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Constructing self-similar subsets within the fractal support of lacunary wavelet series for their multifractal analysis

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Abstract

Given a fractal $\mathcal{I}$ whose Hausdorff dimension matches with the upper-box dimension, we propose a new method which consists in selecting inside $\mathcal{I}$ some subsets (called quasi-Cantor sets) of almost same dimension and with controlled properties of self-similarities at prescribed scales. It allows us to estimate below the Hausdorff dimension $\mathcal{I}$ intersected with limsup sets of contracted balls selected according to a Bernoulli law, in contexts where classical mass transference principles cannot be applied. We apply this result to the computation of the increasing multifractal spectrum of lacunary wavelet series supported on $\mathcal{I}$.

Keywords: multifractal analysis, lacunary wavelet series, fractal sets, fractal dimensions

Mathematics Subject Classification numbers: 42C40, 28A78, 28A80, 26A16, 60G17

1. Introduction and statement of the main results

Multifractal analysis has become a pivotal tool for analysing irregular signals. This field has been extensively studied across a wide range of functions, measures and stochastic processes, see e.g. [1, 2, 4, 7, 8, 10, 20, 21, 24]. It provides a useful framework for examining the structures of sets and functions, giving interesting perspectives on many mathematical areas like stochastic analysis, number theory, ergodic theory, and functional analysis for example. One of

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the key challenges in multifractal analysis is determining the Hausdorff dimension of subsets of $\mathbb{R}$ (or $\mathbb{R}^d$), known as iso-Hölder sets. These sets correspond to the level sets of the local regularity exponents of functions. Often, these subsets are described as limsup sets of balls, a connection that naturally arises from the characterization of regularity based on wavelet coefficients, which involves a liminf condition (see theorem 2.7 and proposition 2.8 below).

Estimating the dimensions of such limsup sets is crucial for understanding the geometric properties of fractals. Obtaining an upper bound for the Hausdorff dimension of these sets is often much easier to establish than the corresponding lower bounds. A common approach to obtain lower bounds relies on the mass transference principle (MTP), a powerful result that establishes a link between measure-theoretic and geometric properties.

Theorem 1.1 (MTP) [14, theorem 2]. *Let X be a compact set in $\mathbb{R}^d$ and assume that there exist $s \in [0,d]$ and $a,b,r_0 > 0$ such that*

$$ar^s \leqslant \mathcal{H}^s(B \cap X) \leqslant br^s \tag{1}$$

for any ball B of centre $x \in X$ and of radius $r \leqslant r_0$. Let $\delta > 0$. Given a ball $B = B(x,r)$ with centre in X, we set $B^\delta = B\left(x, r^\delta\right)$. Assume that $(B_n)_{n\in\mathbb{N}}$ is a sequence of balls with centres in X and radii r_n such that the sequence $(r_n)_{n\in\mathbb{N}}$ converges to 0. If

$$\mathcal{H}^s\left(X \cap \limsup_{n\to+\infty} B_n\right) = \mathcal{H}^s(X),$$

then

$$\mathcal{H}^{\frac{s}{\delta}}\left(X \cap \limsup_{n\to+\infty} B_n^\delta\right) = \mathcal{H}^{\frac{s}{\delta}}(X).$$

Here, $\mathcal{H}^s$ denotes the Hausdorff measure of dimension s. Historically, the first result of this type was obtained by Jaffard [22, theorem 2] in the context of multifractal analysis of lacunary wavelet series (LWS) on $[0,1]$, and states that if $\mathcal{L}(\limsup_{n\to+\infty} B_n) = 1$, then

$$\dim_{\mathcal{H}}\left(\limsup_{n\to+\infty} B_n^\delta\right) \geqslant \frac{1}{\delta},$$

where $\dim_{\mathcal{H}}$ is the Hausdorff dimension, see definition 2.1. The key argument of both proofs is to construct a generalized Cantor set included in the limsup set of the contracted balls and, simultaneously, a probability measure supported by this Cantor set with prescribed scaling properties. The MTP and its extensions have proven effective in a variety of settings, making it a key tool for studying the dimensions of fractal sets and their associated structures, see e.g. [9, 14, 17, 27].

Regarding the multifractal analysis of functions, despite significant advances in both theory and applications, important challenges remain. Among them are the refinement of large deviation methods, the multivariate analysis of functions, and the multifractal analysis of signals with lacunar supports. The analysis of functions or stochastic processes defined on fractal sets is a growing area of research, particularly in fields where data is naturally supported on fragmented or irregular structures. This is especially relevant in geographical studies, where the multifractal analysis of parcel-based data is essential ([28, 29]). Recent advances have focused on defining and studying random fields on fractal supports, such as Sierpinski carpets [13], which serve as prototypical examples of fractal domains. These works extend classical models of random fields to accommodate the intricate geometry and self-similarity of fractals.

The multivariate analysis of functions also naturally makes emerge fractal supports, since it aims to describe the intersection of iso-Hölder sets ([25, 35]).

Building on this context, we have investigated the overestimation of the Hausdorff dimension of iso-Hölder sets by classical large deviation methods [18]. Focusing on functions represented in a wavelet basis, we have proposed a method based on 'lacunarized expansions' of the function. Specifically, we have shown that for the so-called 'α-wavelet series', large deviation methods for the lacunarized wavelet series provides a criterion for detecting such overestimations. We refer to section 2 for the definition of wavelet bases and α-wavelet series. This approach offers a computationally feasible way to refine the multifractal analysis of signals and images.

Based on this work, the present paper further explores the multifractal properties of wavelet series defined over fractal subsets, where the upper box-counting dimension matches the Hausdorff dimension.

Let us be more precise in the description of the model under study. Throughout, we work on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A property is said to hold *almost surely* if it holds on a subset of Ω of probability one. Let $\mathcal{I} \subset [0,1]$ be a compact set satisfying

$$\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) > 0, \tag{2}$$

let $\alpha > 0$ and let $\eta \in (0, \dim_{\mathcal{H}}(\mathcal{I}))$. Given a wavelet ψ, the *lacunary wavelet series f on $\mathcal{I}$ of parameters* (α, η) is the random series defined on $[0,1] \times \Omega$ by

$$f(x,\omega) = \sum_{j \geqslant 0} \sum_{k \in \mathcal{I}_j} 2^{-\alpha j} \xi_{j,k}(\omega) \, \psi\left(2^j x - k\right)$$

where

$$\mathcal{I}_j = \left\{k \in \left\{0, \ldots, 2^j - 1\right\} : \left[k 2^{-j}, (k+1) 2^{-j}\right] \cap \mathcal{I} \neq \emptyset\right\} \tag{3}$$

and where $(\xi_{j,k})_{j,k}$ is a sequence of independent Bernoulli random variables with parameter $2^{(\eta - \dim_{\mathcal{H}}(\mathcal{I}))j}$. Basic definitions regarding wavelets are provided in section 2. As mentioned above, the multifractal analysis of the basic case corresponding to $\mathcal{I} = [0,1]$ was studied by Jaffard [22]. Despite its very simple structure, this wavelet series already reveals a rich multifractal structure. When a fractal set $\mathcal{I}$ satisfies the assumptions of more general versions of the MTP, the multifractal study of LWS defined on $\mathcal{I}$ becomes feasible, using similar arguments to those of Jaffard for the case $[0,1]$. This is the case for Cantor subsets and more generally for some attractors of iterated function systems (see [18] for the study of LWS on Cantor sets).

In contexts where classical MTPs cannot be applied, alternative strategies are required to obtain lower bounds for the dimension of limsup sets. To address this issue and enable multifractal analysis of lacunary wavelet series on fractals $\mathcal{I}$—which satisfy the equality between their Hausdorff and box-counting dimensions but, for example, lack separation properties—we use another strategy. Before considering the limsup of balls, we perform some 'preconditioning' of the fractal set. This consists in constructing compact subsets $\mathcal{K}$, which we call quasi-Cantor sets, within $\mathcal{I}$. These subsets exhibit controlled self-similarity on specific dyadic scales and have a dimension that can be arbitrarily close to that of $\mathcal{I}$. This controlled self-similarity facilitates accurate dimension estimation of limsup sets of balls centred at random points in $\mathcal{K}$. This approach avoids the limitations of classical MTPs and allows us to obtain useful geometric information even in cases where standard assumptions fail. Moreover, it not only generalizes classical results on lacunary series but also offers a deeper understanding of the fractal geometry of the set $\mathcal{I}$.

Let us state our main results. The first provides the announced construction of quasi-Cantor subsets of $\mathcal{I}$, by controlling the duplication rate up to some $\varepsilon > 0$, from a scale J to scales $(1+b)^\ell J$ for any $\ell \geqslant 0$.

Theorem 1.2. *Let $\mathcal{I} \subset [0,1]$ be a compact set satisfying*

$$\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) > 0$$

and denote by $\mathcal{I}_j$ the set of dyadic intervals of size 2^{-j} that intersect $\mathcal{I}$. Let $\varepsilon > 0$ and $b \in (0,1)$. There exist a set $\mathcal{K}_\varepsilon(b) \subset \mathcal{I}$ and $J, \ell_0 \in \mathbb{N}$ such that

1. $\dim_{\mathcal{H}}(\mathcal{I}) - \varepsilon \leqslant \dim_{\mathcal{H}}(\mathcal{K}_\varepsilon(b)) \leqslant \dim_{\mathcal{H}}(\mathcal{I})$
2. *for all $j = \lfloor (1+b)^\ell J \rfloor$ with $\ell \geqslant \ell_0$, one has*

$$2^{j(\dim_{\mathcal{H}}(\mathcal{I})-\varepsilon)} \leqslant \#\{\lambda \in \mathcal{I}_j : \lambda \cap \mathcal{K}_\varepsilon(b) \neq \emptyset\} \leqslant 2^{j(\dim_{\mathcal{H}}(\mathcal{I})+\varepsilon)},$$

3. *for each dyadic interval $\lambda \in \mathcal{I}_j$ such that $\lambda \cap \mathcal{K}_\varepsilon(b) \neq \emptyset$, with $j = \lfloor (1+b)^\ell J \rfloor$ and $\ell \geqslant \ell_0$, and for any $\ell' \geqslant 1$, one has*

$$\#\left\{\lambda' \in \mathcal{I}_{\lfloor (1+b)^{\ell+\ell'} J \rfloor} : \lambda' \subset \lambda, \lambda' \cap \mathcal{K}_\varepsilon(b) \neq \emptyset\right\} \geqslant 2^{\left((1+b)^{\ell'}-1\right)(1+b)^\ell J\left(\dim_{\mathcal{H}}(I) - \frac{5\varepsilon}{b}\right)}.$$

Such a set is called an (ε, b)-quasi-Cantor subset of $\mathcal{I}$.

The previous theorem is the main tool for studying the multifractal properties of lacunary wavelet series on $\mathcal{I}$, as stated in our second main result.

Theorem 1.3. *Let $\mathcal{I} \subset [0,1]$ be a compact set satisfying $\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) > 0$, let $\alpha > 0$ and let $\eta \in (0, \dim_{\mathcal{H}}(\mathcal{I}))$. Assume that the regularity $r(\psi)$ of the wavelet is greater than $\frac{\dim_{\mathcal{H}}(\mathcal{I})}{\eta} + 1$. If f is the lacunary wavelet series on $\mathcal{I}$ of parameters (α, η), then almost surely, one has*

$$\dim_{\mathcal{H}}(\{x \in [0,1] : h_f(x) \leqslant h\}) = \frac{\eta}{\alpha} h, \quad \forall h \in \left[\alpha, \alpha \frac{\dim_{\mathcal{H}}(\mathcal{I})}{\eta}\right] \tag{4}$$

where $h_f(x)$ denotes the Hölder exponent of f at x, see definition 2.4.

