Simulation of multibody systems with switching constraints: Formulation and time integration

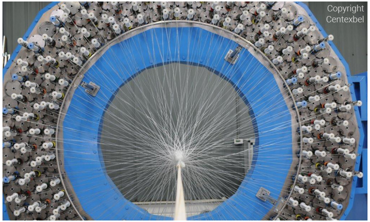
Indrajeet Patil[1,2] and Olivier Brüls[1]

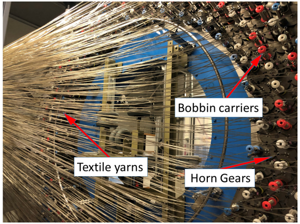
[1]Department of Aerospace and Mechanical Engineering, University of Liège, Belgium
[2]Fraunhofer Institute for Industrial Mathematics ITWM, Kaiserslautern, Germany

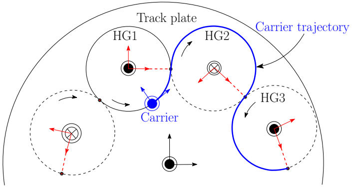
ECCOMAS Thematic Conference on Multibody Dynamics

13-18 July 2025, Innsbruck, Austria

Motivating application: Overbraiding process



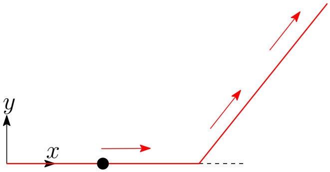




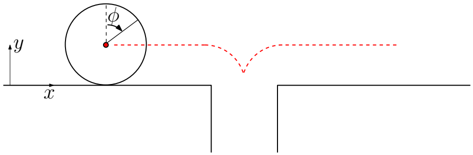
Each bobbin carrier follows a complex trajectory, with periodic transitions (**switchings**) from one horn gear to the next.

At switching, **jumps in angular velocity** occur.

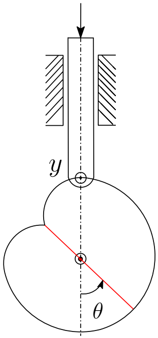
More examples of systems with switching constraints



Point on track



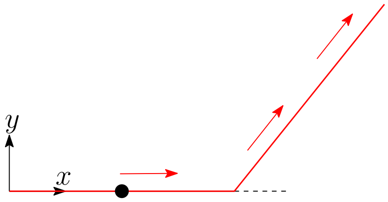
Wheel over pothole



Cam-follower

At every time instant, the number of active constraints is the same.

Modelling approaches



Point on track:
at every time,
a single constraint
is active

Two possible approaches to handle the different modes

- Use two constraints + an activation criterion for each of them.
  - Number of constraints (and multipliers) increases with the number of modes
  - No global coordination of the activations $\Rightarrow$ risk of over-/under-constrainments
- Use a **single constraint** whose mathematical expression is **switching**
  - How can we formulate the switching constraint?
  - How can we obtain the equations of motion?
  - How can we solve the equations of motion?

Related work

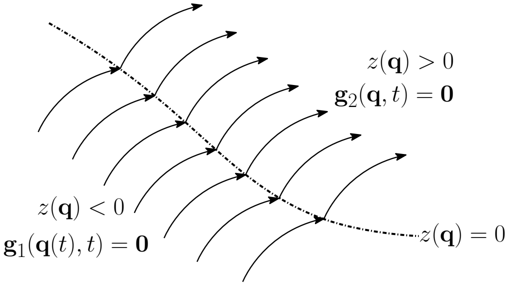
Switching dynamical systems, which fall in the broader class of **hybrid systems**, have been studied on various aspects: stability, control, relation to DAE theory, numerical methods, applications in electrical engineering and power electronics[1,2,3,4,5,6,7,8]

Open question:
can we adapt these theories to **mechanical systems with switching bilateral constraints**?

[1] A. J. van der Schaft, J. M. Schumacher, *IEEE Transactions on Automatic Control* **43**, 483–490 (1998).
[2] J. Cortes, *IEEE Control systems magazine* **28**, 36–73 (2008).
[3] V. Mehrmann, L. Wunderlich, *Journal of Process Control* **19**, 1218–1228 (2009).
[4] R. Goebel *et al.*, *IEEE control systems magazine* **29**, 28–93 (2009).
[5] V. Acary *et al.*, *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* **29**, 1042–1055 (2010).
[6] Z. Sun, S. S. Ge, *Stability theory of switched dynamical systems*, (Springer, London, 2011).
[7] S. Trenn, *Dynamics and Control of Switched Electronic Systems: Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, 189–216 (2012).
[8] A. Rocca *et al.*, *IFAC-PapersOnLine* **53**, 1888–1893 (2020).

