Models for Network Flow and Network Design Problems with Piecewise Linear Costs

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Abstract In modeling real-world applications of network design, a difficulty often encountered is the fact that capacity on the arcs are not ”all-or-nothing” but incur a more complex cost. Often, the associated cost function can be approximated with a piecewise linear function (possibly non-convex).

We review in this chapter some pioneering work of Bernard Gendron on multicommodity flow and network design problems with piecewise linear costs, and we also briefly address related recent works that build on the ideas developed by Bernard Gendron.

1 Introduction

Network flow problems are fundamental optimization problems that involve finding the optimal flow of resources through a network, subject to constraints such as capacity limits and cost considerations. A natural extension is multicommodity network flow problems, where there may be multiple types of resources flowing through the network, each with its own cost function. These models are also useful when considering origin-destination based flows, where it is important that the flow served to a node comes from a particular other node in the network. For example, in a telecommunications network, there may be multiple types of data traffic flowing

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through the network, each with a different cost function based on the bandwidth and other parameters. Often the complex structure of the cost of using a particular network edge or link can be modeled (or approximated) using a piecewise linear function. This means that the cost of using the link changes at certain thresholds or breakpoints, and may not increase at a constant rate. For example, in transportation planning, the cost of using a road may be modeled as a piecewise linear function of the traffic volume on the road. As the traffic volume increases, the cost of using the road may increase at a faster rate, due to congestion or other factors.

Efficiently solving network design and multicommodity network flow problems with piecewise linear costs can be challenging, especially when these cost functions are non-convex. Bernard Gendron has made significant contributions on solving this class of problems, namely on developing models and algorithms that can effectively handle non-convex cost functions, allowing for more accurate and efficient resource allocation in these types of networks. Several references to Bernard's contributions will appear on this review.

In this chapter, we focus on mixed-integer linear models for variants of multicommodity network flow problems with piecewise linear costs. The formulations presented in this chapter are, in general, extended formulations of large size. The formulations will presented in the context of three related problems, the continuous multicommodity network flow problem, the integer multicommodity network flows problem and the unsplittable multicommodity network flow problem. The first problem is a relaxation of the second one, which in turn is a relaxation of the third one.

The presentation of this chapter intends to review Bernard Gendron's contributions, also emphasizing which modeling techniques are needed to better solve the more difficult versions. We observe and emphasize that our work is a tour of known modeling approaches that might be helpful for a researcher wanting to study these or related problem. To the best of our knowledge, this chapter is the first review paper to systematically present models for different cases that have been studied in the literature, unifying notation and presenting the main similarities and differences between the models. However, it is not our intention to make a computational study. We refer the reader to Croxton et al. (2007); Frangioni and Gendron (2009); Gendron and Gouveia (2017) and the recent survey by Frangioni and Gendron (2021) for algorithmic approaches based on columns and rows generation or Lagrangian relaxation to solve the models presented in this chapter efficiently.

In addition to the large scale models presented here, a different perspective is followed by some authors who focus on direct approaches to solve network flow problems with piecewise linear costs with specialized algorithms, see e.g. Murthy and Helgason (1994); Pinto and Shamir (1994).

After an introduction of the notation and the definition of the main concepts used throughout the chapter in Section 2, we review models for multicommodity network flows with piecewise linear costs, starting in Section 3 with the case of continuous flows. In Section 4, we extend these models to the case where flow have to be integral, while Section 5 covers the case where flows need to follow a single path for each commodity. We conclude in Section 6.

2 Notation and definitions

Let $G = (V, A)$ be a directed graph, with a set of nodes V, and a set of arcs A. Consider as well a set of commodities K, where each commodity $k \in K$ has a given origin o_k, a destination d_k, and a demand of d_k units of flow to be routed from o_k to d_k. In order to simplify the writing of classical flow conservations that will appear in all our formulation, we define, for each node $i \in V$ and commodity $k \in K$, λ_i^k such that $\lambda_{o_k}^k = 1$, $\lambda_{d_k}^k = -1$, and $\lambda_i^k = 0$ for every $i \neq \{o_k, d_k\}$. For any proper subset $S \subsetneq V$, $S \neq \emptyset$, we define the forward cut $\delta^+(S)$ associated to S as the set of arcs leaving S, i.e. $\delta^+(S) = \{(i, j) \in A : i \in S, j \notin S\}$, and the backward cut $\delta^-(S)$ as the set of arcs entering S, i.e. $\delta^-(S) = \{(i, j) \in A : i \notin S, j \in S\}$. When $S = \{i\}$, we write $\delta^+(i)$ and $\delta^-(i)$ instead of $\delta^+(\{i\})$ and $\delta^-(\{i\})$.