Fractal sets which fail to satisfy the assumption (1)—known as the Ahlfors regularity—but satisfy (2) are numerous in classical fractal geometry. To conclude this Introduction, let us consider in detail the notable examples of attractors of IFS. Let $S = \{f_1, \ldots, f_n\}$ denote a finite set of contracting similitudes on $\mathbb{R}$ with respective Lipschitz constants $r_1, \ldots, r_n$. Then, there exists a unique compact and non-empty set $\mathcal{I}$ such that

$$\mathcal{I} = \bigcup_{j=1}^{n} f_j(\mathcal{I}),$$

which is called the attractor of the IFS S. The attractor $\mathcal{I}$ easily satisfies the relation (2). To S, one can also associate its similarity dimension $\dim(S)$, defined as the unique α solution of the equation

$$\sum_{j=1}^{n} r_j^\alpha = 1.$$

When the smaller copies of $\mathcal{I}$ are sufficiently well-separated, the fine-scale structure of the set becomes relatively straightforward to analyse, and, in particular, the Hausdorff dimension can be determined explicitly using the properties of the defining similitudes. Specifically, if the IFS S satisfies the open set condition (OSC), the Hausdorff dimension of $\mathcal{I}$ is equal to $\dim(S)$. The OSC requires the existence of a non-empty open set V such that the images $f_j(V)$, for $j \in 1,\ldots,n$, are mutually disjoint and entirely contained within V. The following result, established in [34], demonstrates that the OSC is also the key separation condition for ensuring that self-similar fractals possess a positive Hausdorff measure.

Proposition 1.4 ([34, theorem 2.2]). *Let $\mathcal{I}$ be the attractor of an IFS S such that $\dim(S) = \dim_{\mathcal{H}}(\mathcal{I})$. Then, S satisfies the OSC if and only if*

$$\mathcal{H}^{\dim_{\mathcal{H}}(\mathcal{I})}(\mathcal{I}) > 0.$$

The previous result shows that for any IFS that does not satisfy the OSC but for which the expected equality $\dim_{\mathcal{H}}(\mathcal{I}) = \dim(S)$ is satisfied, the assumption (1) is not fulfilled. Note that the OSC is known to be very restrictive, see e.g. [30]. To give a tangible example, the IFS $S = \{x/3, (x+1)/3, (x+u)/3\}$ does not satisfy the OSC if u is irrational [26, theorem 2]. Thus, if $\dim_{\mathcal{H}}(\mathcal{I}) = \dim(S)$ for the fractal set $\mathcal{I}$ associated with the IFS S, theorem 1.3 applies.

The paper is organized as follows. Section 2 provides a summary of key definitions related to dimensions, wavelets, Hölder regularity, and multifractal analysis. Section 3 focuses on results [18], which estimate the number of dyadic intervals that replicate at the expected rate—relatively to the dimension of the fractal set—at the subsequent scale. These estimates serve as the foundation for the construction of the fractal set $\mathcal{K}_\varepsilon(b)$ in section 4, where theorem 1.2 is proved. Finally, section 5 is dedicated to the proof of theorem 1.3.

In the remainder of the paper, we use the notation $\lfloor x \rfloor$ to denote the entire part of the real x.

2. Basic definitions and classical results on multifractal analysis

2.1. Fractal dimensions

In this subsection, we introduce the fundamental concepts of fractal dimensions needed in this paper. We refer to [19], for example, for more details.

Definition 2.1 (Hausdorff measure and Hausdorff dimensions). Let $\mathcal{I} \subset \mathbb{R}$ and $\delta > 0$. For $s \in [0,1]$, set

$$\mathcal{H}^s_\delta(\mathcal{I}) = \inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(A_i)^s : \mathcal{I} \subset \bigcup_{i\in\mathbb{N}} A_i \text{ and } \operatorname{diam}(A_i) < \delta \; \forall i \in \mathbb{N}\right\}.$$

The δ-dimensional Hausdorff measure of $\mathcal{I}$ is $\mathcal{H}^s(\mathcal{I}) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(\mathcal{I})$ and the Hausdorff dimension of $\mathcal{I}$ is given by

$$\dim_{\mathcal{H}}(\mathcal{I}) = \inf\{s \geqslant 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(\mathcal{I}) = +\infty\}.$$

We take the usual convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

Definition 2.2. Let $\mathcal{I} \subset \mathbb{R}$. The upper-box dimension of $\mathcal{I}$ is given by

$$\overline{\dim_B}(\mathcal{I}) := \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{-\log(\varepsilon)}$$

where $N(\varepsilon)$ is the smallest number of intervals of length ε which can cover $\mathcal{I}$.

The following relationship between the two dimensions always holds.

Lemma 2.3. *For any set $\mathcal{I}$, one has*

$$\dim_{\mathcal{H}}(\mathcal{I}) \leqslant \overline{\dim_B}(\mathcal{I})\,. \tag{5}$$

2.2. Multifractal analysis and computation via wavelets

The aim of multifractal analysis is to describe the local change of smoothness of a function (or a measure). The origin of the study of the pointwise regularity goes back to the 19th century, when the question of whether a continuous function necessarily has points of differentiability was raised. Weierstrass provided a pioneer family of counterexamples (and thus answered negatively to the above question). Since then, the multifractal analysis of functions has been developed within both theoretical (see [5, 6, 11, 12, 21] among others) and numerical ([2, 3, 33], ...) frameworks, with applications in areas such as turbulence modelling, medical image processing, art history among others.

For functions, the smoothness is characterized by the pointwise Hölder exponent, defined as follows.

Definition 2.4. Let $x_0 \in \mathbb{R}$ and $h > 0$. A locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}^h(x_0)$ if there exists $C > 0$ and a polynomial P_{x_0} with $\deg P_{x_0} < \lfloor h \rfloor$ such that

$$|f(x) - P_{x_0}(x)| \leqslant C|x - x_0|^h$$

on a neighbourhood of x_0. The *pointwise Hölder exponent* of f at x_0 is

$$h_f(x_0) = \sup\left\{h \geqslant 0 : f \in \mathcal{C}^h(x_0)\right\}.$$

In many situations and in particular for stochastic processes for which it can change at each realization, we are not able to compute the pointwise exponent at each point. However, a relevant information is given by the dimension of the iso-Hölder set for any level $h \in [0, +\infty]$, which consists of all points where the function f has a Hölder exponent exactly equal to h. This information is provided by the multifractal spectrum, which can be regarded as a signature of the function. For instance, one may recall the classical problem of turbulence introduced by Kolmogorov in the 1940s for which various models have since been proposed, each leading to different theoretical multifractal spectra, which can then be compared with those observed in real experiments.

Definition 2.5. The *multifractal spectrum* $\mathcal{D}_f$ of a locally bounded function f is the function

$$\mathcal{D}_f : h \in [0, +\infty] \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R} : h_f(x_0) = h\})\,.$$

The *increasing multifractal spectrum* of f is the function

$$\mathcal{D}_{f,\leq} : h \in [0, +\infty] \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R} : h_f(x_0) \leqslant h\})\,.$$

The pointwise regularity of a locally bounded function f can be characterized through a wavelet analysis of f. Before recalling this classical result, let us give the definition of a wavelet basis. We refer to [16, 31, 32] e.g. for more information on wavelets.

An orthonormal wavelet basis on $\mathbb{R}$ is given by two functions φ and ψ with the property that the family

$$\{\varphi(\cdot-k) : k\in\mathbb{Z}\}\cup\left\{2^{\frac{j}{2}}\psi\left(2^j\cdot-k\right) : j\in\mathbb{N}, k\in\mathbb{Z}\right\}$$

forms an orthonormal basis of $L^2(\mathbb{R})$. Therefore, for all $f\in L^2(\mathbb{R})$, we have the following decomposition

$$f=\sum_{k\in\mathbb{Z}}C_k\varphi(\cdot-k)+\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}}c_{j,k}\psi\left(2^j\cdot-k\right)$$

where the wavelet coefficients of f are given by

$$C_k=\int_{\mathbb{R}}f(x)\,\varphi(x-k)\,\mathrm{d}x$$

and

$$c_{j,k}=2^j\int_{\mathbb{R}}f(x)\,\psi\left(2^jx-k\right)\mathrm{d}x.$$

Note that we do not use the L^2-normalization to avoid a rescaling in the definition of the wavelet leaders, see definition 2.6 below. Note also that the definition of the wavelet coefficients makes sense even if f does not belong to $L^2(\mathbb{R})$.

We say in addition that the wavelet basis is r-smooth if φ and ψ have partial derivatives up to order r and if these partial derivatives decay rapidly, that is, faster than any polynomial. In this case, the wavelet ψ has a corresponding number of vanishing moments. Constructions of wavelet bases with arbitrarily large r have been proposed in [16]. One can also chose a wavelet basis with infinitely smooth functions [32].

Usually, the following compact notation using dyadic intervals are used for indexing wavelets. If $\lambda=\lambda_{j,k}=[k2^{-j},(k+1)2^{-j}[$, we write $c_\lambda=c_{j,k}$ and $\psi_\lambda=\psi_{j,k}=\psi(2^j\cdot-k)$. This notation is justified by the fact that the wavelet ψ_λ is essentially localized on the interval λ in the following way: if the wavelets are compactly supported then

$$\exists C>0 \text{ such that } \forall\lambda\quad \operatorname{supp}(\psi_\lambda)\subset C\lambda$$

where $C\lambda$ denotes the interval with the same centre as λ and C times the diameter.

Definition 2.6. Let λ be a dyadic cube and 3λ the cube of same centre and three times wider. If f is a bounded function, the wavelet leader d_λ of f is given by

$$d_\lambda=\sup_{\lambda'\subset3\lambda}|c_{\lambda'}|.$$

The pointwise Hölder regularity of the function f at a point x_0 can be determined using the wavelet leaders. Letting $x_0\in\mathbb{R}$, the notation $\lambda_j(x_0)$ refers to the dyadic cube of width 2^{-j} which contains x_0 and

$$d_j(x_0)=d_{\lambda_j(x_0)}=\sup_{\lambda'\subset3\lambda_j(x_0)}|c_{\lambda'}|.$$

Theorem 2.7 ([24, theorem 1]). *Let $f \in \mathcal{C}^{\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$ and let $x_0 \in \mathbb{R}$. Then*

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log(d_j(x_0))}{\log(2^{-j})}$$

provided that the analysing wavelet is r-smooth with $r > \lfloor h_f(x_0) \rfloor + 1$.

In what follows, we will thus always assume that the wavelet ψ is r-smooth with $r = r(\psi)$ large enough.

Consider now an α-*wavelet series*, by which we mean a wavelet series f of the form

$$f = \sum_{j \in \mathbb{N}} \sum_{0 \leqslant k \leqslant 2^j - 1} c_{j,k} \psi_{j,k}$$

where the coefficients are defined by

$$c_{j,k} = \begin{cases} 2^{-\alpha j} & \text{if } k \in F_j \\ 0 & \text{otherwise} \end{cases}$$

for some subsets $F_j \subset \{0, \ldots, 2^j - 1\}$. For $\delta \in (0,1]$, let us define the sets $E_\delta(f)$ by

$$E_\delta(f) := \limsup_{j \to +\infty} \bigcup_{k \in F_j} B\left(k2^{-j}, 2^{-\delta(1-\varepsilon_j)j}\right)$$

where (ε_j) is a sequence tending to 0 as $j \to \infty$. As a direct consequence of theorem [24], the sets $E_\delta(f)$ provide a characterization of the points where the regularity of f is less than h.

Proposition 2.8. *Let f be a α-wavelet series and $\delta \in (0,1]$.*

(a) If $x \in E_\delta(f)$, then $h_f(x) \leqslant \frac{\alpha}{\delta}$.
(b) If $x \notin E_\delta(f)$, then $h_f(x) \geqslant \frac{\alpha}{\delta}$.

The proof is straightforward and is provided in [18, lemma 3.11].

As a consequence, the study of the increasing multifractal spectrum of a lacunary wavelet series, which is a random α-wavelet series, can be obtained by the computation of the Hausdorff dimension of some limsup of random balls.

When in addition, one is able to construct a positive measure on the sets $E_\delta(f)$, it becomes possible to obtain the dimension of the iso-hölder sets and hence compute the spectrum of singularities. It has been established for lacunary wavelet series on $[0,1]$ by Jaffard in [23].