Outline

Switching function and switching surface

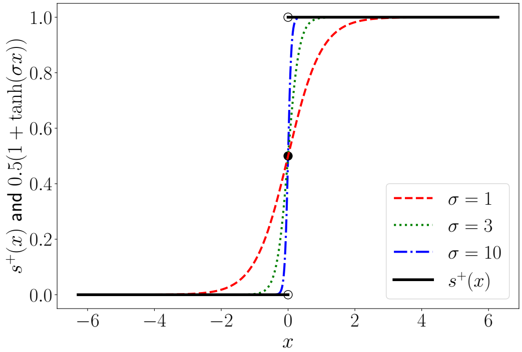


Switching function: $z(\mathbf{q}) : \mathbb{R}^n \to \mathbb{R}$
Switching surface: $\{\mathbf{q} : z(\mathbf{q}) = 0\}$

Switching constraints: $\mathbf{g}(\mathbf{q}, t) = \mathbf{0}$ with

$$\mathbf{g}(\mathbf{q}, t) \triangleq \begin{cases} \mathbf{g}_1(\mathbf{q}, t) & \text{if} \quad z(\mathbf{q}) < 0 \\ \frac{1}{2}(\mathbf{g}_1(\mathbf{q}, t) + \mathbf{g}_2(\mathbf{q}, t)) = \mathbf{0} & \text{if} \quad z(\mathbf{q}) = 0 \\ \mathbf{g}_2(\mathbf{q}, t) & \text{if} \quad z(\mathbf{q}) > 0 \end{cases}$$

Compact reformulation using the Heaviside step function



Heaviside step function $s^+(x)$ vs smooth approximations $(1 + \tanh(\sigma x))/2$

$$\mathbf{g}(\mathbf{q}, t) \triangleq (1 - s^+(z(\mathbf{q})))\, \mathbf{g}_1(\mathbf{q}, t) + s^+(z(\mathbf{q}))\, \mathbf{g}_2(\mathbf{q}, t)$$

- We keep $s^+(x)$ as it is (no regularization or smooth approximation)
- The formulation can be extended to multiple switching surfaces

Switching function: Technical conditions

The two portions of the constraint space should intersect at the switching surface

- $\forall \mathbf{q}$ satisfying both $\mathbf{g}_1(\mathbf{q}, t) = \mathbf{0}$ and $z(\mathbf{q}) = 0$, we require $\mathbf{g}_2(\mathbf{q}, t) = \mathbf{0}$
- $\forall \mathbf{q}$ satisfying both $\mathbf{g}_2(\mathbf{q}, t) = \mathbf{0}$ and $z(\mathbf{q}) = 0$, we require $\mathbf{g}_1(\mathbf{q}, t) = \mathbf{0}$

The switching function cannot be tangent to the constraint space

On the switching surface, the gradient matrices $\begin{bmatrix} \mathbf{G}_1(\mathbf{q}, t) \\ \mathbf{Z}(\mathbf{q}) \end{bmatrix}$ and $\begin{bmatrix} \mathbf{G}_2(\mathbf{q}, t) \\ \mathbf{Z}(\mathbf{q}) \end{bmatrix}$ are full rank

The gradient of the switching constraint $\mathbf{g}(\mathbf{q}, t)$ can be discontinuous

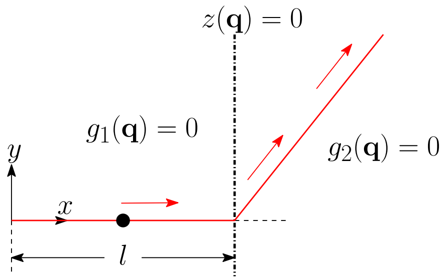
On the switching surface, we allow $\mathbf{G}_1(\mathbf{q}, t) \neq \mathbf{G}_2(\mathbf{q}, t)$

The switching function shapes the constraint manifold

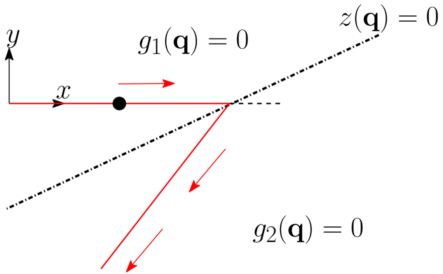
Point on track example: $\mathbf{q} = [x\ y]^T$

$$
\begin{aligned}
g_1(\mathbf{q}) &\triangleq y \\
g_2(\mathbf{q}) &\triangleq y - a\,(x - l)
\end{aligned}
$$