Each arc $a \in A$ has an associated cost function $g_a(l_a)$ of the load flowing through the arc, l_a. This cost function is piecewise linear, with the segments being indexed by the finite set $S_a = \{1, 2, ..., |S_a|\}$. Each segment $s \in S_a$ has a lower and upper bound on the flow, represented by the breakpoints b_a^{s-1} and b_a^s. If finite, the breakpoint of the last segment of each arc $a \in A$, $b_a^{|S_a|}$, can be interpreted as the capacity of the arc. However, the case where $b_a^{|S_a|} = \infty$ also holds. A segment is also characterized by a slope c_a^s and an intercept f_a^s. In conclusion, the cost function g_a is such that $g_a(l_a) = f_a^s + c_a^s l_a$ if $b_a^{s-1} < l_a \leq b_a^s$ for $s \in S_a$. We assume that the function is lower semi-continuous, i.e. $g(l) \leq \liminf_{l' \to l} g(l')$ and that $g(0) = 0$. Figure 1 illustrates this notation. When the objective function is convex, these values are such that $c_a^1 \geq 0$, $c_a^s > c_a^{s-1}$ and $f_a^s \leq 0$, $f_a^s < f_a^{s-1}$. Moreover, when the cost function is continuous, $b_a^s c_a^s + f_a^s = b_a^s c_a^{s+1} + f_a^{s+1}$ must hold.

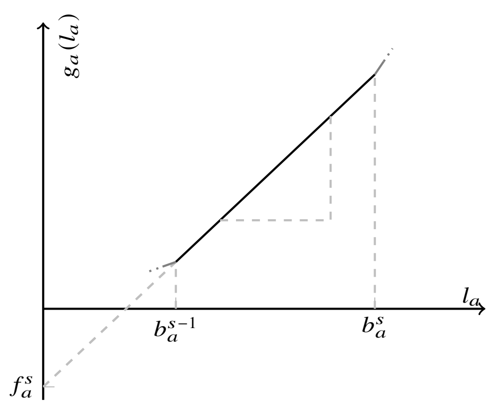

Fig. 1 Notation for each segment of $g_a(l_a)$

3 Continuous multicommodity network flows

In this section, we focus on problems where flows are continuous. The classical multicommodity flow problem in this case can be formulated as a linear program, as described in Section 2, and is polynomially solvable. We start with a basic multicommodity formulation for the case when flows are continuous and the cost function is convex. Then, we consider the case when costs are not convex, point out why the previous model cannot be used directly, and as a consequence introduce the classical multiple choice formulation. Finally, based on the fact the previous formulation has a weak LP relaxation, the concept of disaggregation is introduced to lead to a formulation with a much tighter LP bound.

3.1 Convex costs and a basic multicommodity flow model

The easiest case of multicommodity network flow problems with piecewise linear costs arises when flow are continuous and the cost function is convex. In this case, the problem can be formulated by explicitly introducing a variable g_a for each arc $a \in A$ to represent the cost $g_a(l_a)$ associated with the load of arc a. As the piecewise linear functions are convex, it is sufficient to impose that g_a is the maximum of the linear functions defining the cost for each segment. This approach was among others used by Fortz and Thorup (2000).

$$
\begin{aligned}
\min \quad & \sum_{a \in A} g_a && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{a \in \delta^+(i)} x_a^k - \sum_{a \in \delta^-(i)} x_a^k = \lambda_i^k, && i \in V, k \in K, && \text{(1b)}\\
& l_a = \sum_{k \in K} d^k x_a^k, && a \in A, && \text{(1c)}\\
& g_a \geq f_a^s + c_a^s l_a, && a \in A, s \in S_a, && \text{(1d)}\\
& 0 \leq x_a^k \leq 1, && a \in A, k \in K, && \text{(1e)}\\
& l_a \geq 0, && a \in A, s \in S_a, && \text{(1f)}\\
& g_a \geq 0, && a \in A. && \text{(1g)}
\end{aligned}
$$

In this formulation, (1b) are the flow conservation constraints, (1c) define the load on each arc and (1d) bounds the cost on each arc from below. Constraints (1e) to (1g) define the bounds on the variables. As for the basic multicommodity flow problem, this is a linear programming model that can be solved in polynomial time.

