtt{a\,a\,a} \\ \mathtt{a\,b\,a} \\ \mathtt{b\,a\,b} \end{pmatrix}, \quad P = \begin{pmatrix} \mathtt{a\,a} \\ \mathtt{b\,b} \end{pmatrix}.$$

Then

$$\binom{M}{P}_{q,t} = (q^2+q)\cdot 1 + t\cdot(q^2+q)\cdot(q^2+q) + t^2\cdot 1\cdot(q^2+q)$$

which is equal to $q^4t + 2q^3t + q^2t^2 + q^2t + q^2 + qt^2 + q$. The three terms of the sum correspond, respectively, to the choices of columns $1,2$; $1,3$; and $2,3$.

Remark 2. If M and P both have a single row, then the polynomial $\binom{M}{P}_{q,t}$ only contains the variable t and is equal to $\binom{M}{P}_q(t)$ (i.e., replace the variable q by t). This shows the coherence of the definition. Indeed, let $M = \mathtt{m}_1 \cdots \mathtt{m}_k$ and $P = \mathtt{p}_1 \cdots \mathtt{p}_\ell$ be words. The definition reduces to

$$\binom{M}{P}_{q,t} = \sum_{1 \le i_1 < \cdots < i_\ell \le k} t^{\sum_{j=1}^{\ell}(i_j - j)} \delta_{\mathtt{m}_{i_1},\mathtt{p}_1} \cdots \delta_{\mathtt{m}_{i_\ell},\mathtt{p}_\ell}.$$

Observe that if P appears as a subword of M with the letters occurring in position $i_1, \ldots, i_\ell$, i.e., $M = u_1\mathtt{p}_1 \cdots u_\ell \mathtt{p}_\ell u_{\ell+1}$ with $i_j = \sum_{i=1}^{j} |u_i| + j$ for all j, then

$$\sum_{j=1}^{\ell}(i_j - j) = \sum_{j=1}^{\ell}\sum_{i=1}^{j} |u_i| = \sum_{i=1}^{\ell}(\ell + 1 - i)|u_i|.$$

We conclude with Theorem 3.

If M and P both have a single column, then the polynomial $\binom{M}{P}_{q,t}$ only contains the variable q and it is trivially equal to $\binom{M}{P}_q$.

For classical binomial coefficients of words, the *reconstruction problem* is fundamental. Given some binomial coefficients associated with a word u, one asks whether u can be uniquely reconstructed i.e., whether the knowledge of certain coefficients uniquely determines the word [5]. As observed in [11], q-binomials contain more precise information. The same observation holds for (q,t)-binomials. For example,

$$\text{with } M = \begin{pmatrix} \mathtt{a\,a\,b} \\ \mathtt{a\,b\,b} \\ \mathtt{b\,a\,b} \end{pmatrix}, \quad \binom{M}{(\mathtt{b})}_{q,t} = q^2 + tq + t^2\left(q^2+q+1\right).$$

We clearly see that $q^{i-1}t^{j-1}$ occurs in the (q,t)-binomial if and only if there is a letter $\mathtt{b}$ in position (i,j) (counted from 1 in the upper-left corner). This observation is general. To be consistent with our convention of processing from the left, we concatenate a column to the left of an array. The exponent $p+1$ occurring in the statement is similar to the exponent $|\mathtt{b}v|$ in (4).

Proposition 5. *Let $M \in A^{r\times m}$, $P \in A^{s\times p}$, $C \in A^{r\times 1}$, and $D \in A^{s\times 1}$. Then*

$$\binom{C \oplus M}{D \oplus P}_{q,t} = \binom{M}{D \oplus P}_{q,t} \cdot t^{p+1} + \binom{C}{D}_q \binom{M}{P}_{q,t}.$$

Proof. By definition of the (q,t)-column-binomial coefficient, we have

$$\binom{C \oplus M}{D \oplus P}_{q,t} = \sum_{1\le i_1<\cdots<i_{p+1}\le m+1} t^{\sum_{j=1}^{p+1}(i_j-j)} \binom{(C\oplus M)_{i_1}}{(D\oplus P)_1}_q \cdots \binom{(C\oplus M)_{i_{p+1}}}{(D\oplus P)_{p+1}}_q.$$

We partition the selection of columns in the concatenated array $C \oplus M$ into two cases, depending on whether the first column C is included. If $i_1 = 1$, the contribution is $\binom{C}{D}_q\binom{M}{P}_{q,t}$. Otherwise, the first column C is not selected. All $p+1$ columns are selected from M. By reindexing with $\ell_j = i_j - 1$, the contribution becomes:

$$t^{p+1} \cdot \sum_{1\le \ell_1<\cdots<\ell_{p+1}\le m} t^{\sum_{j=1}^{p+1}(\ell_j-j)} \prod_{k=1}^{p+1} \binom{M_{\ell_k}}{(D\oplus P)_k}_q = t^{p+1}\cdot \binom{M}{D\oplus P}_{q,t}.$$