$\boxed{z(\mathbf{q}) \triangleq x - l}$



$\boxed{z(\mathbf{q}) \triangleq y - 0.5\,a\,(x - l)}$



Hybrid DAE

For almost every time: $\begin{cases} \dot{\mathbf{q}} &= \mathbf{v} \\ \mathbf{M}(\mathbf{q})\,\dot{\mathbf{v}} + \mathbf{G}^T(\mathbf{q},t)\,\boldsymbol{\lambda} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) \\ \mathbf{g}(\mathbf{q},t) &= \mathbf{0} \end{cases}$

At the switching time t_i, we expect a **velocity jump** and a **reaction impulse** (**impact**):

$$\begin{cases} \mathbf{M}(\mathbf{q}_i)\,(\mathbf{v}_i^+ - \mathbf{v}_i^-) + \mathbf{R} &= \mathbf{0} \\ \mathbf{G}^+(\mathbf{q}_i,t_i)\,\mathbf{v}_i^+ &= -\mathbf{g}_t^+(\mathbf{q}_i,t_i) \end{cases}$$

Assumption: $\mathbf{R} = \mathbf{G}_{\mathcal{E}}^T\,\boldsymbol{\Lambda}$, where $\mathbf{G}_{\mathcal{E}}$ is an **intermediate constraint gradient**

At switching time t_i: $\begin{cases} \mathbf{M}(\mathbf{q}_i)\,(\mathbf{v}_i^+ - \mathbf{v}_i^-) + \mathbf{G}_{\mathcal{E}}^T\,\boldsymbol{\Lambda} &= \mathbf{0} \\ \mathbf{G}^+(\mathbf{q}_i,t_i)\,\mathbf{v}_i &= -\mathbf{g}_t^+(\mathbf{q}_i,t_i) \end{cases}$

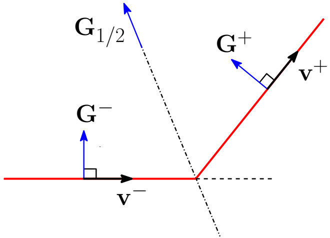
Reformulation as an equality of differential measures

$$\dot{\mathbf{q}} = \mathbf{v}$$
$$\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} - \mathbf{G}_{\mathcal{E}}^{T}\,\mathrm{d}\mathbf{i} = \mathbf{f}(\mathbf{q}, \mathbf{v}, t)\,\mathrm{d}t$$
$$\mathbf{g}(\mathbf{q}, t) = \mathbf{0}$$

How to define the intermediate constraint gradient?

In the case of a **single constraint**, $\mathbf{G}_{\mathcal{E}}$ can be defined by interpolation between the vectors $\mathbf{G}^-$ and $\mathbf{G}^+$ (with $\mathcal{E} \in [0, 1]$)

- the interpolation should be insensitive to scaling and sign inversion of the two vectors
  $\Rightarrow$ **normalization steps** are needed
- if $\mathbf{G}_{\mathcal{E}} = \mathbf{G}^+$ (**post-switch gradient**), then **energy is dissipated** at switching
  (the velocity component orthogonal to the post-switch constraint is annihilitated)
- if $\mathbf{G}_{\mathcal{E}} = \mathbf{G}_{1/2}$ (**mid-gradient**), then **energy is preserved** at switching



$\Rightarrow$ **The intermediate gradient drives the energy behaviour at switching**

How to define the intermediate constraint gradient?

In the general case, $\mathbf{G}_\mathcal{E}$ can still be defined by interpolation

- $\mathbf{G}^-$ and $\mathbf{G}^+$ are **linear subspaces** of $\mathbb{R}^n$
  $\Rightarrow$ the interpolation is performed on **a Grassmann manifold**, see also[a]
- if $\mathbf{G}_\mathcal{E} = \mathbf{G}^+$ (**post-switch gradient**), then **energy is dissipated** at switching
  (the velocity component orthogonal to the post-switch constraint is annihilitated)
- if $\mathbf{G}_\mathcal{E} = \mathbf{G}_{1/2}$, ??

[a] D. Amsallem, C. Farhat, *AIAA Journal* **46**, 1803–1813 (July 2008).