Theorem 2.9 ([23, theorem 1]). *Let f be a LWS on $[0,1]$ of parameters $\alpha > 0$ and $\eta \in (0,1)$. Almost surely, one has*

$$\mathcal{D}_f(h) = \begin{cases} \frac{\eta}{\alpha} h & \text{if } h \in \left[\alpha, \frac{\alpha}{\eta}\right], \\ -\infty & \text{otherwise.} \end{cases}$$

Using the more general MTP of [14], the result can be easily extended to symmetric binary Cantor sets; see [18]. We denote by $\mathcal{C}(r)$ the symmetric binary Cantor set with a dissection ratio $r < \frac{1}{2}$, having Hausdorff dimension given by

$$\dim_{\mathcal{H}}(\mathcal{C}(r)) = -\frac{1}{\log_2 r}.$$

Proposition 2.10 ([18, proposition 1.7]). *Let f be a LWS on* $\mathcal{C}(r)$ *of parameters* $\alpha > 0$ *and* $\eta \in (0, \dim_{\mathcal{H}}(\mathcal{C}(r)))$. *Almost surely, one has*

$$\mathcal{D}_f(h) = \begin{cases} \frac{\eta}{\alpha} h & \text{if } h \in \left[\alpha, \frac{\alpha \dim_{\mathcal{H}}(\mathcal{C}(r))}{\eta}\right], \\ 1 & \text{if } h = r(\psi), \\ -\infty & \text{otherwise.} \end{cases}$$

3. Sets with expected rate of duplication

In this section, we recall two results from [18] that provide an estimation of the number of dyadic intervals that duplicate at the expected rate at the next scale, with respect to the dimension of the fractal set. Since these results will serve as the foundation for the iterative procedure used to construct the quasi-Cantor set $\mathcal{K}_\varepsilon(b)$ in theorem 1.2 in the next section, we have included the proofs of these results in the appendix. First, let us introduce some notation.

Notation:

- In what follows, we assume that $\mathcal{I}$ is a compact subset of $[0,1]$ such that

$$\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) > 0.$$

  We denote this common dimension by H.
- Let

$$\mathcal{I}_j = \left\{\lambda = \left[2^{-j}k, 2^{-j}(k+1)\right] : \lambda \cap \mathcal{I} \neq \emptyset\right\},$$

  be the collection of dyadic intervals of size 2^{-j} that intersect the fractal set $\mathcal{I}$.
- For a collection $U = U(J)$ of dyadic intervals at a scale J, we denote the collection of all their children at the scale $(1+b)J$ with $b > 0$, by

$$C_b U := \left\{\lambda' \in \mathcal{I}_{\lfloor (1+b)J \rfloor} : \lambda' \subset \lambda \text{ with } \lambda \in U\right\}.$$

- We also denote by $\widetilde{U}$ the points $x \in [0,1]$ in the union of the dyadic intervals which belong to U.

Remark 3.1. Clearly, the scale $(1+b)J$ is not necessarily an integer, which is why we consider instead its integer part. The same convention will be adopted throughout this section, although it does not play a crucial role. For the sake of readability in section 4, where the set $\mathcal{K}_\varepsilon(b)$ is constructed, we will omit the integer parts of the scales in the remainder of the paper.

Lemma 3.2 ([18, lemma 4.1]). *For every* $\varepsilon > 0$, *there exists* $J \in \mathbb{N}$ *such that for all* $j \geqslant J$, *one has*

$$2^{j(H-\varepsilon)} \leqslant \#\mathcal{I}_j \leqslant 2^{j(H+\varepsilon)}.$$

We now introduce different collections of intervals λ of scale J by considering the number of their children at scale $\lfloor (1+b)J \rfloor$ for a fixed $b \in (0,1)$ and up to some $\varepsilon > 0$.

Definition 3.3. Let $\varepsilon > 0$, $b \in (0,1)$ and $j \in \mathbb{N}$.

1. We say that $\lambda \in \mathcal{I}_j$ has a *slow duplication rate* if

$$\#\left\{\lambda' \subset \lambda : \lambda' \in \mathcal{I}_{\lfloor (1+b)j \rfloor}\right\} \leqslant 2^{j\,(bH-4\varepsilon)} \tag{6}$$

   and we denote by $\mathrm{SD}(j,b,\varepsilon)$ the collection of such intervals.
2. we say that $\lambda \in \mathcal{I}_j$ has an *expected duplication rate* if

$$2^{j(bH-4\varepsilon)} \leqslant \#\left\{\lambda' \subset \lambda \cap \mathcal{I}_{\lfloor (1+b)j \rfloor}\right\} \leqslant 2^{j(bH+4\varepsilon)}, \tag{7}$$

   and we denote by $\mathrm{ED}(j,b,\varepsilon)$ the collection of such intervals.
3. We say that $\lambda \in \mathcal{I}_j$ has a *fast duplication rate* if

$$\#\left\{\lambda' \subset \lambda : \lambda' \in \mathcal{I}_{\lfloor (1+b)j \rfloor}\right\} \geqslant 2^{j\,(bH+4\varepsilon)} \tag{8}$$

   and we denote by $\mathrm{FD}(j,b,\varepsilon)$ the collection of such intervals.

In [18], we obtained the following lower and upper bounds for the cardinalities of each set. This proposition is crucial for the construction of the quasi-Cantor subsets in the following section. Note that the derivation of (9) relies on the assumption $\dim_{\mathcal{H}} \mathcal{I} = \overline{\dim}_B \mathcal{I}$, and in fact, this is the only place where this assumption is used.

Proposition 3.4 ([18, theorem 1.13]). *Let $\varepsilon > 0$, $b \in (0,1)$. There exists $J_0 \in \mathbb{N}$ such that for all $j \geqslant J_0$, one has*

$$2^{j(H-2\varepsilon)} \leqslant \#ED\,(j,b,\varepsilon) \leqslant 2^{j(H+\varepsilon)} \tag{9}$$

$$\#FD\,(j,b,\varepsilon) \leqslant 2^{j(H-2\varepsilon)} \tag{10}$$

and

$$\#C_\beta SD\,(j,b,\varepsilon) \leqslant 2^{j(1+b)(H-3\varepsilon)}. \tag{11}$$

4. Quasi-Cantor subsets of $\mathcal{I}$

The aim of this section is to prove theorem 1.2. The results of section 3 do not establish the existence of sets that duplicate at the 'expected rate' at all scales, as is the case for a Cantor set. Indeed,the children of such sets may exhibit either slow or fast duplication. Here, we propose an iterative procedure to retain only the sets that exhibit an 'infinite expected rate' of duplication.

We now consider a positive number $b \in (0,1)$ and a sufficiently small $\varepsilon > 0$ such that

$$bH - 5\varepsilon > 0.$$

We introduce sub-collections of $\mathrm{ED}(J,b,\varepsilon)$, $J \in \mathbb{N}$, in order to accurately analyze if the rate of duplication is preserved through the scales for an expected number of dyadic intervals. We fix J_0 large enough such that the estimations of the cardinality of $\mathrm{FD}(J,b,\varepsilon)$, $\mathrm{ED}(J,b,\varepsilon)$ and $C_b\mathrm{SD}(J,b,\varepsilon)$ given in proposition 3.4 hold for every $J \geqslant J_0$. We now define a succession of nested sets. As mentioned in remark 3.1, we will deliberately omit the minor adjustments required to account for the fact that the scales $(1+b)^\ell J$ should be considered in their integer part. For all $J \geqslant J_0$, we set

$$T_1(J,b,\varepsilon) = \mathrm{ED}(J,b,\varepsilon),$$

and

$$T_2(J,b,\varepsilon) := \left\{\lambda \in T_1(J,b,\varepsilon) : \#\{\lambda' \subset \lambda : \lambda' \in T_1((1+b)J,b,\varepsilon)\} \geqslant 2^{J(bH-5\varepsilon)}\right\}.$$

Next, we define recursively for $\ell \geqslant 3$,

$$T_\ell(J,b,\varepsilon) := \left\{\lambda \in T_{\ell-1}(J,b,\varepsilon) : \#\{\lambda' \subset \lambda : \lambda' \in T_\ell((1+b)J,b,\varepsilon)\} \geqslant 2^{J(bH-5\varepsilon)}\right\}.$$

We also define the complementary sets

$$U_\ell(J,b,\varepsilon) = T_{\ell-1}(J,b,\varepsilon) \setminus T_\ell(J,b,\varepsilon)$$

for all $\ell \geqslant 2$. The following lemma provides an estimate of the maximal number of intervals required to cover the complementary sets.

Lemma 4.1. *For every $J \geqslant J_0$, the set $\widetilde{U}_2(J,b,\varepsilon) \cap \mathcal{I}$ is covered by the union of*

1. *at most $\#U_2(J,b,\varepsilon) \times 2^{J(bH-5\varepsilon)}$ intervals of length $2^{-(1+b)J}$,*
2. *intervals of $FD((1+b)J,b,\varepsilon)$*
3. *intervals of $C_bSD((1+b)J,b,\varepsilon)$.*

Proof. Let us cover the set $\widetilde{U}_2(J,b,\varepsilon) \cap \mathcal{I}$ by considering the set of children of $U_2(J,b,\varepsilon)$ at the scale $(1+b)J$ and by decomposing this set following the rate of duplication of the children at the scale $(1+b)^2 J$. We obtain

$$C_bU_2(J,b,\varepsilon) \subset \mathrm{FD}((1+b)J,b,\varepsilon) \cup \mathrm{SD}((1+b)J,b,\varepsilon) \cup (\mathrm{ED}((1+b)J,b,\varepsilon) \cap C_bU_2(J,b,\varepsilon)).$$

The set $\widetilde{\mathrm{SD}}((1+b)J,b,\varepsilon)$ can be covered by its children at the next generation, that is by the union of the intervals of $C_b\mathrm{SD}((1+b)J,b,\varepsilon)$. Finally, by definition of $U_2(J,b,\varepsilon)$, any dyadic interval $\lambda \in U_2(J,b,\varepsilon)$ has less than $2^{J(bH-5\varepsilon)}$ children in $\mathrm{ED}((1+b)J,b,\varepsilon)$. It follows that $\widetilde{\mathrm{ED}}((1+b)J,b,\varepsilon) \cap \widetilde{C_bU_2}(J,b,\varepsilon)$ is covered by at most $\#U_2(J,b,\varepsilon) \times 2^{J(bJ-5\varepsilon)}$ intervals of length $2^{-(1+b)J}$. □

The previous lemma is the first step towards obtaining a covering for any $\widetilde{U}_\ell(J,b,\varepsilon)$ with $\ell \geqslant 2$.

Lemma 4.2. *Let $\ell \geqslant 2$. For every $J \geqslant J_0$, the set $\widetilde{U}_\ell(J,b,\varepsilon) \cap \mathcal{I}$ is covered by the union of*

1. *at most $\#U_\ell(J,b,\varepsilon) \times 2^{J(bH-5\varepsilon)}$ intervals of length $2^{-(1+b)J}$,*
2. *at most, for each $p \in \{2,\ldots,\ell-1\}$ and each $q \in \{p+1,\ldots,\ell\}$,*

$$\#U_{q-p+1}\left((1+b)^{p-1},b,\varepsilon\right) \times 2^{(1+b)^{p-1}J(bH-5\varepsilon)}$$

   intervals of length $2^{-(1+b)^pJ}$,
3. *for each $p \in \{2,\ldots,\ell\}$, intervals of $FD((1+b)^{p-1}J,b,\varepsilon)$ and of $C_bSD((1+b)^{p-1}J,b,\varepsilon)$.*

Proof. Lemma 4.1 gives the result for $\ell = 2$. Suppose that $\ell \geqslant 2$ and that the result is true for any $J \geqslant J_0$ and for all the sets $U_p(J,b,\varepsilon)$ with $p \leqslant \ell$. We want to obtain the result for $U_{\ell+1}(J,b,\varepsilon)$. As already mentioned, we observe that

$$C_b U_{\ell+1}(J,b,\varepsilon) \subset \mathrm{FD}((1+b)J,b,\varepsilon) \cup \mathrm{SD}((1+b)J,b,\varepsilon) \cup (\mathrm{ED}((1+b)J,b,\varepsilon) \cap C_b U_{\ell+1}(J,b,\varepsilon)).$$

The intervals belonging to $\mathrm{SD}((1+b)J,b,\varepsilon)$ form a set that can be covered by the intervals of $C_b\mathrm{SD}((1+b)J,b,\varepsilon)$.

Moreover, since

$$\mathrm{ED}((1+b)J,b,\varepsilon) = T_\ell((1+b)J,b,\varepsilon) \cup \bigcup_{p=3}^{\ell+1} U_{p-1}((1+b)J,b,\varepsilon),$$

we obtain that

$$\mathrm{ED}((1+b)J,b,\varepsilon) \cap C_b U_{\ell+1}(J,b,\varepsilon) \subset (T_\ell((1+b)J,b,\varepsilon) \cap C_b U_{\ell+1}(J,b,\varepsilon)) \cup \bigcup_{p=3}^{\ell+1} (U_{p-1}((1+b)J,b,\varepsilon).$$

For all the terms appearing in the union, we use our assumption that at the scale $(1+b)J$ and with different generations $(1+b)^s$, we have $2 \leqslant s \leqslant q+1$.