□

We also have an analogue of Chu–Vandermonde identity:

Proposition 6. *Let $M \in A^{r\times m}$, $N \in A^{r\times n}$, and $P \in A^{s\times p}$. We have*

$$\binom{M \oplus N}{P}_{q,t} = \sum_{P_1\oplus P_2 = P} t^{(m-|P_1|)|P_2|} \binom{M}{P_1}_{q,t} \binom{N}{P_2}_{q,t},$$

where $|P|$ is the number of columns of P.

Proof. By definition of the (q,t)-binomial coefficient, we have

$$\binom{M \oplus N}{P}_{q,t} = \sum_{1 \le i_1 < \cdots < i_p \le m+n} t^{\sum_{j=1}^{m+n}(i_j - j)} \binom{(M \oplus N)_{i_1}}{P_1}_q \cdots \binom{(M \oplus N)_{i_p}}{P_p}_q .$$

Partitioning the sum by how many elements of the sequence $i_1 < \cdots < i_p$ are less than or equal to m, the above coefficient can be written

$$\sum_{k=0}^{p} \left(\sum_{1 \le i_1 < \cdots < i_k \le m} t^{\sum_{j=1}^{k}(i_j - j)} \prod_{j=1}^{k} \binom{(M \oplus N)_{i_j}}{P_j}_q \right) \cdot \left(\sum_{m < i_{k+1} < \cdots < i_p \le m+n} t^{\sum_{j=k+1}^{p}(i_j - j)} \prod_{j=k+1}^{p} \binom{(M \oplus N)_{i_j}}{P_j}_q \right).$$

Observe that $\big(M \oplus N\big)_j = M_j$ if $j \le m$ and $\big(M \oplus N\big)_j = N_{j-m}$ if $j > m$. Hence, the second factor in the above formula can be written

$$\sum_{m < i_{k+1} < \cdots < i_p \le m+n} t^{\sum_{j=k+1}^{p}(i_j - m - (j-k) + m - k)} \prod_{j=k+1}^{p} \tbinom{N_{i_j - m}}{P_j}_q$$
$$= t^{\sum_{j=k+1}^{p}(m-k)} \tbinom{N}{P_{[(k+1)..p]}}_{q,t} = t^{(m-|P_1|)|P_2|} \tbinom{N}{P_{[(k+1)..p]}}_{q,t}.$$

where $P_{[(k+1)..p]}$ is made of the last $p-k$ columns. □

Flipping the arrays around a vertical axis reverses the order of coefficients of the (q,t)-binomial, when it is viewed as a polynomial in t with coefficients in $\mathbb{N}[q]$:

$$[t^j] \binom{M_1 \cdots M_m}{P_1 \cdots P_p}_{q,t} = [t^{p(m-p)-j}] \binom{M_m \cdots M_1}{P_p \cdots P_1}_{q,t}.$$

Similarly, if $\widetilde{M}$ denotes a flip of M around a horizontal axis and $\widetilde{P}$ is defined correspondingly, then when we view the (q,t)-binomial as a polynomial in q with coefficients in $\mathbb{N}[t]$, we have

$$[q^j] \binom{\widetilde{M}}{\widetilde{P}}_{q,t} = [q^{ps(r-s)-j}] \binom{M}{P}_{q,t}$$

where $M \in A^{r \times m}$, $P \in A^{s \times p}$. For the two arrays in Example 4, with $r = m = 3$ and $s = p = 2$, we have

$$\binom{M}{P}_{q,t} = q^4 t + 2tq^3 + \left(t^2 + t + 1\right) q^2 + \left(t^2 + 1\right) q,$$

and

$$\binom{\widetilde{M}}{\widetilde{P}}_{q,t} = \left(t^2 + 1\right) q^3 + \left(t^2 + t + 1\right) q^2 + 2tq + t.$$

4 Conclusions

In this short paper, we have introduced a natural generalization of binomial coefficients of words to multidimensional arrays. Expected properties extend without much effort, thus justifying our definition. Moreover, this opens the way to further developments, as coefficients of words already appear in numerous contexts (formal language theory, algebra, theoretical computer science, combinatorics, etc.). Based on a robust definition, one can hope that other non-trivial combinatorial properties may also extend to multidimensional arrays. Furthermore, as we have shown in the two-dimensional case, one can work either rows or columns. In higher dimension d, it is necessary to specify the order in which sections are selected, potentially leading to $d!$ distinct (but related) coefficients. The multivariate polynomials we obtain are also worth to be studied further. For instance, it is straightforward to adapt [12] to this context.

