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Symmetric Absolutely Separable states

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Separable vs entangled mixed state

Separable state [Werner (1989)]

A state $\rho_{\text{sep}} \in \mathcal{S}(\mathcal{H})$ is separable if it can be written as a convex combination of product states:

$$\rho_{\text{sep}} = \sum_k w_k \left(\rho_k^{(1)} \otimes \cdots \otimes \rho_k^{(N)}\right)$$

with $w_k \geq 0$, $\sum_k w_k = 1$. Otherwise, it is entangled.

Negativity [Peres (1996)], [Horodecki et al. (1996)]

The negativity is an entanglement witness (sometimes a measure), defined as

$$\mathcal{N}(\rho) = -2 \sum_{\mu_k < 0} \mu_k$$

where μ_k are the eigenvalues of the partial transposition ρ^{T_A}

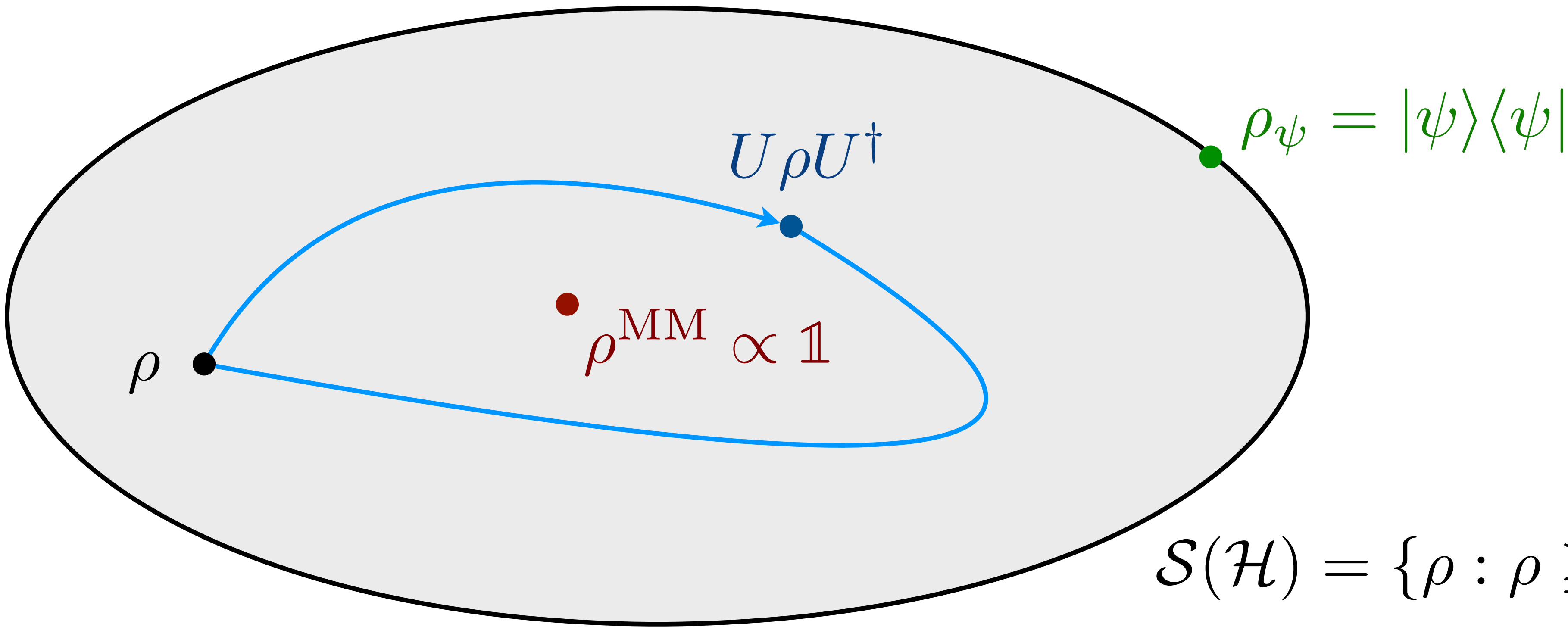
- $\mathcal{N}\left(\rho_{\text{sep}}\right) = 0$ for any separable state
- Invariant under local unitary transformations

Maximal negativity of two-qubit states

[Verstraete, Audenaert & De Moor (2001)] $\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$

- Goal : Find the maximum entanglement (negativity) of ρ in its SU(4)-orbit

Unitary orbit of ρ: $\{U\rho U^\dagger : U^{-1} = U^\dagger\}$



- Spectrum $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq$ of ρ invariant along the orbit

- Entanglement varies along the orbit

Maximal negativity of two-qubit states

[Verstraete, Audenaert & De Moor (2001)] $\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$

- Result : $\max_{U \in \mathrm{SU}(4)} \mathcal{N}\left(U\rho U^\dagger\right) = \max\left(0, \sqrt{(\lambda_0 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2} - \lambda_1 - \lambda_3\right)$

AS state [Kuś, Życzkowski (2001)]