Regarding the first term, if λ belongs to $U_{\ell+1}(J,b,\varepsilon)$, there are no more than $2^{J(bH-5\varepsilon)}$ of its children in $T_\ell((1+b)J,b,\varepsilon)$. Hence, the number of dyadic intervals in $T_\ell((1+b)J,b,\varepsilon) \cap C_b U_{\ell+1}(J,b,\varepsilon)$ is smaller than $\#U_{\ell+1}(J,b,\varepsilon) \times 2^{J(bH-5\varepsilon)}$. □

Finally, for any $J \geqslant J_0$, we introduce the set

$$U_\infty(J,b,\varepsilon) := \bigcup_{\ell=1}^{+\infty} U_\ell(J,b,\varepsilon),$$

and obtain the following proposition.

Proposition 4.3. *For every $J \geqslant J_0$, the set $\widetilde{U_\infty}(J,b,\varepsilon) \cap \mathcal{I}$ is covered by the union of*

1. *at most, for each $p \geqslant 1$,*

$$\#ED\left((1+b)^{p-1}J,b,\varepsilon\right) \times 2^{(1+b)^{p-1}J(bH-5\varepsilon)}$$

intervals of length $2^{-(1+b)^p J}$,

2. *for each $p \geqslant 2$, intervals of $FD((1+b)^{p-1}J,b,\varepsilon)$,*
3. *for each $p \geqslant 2$, intervals of $C_bSD((1+b)^{p-1}J,b,\varepsilon)$.*

Proof. Let us notice that the sets $U_\ell(J,b,\varepsilon)$, $\ell \geqslant 2$ are disjoint and included in the sets $\mathrm{ED}(J,b,\varepsilon)$. Hence

$$\sum_{\ell=2}^{+\infty} \#U_\ell(J,b,\varepsilon) \leqslant \#\mathrm{ED}(J,b,\varepsilon).$$

Similarly, for every $p \geqslant 2$, one has

$$\sum_{q=p+1}^{+\infty} \#U_{q-p+1}\left((1+b)^{p-1}J,b,\varepsilon\right) \leqslant \#\mathrm{ED}\left((1+b)^{p-1}J,b,\varepsilon\right).$$

We obtain then directly the conclusion by applying lemma 4.2. □

The covering obtained in the previous result allows us to estimate the Hausdorff dimension of intersection of sets defined by

$$T_\infty(J,b,\varepsilon) := \bigcap_{\ell \geqslant 1} T_\ell(J,b,\varepsilon)$$

for $J \geqslant J_0$. Before estimating the dimension, we first show that this set (which will be proved to be non-empty in theorem 4.5) possesses a nice (almost) self-similar structure.

Proposition 4.4. *Fix $\ell \geqslant 1$. Any dyadic interval $\lambda \in T_\infty(J,b,\varepsilon)$ satisfies*

$$\#\left\{\lambda' \subset \lambda : \lambda' \in T_\infty\left((1+b)^\ell J,b,\varepsilon\right)\right\} \geqslant 2^{\left((1+b)^\ell-1\right)J\left(H-5\frac{\varepsilon}{b}\right)}.$$

Proof. We proceed by induction. For the case $\ell = 1$, suppose for a contradiction that

$$\#\{\lambda' \subset \lambda : \lambda' \in T_\infty((1+b)J,b,\varepsilon)\} < 2^{J(bH-5\varepsilon)}.$$

Since λ has a finite number of other children (i.e. those not in $T_\infty((1+b)J,b,\varepsilon)$), we can choose

$$\ell_0 > \max_{\lambda' \subset \lambda, \lambda' \notin T_\infty((1+b)J,b,\varepsilon)} \max\{\ell \in \mathbb{N} : \lambda' \in T_\ell((1+b)J,b,\varepsilon)\}.$$

This choice of ℓ_0 ensures that there are strictly less than $2^{J(bH-5\varepsilon)}$ children of λ belonging to the set $T_{\ell_0}((1+b)J,b,\varepsilon)$. This contradicts the fact that λ belongs to $T_{\ell_0+1}(J,b,\varepsilon)$.

Now, assume that

$$\#\left\{\lambda' \subset \lambda : \lambda' \in T_\infty\left((1+b)^\ell J,b,\varepsilon\right)\right\} \geqslant 2^{\left((1+b)^\ell-1\right)J\left(H-5\frac{\varepsilon}{b}\right)}$$

and let us prove the result for $\ell+1$. By the first case, each child λ' of an interval λ that belongs to $T_\infty((1+b)^\ell J,b,\varepsilon)$ gives at least $2^{(1+b)^\ell J(bH-5\varepsilon)}$ children at scale $(1+b)^{\ell+1}J$ belonging to $T_\infty((1+b)^{\ell+1}J,b,\varepsilon)$. It follows that

$$\begin{aligned}\#\left\{\lambda' \subset \lambda : \lambda' \in T_\infty\left((1+b)^{\ell+1}J,b,\varepsilon\right)\right\} &\geqslant 2^{(1+b)^\ell J(bH-5\varepsilon)}2^{\left((1+b)^\ell-1\right)J\left(H-5\frac{\varepsilon}{b}\right)}\\ &= 2^{\left((1+b)^{\ell+1}-1\right)J\left(H-5\frac{\varepsilon}{b}\right)}.\end{aligned}$$

□

Theorem 4.5. *Assume that $J \geqslant J_0$. There exists $\ell_0 \in \mathbb{N}$ such that*

$$\dim_{\mathcal{H}}\left(\bigcap_{\ell \geqslant \ell_0} \widetilde{T_\infty}\left((1+b)^\ell J,b,\varepsilon\right) \cap \mathcal{I}\right) \geqslant H-\varepsilon$$

and for all $\ell \geqslant \ell_0$,

$$2^{(1+b)^\ell J(H-\varepsilon)} \leqslant \#T_\infty\left((1+b)^\ell J,b,\varepsilon\right) \leqslant 2^{(1+b)^\ell J(H+\varepsilon)}.$$

Proof. For all $\ell \geqslant 1$, let us start by setting

$$R\left((1+b)^{\ell} J, b, \varepsilon\right) = FD\left((1+b)^{\ell} J, b, \varepsilon\right) \cup SD\left((1+b)^{\ell} J, b, \varepsilon\right) \cup U_{\infty}\left((1+b)^{\ell} J, b, \varepsilon\right).$$

Note that $R((1+b)^{\ell}J,b,\varepsilon)$ is exactly the set of dyadic intervals of $\mathcal{I}_{(1+b)^{\ell}J}$ that do not belong to the set $T_{\infty}((1+b)^{\ell}J,b,\varepsilon)$. Using proposition 4.3, we obtain a covering of the set $\widetilde{R}((1+b)^{\ell}J,b,\varepsilon)\cap\mathcal{I}$ by

1. at most, for each $p \geqslant 1$,

$$\#\mathrm{ED}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{(1+b)^{\ell+p-1} J(bH-5\varepsilon)}$$

   intervals of length $2^{-(1+b)^{\ell+p}J}$,
2. for each $p \geqslant 1$, intervals of $\mathrm{FD}((1+b)^{\ell+p-1}J,b,\varepsilon)$,
3. for each $p \geqslant 1$, intervals of $C_b\mathrm{SD}((1+b)^{\ell+p-1}J,b,\varepsilon)$.

This covering provides an upper bound of the r-dimensional Hausdorff measure of the set $\widetilde{R}((1+b)^{\ell}J,b,\varepsilon)\cap\mathcal{I}$ for $r \leqslant 2^{-(1+b)^{\ell}J}$ as follows. One has

$$\begin{aligned}
&\mathcal{H}_r^{H-\varepsilon}\left(\widetilde{R}\left((1+b)^{\ell} J, b, \varepsilon\right) \cap \mathcal{I}\right) \\
&\quad \leqslant \sum_{p\geqslant 1} \#\mathrm{ED}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{(1+b)^{\ell+p-1} J(bH-5\varepsilon)} 2^{-(1+b)^{\ell+p} J(H-\varepsilon)} \\
&\qquad + \sum_{p\geqslant 1} \#\mathrm{FD}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{-(1+b)^{\ell+p-1} J(H-\varepsilon)} \\
&\qquad + \sum_{p\geqslant 1} \#C_b\mathrm{SD}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{-(1+b)^{\ell+p} J(H-\varepsilon)}.
\end{aligned} \tag{12}$$

Let us examine the three sums separately, using proposition 3.4 for each estimation. First, we have

$$\begin{aligned}
&\sum_{p\geqslant 1} \#\mathrm{ED}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{(1+b)^{\ell+p-1} J(bH-5\varepsilon)} 2^{-(1+b)^{\ell+p} J(H-\varepsilon)} \\
&\quad \leqslant \sum_{p\geqslant 1} 2^{(1+b)^{\ell+p-1} J(H+\varepsilon)} 2^{(1+b)^{\ell+p-1} J(bH-5\varepsilon)} 2^{-(1+b)^{\ell+p} J(H-\varepsilon)} \\
&\quad = \sum_{p\geqslant 1} 2^{(1+b)^{\ell+p-1} J\varepsilon(-3+b)} \\
&\quad \leqslant \sum_{p\geqslant 1} 2^{-(1+b)^{\ell+p} J\varepsilon}
\end{aligned} \tag{13}$$

since $b-3 \leqslant -(1+b)$. Secondly, we have

$$\begin{aligned}
\sum_{p\geqslant 1} \#\mathrm{FD}\left((1+b)^{\ell+p-1} J, b, \varepsilon\right) \times 2^{-(1+b)^{\ell+p-1} J(H-\varepsilon)} &\leqslant \sum_{p\geqslant 1} 2^{(1+b)^{\ell+p-1} J(H-2\varepsilon)} 2^{-(1+b)^{\ell+p-1} J(H-\varepsilon)} \\
&= \sum_{p\geqslant 1} 2^{-(1+b)^{\ell+p-1} J\varepsilon}
\end{aligned} \tag{14}$$

and similarly

$$\sum_{p\geqslant 1}\#C_b\mathrm{SD}\left((1+b)^{\ell+p-1}J,b,\varepsilon\right)\times 2^{-(1+b)^{\ell+p}J(H-\varepsilon)}\leqslant\sum_{p\geqslant 1}2^{(1+b)^{\ell+p}J(H-3\varepsilon)}2^{-(1+b)^{\ell+p}J(H-\varepsilon)}$$
$$=\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p}J2\varepsilon}$$
$$\leqslant\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p}J\varepsilon}. \tag{15}$$

Gathering (12)–(15), we obtain

$$\mathcal{H}_r^{H-\varepsilon}\left(\widetilde{R}\left((1+b)^{\ell}J,b,\varepsilon\right)\cap\mathcal{I}\right)\leqslant 2\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p}J\varepsilon}+\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p-1}J\varepsilon}$$
$$\leqslant 2\sum_{p\geqslant 1}2^{-(1+b)(1+b)^{\ell+p-1}J\varepsilon}+\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p-1}J\varepsilon}$$
$$\leqslant 3\sum_{p\geqslant 1}2^{-(1+b)^{\ell+p-1}J\varepsilon}$$
$$\leqslant C\sum_{p\geqslant 1}2^{-(1+b)^{\ell}(p-1)J\varepsilon}$$
$$\leqslant C2^{-(1+b)^{\ell}J\varepsilon} \tag{16}$$

for some constant $C>0$ and where we have used that $(1+b)^{p-1}\geqslant p$ for p large enough.

