The $SE(3)$ Lie group framework for flexible multibody systems with contact

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September 2021

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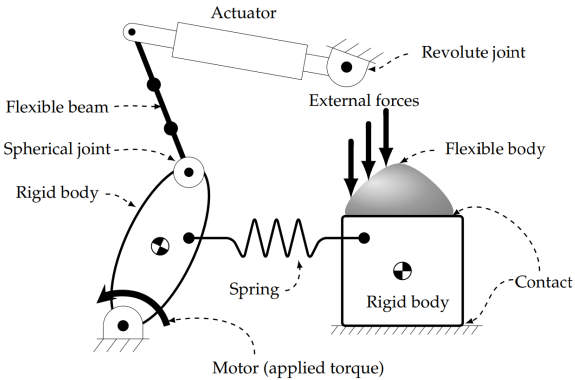
Outline

- **Introduction**
  - Needs of the MB community
  - Engineering approach
  - Lie group theory helps
- Examples of components on $SE(3)$
  - Rigid body
  - Beam
- Contact elements
  - Invariance properties and example of beam contact

Needs of the MB community

Modeling of systems using different structural components such as RBs, beams & shells, joints, actuators and with contact. (FEM approach [1])



- Flexibility
- Large amplitude motion (Geom. non-linearity)
- Kinematic hypotheses introduce orientation (and discontinuities)
  - non-linear configuration space

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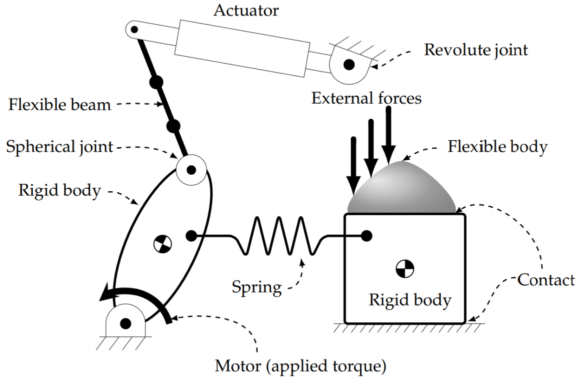

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Classical engineering approach

Formulation in the inertial frame with global parametrization of rotation and subsequent separate discretization of translation and rotation.

- No need for abstract mathematical concepts

Problems and difficulties

- Parametrization induced non-linearity, definition of strains

$$\mathbf{R} = \mathbf{I} + \frac{\sin \|\omega\|}{\|\omega\|}\widetilde{\omega} + \frac{1 - \cos \|\omega\|}{\|\omega\|}\widetilde{\omega}\widetilde{\omega}$$

- Clever tricks to
  - avoid singularities
  - fix locking
  - ensure invariance (strain measures, discretization)

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Lie group theory helps

A nice and well established mathematical toolbox that may be used in a systematic manner for the continuous formulation, space discretization and time integration of MBS with flexible components (beams, shells etc.). It provides the insight that naturally leads to the local frame approach. [2, 3]

- Reduced non-linearity (more intrinsic equations)
- No parametrization singularities
- No locking
- Automatic invariance properties

Lie group toolbox

The coordinates of a frame (location + orientation) with respect to another frame can be represented using the matrix Lie group

$$SE(3) = \left\{ \mathbf{H} : \mathbb{R}^4 \to \mathbb{R}^4 | \ \mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{O}_{1\times 3} & 1 \end{bmatrix}, \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det(\mathbf{R}) = 1, \mathbf{x} \in \mathbb{R}^3 \right\}$$

Coupling through group operation. Left translation is a superimposed RBM

$$\mathbf{H}_A\mathbf{H}_B = \begin{bmatrix} \mathbf{R}_A\mathbf{R}_B & \mathbf{x}_A + \mathbf{R}_A\mathbf{x}_B \\ \mathbf{O} & 1 \end{bmatrix}$$

Derivative w.r.t. parameter a (velocity, deformation, variation)

$$d_a(\mathbf{H}) = \mathbf{H}\widetilde{\mathbf{a}}, \quad \widetilde{\mathbf{a}} \in \mathfrak{se}(3)$$

$\widetilde{\mathbf{a}}$ is left invariant. Indeed

$$\widetilde{\mathbf{a}}^{tr} = (\mathbf{H}^*\mathbf{H})^{-1} d_a(\mathbf{H}^*\mathbf{H}) = \mathbf{H}^{-1}\left(\mathbf{H}^{*^{-1}}\mathbf{H}^*\right) d_a(\mathbf{H}) = \mathbf{H}^{-1} d_a(\mathbf{H}) = \widetilde{\mathbf{a}}$$

We interpret it as insensitive to superimposed RBM. It is a local frame quantity.