3.2 The non-convex case and the multiple choice model

Unfortunately, when costs are not convex and there are multiple commodities, the problem becomes NP-hard (Fortz et al., 2017). Indeed, the difficulty lies in the identification of the segment to which the load of an arc belongs. A classical approach to handle this, is to extend the previous model with a multiple choice structure, leading to the so-called multiple choice model that was first introduced in the context of multicommodity flow problems with piecewise linear costs by Balakrishnan and Graves (1989). We introduce binary variables y_a^s, with $y_a^s = 1$, if arc $a \in A$ contains a non-zero flow on segment $s \in S_a$, and $y_a^s = 0$ otherwise. For the sake of simplicity, if arc a contains a non-zero flow on segment $s \in S_a$, we say that arc a is on segment $s \in S_a$. We also define continuous variables l_a^s, that indicate the load going through arc $a \in A$ on segment $s \in S_a$. The problem can then be formulated as follows.

$$
\begin{aligned}
\min \quad & \sum_{a \in A} \sum_{s \in S_a} \left(f_a^s y_a^s + c_a^s l_a^s \right) && && \text{(2a)}\\
\text{s.t.} \quad & \sum_{a \in \delta^+(i)} x_a^k - \sum_{a \in \delta^-(i)} x_a^k = \lambda_i^k, && i \in V, k \in K, && \text{(2b)}\\
& \sum_{s \in S_a} y_a^s \leq 1, && a \in A, && \text{(2c)}\\
& \sum_{s \in S_a} l_a^s = \sum_{k \in K} d^k x_a^k, && a \in A, && \text{(2d)}\\
& b_a^{s-1} y_a^s \leq l_a^s \leq b_a^s y_a^s, && a \in A, s \in S_a, && \text{(2e)}\\
& 0 \leq x_a^k \leq 1, && a \in A, k \in K, && \text{(2f)}\\
& y_a^s \in \{0, 1\}, && a \in A, s \in S_a, && \text{(2g)}\\
& l_a^s \geq 0, && a \in A, s \in S_a. && \text{(2h)}
\end{aligned}
$$

Constraints (2b) are flow conservation constraints. Constraint sets (2c)-(2e) identify the segment each arc is on. Naturally, only a single segment per arc may be selected (2c). The choice of segment is implied by the load flowing through the respective arc. This load is given by constraints (2d) and its value is assigned to one of the variables l. To ensure that only the appropriate load variable is non-zero, in (2e) we either bound l_a^s by the breakpoints of the corresponding segment, if $y_a^s = 1$, or we force it to zero, if $y_a^s = 0$.

Note that in the case of convex cost functions, Fortz et al. (2017) showed the Linear Programmang (LP) relaxation of the basic formulation (1) has integer y_a^s variables, and the basic formulation (1) becomes equivalent to the multiple choice formulation (2) in the sense that they have the same optimal objective value. Essentially, this means that in the convex costs case, there is no gain in augmenting the original basic model with the multiple choice structure.

In the general (non-convex) case, however, the value of the optimal solution of the LP relaxation of the multiple choice formulation (2) can be arbitrarily far from the optimal solution, even for a single commodity. To illustrate this, consider a toy network with only two nodes v and w, a single arc a from v to w and a unit demand between v and w. Obviously, the only feasible solution is to send one unit of flow on a. Consider an arbitrarily large value $c > 0$. If the cost function is given by two segments, with breakpoints $b_a^0 = 0$, $b_a^1 = 1$ and $b_a^2 = 2$, intercepts $f_a^1 = 0$ and $f_a^2 = 2c$, and slopes $c_a^1 = c$ and $c_a^2 = -c$, the optimal solution of (2a)-(2h) is obtained with $y_a^1 = y_a^2 = 0.5$ and $l_a^1 = 0$, $l_a^2 = 1$ with a total cost of 0, while the optimal solution of the problem has objective value c which can be made arbitrarily large.