Now, let us fix $\ell_0\geqslant 1$. The previous upper bounds imply that for every $r<2^{-(1+b)^{\ell_0}J}$, one has

$$\mathcal{H}_r^{H-\varepsilon}\left(\bigcup_{\ell\geqslant\ell_0}\widetilde{R}\left((1+b)^{\ell}J,b,\varepsilon\right)\cap\mathcal{I}\right)\leqslant C\sum_{\ell\geqslant\ell_0}2^{-(1+b)^{\ell}J\varepsilon}\leqslant C2^{-(1+b)^{\ell_0}J\varepsilon}$$

which in turn implies that

$$\dim_{\mathcal{H}}\left(\bigcap_{\ell_0\geqslant 1}\bigcup_{\ell\geqslant\ell_0}\widetilde{R}\left((1+b)^{\ell}J,b,\varepsilon\right)\cap\mathcal{I}\right)\leqslant H-\varepsilon.$$

Recalling that $\dim_{\mathcal{H}}\mathcal{I}=H$ and noting that

$$\mathcal{I}=\left(\bigcup_{\ell_0\geqslant 1}\bigcap_{\ell\geqslant\ell_0}\widetilde{T_\infty}\left((1+b)^{\ell}J,b,\varepsilon\right)\cap\mathcal{I}\right)\cup\left(\bigcap_{\ell_0\geqslant 1}\bigcup_{\ell\geqslant\ell_0}\widetilde{R}\left((1+b)^{\ell}J,\varepsilon\right)\cap\mathcal{I}\right),$$

we obtain

$$\dim_{\mathcal{H}}\left(\bigcup_{\ell_0\geqslant 1}\bigcap_{\ell\geqslant\ell_0}\widetilde{T_\infty}\left((1+b)^{\ell}J,b,\varepsilon\right)\cap\mathcal{I}\right)=H.$$

In particular, there is $\ell_0 \in \mathbb{N}$ such that

$$\dim_{\mathcal{H}}\left(\bigcap_{\ell \geqslant \ell_0} \widetilde{T_\infty}\left((1+b)^\ell J, b, \varepsilon\right) \cap \mathcal{I}\right) \geqslant H - \varepsilon.$$

It gives the conclusion of the first part of the theorem.

For the second part of the theorem, note first that one has

$$T_\infty\left((1+b)^\ell J, b, \varepsilon\right) \subset \mathrm{ED}\left((1+b)^\ell J, b, \varepsilon\right)$$

by definition. Hence, proposition 3.4 directly gives

$$\# T_\infty\left((1+b)^\ell J, b, \varepsilon\right) \leqslant 2^{(1+b)^\ell J(H+\varepsilon)}.$$

The lower bound is obtained by contradiction using the first part of the proof : if there were infinitely many $q \geqslant q_0$ such that

$$\# T_\infty\left((1+b)^\ell J, b, \varepsilon\right) < 2^{(1+b)^\ell J(H-\varepsilon)},$$

it would provide a sequence of coverings of $\mathcal{I}$ implying that $\dim_{\mathcal{H}}(\mathcal{I}) \leqslant H - \varepsilon$. □

For a fixed $J \geqslant J_0$, if ℓ_0 is the number given by theorem 4.5, we set

$$\mathcal{K}_\varepsilon(b) = \bigcap_{\ell \geqslant \ell_0} \widetilde{T_\infty}\left((1+b)^\ell J, b, \varepsilon\right) \cap \mathcal{I}. \tag{17}$$

It remains to prove the third point of theorem 1.2, which follows directly from proposition 4.4.

Proof. Let $\lambda \subset \mathcal{I}_j$ be such that $\lambda \cap \mathcal{K}_\varepsilon(b) \neq \emptyset$ with $j = (1+b)^\ell J$ and $\ell \geqslant \ell_0$. Such a λ belongs by definition to $T_\infty((1+b)^\ell J, b, \varepsilon)$ and it follows by proposition 4.4 that

$$\#\left\{\lambda' \subset \lambda : \lambda' \in T_\infty((1+b)^{\ell+\ell'} J, b, \varepsilon)\right\} \geqslant 2^{((1+b)^{\ell'}-1)(1+b)^\ell J(H-5\frac{\varepsilon}{b})}$$

for any $\ell' \geqslant 1$. Such a λ' intersects obviously the set $\mathcal{K}_\varepsilon(b)$ since it contains children in $T_\infty((1+b)^{\ell''} J, b, \varepsilon)$ at any scale $\ell'' \geqslant \ell + \ell' + 1$. □

5. LWS on the compact fractal set $\mathcal{I}$

In this section, our main objective is to establish theorem 1.3, which provides the increasing multifractal spectrum of lacunary wavelet series on the compact fractal set $\mathcal{I}$. To this end, we apply the results from the previous section and prove a MTP for the limsup of dyadic balls with centres chosen according to a Bernoulli law.

We continue to consider a compact subset $\mathcal{I}$ of $[0,1]$ satisfying

$$\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) = H > 0.$$

Fix $\alpha > 0$ and $\eta \in (0,H)$. Recall that the lacunary wavelet series on $\mathcal{I}$ with parameters (α,η) is defined by

$$f = \sum_{j\in\mathbb{N}} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k} \quad \text{with} \quad c_{j,k} = \begin{cases} 2^{-\alpha j}\xi_{j,k} & \text{if } \lambda_{j,k}\cap\mathcal{I}\neq\emptyset \\ 0 & \text{otherwise,} \end{cases}$$

where $(\xi_{j,k})_{j,k}$ denotes a sequence of independent Bernoulli random variables with parameter $2^{(\eta-H)j}$. Using lemma 3.2, we know that the average number of nonzero coefficients at each large scale j is comparable to $2^{\eta j}$. In particular, one might expect that the lowest possible regularity, clearly given by α, is reached on an iso-Hölder set of dimension η. Moreover, it is clear that points outside $\mathcal{I}$ have a Hölder exponent equal to the regularity $r(\psi)$ of the wavelet.

Let us now show that the regularity of 'almost every' point belonging to $\mathcal{I}$ is controlled by the lacunarity parameter η: larger the η, the more regular the function. For every fixed $b \in (0,1)$, we consider as before J_0 large enough such that the estimations of the cardinalities of $\mathrm{FD}(J,b,\varepsilon)$, $\mathrm{ED}(J,b,\varepsilon)$ and $C_b\mathrm{SD}(J,b,\varepsilon)$ given in proposition 3.4 hold for every $J \geqslant J_0$, and we consider k large enough so that $J = (1+b)^k \geqslant J_0$. The corresponding quasi-Cantor set given in equation (17) is then of the form

$$\mathcal{K}_{\varepsilon_n}(b) = \bigcap_{p\geqslant p_0} \widetilde{T_\infty}\left((1+b)^p, b, \varepsilon_n\right) \cap \mathcal{I}$$

for some $p_0 > 0$.

Proposition 5.1. *Let $n \in \mathbb{N}_0$ and fix $\beta_n = \frac{1}{\eta}\left(H + \frac{1}{n}\right) - 1$. Consider $\ell \in \mathbb{N}$ such that*

$$(1+\beta_n) = (1+b_n)^\ell$$

for some $b_n > 0$ with $b_n^2 < \frac{1}{10n\beta_n}$ and $\beta_n H - 5b_n^2 > 0$. Finally, select $\varepsilon_n = b_n^2$ and consider $\mathcal{K}_{\varepsilon_n}(b_n)$ as defined in equation (17). Almost surely, there exists $p_0 \in \mathbb{N}$ such that

$$\sup_{\lambda'\subset\lambda} |c_{\lambda'}| \geqslant 2^{-\frac{\alpha}{\eta}\left(H+\frac{1}{n}\right)j}$$

for every λ of scale $j = \lfloor (1+\beta_n)^p \rfloor$ with $p \geqslant p_0$ and such that $\lambda \cap \mathcal{K}_{\varepsilon_n}(b_n) \neq \emptyset$. In particular, almost surely

1. *$h_f(x) \leqslant \frac{\alpha}{\eta}(H + \frac{1}{n})$ for every $x \in \mathcal{K}_{\varepsilon_n}(b_n)$,*
2. *one has*

$$\mathcal{K}_{\varepsilon_n}(b_n) \subset \bigcap_{p\geqslant p_0} \bigcup_{k\in F_\varepsilon(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2\cdot 2^{-(1+\beta_n)^p}\right)$$

where

$$F_{\varepsilon_n}(p+1) = \left\{k : \lambda_{\lfloor (1+\beta_n)^{p+1}\rfloor,k} \cap \mathcal{K}_{\varepsilon_n}(b_n) \neq \emptyset \text{ and } c_{\lfloor (1+\beta_n)^{p+1}\rfloor,k} \neq 0\right\}. \tag{18}$$

Proof. Remember that the quasi-Cantor set $\mathcal{K}_{\varepsilon_n}(b_n)$ is obtained as

$$\mathcal{K}_{\varepsilon_n}(b_n) = \bigcap_{p\geqslant p_0} \widetilde{T_\infty}\left((1+b_n)^p, b_n, \varepsilon_n\right) \cap \mathcal{I}$$

where p_0 is chosen large enough so that the Hausdorff dimension of $\mathcal{K}_\varepsilon(b_n)$ is larger than $H-\varepsilon_n$ and

$$2^{(1+b_n)^p(H-\varepsilon_n)} \leqslant \# T_\infty\left((1+b_n)^p, b_n, \varepsilon_n\right) \leqslant 2^{(1+b_n)^p(H+\varepsilon_n)}$$

for all $p \geqslant p_0$, by theorem 4.5. Using the equality $(1+\beta_n)^p = (1+b_n)^{\ell p}$, it follows that

$$2^{(1+\beta_n)^p(H-\varepsilon_n)} \leqslant \# T_\infty\left((1+\beta_n)^p, b_n, \varepsilon_n\right) \leqslant 2^{(1+\beta_n)^p(H+\varepsilon_n)}$$

for all $p \geqslant p_0/\ell$. Furthermore, we know from proposition 4.4 that if $\lambda \in T_\infty\left((1+\beta_n)^p, b_n, \varepsilon_n\right)$

$$\#\left\{\lambda' \subset \lambda : \lambda' \in T_\infty\left((1+\beta_n)^{p+1}, b_n, \varepsilon_n\right)\right\} \geqslant 2^{\beta_n(1+\beta_n)^p\left(H-5\frac{\varepsilon_n}{b_n}\right)}.$$

We consider

$$\Omega_p = \left\{\exists \lambda \in \mathcal{I}_{(1+\beta_n)^p} : \lambda \cap \mathcal{K}_{\varepsilon_n}(b_n) \neq \emptyset \text{and } \sup_{\lambda' \subset \lambda} |c_{\lambda'}| < 2^{-\frac{\alpha}{\eta}\left(H+\frac{1}{n}\right)(1+\beta_n)^p}\right\}.$$

At the scale $j_0 = (1+\beta_n)^{p+1}$, one has $2^{-\alpha j_0} \geqslant 2^{-\frac{\alpha}{\eta}(H+\frac{1}{n})(1+\beta_n)^p}$ hence

$$\begin{aligned}
\mathbb{P}(\Omega_p) &\leqslant \sum_{\lambda \in \mathcal{I}_{(1+\beta_n)^{p+1}} : \lambda \cap \mathcal{K}_\varepsilon(b_n) \neq \emptyset} \mathbb{P}\left(\{\xi_{\lambda'} = 0, \forall \lambda' \subset \lambda\}\right) \\
&\leqslant 2^{(1+\beta_n)^{p+1}(H+\varepsilon_n)} \left(1 - 2^{(\eta-H)(1+\beta_n)^{p+1}}\right)^{2^{\beta_n(1+\beta_n)^p\left(H-5\frac{\varepsilon_n}{b_n}\right)}} \\
&\leqslant 2^{(1+\beta_n)^{p+1}(H+\varepsilon_n)} \exp\left(-2^{(\eta-H)(1+\beta_n)^{p+1}} 2^{\beta_n(1+\beta_n)^p\left(H-5\frac{\varepsilon_n}{b_n}\right)}\right) \\
&\leqslant 2^{(1+\beta_n)^{p+1}(H+\varepsilon_n)} \exp\left(-2^{\frac{1}{2n}(1+\beta_n)^p}\right)
\end{aligned}$$

by using the choice of β_n, b_n and ε_n. An application of the Borel–Cantelli lemma gives that almost surely, for every p large enough and every $\lambda \in \mathcal{I}_{(1+\beta_n)^p}$ such that $\lambda \cap \mathcal{K}_{\varepsilon_n}(b_n) \neq \emptyset$, one has

$$d_\lambda \geqslant 2^{-\frac{\alpha}{\eta}\left(H+\frac{1}{n}\right)(1+\beta_n)^p}.$$

□

Corollary 5.2. *Almost surely, for every $\varepsilon > 0$, one has*

$$\dim_{\mathcal{H}}\left(\left\{x : h_f(x) \leqslant \frac{\alpha H}{\eta} + \varepsilon\right\}\right) = H.$$

Proof. For every $n \in \mathbb{N}$, we know by proposition 5.1 that almost surely

$$\dim_{\mathcal{H}}\left(\left\{x : h_f(x) \leqslant \frac{\alpha}{\eta}\left(H + \frac{1}{n}\right)\right\}\right) \geqslant \dim_{\mathcal{H}}\left(\mathcal{K}_{\varepsilon_n}(b_n)\right) \geqslant H - \varepsilon_n$$

with $\varepsilon_n < \frac{1}{10(nH+1)}$. It suffices then to take $n \to +\infty$. □

Theorem 5.3. *Almost surely, for every $h \in [\alpha, \frac{\alpha H}{\eta}]$, one has*

$$\dim_{\mathcal{H}}(\{x : h_f(x) \leqslant h\}) \geqslant h\frac{\eta}{\alpha}.$$

Before giving the proof of theorem 5.3, let us recall the useful mass distribution principle.