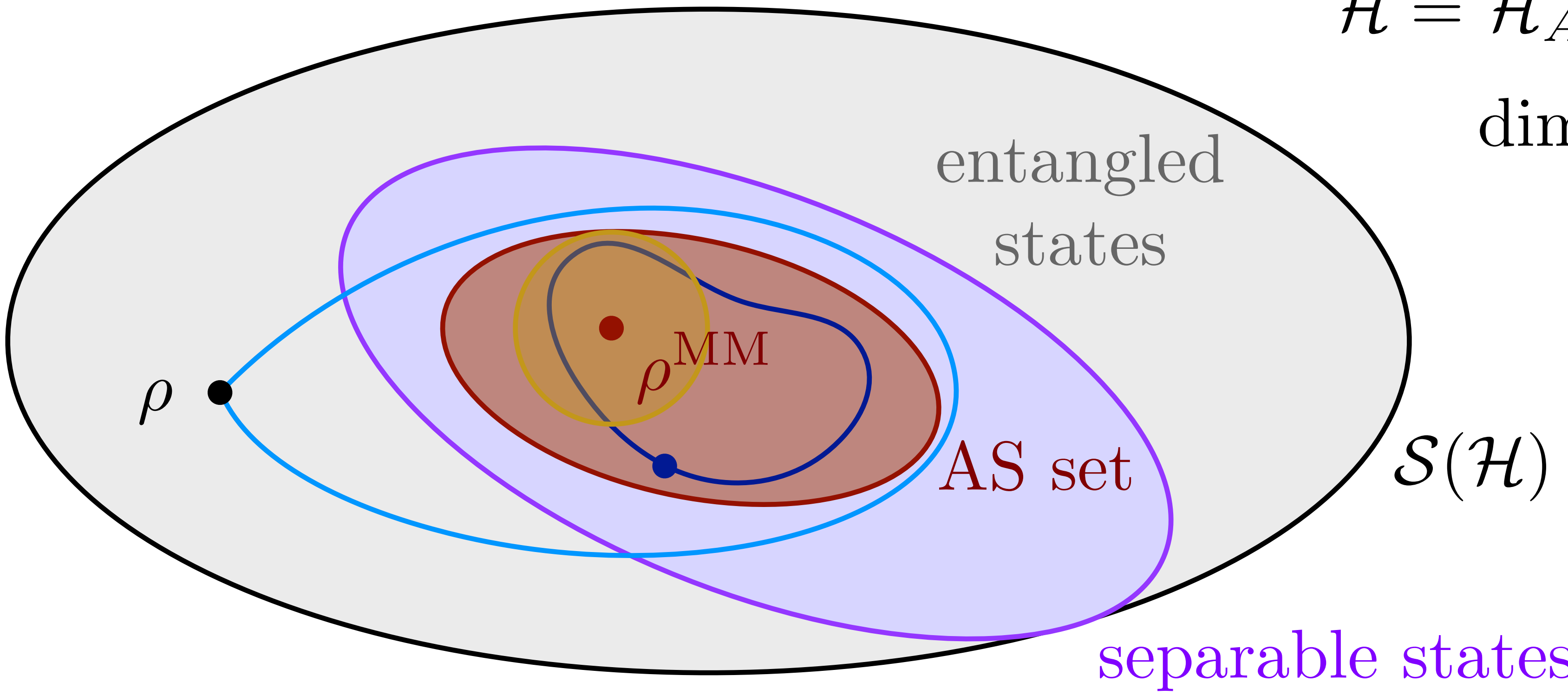
A state $\rho \in \mathcal{S}(\mathcal{H})$ is Absolutely Separable (AS) if

$$\rho' = U\rho U^\dagger$$

is separable for any unitary transformation U.

- Corollary : ρ is AS iff $\lambda_0 \leqslant \lambda_2 + 2\sqrt{\lambda_1 \lambda_3}$

AS states for bipartite systems

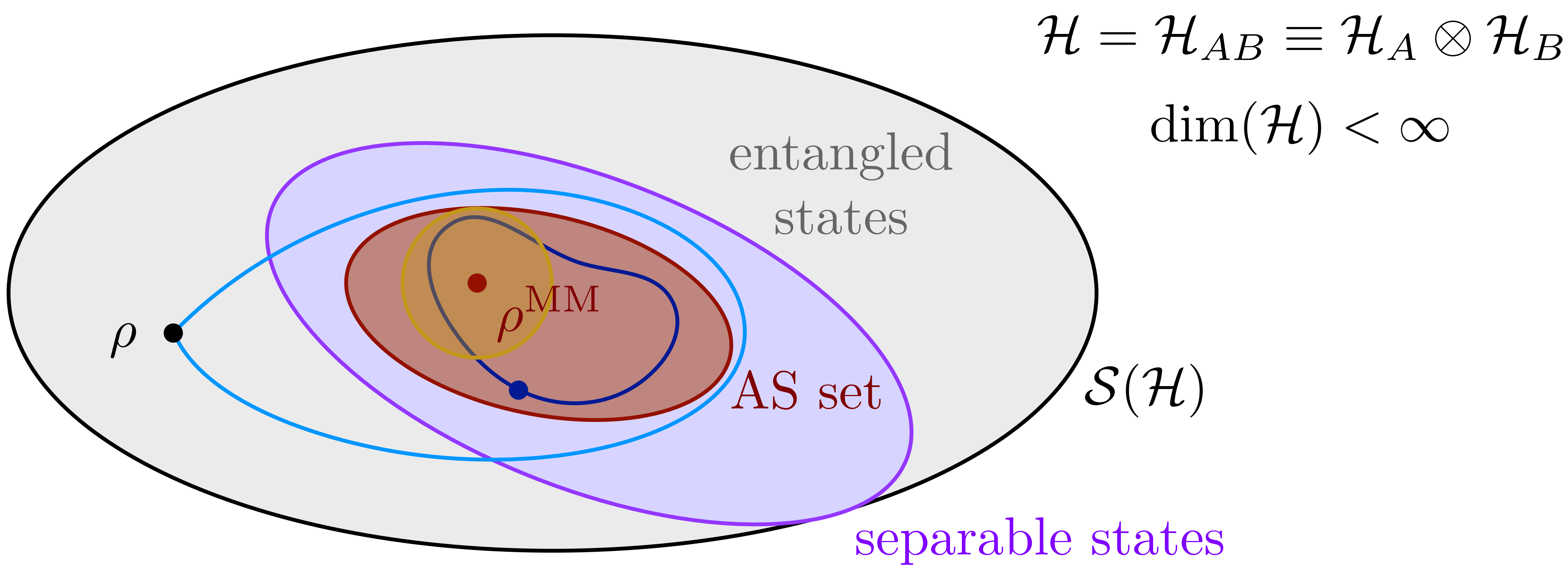


$$\mathcal{H} = \mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\dim(\mathcal{H}) < \infty$$

- $\mathcal{S}(\mathcal{H})$ is a convex and compact set
- Separable states form a convex and compact set $\mathcal{S}_{\mathrm{sep}} \subset \mathcal{S}$
- AS states form a convex and compact set $\mathcal{A}_{\mathrm{sep}} \subset \mathcal{S}_{\mathrm{sep}}$
- Ball of AS states around the MMS [Życzkowski (1998), Gurvits & Barnum (2002)]

AS states: some properties



- Noise Resilience $\rightarrow$ remain separable under arbitrary unitary noise
- Quantum Control $\rightarrow$ cannot be entangled via any global control operation
- Quantum Thermodynamics $\rightarrow$ relevant in studies of passive states and thermalization

Known results for bipartite systems

Exact results for maximum entanglement in larger systems remain incomplete

Partial results for qubit-qutrit: Mendonça, Marchiolli, Hedemann (2017)

... but exact results for absolute separability

- $\boxed{\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2}$: ρ is AS iff $\lambda_0 \leqslant \lambda_2 + 2\sqrt{\lambda_1 \lambda_3}$ [VAD (2001)]
- $\boxed{\mathcal{H} \simeq \mathbb{C}^2 \otimes \mathbb{C}^m}$: ρ is AS iff $\lambda_0 \leq \lambda_{2m-2} + 2\sqrt{\lambda_{2m-3}\lambda_{2m-1}}$ [Johnston (2013)] based on PPT
- $\boxed{\mathcal{H} \simeq \mathbb{C}^n \otimes \mathbb{C}^m}$: ? (only sufficient conditions)

$\rightarrow$ what about symmetry-constrained systems ?