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Relative transformation

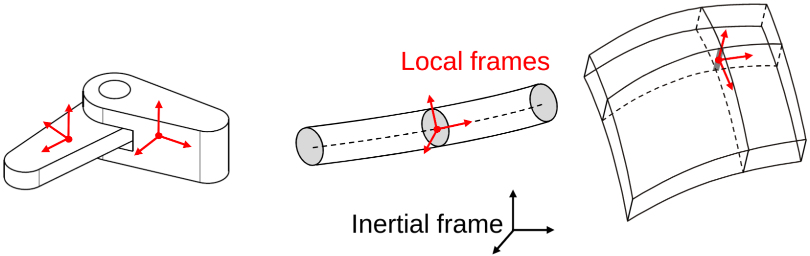
The $SE(3)$ element, $\mathbf{H}_{AB} = \mathbf{H}_A^{-1}\mathbf{H}_B$, which expresses the relative transformation between two elements of $SE(3), \mathbf{H}_A(s)$ and $\mathbf{H}_B(s)$ is invariant under a superimposed Euclidean transformation i.e. a left translation on $SE(3)$.

Indeed, by applying a left translation of $\mathbf{H}^*$ to both elements the relative transformation remains unchanged:

$$\begin{aligned}
\mathbf{H}_{AB}^{\text{tr}} = (\mathbf{H}^*\mathbf{H}_A)^{-1}\,\mathbf{H}^*\mathbf{H}_B &= \mathbf{H}_A^{-1}\left(\mathbf{H}^{*^{-1}}\mathbf{H}^*\right)\mathbf{H}_B \\
&= \mathbf{H}_A^{-1}\mathbf{H}_B \\
&= \mathbf{H}_{AB}
\end{aligned}$$

Development strategy: Geometric FEM

- Represent finite motions as frame transformations
- Consider these frame transformations as elements of $SE(3)$
- Joints interpreted as subgroups of $SE(3)$
- Work with left invariant derivatives and solve the dynamics in the local frame ($\neq$ floating frame)
- Exploit modern numerical methods on Lie groups



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Free Rigid body

Equations of motion in the local frame (on the Lie group)

$$\dot{\mathbf{H}} = \mathbf{H}\widetilde{\mathbf{v}}$$
$$\mathbf{f}(\mathbf{H}, \mathbf{v}, \dot{\mathbf{v}}) = \mathbf{M}\dot{\mathbf{v}} - \widehat{\mathbf{v}}^T \mathbf{M}\mathbf{v} = \mathbf{0}_{6\times 1}$$

- Velocity on $SE(3)$: $\begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{u} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} = \widetilde{\mathbf{v}} \in \mathfrak{se}(3)$
  Linear velocity: $\mathbf{u} = \mathbf{R}^T \dot{\mathbf{x}}$, Angular velocity: $\widetilde{\boldsymbol{\omega}} = \mathbf{R}^T \dot{\mathbf{R}}$
- $\mathbf{M}$ independent of position and orientation $\rightarrow$ invariance of inertia forces
- Implicit Lie group TI using a constant velocity field

$$\mathbf{H}_{n+1} = \mathbf{H}_n \exp\left(h\widetilde{\mathbf{v}}_{n+1}\right) \text{ (local parametrization)}$$
$$\mathbf{f}(\mathbf{H}_{n+1}, \mathbf{v}_{n+1}, \dot{\mathbf{v}}_{n+1}) = \mathbf{0}$$

Generalized-α, Runge-Kutta, Multistep... [4, 5, 6]

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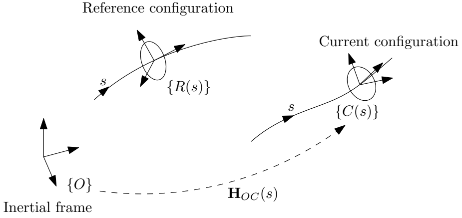
Geometrically exact beam

Objective strains for free:

$$\mathbf{H}'_{OC}(s) = \mathbf{H}_{OC}(s)\tilde{\boldsymbol{f}}_C$$

$$\epsilon_C = \boldsymbol{f}_C - \boldsymbol{f}_R = \begin{bmatrix} \mathbf{R}^T_{OC}\mathbf{x}'_{OC} \\ \mathbf{R}^T_{OC}\mathbf{R}'_{OC} \end{bmatrix} - \boldsymbol{f}_R$$