Time integration

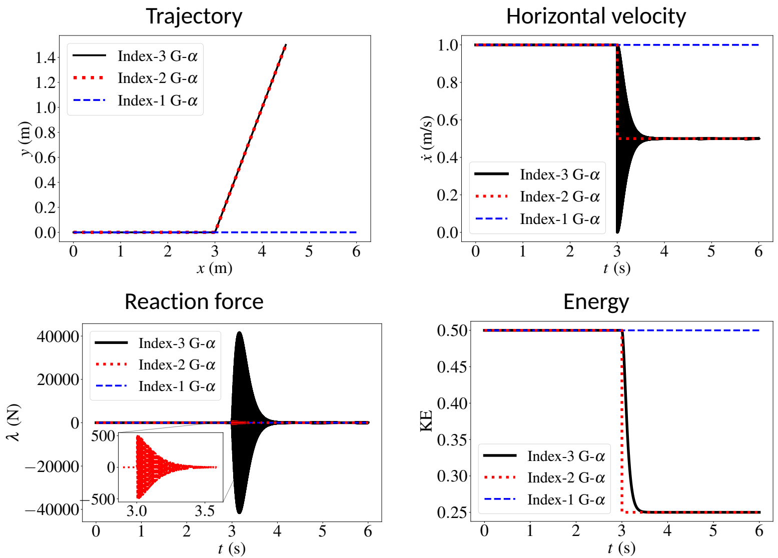
At switching, we expect a nonsmooth behaviour with velocity jumps

- Classical time integration schemes may fail to handle such discontinuous behaviours.
- Methods from nonsmooth dynamics should rather be considered
  - Event-driven methods
  - Event-capturing methods (e.g., Moreau-Jean & nonsmooth generalized-α schemes)

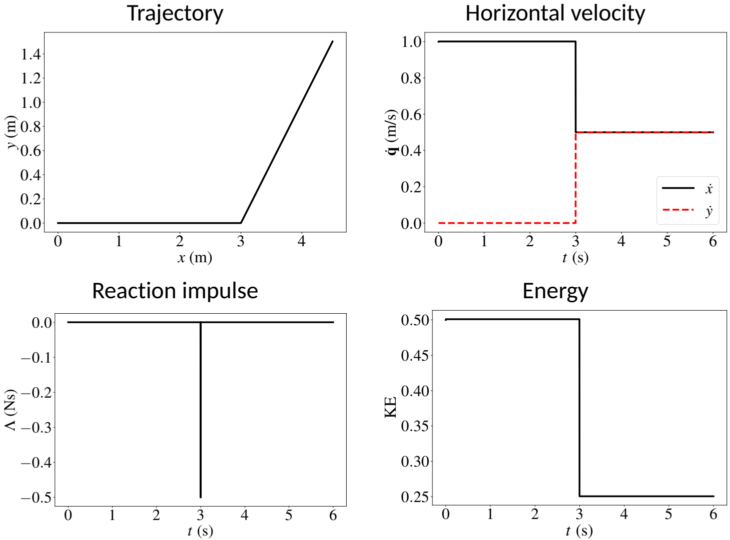
4 different versions of the generalized-α will be compared

- Index-3 G-α: classical generalized-α with constraints at position level
- Index-2 G-α: classical generalized-α with constraints at velocity level
- Index-1 G-α: classical generalized-α with constraints at acceleration level
- NSGA: nonsmooth generalized-α with constraints both at position and velocity levels

Point on track with $\mathsf{G}_{\mathcal{E}} = \mathsf{G}^{+}$: Classical integration scheme



Point on track with $\mathbf{G}_{\mathcal{E}} = \mathbf{G}^{+}$: Nonsmooth integration scheme (NSGA)

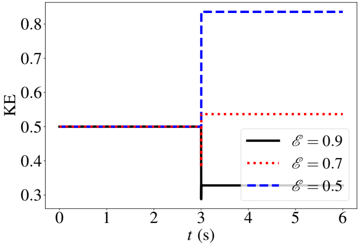


Point on track: Choice of intermediate gradient

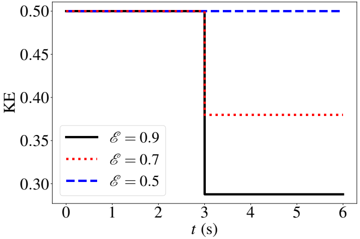
Interpolation parameter

- $\mathcal{E} = 1 \rightarrow \mathbf{G}_{\mathcal{E}} = \mathbf{G}^{+}$ (energy dissipation at switching, as in previous simulation)
- $\mathcal{E} = 0.5 \rightarrow \mathbf{G}_{\mathcal{E}} = \mathbf{G}_{1/2}$ (energy conservation at switching)