APPT states

APPT STATE

A state $\rho \in \mathcal{S}(\mathcal{H}_{\mathrm{AB}})$ is Absolutely PPT (APPT) if

$$\rho' = U \rho U^\dagger$$

is PPT for all unitary transformation U, i.e.,

$$\min_U \lambda_{\min}\left(U \rho^{T_A} U^\dagger\right) \geq 0$$

- The set of APPT states is convex and compact ($\mathcal{A}_{\mathrm{PPT}} \subseteq \mathcal{A}_{\mathrm{sep}}$)
- The set of APPT states is fully characterized in terms of LMI (Linear Matrix Inequalities) Hildebrand (2007)
- Open question: AS set $\overset{?}{=}$ APPT set $2 \times n$ systems ✓ Johnston (2013)

APPT states

- The set of APPT states is fully characterized in terms of LMI (Linear Matrix Inequalities) Hildebrand (2007)

- State ρ with eigenspectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{m\times n}$

<u>$2 \times n$</u>:

$$\begin{pmatrix} 2\lambda_{2n} & \lambda_{2n-1}-\lambda_1 \\ \lambda_{2n-1}-\lambda_1 & 2\lambda_{2n-2} \end{pmatrix} \succeq 0$$

<u>$3 \times n$</u>:

$(n \geq 3)$

$$\begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1}-\lambda_1 & \lambda_{3n-3}-\lambda_2 \\ \lambda_{3n-1}-\lambda_1 & 2\lambda_{3n-2} & \lambda_{3n-4}-\lambda_3 \\ \lambda_{3n-3}-\lambda_2 & \lambda_{3n-4}-\lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \succeq 0$$

$$\begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1}-\lambda_1 & \lambda_{3n-2}-\lambda_2 \\ \lambda_{3n-1}-\lambda_1 & 2\lambda_{3n-3} & \lambda_{3n-4}-\lambda_3 \\ \lambda_{3n-2}-\lambda_2 & \lambda_{3n-4}-\lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \succeq 0$$

Symmetric and SAS states

Symmetric state

A state ρ_S is symmetric if it is supported on $\mathcal{H}_S$:

$$\rho_S = P_S \rho_S P_S^\dagger$$

where P_S is the projector onto the symmetric subspace $\mathcal{H}_S \subset \mathcal{H}$ spanned by the Dicke states.

SAS state

A state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ is Symmetric Absolutely Separable (SAS) if

$$\rho_S' = U_S \rho_S U_S^\dagger$$

is separable for any symmetry-preserving unitary transformation U_S.

SAS and SAPPT states

SAPPT STATE

A state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ is Symmetric Absolutely PPT (SAPPT) if

$$\rho'_S = U_S \rho_S U_S^\dagger$$

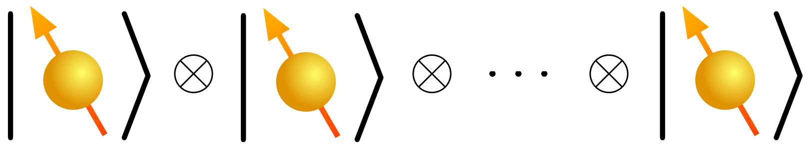
is PPT for any symmetry-preserving unitary transformation U_S.

- The AS and SAS states are fundamentally different:
  - SAS states are low-rank states
  - no SAS state is also AS (except the symmetric MM state for two qubits)
  - SAS states $\equiv$ Absolutely classical spin-j states

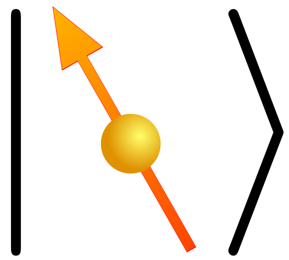
[Bohnet-Waldraff, Giraud, Braun (2017)]

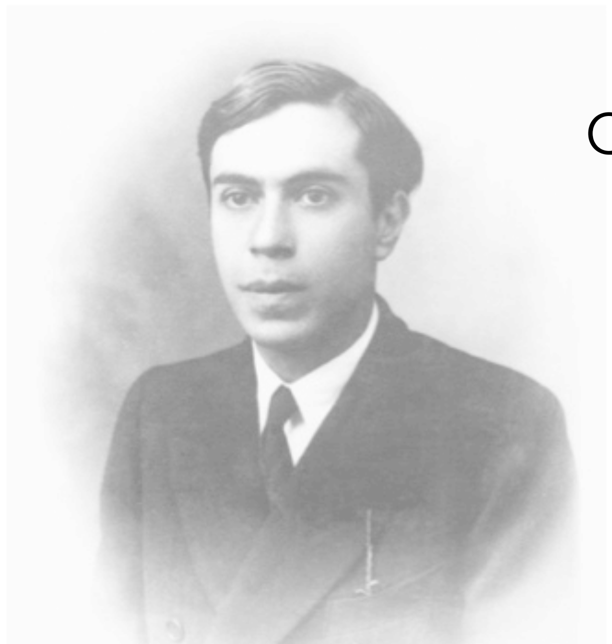
Majorana's representation

Multipartite state of $2j$ spin-$\frac{1}{2}$



Spin-j state



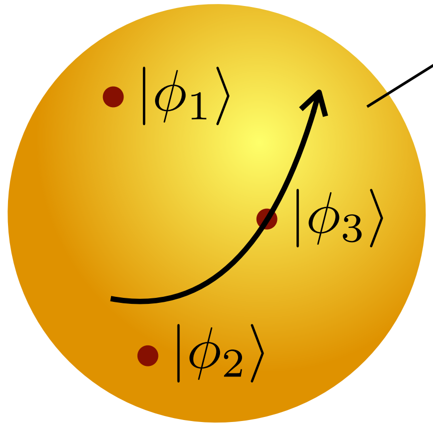


Ettore Majorana
(1906–1938?)