3.3 The disaggregated formulation

In order to strengthen the LP relaxation of Model (2), Croxton et al. (2003, 2007) use disaggregation, a common technique to strengthen the LP relaxation of MIPs. We present here a slightly adapted version of it, to keep notation consistent. Let us introduce variables x_a^{ks} to represent the proportion of commodity k routed on arc $a \in A$ on segment $s \in S_a$. These new variables are linked to those in the multiple choice formulation (2) by the relations

$$l_a^s = \sum_{k \in K} d^k x_a^{ks} \qquad a \in A, s \in S_a, \tag{3}$$

$$x_a^k = \sum_{s \in S_a} x_a^{ks} \qquad a \in A, k \in K. \tag{4}$$

and a straightforward substitution leads to

$$\min \quad \sum_{a \in A} \sum_{s \in S_a} \left(f_a^s y_a^s + c_a^s \sum_{k \in K} d^k x_a^{ks} \right) \tag{5a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(i)} \sum_{s \in S_a} x_a^{ks} - \sum_{a \in \delta^-(i)} \sum_{s \in S_a} x_a^{ks} = \lambda_i^k, \qquad i \in V, k \in K, \tag{5b}$$

$$\sum_{s \in S_a} y_a^s \leq 1, \qquad a \in A, \tag{5c}$$

$$b_a^{s-1} y_a^s \leq \sum_{k \in K} d^k x_a^{ks} \leq b_a^s y_a^s, \qquad a \in A, s \in S_a, \tag{5d}$$

$$\sum_{s \in S_a} x_a^{ks} \leq 1, \qquad k \in K, a \in A, \tag{5e}$$

$$y_a^s \in \{0, 1\}, \qquad a \in A, s \in S_a. \tag{5f}$$

This formulation has the same LP relaxation value as formulation (2), but the introduction of disaggregated variables x_a^{ks} in the model allows an enhancement based on the observation that the flow on arc a can be on segment s if and only if $y_a^s = 1$. This observation leads to the following valid inequalities:

$$x_a^{ks} \leq y_a^s, \qquad k \in K, a \in A, s \in S_a. \tag{5g}$$

To see the effect of this enhancement on the LP relaxation of the original model, let us consider again the toy example described before. Since there is a single commodity ($K = \{1\}$), the LP relaxation of this model corresponds to the vector of disaggregated variables $x_a^{11} = 0$, $x_a^{12} = 1$. But this vector violates the new valid inequality (5g) as $x_a^{12} = 1 > 0.5 = y_a^2$. On the other hand, it is quite easy to see that the optimal solutions of the LP relaxation of (5) are all the convex combinations of $x_a^{11} = y_a^1 = 1; x_a^{12} = y_a^2 = 0$ and $x_a^{11} = y_a^1 = 0; x_a^{12} = y_a^2 = 1$, with optimal objective function value c, and the LP relaxation is integral. Intuitively, the gain obtained by (5g) is due to the fact that these constraints impose the value of variable y_a^s to be at least equal to the maximum (over k) of the values of variables x_a^{ks} whereas the previous constraint (5d) states that the value of the same variable should be at least equal to a weighted average variables x_a^{ks}.

This clearly shows that the LP relaxation of the strong disaggregated multiple choice formulation (5) can be much better than the LP relaxation of the original multiple choice formulation (2). Croxton et al. (2007) show that the LP relaxation of the disaggregated multiple choice formulation (5) approximates the cost function on each arc with its lower convex envelope, which is the best one can hope for with linear constraints.

Note that the disaggregated multiple choice formulation also strenghtens the LP bound in the convex case, but this reformulation is only possible in the presence of binary y_a^s variables. The same technique cannot be applied to the basic model (1), hence justifying furthermore the interest of the multiple choice model.