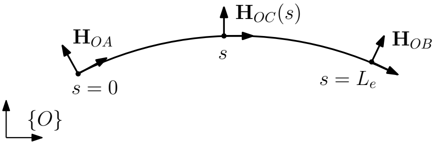


Continuous formulation

- The left invariant derivative on $SE(3)$ automatically yields the classical deformation measures [7]
- EOM represented in the local frame

Geometrically exact beam

Local parametrization element by element [8]



- Interpolation formula

$$\mathbf{H}_{OC}(s) = \mathbf{H}_{OA} \exp_{SE(3)}(s\tilde{\epsilon}_C)$$

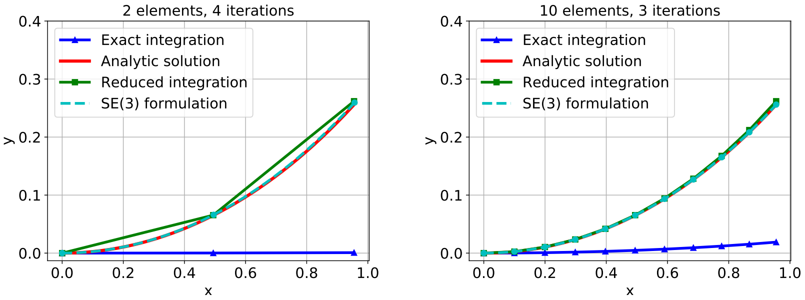
- Discrete deformation

$$\tilde{\epsilon}_C = \frac{\log_{SE(3)}(\mathbf{H}_{AB})}{L}$$

Discretization

- Extension of 1st order interpolation to a non-linear space
- Natural coupling of translation and rotation (no locking)
- Depends on relative configuration (frame invariant + implicit)
- Geometric nonlinearities decrease with mesh refinement

Pure bending example



$$L = 1\text{m},\ r = 0.01\text{m},\ E = 210\text{GPa},\ \nu = 0.3,\ M = \frac{EI\pi}{6L}\text{Nm}$$

- Classical reduced integration trick to avoid shear locking not needed
- Less iterations with finer mesh (less non-linear)

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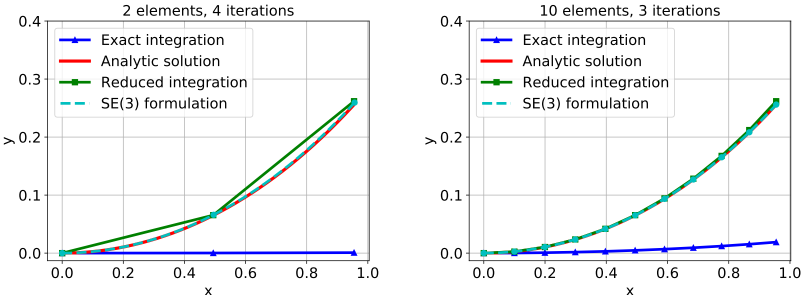

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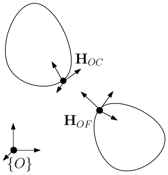
Contact kinematics and constraint gradient

Association of two frames $\mathbf{H}_{OC}, \mathbf{H}_{OF}$ at the potential contact location. The variations are

$$\delta\mathbf{H}_{OC} = \mathbf{H}_{OC}\widetilde{\delta\boldsymbol{\pi}}_C, \quad \delta\mathbf{H}_{OF} = \mathbf{H}_{OF}\widetilde{\delta\boldsymbol{\pi}}_F$$

$$\widetilde{\delta\boldsymbol{\pi}}_C, \widetilde{\delta\boldsymbol{\pi}}_F \in \mathfrak{se}(3), \quad \delta\boldsymbol{\pi} = \begin{bmatrix}\delta\boldsymbol{\pi}_C \\ \delta\boldsymbol{\pi}_F\end{bmatrix} \in \mathbb{R}^{12}$$



Unilateral restriction of relative motion

$$g(\mathbf{H}_{CF}) \geq 0$$

Direction of the contact force in the local frame given by constraint gradient

$$\delta g(\mathbf{H}_{CF}) = \mathbf{G}(\mathbf{H}_{CF})\delta\boldsymbol{\pi}$$

Invariance property

- The local constraint gradient only depends on relative quantities ($\mathbf{H}_{CF}$)

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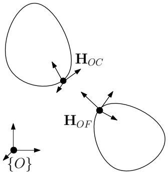