Quantum Information Theory Toolbox

[entanglement; qubits; Von Neumann entropy; …]

Bloch sphere



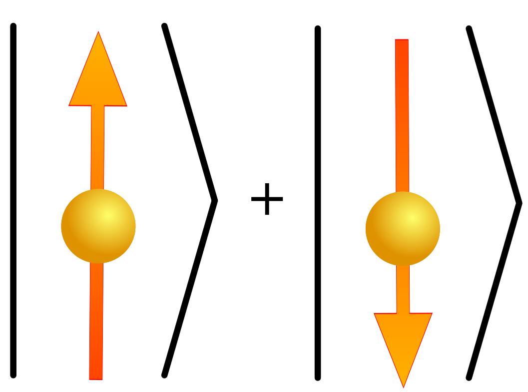
Geometry Toolbox

[symmetry; geometric phase; topology; …]

J. H. Hannay, J. Mod. Opt. **45**, 1001 (1998); J. Phys. A: Math. Gen. **31** L53 (1998)
P. Bruno, Phys. Rev. Lett. **108**, 240402 (2012)
C. Chryssomalakos *et al.*, J. Phys. A: Math. Theor. **51**, 165202 (2018)
P. Aguilar *et al.*, J. Phys. A: Math. Theor. **53**, 065301 (2020)

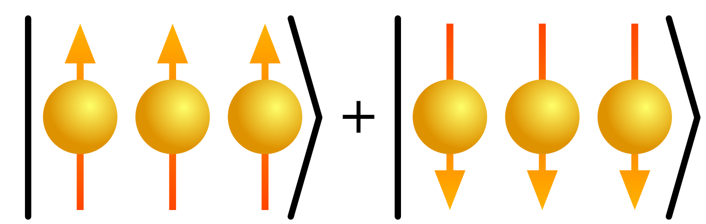
One-to-one mapping

Single spin-j state $\|\psi_j\rangle$	$N \equiv 2j$-qubit symmetric state $\|\psi_S\rangle$
spin operators $\mathbf{J}^2, J_z$	collective spin operators $\mathbf{S}^2, S_z$
standard basis $\{\|j,m\rangle\}$	symmetric Dicke basis $\{\|D_N^{(j-m)}\rangle\}$
full Hilbert space $\mathcal{H}$	symmetric subspace $\mathcal{H}_S$
coherent state	symmetric separable state
rotation	local unitary transf. $U^{\otimes N}$
anticoherent state	**maximally entangled symmetric state**



spin-3/2, $|\frac{3}{2},\frac{3}{2}\rangle + |\frac{3}{2},-\frac{3}{2}\rangle$ state

$\longleftrightarrow$



3 spin-$\frac{1}{2}$ or qubits, $|\mathrm{GHZ}\rangle$ state

[Baguette, Bastin, Martin (2014)]

SAS and SAPPT states: open questions

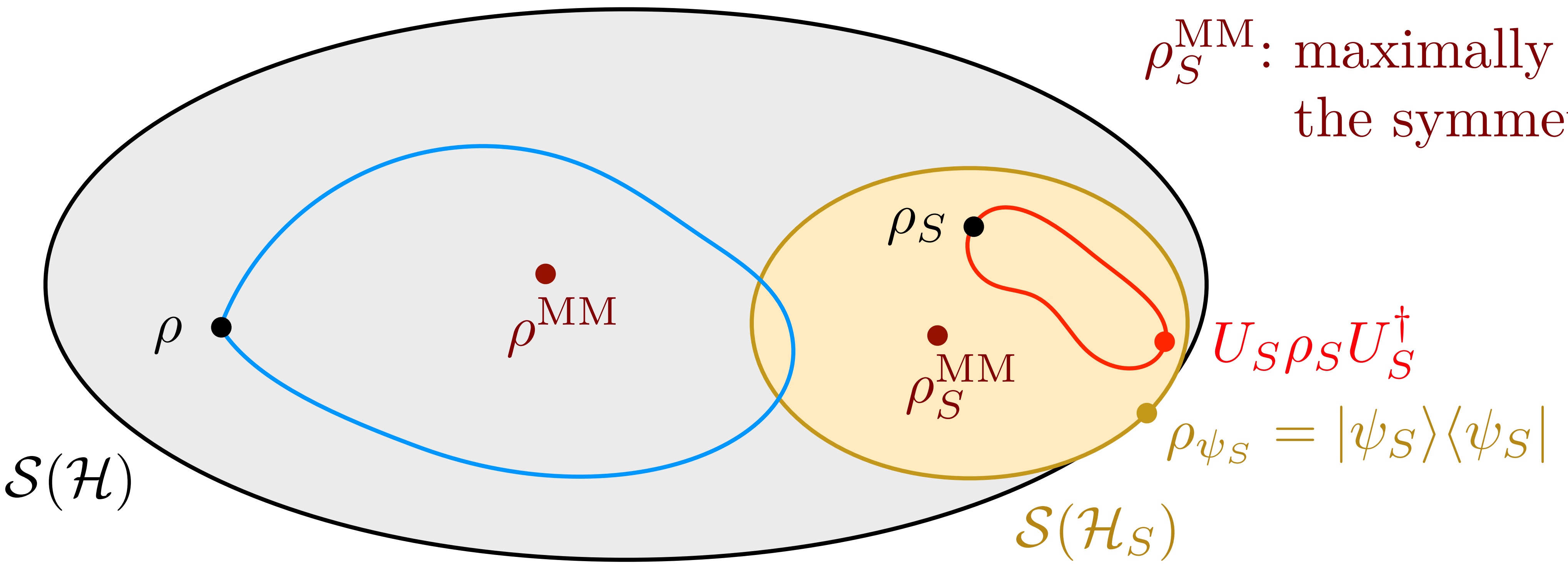
- The set of SAPPT states remains to be fully characterized — open question