Lemma 5.4 (mass distribution principle [19]). *Let F be a Borel subset of $\mathbb{R}$ and μ be a mass distribution on F. Suppose that for some s there exists $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \leqslant c|U|^s$$

for all sets U with $|U| \leqslant \delta$. Then $\mathcal{H}^s(F) \geqslant \mu(F)/c$ and in particular,

$$s \leqslant \dim_{\mathcal{H}}(F).$$

Proof of theorem 5.3. Using lemma 2.8, the result boils down to establishing a lower bound for the Hausdorff dimension of the sets

$$E_\delta(f) = \limsup_{j \to +\infty} \bigcup_{k,\, \xi_{j,k} \neq 0} B\left(k2^{-j}, 2^{-\delta(1-\varepsilon_j)j}\right).$$

We fix again $n \in \mathbb{N}$, $\beta_n = \frac{1}{\eta}(H + \frac{1}{n}) - 1$. We consider as previously $\ell \in \mathbb{N}$ such that

$$(1+\beta_n) = (1+b_n)^\ell$$

for some $b_n > 0$ with $b_n^2 < \frac{1}{10n\beta_n}$ and $\beta_n H - 5b_n^2 > 0$. Note that, as mentioned earlier, we should actually consider the integer parts so that all scales under consideration are integers. Finally set $\varepsilon_n = b_n^2$. We consider $\mathcal{K}_{\varepsilon_n}(b_n)$ as defined in equation (17). Using proposition 5.1, we can fix p_0 large enough such that

$$\mathcal{K}_{\varepsilon_n}(b_n) \subset \bigcap_{p \geqslant p_0} \bigcup_{k \in F_{\varepsilon_n}(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2 \cdot 2^{-(1+\beta_n)^p}\right),$$

where $F_{\varepsilon_n}(p+1)$ is defined in equation (18). For s such that $1 \leqslant s \leqslant 1 + \beta_n$ that can be written as $s = (1+b_n)^{q_0}$ with $q_0 \in \{0, \ldots, \ell\}$, we claim and prove below that almost surely, one has

$$\dim_{\mathcal{H}}\left(\mathcal{K}_{\varepsilon_n}(b_n) \cap \bigcap_{p \geqslant p_0} \bigcup_{k \in F_{\varepsilon_n}(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2^{-s(1+\beta_n)^p}\right)\right) \geqslant \frac{H - 7b_n}{s(1+b_n)}. \tag{19}$$

An immediate consequence is that, for any $\delta \in [\frac{1}{1+\beta_n}, 1]$, if one sets $t = (1+\beta_n)\delta$ and if $q_0 \in \{0, \ldots, \ell\}$ is chosen so that $(1+b_n)^{q_0} \leqslant t < (1+b_n)^{q_0+1}$, one can write almost surely

$$\begin{aligned}\dim_H(E_\delta(f)) &\geqslant \dim_{\mathcal{H}}\left(\mathcal{K}_{\varepsilon_n}(b_n) \cap \bigcap_{p \geqslant p_0} \bigcup_{k \in F_{\varepsilon_n}(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2^{-t(1+\beta_n)^p}\right)\right) \\ &\geqslant \dim_{\mathcal{H}}\left(\mathcal{K}_{\varepsilon_n}(b_n) \cap \bigcap_{p \geqslant p_0} \bigcup_{k \in F_{\varepsilon_n}(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2^{-s(1+\beta_n)^p}\right)\right)\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{H-7b_n}{s(1+b_n)} \\
&\geqslant \frac{H-7b_n}{t(1+b_n)^2} \\
&\geqslant \frac{H-7b_n}{\delta(1+\beta_n)(1+b_n)^2} \\
&= \eta \frac{H-7b_n}{\delta\left(H+\frac{1}{n}\right)(1+b_n)^2}
\end{aligned}$$

where $s=(1+b_n)^{q_0}$. Taking a dense set $\{h_k : k\in\mathbb{N}\}$ of $[\alpha, \frac{\alpha}{\eta}H]$ and the limit as $n\to+\infty$, lemma 2.8 implies that, almost surely, for any $k\in\mathbb{N}$,

$$\dim_{\mathcal{H}}\left(\{x : h_f(x)\leqslant h_k\}\right) \geqslant \dim_{\mathcal{H}}\left(E_{\frac{\alpha}{h_k}}(f)\right) \geqslant \eta\frac{h_k}{\alpha}.$$

Since $\dim_{\mathcal{H}}\{x : h_f(x)\leqslant h\}$ is an increasing function, it follows that, almost surely,

$$\mathcal{D}_{f,\leqslant}(h) = \dim_{\mathcal{H}}\left(\{x : h_f(x)\leqslant h\}\right) \geqslant \eta\frac{h}{\alpha}, \quad \forall h\in\left[\alpha, \frac{\alpha}{\eta}H\right]$$

and it will yield the stated result.

To obtain the lower bound of the dimension stated in equation (19), following the usual approach (see, for instance, [15, 23]), we will construct a Cantor set inside

$$\mathcal{K}_{\varepsilon_n}(b_n)\cap\bigcap_{p\geqslant p_0}\bigcup_{k\in F_{\varepsilon_n}(p+1)} B\left(k2^{-(1+\beta_n)^{p+1}}, 2\cdot 2^{-s(1+\beta_n)^p}\right)$$

and a measure on the Cantor set in order to apply the Mass distribution principle (lemma 5.4). We fix s such that $1\leqslant s\leqslant 1+\beta_n$ and $s=(1+b_n)^{q_0}$ with $q_0\in\{0,\dots\ell\}$. The Cantor set and the measure are constructed via an iterative process. Let us first give a heuristic explanation of the construction of the Cantor set and the associated measure. In the first step, we select a collection of disjoint balls of radius $2^{-s(1+\beta_n)^{p_1}}$, centred at points of $\mathcal{K}_{\varepsilon_n}(b_n)$. Their number is chosen to be roughly $2^{(1+\beta_n)^{p_1}(H-6b_n)}$, in accordance with the abundance of admissible centres guaranteed by theorem 4.5. We then distribute the mass of μ uniformly over these balls. In the next step, each ball from the previous generation is refined. Thanks to the reproduction property of $\mathcal{K}_{\varepsilon_n}(b_n)$ (proposition 4.4), one can again select many admissible sub-balls at a finer scale, still in large enough number but slightly reduced to ensure disjointness. The mass carried by each parent ball is then redistributed uniformly among its children. Iterating this procedure yields a decreasing sequence of collections of disjoint balls. Their intersection defines the Cantor set, while the uniform redistribution rule defines the measure μ. Let us now turn to a detailed step-by-step description of the construction. We recall that $F_{\varepsilon_n}(p+1)$ is defined in equation (18) and, using the relation $(1+\beta_n)^p=(1+b_n)^{\ell p}$, that we have

$$2^{(1+\beta_n)^p(H-\varepsilon_n)} \leqslant \#T_\infty\left((1+\beta_n)^p, b_n, \varepsilon_n\right) \leqslant 2^{(1+\beta_n)^p(H+\varepsilon_n)}$$

for all $p\geqslant p_0/\ell$.

First generation : Fix $p_1 \geqslant p_0$. Since

$$\mathcal{K}_{\varepsilon_n}(b_n) \subset \bigcup_{k \in F_{\varepsilon_n}(p_1+1)} B\left(k2^{-(1+\beta_n)^{p_1+1}}, 2\cdot 2^{-(1+\beta_n)^{p_1}}\right)$$

there are at least $2^{(1+\beta_n)^{p_1}(H-\varepsilon_n)}$ elements in $F_{\varepsilon_n}(p_1+1)$ since

$$\# T_\infty\left((1+\beta_n)^{p_1}, b_n, \varepsilon_n\right) \geqslant 2^{(1+\beta_n)^{p_1}(H-\varepsilon_n)}$$

by theorem 4.5. We select exactly $2^{(1+\beta_n)^{p_1}(H-6\frac{\varepsilon_n}{b_n})} = 2^{(1+\beta_n)^{p_1}(H-6b_n)}$ elements corresponding to dyadic points $x_r^{(1)} = k_r 2^{-(1+\beta_n)^{p_1+1}}$, with $1 \leqslant r \leqslant 2^{(1+\beta_n)^{p_1}(H-6b_n)}$ and so that the balls

$$B\left(x_r^{(1)}, 2^{-s(1+\beta_n)^{p_1}}\right), \quad 1 \leqslant r \leqslant 2^{(1+\beta_n)^{p_1}(H-6b_n)},$$

are pairwise disjoint. Note that this choice is possible if p_1 is large enough. We set

$$B_r^{(1)} = B\left(x_r^{(1)}, 2^{-(1+b_n)^{q_0}(1+\beta_n)^{p_1}}\right), \quad 1 \leqslant r \leqslant 2^{(1+\beta_n)^{p_1}(H-6b_n)}.$$

These balls give the first generation of our Cantor set and we define a measure μ_1 by distributing the mass uniformly over them, i.e.

$$\mu_1\left(B_r^{(1)}\right) = 2^{-(H-6b_n)(1+\beta_n)^{p_1}} = |B_r^{(1)}|^{(H-6b_n)/(1+b_n)^{q_0}} = |B_r^{(1)}|^{(H-6b_n)/s}$$

for all $1 \leqslant r \leqslant 2^{(1+\beta_n)^{p_1}(H-6b_n)}$. Note that by construction,

$$B_r^{(1)} \in T_\infty\left((1+b_n)^{q_0}(1+\beta_n)^{p_1}, b_n, \varepsilon_n\right). \tag{20}$$

Choice of scales : The next generations of the Cantor set will be constructed at scales $(1+b_n)^{q_0}(1+\beta_n)^{p_N}$, with $N \geqslant 1$, where the sequence $(p_N)_N$ of integers is chosen so for all $N \geqslant 2$

$$2^{(1+\beta_n)^{p_{N-1}}((1+b_n)^{q_0}-1)(H-6b_n)} < 2^{(1+\beta_n)^{p_N} b_n 2^{-N}}. \tag{21}$$

Second generation : Let us pick one of the balls $B_{r_1}^{(1)}$ of the first generation. Using (20) and the self-similarity given in proposition 4.4 (with $\varepsilon_n = b_n^2$), we know that, for all the scales of the form $(1+b_n)^p$ between $(1+b_n)^{q_0+1}(1+\beta_n)^{p_1}$ and $(1+\beta_n)^{p_2}$, the ball $B_{r_1}^{(1)}$ contains at least

$$2^{((1+b_n)^p-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-5b_n)}$$

dyadic intervals of $\mathcal{K}_{\varepsilon_n}(b_n)$. We select exactly $2^{((1+b_n)^p-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}$ such intervals. At the last scale $p = p_2$ each of these intervals corresponds to a ball $B(x_r^{(2)}, 2^{-(1+\beta_n)^{p_2}})$ where $x_r^{(2)} = k2^{(1+\beta_n)^{p_2+1}}$ belongs to $F_{\varepsilon_n}(p_2+1)$, since

$$\mathcal{K}_{\varepsilon_n}(b_n) \subset \bigcup_{k \in F_{\varepsilon_n}(p_2+1)} B\left(k2^{-(1+\beta_n)^{p_2+1}}, 2\cdot 2^{-(1+\beta_n)^{p_2}}\right).$$

Our second generation of balls in $B_{r_1}^{(1)}$ is then given by $2^{((1+\beta_n)^{p_2}-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}$ disjoint balls $B(k2^{-(1+\beta_n)^{p_2+1}}, 2^{-(1+b_n)^{q_0}(1+\beta_n)^{p_2}})$ included in $B_{r_1}^{(1)}$.

We label these balls

$$B^{(2)}_{r_1,r_2} \quad \text{with} \quad 1 \leqslant r_2 \leqslant 2^{((1+\beta_n)^{p_2}-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}.$$

Note that for each dyadic interval of $\mathcal{K}_\varepsilon(b_n)$ of scale $(1+\beta_n)^{p_2}$, we select only one ball $B(x_r^{(2)}, 2^{-(1+\beta_n)^{p_2}})$ with $x_r^{(2)} = k2^{(1+\beta_n)^{p_2+1}} \in F_{\varepsilon_n}(p_2+1)$. Therefore, the balls with the same center but different dilated radii can intersect at most pairwise. The selection of exactly $2^{((1+\beta_n)^{p_2}-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}$ ensures the possibility of choosing a collection of disjoint balls.