4 Integer multicommodity network flows

Gendron and Gouveia (2017) study the same problem with the additional restriction that the flow on each arc is integer. The basic formulation (1) (in the convex case) and the multiple choice formulation (2) (for the general case) immediately extend to this case by adding the restriction that variables l_a or l_a^s are integer. The disaggregated formulation (5) can also be used if variables l_a are re-introduced in the model as integer variables, together with linking constraints (3). These formulations have a very weak LP relaxation, as their continuous counterparts. Moreover, as we replace continuous variables by integer variables, the continuous flow models are relaxations of the integer ones, indicating that these latest problems are more difficult to solve.

4.1 The Point-Based Model

In order obtain stronger models, Gendron and Gouveia (2017) propose several formulations, based on discretization. The first one leads to the so-called Point-Based Model. This model is based on the assumption that the flow in each arc is integer, allowing the definition of a set of possible values the flow an arc a can carry, $Q_a = \{1, 2, \ldots, u_a\}$ where u_a is an upper bound on the maximal flow that can be sent on a in a feasible solution. Note that in the uncapacitated case considered here, we might take $u_a = \sum_{kinK} d^k$, but this approach also allows to model cases with explicit arc capacities. Sets Q_a can also be partitioned by segments, by defining $Q_a^s = \{q \in \mathcal{Z} : b_a^{s-1} < q \leq b_a^s$. The discretization of the the value of the flow on any arc allows the introduction of new binary variables y_a^q such that $y_a^q = 1$ if and only if the total flow on arc a is equal to q.

Observe that these new variables are linked to original variables y_a^s and l_a^s in (2) by the following relations:

$$\begin{aligned} y_a^s &= \sum_{q \in Q_a^s} y_a^q && s \in S_a, a \in A, \\ \sum_{s \in S_a} l_a^s &= \sum_{q \in Q_a} q y_a^q && a \in A. \end{aligned}$$

Substituting in (2) leads immediately to the new formulation

$$\min \quad \sum_{a \in A} \sum_{s \in S_a} \sum_{q \in Q_a^s} \left(f_a^s + c_a^s q \right) y_a^q \tag{6a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(i)} x_a^k - \sum_{a \in \delta^-(i)} x_a^k = \lambda_i^k, \qquad i \in V, k \in K, \tag{6b}$$

$$\sum_{q \in Q_a} y_a^q \leq 1, \qquad a \in A, \tag{6c}$$

$$\sum_{q \in Q_a} q y_a^q = \sum_{k \in K} d^k x_a^k, \qquad a \in A, \tag{6d}$$

$$0 \leq x_a^k \leq 1, \qquad a \in A, k \in K, \tag{6e}$$

$$y_a^q \in \{0, 1\}, \qquad a \in A, q \in Q_a. \tag{6f}$$

As far as we know, this discretization technique was first introduced in Gouveia (1995). One of the motivations for this technique, as noted in the earlier work, is also seen here, namely that transformed constraints (5d) become redundant with the new variables and therefore are not needed in the new model. Also it can be easily shown that the LP bounds of the disaggregated multiple choice formulation (5) and of the discretized point-based formulation (6) are equal. The LP relaxation of this model can be enhanced in two different ways described next, that, when combined together, lead to the formulation with the strongest LP bound.

4.2 Disaggregating the Point-Based Formulation

As noticed in Gendron and Gouveia (2017), the disaggregation technique used in the previous section can also be used together with the discretization technique used to derive the previous model. However, the load variables l_a^s must be included and defined as integer, and constraints (3) need to be added to the disaggregated multiple choice formulation (5). Based on this observation, they propose to apply the same ideas of disaggregation to the point-based model.