- SAS set $\stackrel{?}{=}$ SAPPT set — open question

Maximal negativity of symmetric two-qubit states

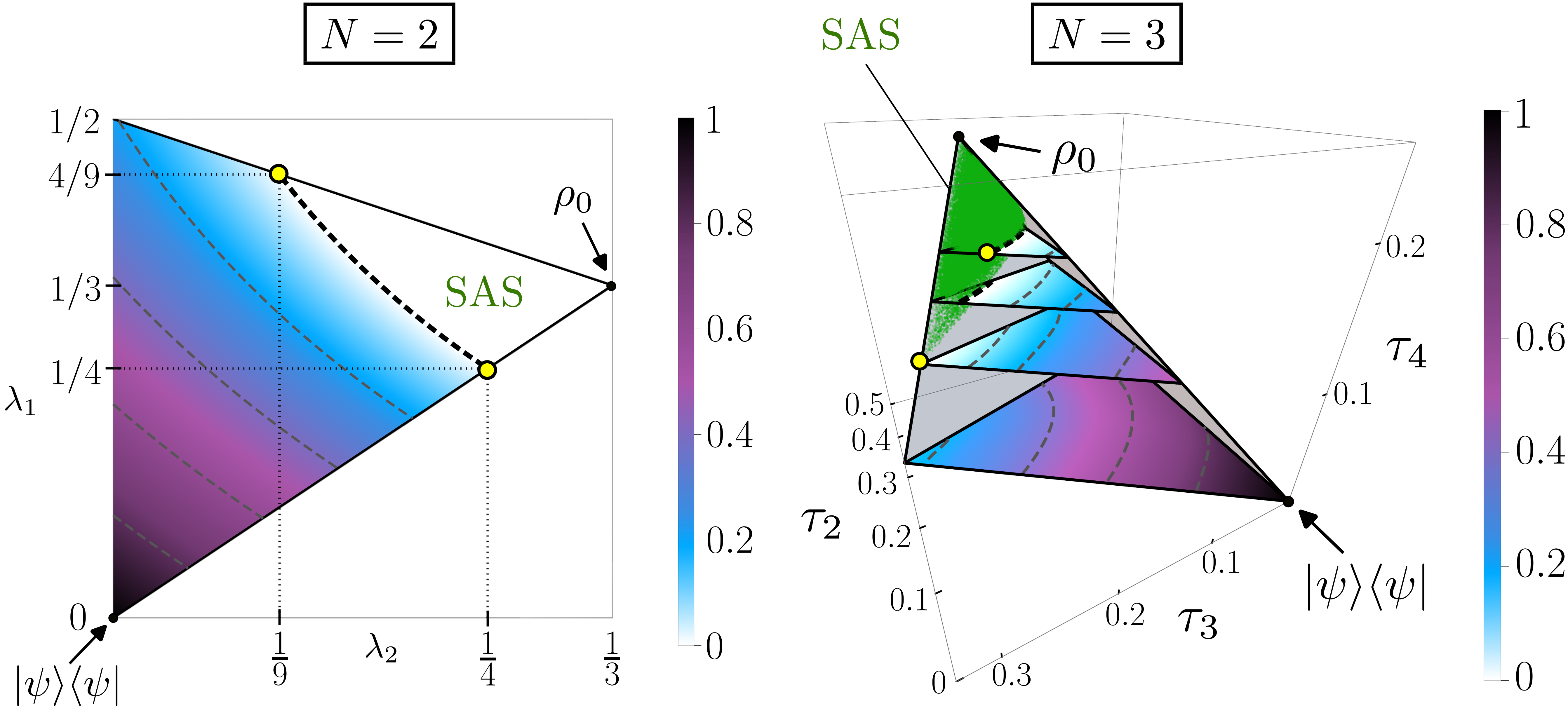
[Martin, Serrano Ensástiga (2023)]

- Goal : Find the maximum entanglement (negativity) of ρ_S in its SU(3)-orbit
- Result : $\max_{U_S \in SU(3)} \mathcal{N}\left(U_S \rho_S U_S^\dagger\right) = \max\left(0, \sqrt{\lambda_0^2 + (\lambda_1 - \lambda_2)^2} - \lambda_1 - \lambda_2\right)$
- Corollary : ρ is SAS iff $\sqrt{\lambda_1} + \sqrt{\lambda_2} \geq 1$ see also [Champagne (2022)]



Maximal negativity in the unitary orbit

[Martin, Serrano Ensástiga (2023)]