Next, we uniformly distribute the mass of $B^{(1)}_{r_1}$ on the disjoint smaller balls, thereby defining a measure μ_2 by

$$\begin{aligned}
\mu_2\left(B^{(2)}_{r_1,r_2}\right) &= \frac{\mu_1\left(B^{(1)}_{r_1}\right)}{2^{((1+\beta_n)^{p_2}-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}} \\
&= \frac{2^{-(1+\beta_n)^{p_1}(H-6b_n)}}{2^{((1+\beta_n)^{p_2}-(1+b_n)^{q_0}(1+\beta_n)^{p_1})(H-6b_n)}} \\
&\leqslant 2^{-(1+\beta_n)^{p_2}\left(H-6b_n-\frac{b_n}{4}\right)} \\
&= |B^{(2)}_{r_1,r_2}|^{\frac{H-6b_n-\frac{b_n}{4}}{(1+b_n)^{q_0}}} \\
&= |B^{(2)}_{r_1,r_2}|^{\frac{H-6b_n-\frac{b_n}{4}}{s}},
\end{aligned}$$

where the inequality holds due to the choice of p_2 according to equation (21).

***N*th generation :** We assume that the balls $B^{(N-1)}_{r_1,\ldots,r_{N-1}}$ and the measure μ_{N-1} have been constructed so that

$$\mu_{N-1}\left(B^{(N-1)}_{r_1,\ldots,r_{N-1}}\right) \leqslant 2^{-(1+\beta_n)^{p_{N-1}}\left(H-b_n\left(6+\sum_{m=2}^{N-1}2^{-m}\right)\right)} = |B^{(N-1)}_{r_1,\ldots,r_{N-1}}|^{\frac{H-b_n\left(6+\sum_{m=2}^{N-1}2^{-m}\right)}{(1+b_n)^{q_0}}}.$$

Let us consider one of these balls. It contains at least

$$2^{((1+\beta_n)^{p_N}-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-5b_n)}$$

dyadic intervals of $K_{\varepsilon_n}(b_n)$ at scale $(1+\beta_n)^{p_N}$. Again, each of these intervals corresponds to a ball $B(x_r^{(N)}, 2^{-(1+\beta_n)^{p_N}})$ where $x_r^{(N)} = k2^{(1+\beta_n)^{p_N+1}}$ belongs to $F_{\varepsilon_n}(p_N+1)$. Our Nth generation of balls in $B^{(N)}_{r_1,\ldots,r_N}$ is then given by $2^{((1+\beta_n)^{p_N}-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-6b_n)}$ disjoint balls $B(k2^{-(1+\beta_n)^{p_N+1}}, 2^{-(1+b_n)^{q_0}(1+\beta_n)^{p_N}})$ included in $B^{(N-1)}_{r_1,\ldots,r_{N-1}}$. Again, one may choose it such that at all intermediary scales $(1+b_n)^p$, exactly $2^{((1+b_n)^p-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-6b_n)}$ dyadic intervals of $\mathcal{K}_{\varepsilon_n}(b_n)$ are selected. We label this balls

$$B^{(N)}_{r_1,\ldots,r_N} \quad \text{with} \quad 1 \leqslant r_N \leqslant 2^{((1+\beta_n)^{p_N}-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-6b_n)}.$$

Next, we uniformly distribute the mass of $B^{(N-1)}_{r_1,\ldots,r_{N-1}}$ on the disjoint smaller balls, which leads us to define μ_N by

$$\mu_N\left(B^{(N)}_{r_1,\ldots,r_N}\right) = \frac{\mu_{N-1}\left(B^{N-1}_{r_1,\ldots,r_{N-1}}\right)}{2^{((1+\beta_n)^{p_N}-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-6b_n)}}$$
$$\leqslant \frac{2^{-(1+\beta_n)^{p_{N-1}}\left(H-b_n\left(6+\sum_{m=2}^{N-1}2^{-m}\right)\right)}}{2^{((1+\beta_n)^{p_N}-(1+b_n)^{q_0}(1+\beta_n)^{p_{N-1}})(H-6b_n)}}$$
$$\leqslant 2^{-(1+\beta_n)^{p_N}\left(H-b_n\left(6+\sum_{m=2}^{N}2^{-m}\right)\right)}$$
$$= |B^{(N)}_{r_1,\ldots,r_N}|^{\frac{H-b_n\left(6+\sum_{m=2}^{N}2^{-m}\right)}{(1+b_n)^{q_0}}}$$
$$= |B^{(N)}_{r_1,\ldots,r_N}|^{\frac{H-b_n\left(6+\sum_{m=2}^{N}2^{-m}\right)}{s}},$$

because of (21).

Definition of the Cantor set: We define the Cantor set $\mathcal{Z}_{\varepsilon_n}(b_n) \subset \mathcal{K}_{\varepsilon_n}(b_n)$ by

$$\mathcal{Z}_{\varepsilon_n}(b_n) = \bigcap_{N\geqslant 1}\bigcup_{(r_1,\ldots,r_N)} B^{(N)}_{r_1,\ldots,r_N}.$$

The sequence $(\mu_N)_{N\geqslant 1}$ converges weakly to a Borel probability measure μ supported on $\mathcal{Z}_{\varepsilon_n}(b_n)$, such that for every $N \geqslant 1$ and every ball B appearing in the Nth generation of the construction, we have

$$\mu(B) = \mu_N(B)$$

(see [19], proposition 1.7). In particular, the construction guarantees that for any ball B appearing in the construction of the Cantor set,

$$\mu(B) \leqslant |B|^{\frac{H-7b_n}{s}}. \tag{22}$$

Measure of a ball of $\mathcal{K}_{\varepsilon_n}(b_n)$: Assume first that D corresponds to a dyadic interval that appears in the construction of $\mathcal{K}_{\varepsilon_n}(b_n)$ that intersects $\mathcal{Z}_{\varepsilon_n}(b_n)$. Let N be such that

$$2^{-(1+\beta_n)^{p_{N+1}}} < |D| \leqslant 2^{-(1+\beta_n)^{p_N}}.$$

We consider $m \in \{0,\ldots,q\}$ and $p \in \{p_N,\ldots,p_{N+1}-1\}$ such that

$$|D| = 2^{-(1+\beta_n)^p(1+b_n)^m}.$$

We consider two cases:

- First, assume that

$$2^{-(1+\beta_n)^{p_N}(1+b_n)^{q_0}} \leqslant |D| \leqslant 2^{-(1+\beta_n)^{p_N}},$$

i.e. $p = p_N$ and $m \leqslant q_0$. Then, there exists at most one ball $B^N_{r_1,\ldots,r_N}$ of the Nth generation of $\mathcal{Z}_{\varepsilon_n}(b_n)$ such that $B^{(N)}_{r_1,\ldots,r_N} \cap D \neq \emptyset$. Hence

$$\mu(D) \leqslant \mu\left(B^{(N)}_{r_1,\ldots,r_N}\right) \leqslant |B^{(N)}_{r_1,\ldots,r_N}|^{\frac{H-7b_n}{(1+b_n)^{q_0}}} \leqslant |B^{(N)}_{r_1,\ldots,r_N}|^{\frac{H-7b_n}{s}} \leqslant |D|^{\frac{H-7b_n}{s}}.$$

- Next, assume that

$$|D| = 2^{-(1+\beta_n)^{p_N}(1+b_n)^{q_0+m}}$$

with $m \geqslant 1$ and $qp_N + q_0 + m < p_{N+1}$. Since $D \in T_\infty((1+\beta_n)^{p_N}(1+b_n)^m, b_n, \varepsilon_n)$, we know that there are at most

$$2^{\left((1+\beta_n)^{p_{N+1}}-(1+\beta_n)^{p_N}(1+b_n)^{q_0+m}\right)(H-6b_n)}$$

children of D at scale $(1+\beta_n)^{p_{N+1}}$ in D. Since each of these intervals contains at most one subinterval of the $(N+1)$th generation of the Cantor set, it follows that

$$\begin{aligned}
\mu(D) &\leqslant 2^{\left((1+\beta_n)^{p_{N+1}}-(1+\beta_n)^{p_N}(1+b_n)^{q_0+m}\right)(H-6b_n)} \frac{\mu\left(B^{(N)}\right)}{2^{((1+\beta_n)^{p_{N+1}}-(1+b_n)^{q_0}(1+\beta_n)^{p_N})(H-6b_n)}} \\
&\leqslant 2^{-(1+\beta_n)^{p_N}(1+b_n)^{q_0+m}(H-6b_n)} \frac{\mu\left(B^{(N)}\right)}{2^{-(1+b_n)^{q_0}(1+\beta_n)^{p_N}(H-6b_n)}} \\
&\leqslant 2^{-(1+b_n)^{q_0}((1+b_n)^m-1)(1+\beta_n)^{p_N}(H-6b_n)} \mu(B^{(N)}) \\
&\leqslant 2^{-((1+b_n)^m-1)(1+\beta_n)^{p_N}(H-6b_n)} 2^{-(1+\beta_n)^{p_N}(H-7b_n)} \\
&\leqslant 2^{-(1+b_n)^m(1+\beta_n)^{p_N}(H-7b_n)} \\
&\leqslant |D|^{\frac{(H-7b_n)}{(1+b_n)^{q_0}}} \\
&= |D|^{\frac{(H-7b_n)}{s}}
\end{aligned}$$

where the fourth inequality follows from (22) and from the fact that the balls of the Nth generation have radius $2^{-(1+b_n)^{q_0}(1+\beta_n)^{p_N}}$.

Measure of any ball D of radius smaller than $2^{-(1+\beta)^{p_1}}$: Clearly, if $D \cap \mathcal{Z}_{\varepsilon_n} = \emptyset$, then $\mu(D) = 0$. Hence, we can assume that $D \cap \mathcal{Z}_{\varepsilon_n} \neq \emptyset$. If $|D| = 2^{-\tau}$, we consider $N \geqslant 1$, $m \in \{0, \ldots, q\}$ and $p \in \{p_N, \ldots, p_{N+1}-1\}$ such that

$$(1+\beta_n)^p (1+b_n)^{m+1} < \tau < (1+\beta_n)^p (1+b_n)^m .$$

Such a ball intersects at most two balls B_1, B_2 of $\mathcal{K}_{\varepsilon_n}(b_n)$ of size $2^{-(1+\beta_n)^p(1+b_n)^m}$. It follows that

$$\begin{aligned}
\mu(D) &\leqslant \mu(B_1) + \mu(B_2) \\
&\leqslant 2 \cdot 2^{-(1+\beta_n)^p(1+b_n)^m \frac{H-7b_n}{s}} \\
&\leqslant 2 \cdot 2^{-\tau \frac{H-7b_n}{s(1+b_n)}} \\
&= 2 \cdot |D|^{\frac{H-7b_n}{s(1+b_n)}}.
\end{aligned}$$

Dimension of $\mathcal{Z}_{\varepsilon_n}$: We conclude that $\dim_{\mathcal{H}}(\mathcal{Z}_{\varepsilon_n}) \geqslant \frac{H-7b_n}{s(1+b_n)}$ by applying the mass distribution principle recalled in lemma 5.4. This establishes (19), and thus completes the proof. □

Proposition 5.5. *Almost surely, for every $h \in [\alpha, \frac{\alpha H}{\eta}]$, one has*

$$\dim_{\mathcal{H}}(\{x : h_f(x) \leqslant h\}) \leqslant h\frac{\eta}{\alpha}.$$

Proof. Let us fix $h \in [\alpha, \frac{\alpha H}{\eta}]$. Fix $\varepsilon > 0$ and remark that for every j large enough, one has

$$\mathbb{E}\left[\#\left\{k : c_{j,k} = 2^{-\alpha j}\right\}\right] \leqslant 2^{(\eta - H)j} 2^{j(H+\varepsilon)} = 2^{(\eta+\varepsilon)j}$$

by applying lemma 3.2. Chebyshev's inequality combined with Borel–Cantelli lemma gives directly that almost surely, for j large enough,

$$\#\left\{k : c_{j,k} = 2^{-\alpha j}\right\} \leqslant 2^{(\eta+2\varepsilon)j}.$$

Considering a sequence $(\varepsilon_n)_n$ that decreases to 0, we obtain that

$$\rho_f(\alpha) := \limsup_{j \to +\infty} \frac{\log \#\left\{k : c_{j,k} = 2^{-\alpha j}\right\}}{\log 2^j} \leqslant \eta,$$

almost surely. It has been proved in [6] that

$$\mathcal{D}_f(h) \leqslant h \sup_{h' \in (0,h]} \frac{\rho_f(h')}{h'}.$$

The right term is an increasing function of h, hence it is also an upper-bound of $\mathcal{D}_{f,\leq}(h)$.
Since $\rho_f(h') = -\infty$ if $h' \neq \alpha$, we obtain that

$$\dim_{\mathcal{H}}\left(\{x : h_f(x) \leqslant h\}\right) \leqslant h \frac{\eta}{\alpha}.$$

□

The combination of theorem 5.3 and proposition 5.5 finally yields theorem 1.3.