To do this, let us introduce variables x_a^{kq} to represent the proportion of commodity k routed on arc $a \in A$ if the total flow on arc a is exactly q (i.e. $\sum_{kinK} d^k x_a^k = q$). These new variables are linked to those in formulation (6) by the relations

$$x_a^k = \sum_{q \in Q_a} x_a^{kq} \qquad a \in A, k \in K \tag{7}$$

$$q y_a^q = \sum_{k \in K} d^k x_a^{kq} \qquad a \in A, q \in Q_a. \tag{8}$$

Substituting and adding strong linking constraints then leads to

$$\min \quad \sum_{a \in A} \sum_{s \in S_a} \sum_{q \in Q_a^s} \left(f_a^s y_a^q + c_a^s \sum_{k \in K} d^k x_a^{kq} \right) \tag{9a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^+(i)} \sum_{q \in Q_a} x_a^{kq} - \sum_{a \in \delta^-(i)} \sum_{q \in Q_a} x_a^{kq} = \lambda_i^k, \qquad i \in V, k \in K, \tag{9b}$$

$$\sum_{q \in Q_a} y_a^q \leq 1, \qquad a \in A, \tag{9c}$$

$$\sum_{q \in Q_a} q y_a^q = \sum_{k \in K} d^k \sum_{q \in Q_a} x_a^{kq}, \qquad a \in A, \tag{9d}$$

$$x_a^{kq} \leq y_a^q, \qquad a \in A, q \in Q_a, k \in K, \tag{9e}$$

$$x_a^{kq} \in \{0, 1\}, \qquad a \in A, k \in K, q \in Q_a. \tag{9f}$$

Note that in this last formulation, we have relaxed the fact that y_a^q need to be defined as binary variables as this is implied by the fact that x_a^{kq} are defined as binary variables combined with constraints (9c) and (9d).

4.3 Adding Chvátal-Gomory Rank 1 Inequalities

Another improvement to the point-based formulation (6) can be obtained by considering rounded cut-set inequalities, described next. Multiplying each flow conservation constraint (9b) by the corresponding demand d^k, summing over K and subsituting using (9d) leads to

$$\sum_{a\in\delta^+(i)}\sum_{q\in Q_a} q y_a^q - \sum_{a\in\delta^-(i)}\sum_{q\in Q_a} q y_a^q = \sum_{k\in K}\lambda_i^k d^k, \qquad i\in V. \tag{10}$$

Summing over all nodes in a proper subset $S \subsetneq V$, $S \neq \emptyset$, leads to the following set of (redundant) constraints, called point-based cut-set equations:

$$\sum_{a\in\delta^+(S)}\sum_{q\in Q_a} q y_a^q - \sum_{a\in\delta^-(S)}\sum_{q\in Q_a} q y_a^q = D(S), \qquad \emptyset \neq S\subsetneq V, \tag{11}$$

where $D(S) = \sum_{i\in S}\sum_{k\in K}\lambda_i^k d^k$ is the net supply accross cut S.

Although these point-based cut-set equations do not improve the LP relaxation of the model, Chvátal-Gomory rank 1 inequalities derived from equations (11) turn out to be very effective in improving the corresponding LP relaxation.The idea is to divide equations (11) by each possible integer value in the disagreggated set, then round up or down, leading to:

$$\sum_{a\in\delta^+(S)}\sum_{q\in Q_a}\left\lceil\frac{q}{p}\right\rceil y_a^q - \sum_{a\in\delta^-(S)}\sum_{q\in Q_a}\left\lfloor\frac{q}{p}\right\rfloor y_a^q \geq \left\lceil\frac{D(S)}{p}\right\rceil, \quad \emptyset\neq S\subsetneq V, p\in Q_a, \tag{12a}$$

$$\sum_{a\in\delta^+(S)}\sum_{q\in Q_a}\left\lfloor\frac{q}{p}\right\rfloor y_a^q - \sum_{a\in\delta^-(S)}\sum_{q\in Q_a}\left\lceil\frac{q}{p}\right\rceil y_a^q \leq \left\lfloor\frac{D(S)}{p}\right\rfloor, \quad \emptyset\neq S\subsetneq V, p\in Q_a, \tag{12b}$$

This idea has been used previously in other works using discretization, see e.g, Gouveia and Saldanha-da Gama (2006); Correia et al. (2010); Gouveia and Moura (2012). Similar to the results in these works, the results reported in Gendron and Gouveia (2017) indicate that inequalities (12) considerably improve the LP bound of the point based model (6). The reported results also show that strengthened with inequalities (12), the disaggregated point-based formulation (9) provides the strongest lower bound of all formulations presented in this section.