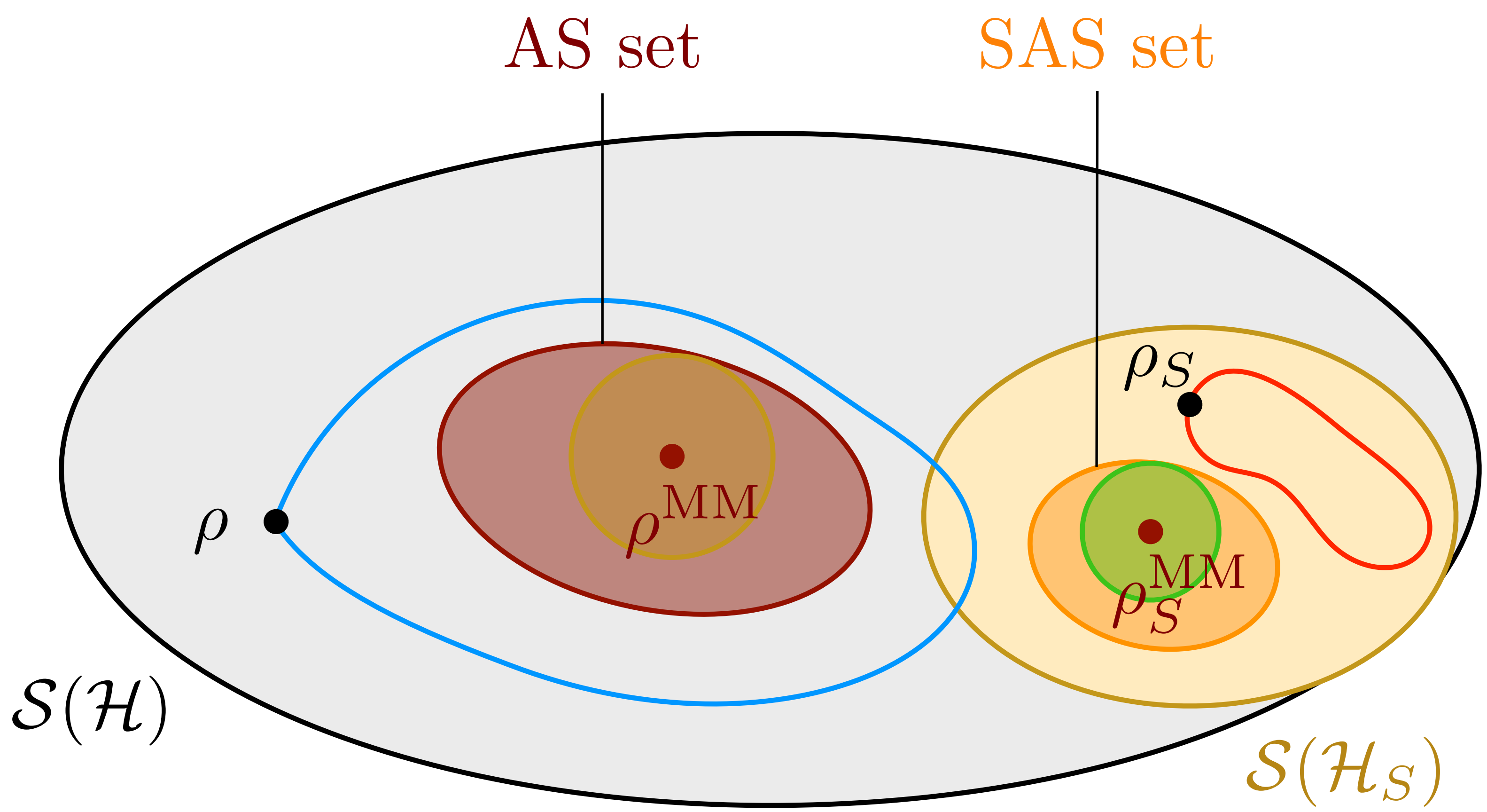


SAS states for any N

[Martin, Serrano Ensástiga (2023), Champagne (2022)]

- Ball of SAS states around the symmetric MMS [Bohnet-Waldraff (2017)]
- Polytope ($\supset$ ball) of SAS states around the symmetric MMS

[Martin, Denis, Serrano Ensástiga (2024)]



SAS states ball and polytope

[Martin, Denis, Serrano Ensástiga (2024)] [Denis, Davis, Mann, Martin (2023)]

SAS states ball

An N–qubit symmetric state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ is SAS if

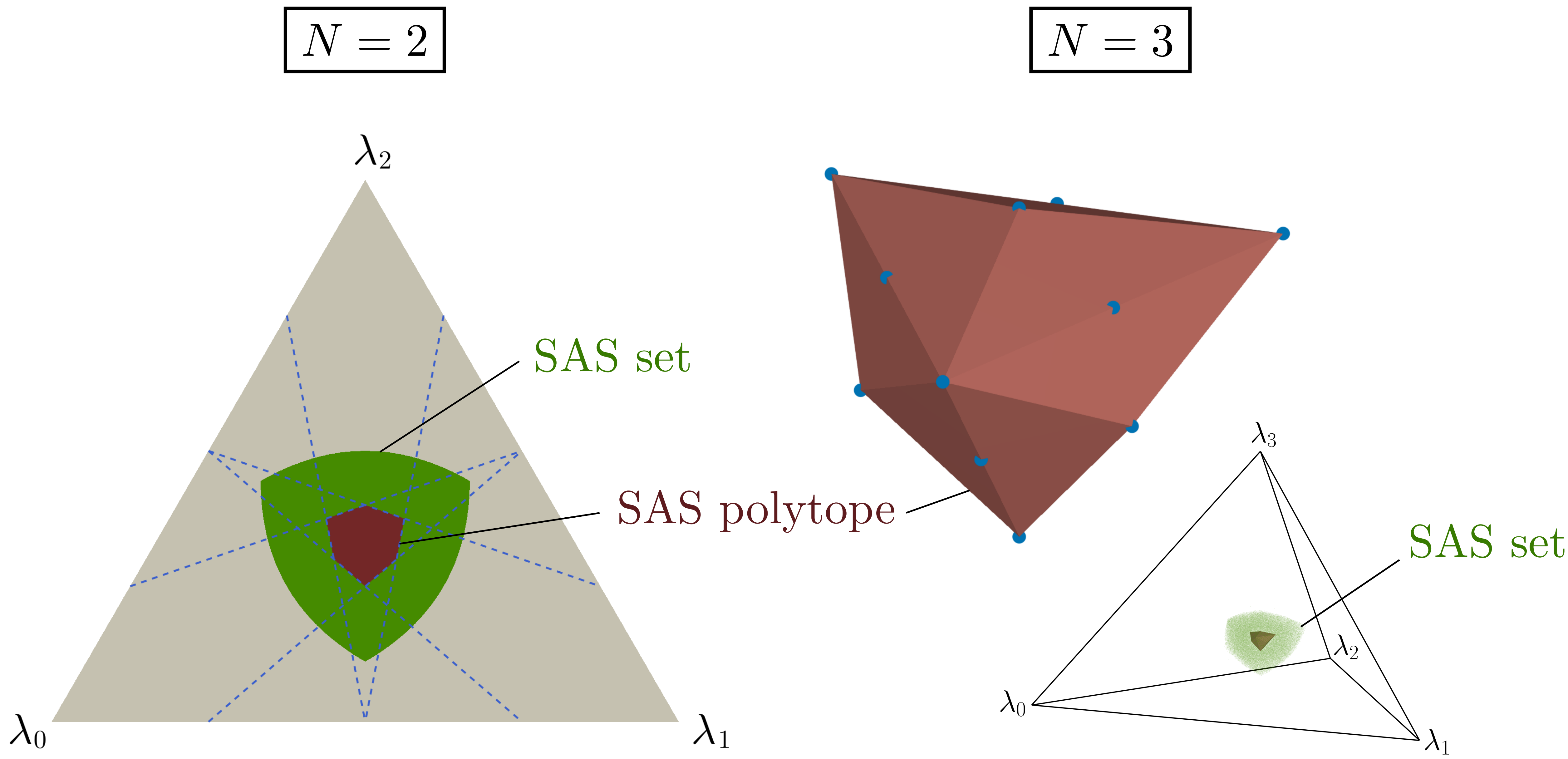
$$\mathrm{Tr}\left(\rho_S^2\right) \leqslant \frac{1}{N+1}\left(1 + \frac{1}{2(2N+1)\binom{2N}{N} - (N+2)}\right)$$

SAS states polytope

An N–qubit symmetric state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ with eigenspectrum $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots)$ is SAS if

$$\boldsymbol{\lambda}^{\downarrow}\boldsymbol{\Delta}^{\uparrow T} \geqslant 0 \quad \text{with} \quad \Delta_k = (-1)^{N-k}\binom{N+1}{k}$$

Polytopes for *N*=2, *3* qubits



[in barycentric coordinates]

SAS witness for arbitrary bipartite systems

[Abellanet-Vidal, Müller-Rigat, Rajchel-Mieldzioć, and Sanpera (2025)]

- P but not CP maps usually provide sufficient conditions for entanglement, but can also be used to provide sufficient conditions for separability

[Lewenstein, Augusiak, Chruściński, Rana, and Samsonowicz (2016)]

- Let $\Lambda : \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^m) \to \mathcal{S}_{\text{sep}}(\mathbb{C}^n \otimes \mathbb{C}^m)$ be an invertible linear map.
  If $\Lambda^{-1}(\sigma) \in \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^m)$, then $\sigma \in \mathcal{S}_{\text{sep}}(\mathbb{C}^n \otimes \mathbb{C}^m)$.