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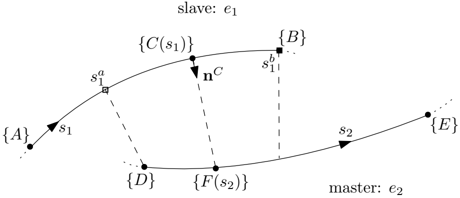
Contact kinematics and constraint gradient

Thin circular beam [9]:

$$g(\mathbf{H}_{OC}, \mathbf{H}_{OF}) = \|\mathbf{x}_{OF} - \mathbf{x}_{OC}\| - 2r$$

$$\mathbf{G}^T(\mathbf{H}_{OC}, \mathbf{H}_{OF}) = \begin{bmatrix} -\mathbf{M}\mathbf{n}^C \\ \mathbf{M}\mathbf{R}_{OF}^T\mathbf{R}_{OC}\mathbf{n}^C \end{bmatrix}$$



Normal in the local frame:

$$\mathbf{n}^C(\mathbf{H}_{OC}, \mathbf{H}_{OF}) = \mathbf{R}_{OC}^T \frac{(\mathbf{x}_{OF} - \mathbf{x}_{OC})}{\|\mathbf{x}_{OF} - \mathbf{x}_{OC}\|}$$

It may be verified that:

$$\begin{aligned}\mathbf{n}^C(\mathbf{H}^*\mathbf{H}_{OC}, \mathbf{H}^*\mathbf{H}_{OF}) &= \mathbf{R}_{OC}^T\mathbf{R}^{*T} \frac{(\mathbf{R}^*\mathbf{x}_{OF} - \mathbf{x}^* - \mathbf{R}^*\mathbf{x}_{OC} + \mathbf{x}^*)}{\|\mathbf{R}^*\mathbf{x}_{OF} - \mathbf{x}^* - \mathbf{R}^*\mathbf{x}_{OC} + \mathbf{x}^*\|} \\ &= \mathbf{R}_{OC}^T \left(\mathbf{R}^{*T}\mathbf{R}^*\right) \frac{(\mathbf{x}_{OF} - \mathbf{x}_{OC})}{\|\mathbf{R}^*(\mathbf{x}_{OF} - \mathbf{x}_{OC})\|} \\ &= \mathbf{n}^C(\mathbf{H}_{OC}, \mathbf{H}_{OF})\end{aligned}$$

and thus the gradient is invariant: $\mathbf{G}^T(\mathbf{H}^*\mathbf{H}_{OC}, \mathbf{H}^*\mathbf{H}_{OF}) = \mathbf{G}^T(\mathbf{H}_{OC}, \mathbf{H}_{OF})$

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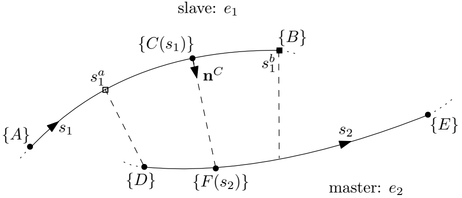

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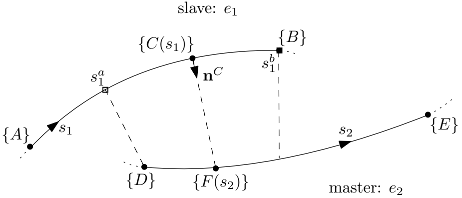

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Beam contact

See [9] and [10]

- Mortar formulation for contact between beams
- Augmented Lagrangian approach

Concluding remarks

Multibody Systems in $SE(3)$:

- Possibility to formulate a wide range of components: rigid, flexible, joints, contact, superelements, structural optimization
- Elegant, intrinsic equations with decreased non-linearities
- Global parametrization issues circumvented, no locking
- Discrete EOMs and iteration matrix are invariant. It also remains true for contact problems

Perspectives on the beam contact side:

- Upscaling towards more complex assemblies
- Dynamic problems with impacts
- Friction

The $SE(3)$ Lie group framework for flexible multibody systems with contact

Thank you for your attention!

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September 2021

LIÈGE université

Appendix: local, floating, corotational frame

- Local frame
  - Follow from the kinematice assumption (rigid, beam, shell)
  - At the continuous level it exists everywhere on the body. Several local frames may coexist in one finite element
  - Nodal quantity (FE assembly)
  - The equilibrium equations are written in these frames
- Corotational frame
  - Additionally defined
  - Unique and internal to each element (no assembly)
  - Ultimately the equations are solved in the inertial frame (corotational frame is only intermediate)
- Floating frame
  - Defined from rigid body coordinates
  - The motion is splitted into rigid and flexible coordinates
  - Unique for the whole body
  - Ultimately the equations are again solved in the inertial frame

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