5 Unsplittable multicommodity network flows

An important extension of the models above is to impose that all the flow for a commodity follows the same path. This arises for example in the context of telecommunications networks, see e.g. the models proposed by Papadimitriou and Fortz (2014a,b), in the context of a complex multi-period design and routing problem. Lower bounds resulting from the LP relaxation of the problem are very weak, and part of this weakness is due to the piecewise linear objective function combined with single-path routing. This can be explained furthermore by observing that the relaxation obtained by relaxing single-path routing is exactly the continous version of the problem studied in Section 3, and both have the same weak LP relaxation.

Again, models for continuous case (see Section 3) can easily be adapted to this extension by imposing that variables x_a^k are defined as binary. Fortz et al. (2017) extend the models from Croxton et al. (2003, 2007), using disaggregation to build a strong model similar to Model (5) where, in addition, x_a^{ks} are also defined as binary.

Exploiting the fact that variables x_a^{ks} are binary, Fortz et al. (2017) present two model enhancements. First, since flows are unsplittable, an arc being traversed by a given commodity cannot be on a segment whose upper breakpoint is smaller than the demand flow. Therefore, some x_a^{ks} variables can be fixed to zero (or removed from the model):

$$x_a^{ks} = 0, \qquad a \in A, k \in K, s \in S_a : b_a^s < d^k, \tag{13}$$

Secondly, since x_a^{ks} and y_a^s are binary, coefficients in the right-side inequality in (5d), namely $b_a^{s-1} y_a^s \le \sum_{k \in K} d^k x_a^{ks}$, can be tightened, leading to valid inequalities

$$b_a^{s-1} y_a^s \le \sum_{k \in K} \min\left(d^k, b_a^{s-1}\right) x_a^{ks}, \qquad a \in A, s \in S_a. \tag{14}$$

Fixing variables to zero (13) is a major improvement since, in addition to reducing the size of the model to be solved, it also strenghtens the LP bound. Similarly, coefficients tightening like the one proposed with (14) is well-known to have a major impact on LP bounds and help reduce considerably the computation times needed to solve the model. The important message here is that these enhancements would not be possible without applying disaggregation.

Fortz et al. (2017) have also shown that the problem is polynomially solvable if there is a single commodity, i.e. when $|K| = 1$, as it reduces to a single path problem, and that the problem is NP-hard if $|K| > 1$. They also showed that the LP relaxation of their strengthened formulation always has an optimal solution with x_a^{ks} taking binary values when $|K| = 1$.

6 Conclusion

Bernard Gendron has made tremendous contributions in modeling and solving large scale transportation and network design problems, with a significant focus on variants of the fixed-charge multi-commodity flow problem. Besides contributions on modeling the problem and variants described in this chapter, other chapters of this book illustrate the careful and innovative ways in which Bernard has developed and applied decomposition techniques such as Lagrangian relaxation or Benders decomposition to these models.

Our objective in this chapter was to emphasize the importance of methodological advances in developing models for these problems. Finding an appropriate (strong) formulation is a necessary and important step to make the development of decomposition algorithms relevant and effective in practice.

We proposed a systematic review on mixed-integer linear models for multicommodity network flow problems with piecewise linear costs, unifying notation and presenting the main similarities while pointing out differences between the models. We have reviewed formulations and enhancements in the context of several variants of the problem, with convex and non-convex costs, and with and without integer flows. The approach taken was to start from the most basic version of continuous multicommodity network flow problems with piecewise linear costs and restrict progressively the set of feasible solutions, by imposing first integer flows, then unsplittable flows.

The review points out and emphasizes relevant model contributions by Bernard on this topic. In particular, the ideas of (variable and constraint) disaggregation, allowing to develop structural valid inequalities or to eliminate variables in a more efficient pre-processing step, are illustrated.

Due to the large size of these disaggregated models that involve a huge number of variables and constraints and thus, are difficult to solve directly by available ILP packages, over the last years, the contributions of Bernard were going in the direction of applying efficient decomposition techniques to solve large-scale instances in an efficient way. There is plenty of room for future research on the subject: on the one hand, advances are still needed to solve large scale instances of these models exactly. On the other hand, these techniques of disaggregation used as a tool to better catch the structure of the underlying problem could be applied to other classes of problems.

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