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  If $\Lambda^{-1}(\sigma) \in \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^m)$, then $\sigma \in \mathcal{A}_{\text{sep}}(\mathbb{C}^n \otimes \mathbb{C}^m)$.

SAS witness based on reduction-like maps

- Unitarily equivariant linear maps $[\Lambda\left(U\rho U^{\dagger}\right) = U\Lambda(\rho)U^{\dagger}]$ provide sufficient criteria to detect AS

- These maps are reduction-like and defined as

[Bardet, Collins, and Sapra (2020)]

$$\Lambda_{\alpha}(\rho) = \mathrm{Tr}(\rho)\mathbb{1} + \alpha\rho,$$

with $\alpha \in \mathbb{R}$. Invertible for $\alpha \neq 0$, the inverse is

$$\Lambda_{\alpha}^{-1}(\sigma) = \frac{1}{\alpha}\left(\sigma - \frac{\mathrm{Tr}(\sigma)\mathbb{1}}{D+\alpha}\right),$$

where D is the Hilbert space dimension.

- For $\alpha \in [-1, 2]$, $\Lambda_{\alpha}(\rho)$ renders any ρ separable. Thus, $\Lambda_{\alpha}^{-1}(\sigma) \geq 0$ ensures separability. Since $\Lambda_{\alpha}^{-1}(\sigma)$'s positivity depends only on σ's spectrum and is unitarily invariant, it provides a sufficient criterion for absolute separability.

SAS witness based on min and max eigenvalues

[Abellanet-Vidal, Müller-Rigat, Rajchel-Mieldzioć, and Sanpera (2025)]

Let ρ be a normalized bipartite state acting on the space $\mathbb{C}^N \otimes \mathbb{C}^M$ with minimal and maximal eigenvalues $\lambda_{\min}(\rho)$ and $\lambda_{\max}(\rho)$ respectively. If

$$\lambda_{\min}(\rho) \geq \frac{1}{N \cdot M + 2} \quad \text{or} \quad \lambda_{\max}(\rho) \leq \frac{1}{N \cdot M - 1}$$

then ρ is absolutely separable.

SAS witness for symmetric states

[Abellanet-Vidal, Müller-Rigat, Rajchel-Mieldzioć, and Sanpera (2025)]

Let ρ_S be a symmetric state of N qubits and $\{\lambda_i\}_{i=0}^{N}$ its eigenvalues in increasing order. If

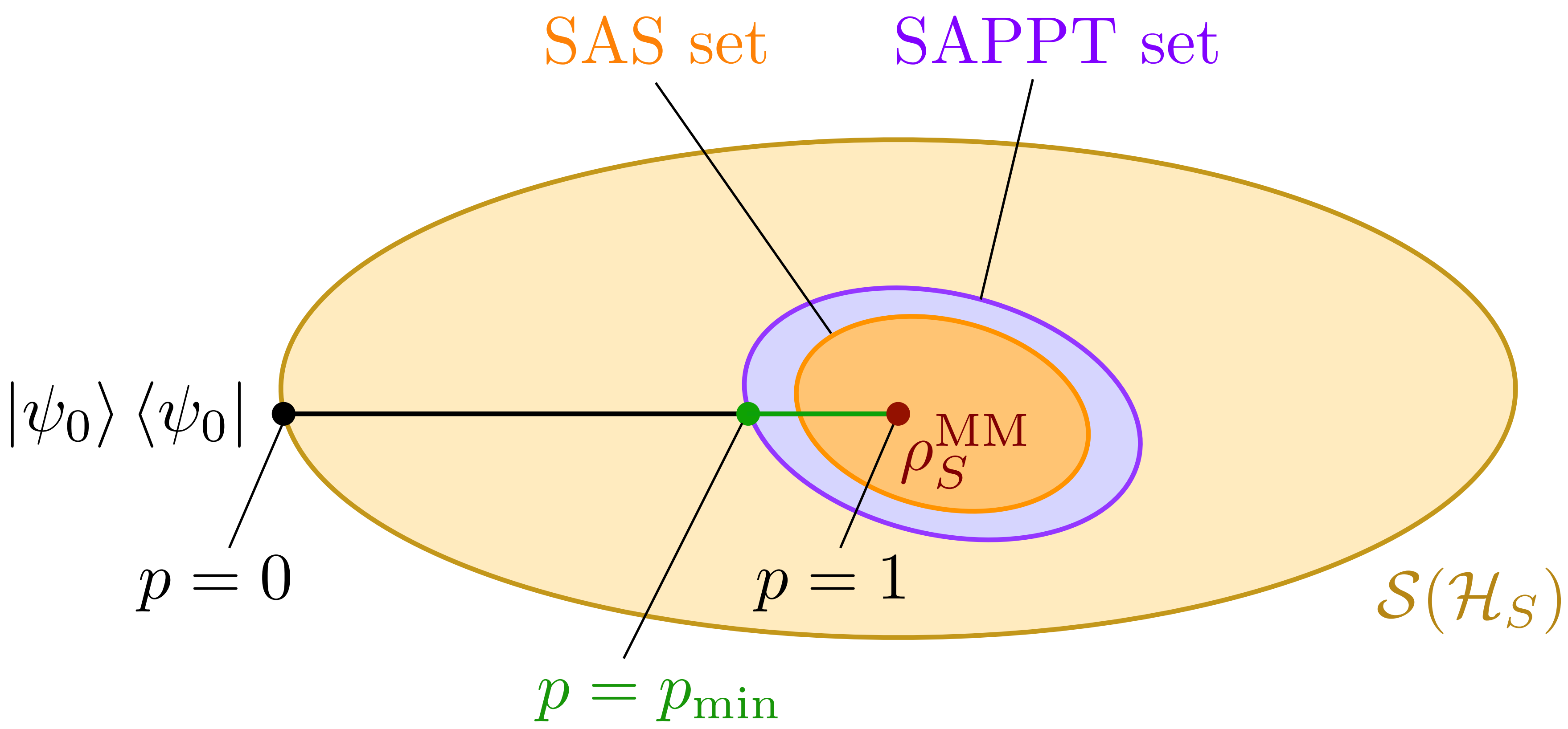
$$3\binom{N}{\lfloor N/2 \rfloor}\sum_{i=0}^{\lfloor \frac{N+1}{3} \rfloor -1} \lambda_i + \left[\binom{N}{\lfloor N/2 \rfloor}\left(N+1-3\left\lfloor \frac{N+1}{3} \right\rfloor\right)+2\right]\lambda_{\lfloor \frac{N+1}{3} \rfloor} \geq \binom{N}{\lfloor N/2 \rfloor}$$

then ρ_S is SAPPT.