6. Conclusion and comments

Let us conclude by presenting an example that demonstrates the optimality of our results given the tools developed in the paper. We consider a sequence $(K_n)_{n\geqslant 1}$ of disjoint Cantor sets of Hausdorff on the real line and of dimension $1 - \frac{1}{n}$, and let us set

$$\mathcal{I} = \bigcup_{n \geqslant 1} K_n.$$

Clearly, $\mathcal{I}$ satisfies the upper-box counting formalism with

$$\dim_{\mathcal{H}}(\mathcal{I}) = \overline{\dim_B}(\mathcal{I}) = 1.$$

Now, consider the lacunary wavelet series f on $\mathcal{I}$. Since the Cantor sets are disjoint, one has

$$\{x : h_f(x) = h\} = \bigcup_{n \in \mathbb{N}} \{x \in K_n : h_{f_n}(x) = h\}$$

where

$$f_n = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j - 1} c_{j,k} \psi_{j,k} \quad \text{with} \quad \begin{cases} 2^{-\alpha j} \xi_{j,k} & \text{if } \lambda_{j,k} \cap K_n \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

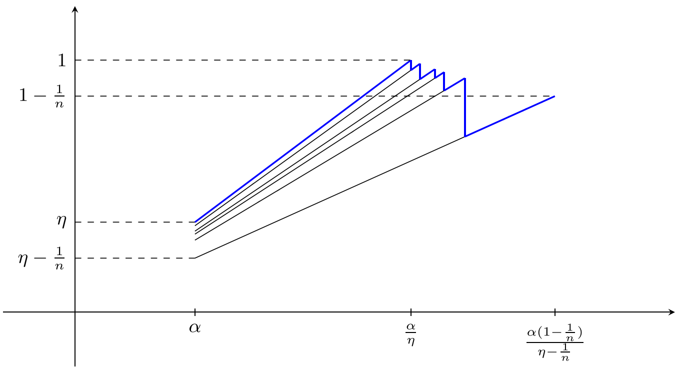

Figure 1. The regularity of a LWS on $\mathcal{I}$ can be larger than the regularity attained on a subset of the dimension of $\mathcal{I}$.

is the corresponding lacunary wavelet series on K_n. Note that $\xi_{j,k}$ is a Bernoulli random variable with parameter $2^{(\eta-1)j} = 2^{(\eta-\frac{1}{n}-\dim_{\mathcal{H}}(K_n))j}$. Hence, the parameter of lacunarity of f_n is given by $\eta-\frac{1}{n}$. From [18], we know that if $\eta-\frac{1}{n}>0$, one has almost surely

$$\dim_{\mathcal{H}}(\{x\in K_n : h_{f_n}(x)=h\}) = h\frac{\eta-\frac{1}{n}}{\alpha}$$

for all $h\in[\alpha,\frac{\alpha(1-\frac{1}{n})}{\eta-\frac{1}{n}}]$, and the iso-Hölder set is empty for all other finite values of h. Consequently,

$$\dim_{\mathcal{H}}(\{x : h_f(x)=h\}) = \sup_{n\geqslant 1} h\frac{\eta-\frac{1}{n}}{\alpha} = h\frac{\eta}{\alpha}$$

for all $h\in[\alpha,\frac{\alpha H}{\eta}]$. Let us however make two remarks.

- First, note that

$$\mathcal{H}^{h\frac{\eta}{\alpha}}(\{x : h_f(x)=h\}) = \sum_{n\in\mathbb{N}} \mathcal{H}^{h\frac{\eta}{\alpha}}(\{x : h_{f_n}(x)=h\}) = 0.$$

  Hence, in general, one cannot expect that the $h\frac{\eta}{\alpha}$-dimensional Hausdorff measure of the iso-Hölder set of level h of a LWS f is strictly positive.
- One cannot hope to obtain precise results about the regularity that exceed the regularity attained 'almost everywhere' on $\mathcal{I}$. Indeed, remark that for all n, one has

$$\frac{\alpha\left(1-\frac{1}{n}\right)}{\eta-\frac{1}{n}} > \frac{\alpha}{\eta}$$

  so that the maximal regularity on each K_n is larger than the regularity $\frac{\alpha}{\eta}$ attained by f on a set of dimension 1, see figure 1.

The strategy of constructing quasi-Cantor sets with self-similarity properties at prescribed scales inside the fractal $\mathcal{I}$ allow us to recover some MTPs for limsup of balls whose centres are chosen according to Bernoulli laws. The counterpart is that we do not obtain the preservation of the positivity of a measure as mentioned above, but only an estimation of the dimension. The usual arguments to obtain the multifractal spectrum, and not just the increasing one, are crucially based on the existence of a positive measure on the sets $E_\delta(f)$. Indeed, for a LWS f of parameter (α,η), and every $\delta \in (0,1)$, we have

$$\left\{x \in \mathcal{I} : h_f(x) = \frac{\alpha}{\delta}\right\} = \mathcal{I} \cap \left(\bigcap_{0<\delta'<\delta} E_{\delta'}(f) \setminus \bigcup_{\delta<\delta'\leqslant 1} E_{\delta'}(f) \right).$$

The example presented above tends also to show that various different behaviours can be expected for the regularities greater than the first regularity h_0 for which the dimension of the fractal set is reached. We do not know whether the multifractal spectrum of a LWS supported on a fractal set $\mathcal{I}$ might be discontinuous even for regularities smaller than h_0. It would be interesting to exhibit LWS on fractal sets $\mathcal{I}$ for which $\dim_{\mathcal{H}}(\mathcal{I})$ is not reached—maybe by considering LWS on the attractor of an IFS which do not satisfy the OSC. The difficulty to obtain the multifractal spectrum in situations where the increasing spectrum is known also occurs in the context of multivariate multifractal analysis. This is the case for the bivariate analysis of two independent LWS on $[0,1]$ (see [25]). Note also that a relaxed version of the singularity spectrum which is more accurate than the increasing one has been defined the following way:

$$\widetilde{D}_f(h) = \lim_{\varepsilon\to 0^+} \dim_{\mathcal{H}}\left(\{x : h-\varepsilon \leqslant h_f(x) \leqslant h+\varepsilon\}\right)$$

which is easier to obtain since it is enough to compute the exact dimensions of the sets $E_\delta(f)$ and for which the validity of the large deviation methods are more robust.

We also expect that our strategy of first constructing quasi-Cantor sets before estimating the dimension of the limsup of balls could be developed in several directions. In particular, in future work, we plan to

- Consider other types of random processes on fractal sets, such as random wavelet series or lacunarized cascades;
- Investigate the optimal conditions on the fractal set $\mathcal{I}$ under which we can construct quasi-Cantor sets inside. In particular, we plan to explore the generalization of this method to iso-Hölder sets, for which the upper-box counting dimension is 1 as soon as there is some homogeneity. This is the case for random cascades or LWS. Such developments would be particularly useful for developing criteria to test the validity of the formalism (see [18]) and for multivariate analysis of functions.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Proofs of the results of section 3

In this appendix, for pedagogical purposes, we recall the proof of the two results of [18] recalled in section 3 on which the construction presented in section 4 is based.

Proof of lemma 3.2. Because the upper-box dimension of $\mathcal{I}$ is equal to H, we have $\log_2 \#\mathcal{I}_j \leqslant H+\varepsilon$, i.e. $\#\mathcal{I}_j \leqslant 2^{j(H+\varepsilon)}$ for j large enough. The lower bound is proved by contradiction. Assume that there is a sequence $(j_n)_n$ such that $\#\mathcal{I}_{j_n} \leqslant 2^{j_n(H-\varepsilon)}$. Since $\mathcal{I} \subset \widetilde{\mathcal{I}_{j_n}}$, it provides a covering of $\mathcal{I}$ by less than $2^{j_n(H-\varepsilon)}$ intervals of size 2^{-j_n}. It follows that $\dim_H(\mathcal{I}) \leqslant H-\varepsilon$, which gives the contradiction. □

Note that lemma 3.2 holds in fact under the weaker assumption that $\underline{\dim}_B(\mathcal{I}) = \overline{\dim}_B(\mathcal{I})$.

Proof of proposition 3.4. Since $\mathcal{I}_j = \mathrm{ED}(j,b,\varepsilon) \cup \mathrm{SD}(j,b,\varepsilon) \cup \mathrm{FD}(j,b,\varepsilon)$, there exists $J \in \mathbb{N}$ such that, for all $j_n \geqslant J$,

$$\#\left(\mathrm{SD}\left(j_n,b,\varepsilon\right) \cup \mathrm{FD}\left(j_n,b,\varepsilon\right)\right) \leqslant 2^{j_n(H+\varepsilon)}.$$

Moreover, since $\#\mathcal{I}_{\lfloor(1+b)j\rfloor}$ is bounded by $2^{(1+b)(H+\varepsilon)j}$, we deduce that

$$\#\mathrm{FD}\left(j,b,\varepsilon\right) \times 2^{j(bH+4\varepsilon)} \leqslant \#\mathcal{I}_{\lfloor(1+b)j\rfloor} \leqslant 2^{j(1+b)(H+\varepsilon)}.$$

This yields (10).

Since $\#\mathrm{SD}(j,b,\varepsilon)$ is bounded above by $2^{j(H+\varepsilon)}$ and since the dyadic intervals of $\mathrm{SD}(j,b,\varepsilon)$ present a slow duplication, we can control the cardinality of $C_b\mathrm{SD}(j,b,\varepsilon)$ by

$$\#C_b\mathrm{SD}\left(j,b,\varepsilon\right) \leqslant 2^{j(H+\varepsilon)}2^{j(Hb-4\varepsilon)} \leqslant 2^{j((1+b)H-3\varepsilon)}$$

which is (11).

We now turn to the set $\mathrm{ED}(j,b,\varepsilon)$. Its upper bound is trivial. Suppose now that there is a sequence $(j_n)_n$ such that, for all n,

$$\#\mathrm{ED}\left(j_n,b,\varepsilon\right) \leqslant 2^{j_n(H-2\varepsilon)}. \tag{23}$$

For each $n \geqslant 0$, we have the following covering of $\mathcal{I}$ with sets of diameter smaller than 2^{-j_n}:

$$\mathcal{I} \subset \{\lambda \in \mathrm{ED}\left(j_n,b,\varepsilon\right)\} \cup \{\lambda \in \mathrm{FD}\left(j_n,b,\varepsilon\right)\} \cup \{\lambda' \in C_\beta\mathrm{SD}\left(j_n,b,\varepsilon\right)\}.$$

Combining (23), (10) and (11), it implies that, for any $r>0$ and $0<s<1$,

$$\mathcal{H}^s_r\left(\mathcal{I}\right) \leqslant 2 \times 2^{j_n(H-2\varepsilon)}2^{-j_ns} + 2^{j_n((1+b)H-3m\varepsilon)}2^{-(1+b)j_ns}$$

for any j_n such that $2^{-j_n} \leqslant r$. Taking $s = H-\varepsilon$, it comes

$$\mathcal{H}^s_r\left(\mathcal{I}\right) \leqslant 2 \times 2^{-\varepsilon j_n} + 2^{-j_nm\varepsilon}.$$

It follows that $\lim_{r\to 0^+}\mathcal{H}^s_r(\mathcal{I}) = 0$ and therefore $\dim_H(\mathcal{I}) \leqslant H-\varepsilon$, which is impossible. Hence, there exists some $J \in \mathbb{N}$ such that for all $j \geqslant J$, $2^{j(H-2\varepsilon)} \leqslant \#\mathrm{ED}(j,b,\varepsilon) \leqslant 2^{j(H+\varepsilon)}$. □

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