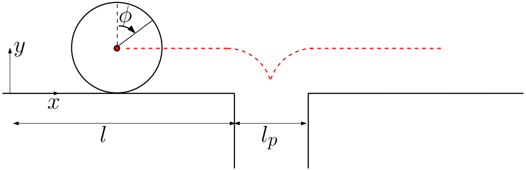
Classical integration scheme (index-2)

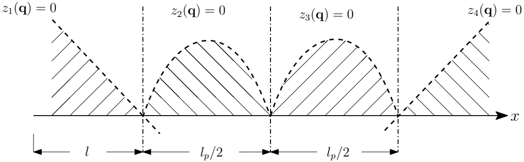


Nonsmooth integration scheme



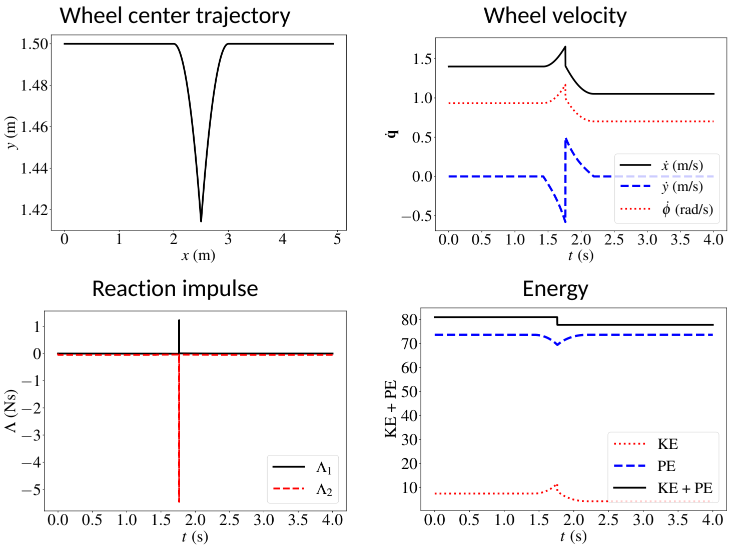
Wheel over pothole: Model definition





- $\mathbf{q} = \begin{bmatrix} x & y & \phi \end{bmatrix}^T$
- Constraint 1: rolling without slipping
- Constraint 2: non-penetration
- 4 switching functions: $\begin{cases} z_1(\mathbf{q}) &= -x + l \\ z_2(\mathbf{q}) &= -(x - l)(x - l - l_p/2) \\ z_3(\mathbf{q}) &= -(x - l - l_p/2)(x - l - l_p) \\ z_4(\mathbf{q}) &= x - l - l_p \end{cases}$
- Constraint formulation: $\mathbf{g}(\mathbf{q}, t) = \sum_{i=1}^{k+1} \left(\prod_{j \neq i} (1 - s^+(z_j(\mathbf{q})) \right) s^+(z_i(\mathbf{q}))\, \mathbf{g}_i(\mathbf{q}, t) = \mathbf{0}$

Wheel over pothole: Numerical results ($G_{\mathcal{E}} = G^{+}$)



Conclusion

- Multibody systems with **switching bilateral constraints**
- The **switching functions** shape the geometry of the constraint manifold
- At switching: discontinuous constraint gradient & velocity jump (impact)
  We postulate that the reaction impulse is along an **intermediate constraint gradient**
  - $\mathbf{G}_\mathcal{E}$ is defined by a subspace interpolation between $\mathbf{G}^-$ and $\mathbf{G}^+$
  - If $\mathbf{G}_\mathcal{E}$ = **post-switch gradient** $\Rightarrow$ **energy dissipation**
  - If $\mathbf{G}_\mathcal{E}$ = **mid-gradient** $\Rightarrow$ **energy conservation** in the single constraint case
- Equations of motion can be formulated either as a hybrid DAE or as an **equality of differential measures**
- Classical time integration schemes fail to deliver acceptable numerical solutions
- **Nonsmooth time integration schemes** are reliable in this case

I. Patil and O. Brüls. Numerical simulation of nonsmooth multibody systems with switching bilateral constraints. *Nonlinear Dynamics*, online since June 2025.

Thank you for your attention!

Simulation of multibody systems with switching constraints: Formulation and time integration

Indrajeet Patil[1,2] and Olivier Brüls[1]

[1]Department of Aerospace and Mechanical Engineering, University of Liège, Belgium
[2]Fraunhofer Institute for Industrial Mathematics ITWM, Kaiserslautern, Germany

Acknowledgement: THREAD EUROPEAN TRAINING NETWORK

ECCOMAS Thematic Conference on Multibody Dynamics

13-18 July 2025, Innsbruck, Austria

LIÈGE université