Construction of SAPPT states

- $\rho(p) = p\,\rho_S^{\mathrm{MM}} + (1-p)\,|\psi_0\rangle\langle\psi_0|$ with $p \in [0,1]$

[Louvet, Serrano-Ensástiga, Bastin, Martin (2025)]

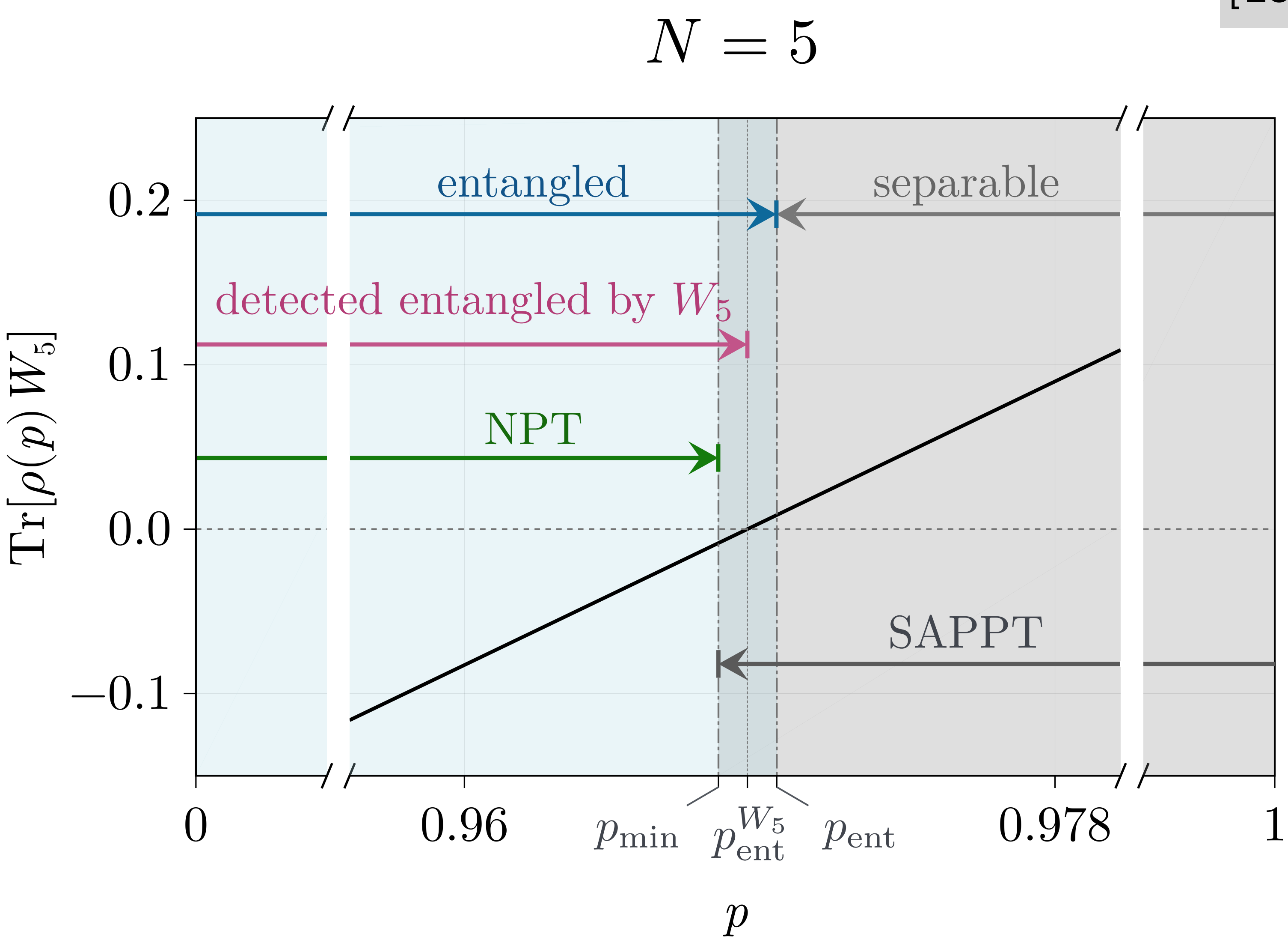


- $\rho(p)$ SAPPT $\Leftrightarrow$ $p \in [p_{\min}, 1]$ with $p_{\min} = \frac{1}{1+2\left[(N+1)\binom{N}{\lfloor N/2 \rfloor}\right]^{-1}}$

- $|\psi_0\rangle = |\mathrm{GHZ}(N)\rangle \Rightarrow \rho(p)$ entangled $\Leftrightarrow$ $p \in [0, p_{\mathrm{ent}}]$ with $p_{\mathrm{ent}} > p_{\min}$ (odd N)

SAS ≠ SAPPT ✓

[Louvet, Serrano-Ensástiga, Bastin, Martin (2025)]



N	$p_{\min}$	p_{ent}	
4	$\frac{15}{16}$	$\frac{15}{16}$	
5	$\frac{30}{31} \approx 0.96774$	0.96953	✓
6	$\frac{70}{71}$	$\frac{70}{71}$	
7	$\frac{140}{141} \approx 0.99291$	0.99329	✓
8	$\frac{315}{316}$	$\frac{315}{316}$	
9	$\frac{630}{631} \approx 0.99842$	0.99849	✓
10	$\frac{1386}{1387}$	$\frac{1386}{1387}$	

Open questions

- The set of SAPPT states remains to be fully characterized — open question
- SAS set $\neq$ SAPPT set because there are entangled SAPPT states
- Existence of entangled SAPPT states with N even ? — open question