References

1. Amihood Amir, Gary Benson, and Martin Farach. An alphabet independent approach to two-dimensional pattern matching. *SIAM J. Comput.*, 23(2):313–323, 1994. `doi:10.1137/S0097539792226321`.
2. Amihood Amir, Oren Kapah, and Dekel Tsur. Faster two-dimensional pattern matching with rotations. *Theor. Comput. Sci.*, 368(3):196–204, 2006. `doi:10.1016/j.tcs.2006.09.012`.
3. Somnath Bera, Sastha Sriram, Atulya K. Nagar, Linqiang Pan, and K. G. Subramanian. Algebraic properties of Parikh matrices of binary picture arrays. *J. Math.*, pages Art. ID 3236405, 7, 2020. `doi:10.1155/2020/3236405`.
4. Traian-Florin Şerbănuţă. Extending Parikh matrices. *Theoret. Comput. Sci.*, 310(1-3):233–246, 2004. `doi:10.1016/S0304-3975(03)00396-7`.
5. Miroslav Dudík and Leonard J. Schulman. Reconstruction from subsequences. *J. Combin. Theory Ser. A*, 103(2):337–348, 2003. `doi:10.1016/S0097-3165(03)00103-1`.
6. Guilhem Gamard, Gwenaël Richomme, Jeffrey Shallit, and Taylor J. Smith. Periodicity in rectangular arrays. *Inf. Process. Lett.*, 118:58–63, 2017. `doi:10.1016/j.ipl.2016.09.011`.
7. K. Janaki, R. Arulprakasam, Meenakshi Paramasivan, and V. Rajkumar Dare. Algebraic properties of Parikh q-matrices on two-dimensional words. In *Combinatorial image analysis*, volume 13348 of *Lecture Notes in Comput. Sci.*, pages 171–188. Springer, Cham, [2023] ©2023. `doi:10.1007/978-3-031-23612-9\_11`.
8. M. Lothaire. *Combinatorics on Words*. Cambridge Mathematical Library. Cambridge University Press, 1997. `doi:10.1017/CBO9780511566097`.
9. Bennet Manvel, Aaron D. Meyerowitz, Allen J. Schwenk, Kenneth W. Smith, and Paul K. Stockmeyer. Reconstruction of sequences. *Discrete Math.*, 94(3):209–219, 1991. `doi:10.1016/0012-365X(91)90026-X`.
10. Shoshana Marcus and Dina Sokol. On two-dimensional Lyndon words. In *String processing and information retrieval. 20th international symposium, SPIRE 2013, Jerusalem, Israel, October 7–9. 2013, Proceedings*, pages 206–217. Berlin: Springer, 2013. `doi:10.1007/978-3-319-02432-5_24`.
11. Antoine Renard, Michel Rigo, and Markus A. Whiteland. Introducing q-deformed binomial coefficients of words. *J. Algebr. Comb.*, 61(2):33, 2025. Id/No 25. `doi:10.1007/s10801-025-01384-9`.

12. Antoine Renard, Michel Rigo, and Markus A. Whiteland. q-Parikh matrices and q-deformed binomial coefficients of words. *Discrete Math.*, 348(5):Paper No. 114381, 2025. doi:10.1016/j.disc.2024.114381.
13. Christophe Reutenauer. Free Lie algebras. In *Handbook of algebra, Vol. 3*, volume 3 of *Handb. Algebr.*, pages 887–903. Elsevier/North-Holland, Amsterdam, 2003. doi:10.1016/S1570-7954(03)80075-X.
14. Olivier Salon. Automatic sequences with multi-indices. Appendix by J. O. Shallit. Sémin. Théor. Nombres, Univ. Bordeaux I 1986-1987, Exp. No. 4, 27 p. (1987); Appendix 29A-36A (1987)., 1987.
15. Traian-Florin Şerbănuţă. Extending Parikh matrices. *Theor. Comput. Sci.*, 310(1-3):233–246, 2004. doi:10.1016/S0304-3975(03)00396-7.
16. K. G. Subramanian, Kalpana Mahalingam, Rosni Abdullah, and Atulya K. Nagar. Two-dimensional digitized picture arrays and Parikh matrices. *Internat. J. Found. Comput. Sci.*, 24(3):393–408, 2013. doi:10.1142/S012905411350010X.